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Dédicace

Je dédie ce travail:

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List of Symbols and Abbreviations

The different symbols and abbreviations used in this thesis.

$a.e$	:	almost evrywhere.
$a.s$	:	almost surely.
$c\grave{a}dl\grave{a}g$	:	right continuous with left limits.
$e.g$	:	for example.
$\mathbb{R}$	:	real numbers.
$\mathbb{R}^n$	:	n-dimensional real Euclidean space.
$\mathbb{R}^{n\times d}$	:	the set of all $(n \times d)$ real matrixes.
$\bar{A}$	:	the closure of the set A.
1_A	:	the indicator function of the set A.
$\sigma(A)$	:	σ-algebra generated by A.
$(\Omega, \mathcal{F})$	:	measurable space..
$(\Omega, \mathcal{F}, \mathbb{P})$	:	probability space.
$\{\mathcal{F}_t\}_{t\in[0,T]}$	:	filtration.
$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$	:	filtered probability space.
$\mathcal{N}$	:	the totality of the $\mathbb{P}$-negligible sets.
$\mathcal{G}_0$	:	the totality of the $\mathbb{P}$-null sets.
$\mathcal{G}_1 \vee \mathcal{G}_2$	:	the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$.
$\mathbb{E}(x)$	:	expectation at x.
$\mathbb{E}(\cdot \mid \mathcal{G})$	:	conditional expectation.

$W_{\cdot} = (W_t)_{t\in[0,T]}$	:	Brownian motion.
$L_{\cdot} = (L_t)_{t\in[0,T]}$	:	$\mathbb{R}$-valued Lévy process.
$\left(H_t^{(i)}\right)_{i=1}^{\infty}$	:	Teugels martingale.
$\mathbb{P}\otimes dt$	:	the product measure of $\mathbb{P}$ with the Lebesgue measure dt.
$(a,\ b)$	:	the inner product in $\mathbb{R}^n$.
$\lvert a\rvert = \sqrt{(a,\ b)}$	:	the norm of $\mathbb{R}^n$.
$(A,\ B)$	:	the inner product in $\mathbb{R}^{n\times d}$.
$\lvert A\rvert = \sqrt{(A,\ B)}$	:	the norm of $\mathbb{R}^{n\times d}$.
l^2	:	the Hilbert space of real-valued sequences $x = (x_n)_{n\geq 0}$ with the norme $\lVert x\rVert = \left(\sum_{i=1}^{\infty} x_i\right)^{\frac{1}{2}} < \infty$.
$\mathcal{P}^2(\mathbb{R}^n)$	:	the space of $\mathbb{R}^n$-valued processes $\{f^i\}_{i\geq 0}$ such that $\left(\sum_{i=1}^{\infty} \lVert f^i\rVert_{\mathbb{R}^n}^2\right)^{\frac{1}{2}} < \infty$.
$l_{\mathcal{F}}^2(0,T,\mathbb{R}^n)$	:	the Banach space of $\mathcal{P}^2(\mathbb{R}^n)$-valued $\mathcal{F}_t$-predictable processes $\{f^i\}_{i\geq 0}$ such that $\left(\mathbb{E}\int_0^T \sum_{i=1}^{\infty} \lVert f^i(t)\rVert_{(\mathbb{R}^n)}^2\, dt\right)^{\frac{1}{2}} < \infty$.
$\mathcal{S}_{\mathcal{F}}^2(0,T,\mathbb{R}^n)$	:	the Banach space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted and càdlàg processes $(f_t)_{t\in[0,T]}$ such that $\left(\mathbb{E}\sup_{0\leq t\leq T} \lvert f_t\rvert^2\right)^{\frac{1}{2}} < \infty$.
$\mathbb{L}_{\mathcal{F}}^2(0,T,\mathbb{R}^n)$	:	the Banach space of $\mathbb{R}^n$-valued, square integrable and $\mathcal{F}_t$-progressively measurable processes $(f_t)_{t\in[0,T]}$ such that $\mathbb{E}\left[\int_0^T \lvert f_t\rvert^2\, dt\right] < \infty$.
$L^2(\Omega,\mathcal{F},\mathbb{P},\mathbb{R}^n)$	:	the Banach space of $\mathbb{R}^n$-valued, square integrable random variables on $(\Omega,\mathcal{F},\mathbb{P})$.
τ	:	stopping time.
$l_{\mathcal{F}}^2(0,\tau,\mathbb{R}^n)$	:	the Banach space of $l^2(\mathbb{R}^n)$-valued $\mathcal{F}_t$-predictable processes $(f_t)_{t\geq 0}$ such that $\left(\mathbb{E}\int_0^{\tau} \sum_{i=1}^{\infty} \lVert f_t^i\rVert_{\mathcal{P}^2(\mathbb{R}^n)}^2\, dt\right)^{\frac{1}{2}} < \infty$.
$\mathcal{L}_{\mathcal{F}}^2(0,\tau,\mathbb{R}^n)$	:	the Banach space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $(f_t)_{t\geq 0}$ such that $\left(\mathbb{E}\int_0^{\tau} \lVert f_t\rVert_{\mathbb{R}^n}^2\, dt\right)^{\frac{1}{2}} < \infty$.

$\mathcal{S}^2_{\mathcal{F}}(0, \tau, \mathbb{R}^n)$	:	the Banach space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted and càdlàg processes $(f_t)_{t\geq 0}$ such that $\left(\mathbb{E} \sup_{0\leq t\leq \tau} \|f_t\|^2\right)^{\frac{1}{2}} < \infty$.
$SDEs$	:	Stochastic differential equations.
$BSDEs$	:	Backward stochastic differential equations.
$FBSDEs$	:	Forward-backward stochastic differential equations.

Résumé

Cette thèse contient deux thèmes. Le premier porte sur le problème de l'existence et l'unicité des solutions pour certain type d'équations différentielles stochastiques progressives rétrogrades fortement couplées dirigées par une famille de martingales de Teugels associées à certains processus de Lévy. Le second est consacré au contrôle stochastique optimal pour des systèmes gouvernés par des équations différentielles stochastiques (EDS en abréger). Dans la première partie qui contient deux documents, nous donnons et prouvons certains résultats d'existence et d'unicité dans deux cas différents: (i) Le temps final est une donnée fixée et grande, (ii) le temps final est supposé aléatoire .

La deuxième partie de cette thèse concerne les problèmes de contrôle stochastique pour optimiser un problème d'entreprise d'assurance dans le cas où son processus de richesse est supposé dirigé par une équation différentielle stochastique gouvernée par une famille de martingales de Teugels. Nous traitons plusieurs cas selon le processus de taux d'intérêt, nous supposons dans un premier temps que la société d'assurance investit uniquement dans un compte monétaire avec un taux d'intérêt composé. Ensuite, nous discutons ce problème de prime optimal, dans le cas où le taux d'intérêt est autorisé à être stochastique. Plus précisément, nous considérons le cas dans lequel la fonction de paiement et le taux d'intérêt stochastique sont donnés par le même mouvement brownien, puis nous considérons le cas où ils sont donnés par des mouvements browniens différents et indépendants.

Mots clés: Equation differentielle stochastique progressive rétrograde; Martingales de Teugels; Processus de Lévy; Temps d'arrêt; Politique de prime optimale.

Abstract

This thesis contains two themes. The first topic considers the problem of the well-posedness for a kind of fully coupled forward backward stochastic differential equations driven by Teugels martingales associated with some Lévy processes. The second one is devoted to the stochastic optimal control for systems driven by stochastic differential equations (SDE for short). In the first part which contains two papers, we provide and prove some existence and uniqueness results in two different cases: (i) The final time is assumed to be fixed and large; (ii) the final time is allowed to be random.

The Second part of this thesis is concerned with the stochastic control problems to optimize an insurance firm problem in the case where its cash-balance process is assumed to be governed by a stochastic differential equation driven by Teugels martingales. We deal with several cases according to the interest rate process; we first suppose that the insurance firm only invests in a money account with compounded interest rate. Then we discuss this optimal premium problem, in the case where the interest rate is allowed to be stochastic. More precisely, we consider the case in which the payment function and the stochastic interest rate are given by the same Brownian motion, in addition to the case where we assume that they are given by different and independent Brownian motions.

Keywords: Forward-backward stochastic differential equations; Teugels Martingale; Lévy process; Stopping time; Optimal premium policy.

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General introduction

This thesis consists of two research topics that can be read independently in one hand and related in the other hand. The first part consists in solving fully coupled forward backward stochastic differential equations (FBSDE for short) driven by Lévy process having moments of all orders in addition to some applications to problems of optimal stochastic control. The second part deals with the resolution of a stochastic optimal control problem, under a type of constraint to optimize an insurance firm problem in the case where its cash-balance process is assumed to be governed by such type of equations. To make the link between the two parts, let us mention that we can translate the control problem into existence and uniqueness result for coupled FBSDE. This done by coupling the controlled state equation with its adjoint equation. Consequently, this technique allow us to prove the existence and uniqueness of an optimal control under suitable conditions.

First of all, let us focus on a backward stochastic differential equations theory, which makes the foundation of FBSDEs theory. It is well known that the first work to prove the existence and uniqueness solution of a nonlinear BSDE under Lipschitz case, began with work of Pardoux and Peng [38]. This work was the starting point for the development of studing this type of equations. Since the purpose of this thesis is to study FBSDEs with jumps rather then thy driven by contionuous Brownian motion, let us recall some very important result in this direction. So, for BSDE driven by the non-continuous martingales, Tang and Li [47] have been discussed the existence and uniqueness theorem for the solution to

BSDE driven by Poisson point process. In the same paper they also proved the maximum principle for optimal control of stochastic systems with random jumps. Later, Situ [46] have studied BSDEs driven by a Brownian motion and Poisson point process. As in [47] and [46], BSDE driven by Poisson random measure have been carried out by Ouknine [37]. It is worth mentioning that the proof of the existence and uniqueness results for FSDE (resp. BSDE) is based on the fixed point theorem. In fact, this is often done using a change of norm which consists in multiplying the solution by $\exp(\alpha t)$, where $|\alpha|$ is assumed to be large enough, with $\alpha < 0$ (resp. $\alpha > 0$). Since then, the theory of BSDEs had found many applications, for example, in the stochastic control, mathematical finance and partial differential equation.

The theory of fully coupled FBSDEs with continuous Brownian motion develops also very dynamically in the last two decades. Comparing with the existing results in the literature for semi-coupled FBSDEs systems, one can say that the main difficulty to deal with the general fully coupled FBSDE appears from the coupling between the forward and the backward equations. This leads to a roundabout reliance between the state variables of both of equations. In general, there are several methods to get around this difficulty. The first one is the method of *contraction mapping.* This method is based on the fixed point theorem, where the final time is assumed to be small enough. Antonelli in its earlier paper [1], have been used it in order to prove an existence and uniqueness result. Then it has been detailed by Pardoux and Tang [39], Delarue [12] and Zhang [53].

The second one is the method of *optimal control.* This method gives the solvability for some classes of FBSDEs in any finite time duration so that the solvability problem is converted to a problem of finding the nodal set of the viscosity solution to a certain Hamilton-Jacobi-Bellman equations. Notice that Ma and Young [29] are the first who are used this method for FBSDE.

The third method is *the Four-Step-Scheme* initiated by Ma, Protter and Young [28]. Such method requires the coefficients to be deterministic and the diffusion coefficient of the forward equation to be nondegenerate. Duffe et all [14] have also used this approach to deal with the case of FBSDEs in infinite time duration. To be more precise, this approach, takes the advantage of some solvable parabolic partial differential equations systems to get construct solution of FBSDEs in large time duration.

The fourth method is inspired by the numerical approaches, we refer the reader to Delarue and Menozzi [13]. In that paper, the authers proposed a time-space discretization scheme for quasi-linear parabolic PDEs. They gave an algorithm that relies on the theory of fully coupled FBSDE, and provides an efficient probabilistic representation for this type of equations. We point out that the derivated algorithm holds for strong solutions defined on any interval of arbitrary length.

Another method is called *the method of continuation*. It has been extensively used in proving the existence of solution to FBSDE for any arbitrary large time duration. This method was initiated by Hu and Peng [23] and Peng and Wu [40] and later was developed by Young [52]. In 1995, Hu and Peng [23], first established an existence and uniqueness result under some monotonicity conditions on the coefficients by considering the case where the forward and backward components have the same dimension. Then, in 1998, Hamadène [20] improved their result by proving it under weaker monotonicity assumptions. Later, in 1999, Peng and Wu [40], provided more general results by extending the two above results, without the restriction on the dimensions of the forward and backward parts. An interesting account with many applications about the four previous methods, is given in the useful book of Ma and Young [30].

Recently, motivated by studying numerical methods for FBSDEs, Jianfeng Zhang [53] have investigated a wellposedness of a class of FBSDEs by imposing some assumptions on the derivatives of the coefficients. She also provided a comparison theorem under the same conditions.

To the best of our knowledge, the first paper deals with the existence and uniqueness problems of fully coupled FBSDEs driven by Teugels martingales, is due to Pereira and

Shamarova [41], where they solve FBSDEs via the solution to the associated PIDEs and prove its uniqueness. As a second result in this direction, Baghery et al. [3], show an existence and uniqueness result for this type of equations, using the so called monotonicity conditions on the coefficients. Throughout this thesis, we extend known results result by Jianfeng Zhang [53], proved for FBSDEs driven by a Brownian motion, to FBSDEs driven by general Lévy processes. We also extend those of Peng and Wu [40], to the jump case in stopping time duration.

This thesis is organized as follows:

In the first chapter, we give some basic proprieties of Lévy processes and Teugels martingales. After this, we show some existing results lie to theory of backward stochastic differential equation driven by Teugels martingales.

In chapter 2, Under some assumptions on the derivatives of the coefficients, we prove the existence and uniqueness of a global solution on an arbitrarily large time interval. After that, we establish stability and comparison theorems for the solutions of such equations.

In chapter 3, we first study a class of backward stochastic differential equation driven by Teugels martingales associated with some Lévy processes having moment of all orders and an independent Brownian motion. We obtain an existence and uniqueness results for this type of BSDEs when the final time is allowed to be random. Then, we prove, under some monotonicity conditions, an existence and uniqueness result for possibly degenerate fully coupled forward-backward stochastic differential equation driven by Teugels martingales in stopping time duration. At the end of this chapter, as an illustration of our previous theoretical results, we deal with a portfolio selection problem in Lévy-type market.

The last chapter in this thesis is devoted to a problem of optimal stochastic control which consists in finding the best strategies of an insurance firm in the case where its cash-balance process is described by a stochastic differential equation with jumps. We treat several cases according to the definition of the interest rate process.

To wind up this introduction, let us recall that the content of this thesis is the subject of the following papers:

1) Guerdouh, D., & Khelfallah, N. (2017). Forward–Backward SDEs Driven by Lévy Process in Stopping Time Duration. Communications in Mathematics and Statistics, 5(2), 141-157.

2) Guerdouh, D., Khelfallah, N., & Mezerdi, B. (2017). On the solvability of forward-backward stochastic differential equations driven by Teugels Martingales. arXiv preprint arXiv:1701.08396.(submit)

3) Guerdouh, D., Khelfallah, N., & Vives, J.(2017). Optimal Control Strategies for Premium Policy of an Insurance Firm with Jump Diffusion Assets and Stochastic Interest Rate. Submitted for publication in Insurance Mathematics and Economics.

Chapter 1

Background on Lévy processes and Teugels martingales

The term Lévy process honours the work of the French mathematician Paul Lévy, this process play a crucial role in several fields of science, such as a mathematical finance. Recall that in many situations we need to show that there is a sudden crashes in finance for instance. It is then quite natural to permit random jumps in the models thus they can describe the observed reality of financial markets in a more accurate.

This chapter consists of four sections. In section 1, we introduce Lévy processes and discuss some of their general properties. In section 2, we give Itô's formula. Section 3 presents power jump process and Teugels martingales. Finally, in Section 4 we show the existence and uniqueness of a solution to backward stochastic differential equation driven by such processes. For more details on Lévy processes, we refer the reader to the books of Bertoin [5], Sato [43], Applebaum [2] and Kyprianou [27]. See also Schoutens [45] and Cont and Tankov [9] for some applications concern Lévy processes. The book of Protter [42] are essential readings for semimartingale theory.

1.1 Lévy processes

1.1.1 Definitions

We first define a complete probability space, $(\Omega, \mathcal{F}, \mathbb{P})$. On this space we define the Lévy processes.

Definition 1.1 (***Lévy process***) *A process $L = (L_t)_{t\geq 0}$ on $\mathbb{R}$ is called a Lévy process if the following conditions are satisfied:*

(1) $\mathbb{P}(L_0 = 0) = 1$;

(2) *L has independent increments, that is, for all $t > 0$ and $s > 0$, the increment $L_{t+s} - L_t$ is independent of L_s and $s \leq t$;*

(3) *L has stationary increments, that is, for all $t > 0$ and $s > 0$, the increment $L_{t+s} - L_t$ has the same distribution as L_s;*

(4) *L_t is continuous in probability, that is, $\lim_{t\to s} L_t = L_s$, where the limit is taken in probability.*

A Lévy processes on $\mathbb{R}^d$ is called a *d-dimensional Lévy process.*

A stochastic process L satisfying (1), (2) and (4) is called an *additive process.*

Remark 1.1 (*i*) *According to the properties of stationary and independent increments, that the Lévy process is a Markov process.*

(*ii*) *Thanks to almost sure right continuity of paths, one may show in addition that Lévy processes are also strong Markov processes.*

Example 1.1 (*i*) *The linear drift of the form $dt = at$, $a \in \mathbb{R}$ is the simplest Lévy process, a deterministic process.*

(*ii*) *The standard Brownian motion and the Poisson process are fundamental example of Lévy process, where the Brownian motion is the only Lévy process with continuous sample paths.*

(iii) The sum of a linear drift, a Brownian motion and a compound Poisson process is again a Lévy process; it is often called a jump-diffusion process. We shall call it a Lévy jump-diffusion process.

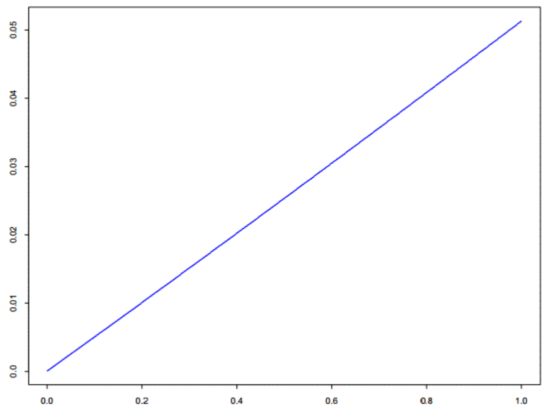

Figure 1.1: Examples of Lévy processes: linear drift.

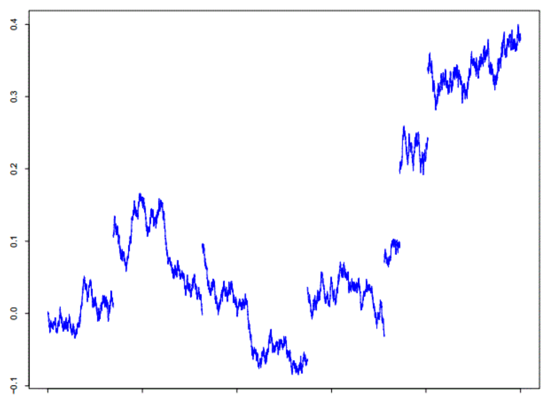

Figure 1.2: Examples of Lévy processes: Lévy jump-diffusion.

Remark 1.2 *There exist jump-diffusion processes which are not Lévy processes.*

We give the definitions of Brownian motion and the Poisson process in the following.

Definition 1.2 (***Brownian motion***) *A stochastic process $W = (W_t)_{t\geq 0}$ on $\mathbb{R}$ is a Brownian motion if it is a Lévy process and if*

(1) *For all $t > 0$, has a Gaussian distribution with mean 0 and covariance t.*

(2) *has continuous sample paths.*

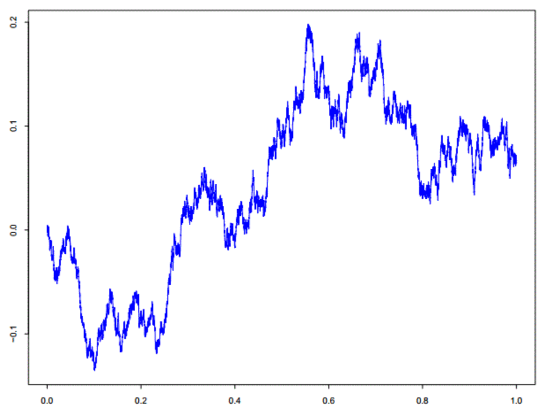

Figure 1.3: Examples of Lévy processes: Brownian motion.

Definition 1.3 (***Poisson process***) *A stochastic process* $N = (N_t)_{t \geq 0}$ *on* $\mathbb{R}$ *such that*

$$P[N_t = n] = \frac{(\beta t)^n}{n!} \exp(-\beta t); \; n = 0, 1...$$

is a Poisson process with parameter $\beta > 0$ *if it is a Lévy process and for* $t > 0$, N_t *has a Poisson distribution with mean* βt.

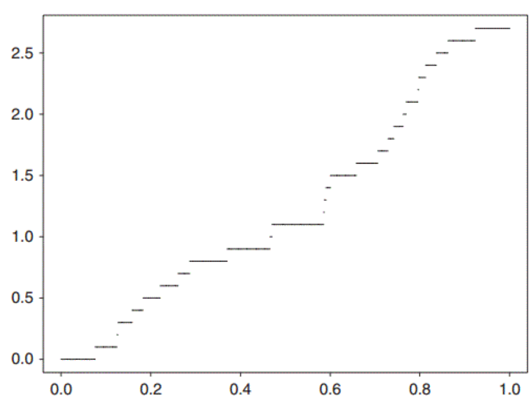

Figure 1.4: Examples of Lévy processes: Poisson process.

1.1.2 Jumps processes and Lévy measure

Let L be a Lévy process and denote by

$$L_{t_-} = \lim_{s\to t, s<t} L_s, \quad t > 0,$$

the left limit process.

The jump process $\Delta L = (\Delta L_t)_{0\leq t\leq T}$ associated to the Lévy process L is defined, for each $0 \leq t \leq T$,

$$\Delta L_t = L_t - L_{t_-}.$$

The condition of stochastic continuity of a Lévy process yields immediately that for any Lévy process L and any fixed $t > 0$, then

$$\Delta L_t = 0, ...a.s,$$

so, a Lévy process has *no fixed times of discontinuity.*

If $\sup_{0\leq t\leq T} |\Delta L_t| \leq c$, where c is a constant, then we say that L_t has *bounded jumps.*

In general, the sum of the jumps of a Lévy process does not converge, in other words, it is possible that

$$\sum_{s\leq t} |\Delta L_s| = \infty \quad a.s.,$$

but we always have that

$$\sum_{s\leq t} |\Delta L_s|^2 < \infty \quad a.s.,$$

which allows us to handle Lévy processes by martingale techniques.

The random measure of jumps of a Lévy process is a convenient tool for analyzing the jumps of this process.

Put $\mathbb{R}_0 = \mathbb{R}\diagdown\{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ be a σ-algebra generated by the family of all Borel subsets $\Gamma \subset \mathbb{R}_0$, such that $\bar{\Gamma} \subset \mathbb{R}_0$. If $A \in \mathcal{B}(\mathbb{R}_0)$ with $\bar{\Gamma} \subset \mathbb{R}_0$ and $t > 0$, we define the random measure of the jumps of the process L by

$$N_t^{\Gamma}(w) = \sum_{s \leq t} 1_{\Gamma}(\Delta L_s(w)),$$

hence, the measure $N_t^{\Gamma}(w)$ *counts the jumps of the process L of size in Γ up to time t.*

We will show that N verify the following properties:

(i) property of independent increment*:*

$$N_t^{\Gamma} - N_s^{\Gamma} \in \sigma(\{L_u - L_v; \quad s \leq v < u \leq t\}),$$

therefore $N_t^{\Gamma} - N_s^{\Gamma}$ is independent of $\mathcal{F}_s$, i.e. $N_{\cdot}^{A}$ has *independent increments.*

(ii) property of stationary increments:

$N_t^{\Gamma} - N_s^{\Gamma}$ is defined as the number of jumps of $L_{s+u} - L_s$ in Γ for $0 \leq u \leq t-s$, hence, by the stationarity of the increment of L , we get $N_t^{\Gamma} - N_s^{\Gamma}$ has the same distribution as N_{t-s}^{Γ}, i.e. $N_{\cdot}^{\Gamma}$ has stationary increments.

Then, by (i) and (ii) we conclude $N_{\cdot}^{\Gamma}$ is a *Poisson process and N is a Poisson random measure.* The *intensity* of this Poisson process is $\nu(\Gamma) = \mathbb{E}\left(N_1^{\Gamma}(w)\right)$.

Next, we give the definitions of Poisson random measure, compensated Poisson random measure and Lévy measure.

Definition 1.4 (***Poisson random measure***) *$N_t : \Gamma \to N_t^{\Gamma}$ is called the Poisson random measure of the Lévy process $(L_t)_{t \geq 0}$.*

Definition 1.5 (***Compensated Poisson random measure***) *We call the random measure $\tilde{N}_t^{\cdot}(w) = N_t^{\cdot}(w) - E\left(N_t^{\cdot}(w)\right) = N_t^{\cdot}(w) - t\nu(\cdot)$ the compensated Poisson random measure of the Lévy process $(L_t)_{t \geq 0}$.*

Definition 1.6 (***Lévy measure***) *The measure* $\nu(\Gamma)$ *defined by*

$$\nu(\Gamma) = \mathbb{E}\left(N_1^{\Gamma}(w)\right) = \mathbb{E}\left[\sum_{s\leq 1} 1_{\Gamma}(\Delta L_s(w))\right],$$

is called the Lévy measure of L, $\nu(\Gamma)$ *is the expected number, per unit time, of jumps whose size belongs to* Γ.

The Lévy measure is a measure on $\mathbb{R}$ that satisfy

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}_0} \min(1, z^2)\, \nu(dz) < \infty. \tag{1.1}$$

Proposition 1.1 *Let* L *be a Lévy process, then*
(i) If $\nu(\mathbb{R}) < \infty$, *then almost all paths of* L *have a finite number of jumps on every compact interval. In that case, the Lévy process has finite activity.*
(ii) If $\nu(\mathbb{R}) = \infty$, *then almost all paths of* L *have an infinite number of jumps on every compact interval. In that case, the Lévy process has infinite activity.*

Proof. See Theorem 21.3 in Sato [43]. ■

To gain the full understanding of how the Lévy measure works, the following relation is useful.

Definition 1.7 *For* $f : \mathbb{R}^d \to \mathbb{R}^d$, *bounded, vanishing in neighborhood of* 0, *that*

$$\mathbb{E}\left\{\sum_{0<s\leq t} f(\Delta L_s)\right\} = t\int_{-\infty}^{+\infty} f(z)\, \nu(dz). \tag{1.2}$$

In the case where $f(z) = 1_S(z)$, where S is some set in $\mathbb{R}^d$ the relation tells us that the expected sum of jump $\Delta L_s \in S$ is the integral over S with the Lévy measure.

1.1.3 Proprieties of Lévy processes

The most important property of the Lévy processes is that any Lévy process can be represented by a triplet consisting of a matrix, a vector and a measure.

Let L be a real valued Lévy process, denotes its characteristic function by ϕ_L and its law by $\mathbb{P}_L$, hence

$$\phi_L(u) = \int_{\mathbb{R}} \exp(iuz)\,\mathbb{P}_L(dz).$$

Let $0 < t \leq T$, for any $n \in \mathbb{N}$ and any $0 = t_0 < t_0 < ... < t_n = t$ is a partition of $[0, t]$ with $t_i - t_{i-1} = \dfrac{1}{n}$ we trivially have that

$$L_t = \left(L_t - L_{t_{n-1}}\right) + \left(L_{t_{n-1}} - L_{t_{n-2}}\right) + ... + \left(L_{t_1} - L_{t_0}\right).$$

The stationarity and independent of the increments yield that $\left(L_{t_i} - L_{t_{i-1}}\right)$ is an i.i.d sequence of random variables, hence we can conclude that the law $\mathbb{P}_L$ of L is infinitely divisible with characteristic function of the form

$$\mathbb{E}\left[\exp(iuL_t)\right] = (\phi(u))^t,$$

where we have denoted by $\phi(u)$, the characteristic function of L_1. If we set $\psi(u) = \log(\phi(u))$, the characteristic exponent function $\psi(u)$ satisfies the following Lévy-khintchine formula.

Theorem 1.1 (***The Lévy-khintchine formula***) *Let $(L_t)_{t \geq 0}$ be a Lévy process on $\mathbb{R}^d$ with characteristic triplet (A, ν, α) (or Lévy triplet). Then*

$$\mathbb{E}\left[e^{izL_t}\right] = e^{t\psi(z)}, \quad z \in \mathbb{R}^d \ \left(i = \sqrt{-1}\right), \tag{1.3}$$

with the characteristic exponent

$$\psi(z) := i\alpha z - \frac{1}{2} z.Az + \int_{\mathbb{R}^d} \left(e^{izx} - 1 - izx1_{|x|\leq 1}\right) \nu(dx), \tag{1.4}$$

where the parameters $\alpha \in \mathbb{R}^d$, A *is a symmetric nonnegative-definite* $d \times d$ *matrix and* ν *is a* σ*-finite measure on* $\mathcal{B}(\mathbb{R}_0)$ *satisfying* (1.1).

Proof. See Cont and Tankov [9] (Section 3.4). ■

For real-valued Lévy process, the formula (1.3) takes the form

$$E\left[e^{izL_t}\right] = e^{t\psi(z)}, \quad z \in \mathbb{R} \ \left(i = \sqrt{-1}\right),$$

with

$$\psi(z) := i\alpha z - \frac{1}{2}\sigma_0^2 z^2 + \int_{-\infty}^{+\infty} \left(e^{izx} - 1 - izx1_{|x|\leq 1}\right) \nu(dx),$$

where $\alpha \in \mathbb{R}$, is called drift term $\sigma_0^2 \geq 0$, the Gaussian or diffusion coefficient and ν satisfies (1.1). In this case the Lévy triplet is given by $(\sigma_0^2, \nu, \alpha)$.

Example 1.2 (i)- ***Brownian motion with drift:*** L_t *is given by*

$$L_t = L_0 + \sigma_0 W_t + \beta t,$$

where W_t *is standard Brownian motion and* $\sigma_0, \beta \in \mathbb{R}$*. The Lévy triplet given as* $(\sigma_0, 0, \beta)$*, that is* $\nu(S) = 0$ *for all Borel sets* $S \subset \mathbb{R}$.

(ii)- ***Poisson process:*** *is the most pure jump Lévy process with jumps of fixed size* 1 *and intensity* λ*. The Lévy triplet for this process is given as* $(0, \lambda\delta_1(dx), 0)$*.* δ_1 *denotes the Dirac* δ*-measure on* 1.

Whether a Lévy process has finite variation or has not also depends on the Lévy measure (and on the presence or absence of a Brownian part).

Proposition 1.2 (***Finite variation Lévy processes***) *A Lévy process is of finite variation if and only if its characteristic triplet* (A, ν, α) *satisfies:*

$$A = 0 \text{ and } \int_{|z| \leq 1} |z| \, \nu(dz) < \infty.$$

Proof. See Proposition 3.9 in Cont and Tankov [9]. ■

The finiteness of the moments of a Lévy process is related to the finiteness of an integral over the Lévy measure.

Proposition 1.3 *Let L be a Lévy process with triplet $(\sigma_0^2, \nu, \alpha)$. Then*

(i) L_t has finite p-th moment for $p \in \mathbb{R}^+$ $(\mathbb{E}|L_t|^p < \infty)$ if and only if $\int_{|z| \geq 1} |z|^p \, \nu(dz) < \infty.$

(ii) L_t has finite p-th exponential moment for $p \in \mathbb{R}^+$ $(\mathbb{E}[\exp(pL_t)] < \infty)$ if and only if $\int_{|z| \geq 1} [\exp(pL_t)] \, \nu(dz) < \infty.$

Proof. See Theorem 25.3 in Sato [43]. ■

Next, we give another important decomposition formula for Lévy processes. This decomposition expresses sample functions of a Lévy process as a sum of deterministic component, Brownian motion and integrals with respect to non-compensated and compensated Poisson random measures.

Theorem 1.2 (***The Lévy Itô decomposition***). *Let L be a Lévy process. Then $L = (L_t)_{t \geq 0}$, admits the following integral representation*

$$L_t = a_1 t + \beta W_t + \int_0^t \int_{|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z| \geq 1} zN(ds, dz),$$

for some constants a_1, $\beta \in \mathbb{R}$. Here $W = (W_t)_{t \geq 0}$, $(W_0 = 0)$, is a Brownian motion.

1.2 Itô's formula

The following result is fundamental in the stochastic calculus of Lévy process.

Theorem 1.3 (***The one-dimensional Itô formula***). *Let X be a semimartingale and let f be a $\mathcal{C}^2$ real function. Then $f(X)$ is again a semimartingale, and the following formula holds:*

$$\begin{aligned} f(X_t) - f(X_0) &= \int_{0^+}^{t} f'(X_{s_-})\, dX_s + \frac{1}{2}\int_{0^+}^{t} f''(X_{s_-})\, d[X,X]_s^c \\ &+ \sum_{0<s\leq t} \left\{ f(X_s) - f(X_{s_-}) - f'(X_{s_-})\, \Delta X_s \right\}, \end{aligned} \tag{1.5}$$

where $[X,X]^c$ is the continuous part of the quadratic variation $[X,X]$.

Proof. See Theorem 32 in Protter [42] (2004. P. 78 − 79). ∎

We can write

$$\begin{aligned} [X,X]_t &= [X,X]_t^c + X^2(0) + \sum_{0<s\leq t} (\Delta X_s)^2 \\ &= [X,X]_t^c + \sum_{0\leq s\leq t} (\Delta X_s)^2 . \end{aligned}$$

Observe that $[X,X]_0^c = 0$.

Remark 1.3 *If X is already continuos and $X_0 = 0$, then $\langle X,X\rangle = [X,X] = [X,X]^c$.*

Definition 1.8 *A semimartingale X will be called quadratic pure jump if $[X,X]^c = 0$.*

We also note that, in the case where $f(X) = X^2$, formula (1.5) can be written in the form

$$X_t^2 = X_0^2 + \int_0^t 2X_{s_-} dX_s + \int_0^t d[X,X]_s .$$

Next we formulate the corresponding multi-dimensional version of Theorem 1.3.

Theorem 1.4 (***The multi-dimensional Itô formula***) *Let X be an n-tuple of semimartingales, and let f have continuous second order partial derivatives. Then $f(X)$ is again a semimartingale, and the following formula holds:*

$$f(X_t) - f(X_0) = \sum_{i=1}^{n} \int_{0+}^{t} \frac{\partial f}{\partial x_i}(X_{s_-})\, dX_s^i + \frac{1}{2} \sum_{1 \le i,j \le n} \int_{0+}^{t} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s_-})\, d[X^i, X^j]_s^c$$
$$+ \sum_{0<s\le t} \left\{ f(X_s) - f(X_{s_-}) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{s_-})\, \Delta X_s^i \right\}.$$

Proof. See Theorem 4.57 in Jacord and Shiryaev [25]. ■

Lemma 1.1 (***Integration by parts***) *Let X, Y be semimartingale. Then XY is also a semimartingale and*

$$XY = \int X_- dY + \int Y_- dX + [X, Y],$$

where $[X, Y]$ the quadratic covariation of X, Y (or bracket process of X, Y).

Proof. See Corollary II.6.2 in Protter [42]. ■

1.3 Power jump processes and Teugels martingales

Let $(L_t)_{t\ge 0}$ be a Lévy process. In the rest of the this thesis, we assume that all Lévy measures concerned satisfy the following hypothesis:

Hypothesis 1.3

For some $\varepsilon > 0$ and $\lambda > 0$,

$$\int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda |z|)\, \nu(dz) < \infty.$$

This condition implies that

$$\int_{-\infty}^{\infty} |z|^i\, \nu(dz) < \infty, \quad i \ge 2,$$

and

$$\lambda \mapsto \mathbb{E}\left[\exp\left(i\lambda L_t\right)\right],$$

is analytic in a neighborhood of 0. Then, L_t *has moments of all orders,* and the polynomials are dense in $L^2\left(\mathbb{R}, \mathbb{P} \circ L_t^{-1}\right)$ for all $t > 0$.

Power jump processes and Teugels martingales.

For each integer $i \geq 1$, define the process $L^{(i)} = \left(L_t^{(i)}\right)_{t \geq 0}$ by

$$L_t^{(i)} = \begin{cases} L_t & \text{if } i = 1; \\ \sum\limits_{0<s\leq t} \left(\triangle L_s\right)^i & \text{if } i \geq 2. \end{cases}$$

It's called the *power jump processes.* These processes *jump* at the same time as L, but the amplitude of their jumps is possibly different. From this definition, we obtain that

$$\mathbb{E}\left[L_t^{(1)}\right] = \mathbb{E}\left[L_t\right] = tm_1,$$

where $m_1 = \mathbb{E}\left[L_1\right]$, by relation (1.2), we have also

$$\mathbb{E}\left[L_t^{(i)}\right] = \mathbb{E}\left[\sum_{0<s\leq t} \left(\triangle L_s\right)^i\right] = t \int_{-\infty}^{+\infty} z^i \nu\left(dz\right) = tm_i, \quad \text{for } i \geq 2,$$

where $m_i = \int\limits_{-\infty}^{+\infty} z^i \nu\left(dz\right).$

Remark 1.4 *If L is a Brownian motion, then $L^{(i)} = 0$ for all $i \geq 2$, and if L is a Poisson process, then $L^{(i)} = L$ for all $i \geq 1$.*

Now, define the processes

$$Y_t^{(i)} = L_t^{(i)} - \mathbb{E}\left[L_t^{(i)}\right] = L_t^{(i)} - tm_i, \ i \geq 1,$$

the compensated power jump process (or *Teugels martingale*) of order i. $Y^{(i)}$ is a normal martingale, since for an integrable Lévy process L, the process $\{L_t - \mathbb{E}[L_t],\ t \geq 0\}$ is a martingale. This process is introduced by Nualart and Schoutens [34].

The predictable quadratic covariation process of $Y^{(i)}$ and $Y^{(j)}$ is given by

$$\left\langle Y^{(i)}, Y^{(j)} \right\rangle_t = m_{i+j} t, \quad i, j \geq 2.$$

The quadratic covariation of $Y^{(i)}$ and $Y^{(j)}$ is given by

$$\left[Y^{(i)}, Y^{(j)} \right]_t = L_t^{(i+j)} + 1_{\{i=j=1\}} \sigma_0^2 t, \quad i, j \geq 1.$$

Orthogonalization

The idea to *orthogonalize* Teugels martingales is to associate with each a polynomial from the same space then to orthogonalize these polynomials. Specifically, construct a family of martingales $\left(H^{(i)}\right)_{i=1}^{\infty}$ such that its elements are pairwise strongly *orthonormal*.

We denote by $\mathbf{M}^2$ the space of square integrable martingale M such that $\sup_t \mathbb{E}\{M_t^2\} < \infty$, and $M_0 = 0$ *a.s.*

Notice that if $M \in \mathbf{M}^2$, then $\lim_{t\to\infty} \mathbb{E}[M_t^2] = \mathbb{E}[M_\infty^2]$, and , $M_t = \mathbb{E}\{M_\infty \mid \mathcal{F}_t\}$. Thus each $M \in \mathbf{M}^2$ can be identified with its terminal value M_∞.

As in Protter [42], we say that

- a martingale $\tilde{M} \in \mathbf{M}^2$ is *(weakly) orthogonal* to M if $\mathbb{E}\left\{M_\infty \tilde{M}_\infty\right\} = 0$.
- two martingales M, $\tilde{M} \in \mathbf{M}^2$ are said *strongly orthogonal* if their product $N = M \times \tilde{M}$ is a uniformly integrable martingale.

Then M, $\tilde{M} \in \mathbf{M}^2$ are strongly orthogonal, if and only if $\left[M, \tilde{M}\right]$ is a uniformly integrable martingale.

If M and $\tilde{M}$ are strongly orthogonal then

$$\mathbb{E}\left\{M_\infty \tilde{M}_\infty\right\} = \mathbb{E}\{N_\infty\} = \mathbb{E}\{N_0\} = 0,$$

so, strong orthogonality implies orthogonality. The converse is not true however.

Example 1.3 *Let $M \in \mathbf{M}^2$ and $Y \in \mathcal{F}_0$, independent of M, with $\mathbb{P}(Y = 1) = \mathbb{P}(Y = -1) = \frac{1}{2}$. Let $N_t = YM_t$. Then $N_t \in \mathbf{M}^2$ and*

$$\mathbb{E}\{N_\infty M_\infty\} = \mathbb{E}\{YM_\infty^2\} = \mathbb{E}\{Y\}\mathbb{E}\{M_\infty^2\} = 0,$$

then M and N are orthogonal. However, $MN = YM^2$ is not a martingale (unless $M = 0$) because $\mathbb{E}\{YM_t^2 \mid \mathcal{F}_0\} = Y\mathbb{E}\{M_t^2 \mid \mathcal{F}_0\} = YM_0^2 \neq 0$.

Definition 1.9 *Let $X, Y \in L^2(\Omega, \mathcal{F})$, we say that X and Y are weakly orthogonal, if*

$$\mathbb{E}[XY] = 0,$$

this relationship is denoted $X \perp Y$.

Let $\{P_n(x),\ n \in \mathbb{N}\}$ the system of polynomials where $P_n(x)$ is a polynomial of exact degree $n \in \mathbb{N}$.

Definition 1.10 (***Orthogonality relations***) *The system of polynomials $\{P_n(x),\ n \in \mathbb{N}\}$ is an orthogonal system of polynomials with respect to some real positive measure ϕ, if we have the following orthogonality relation*

$$\int_S P_n(x) P_m(x) d\phi(x) = d_n^2 \delta_{n,m}, \quad n, m \in \mathbb{N}, \tag{1.6}$$

where S is the support of the measure ϕ and d_n are nonzero constants.

Remark 1.5 *In the case where $d_n = 1$, we say the system is orthonormal.*

The measure ϕ usually has a density $\rho(x)$ or is discrete measure with weights ρ_i at the points x_i, the relation (1.6) can be written in the form

$$\int_S P_n(x) P_m(x) \rho(x) dx = d_n^2 \delta_{n,m}, \quad n, m \in \mathbb{N},$$

in the former case and

$$\sum_{i=0}^{q} P_n(x_i) P_m(x_i) \rho_i dx = d_n^2 \delta_{n,m}, \quad n, m \in \mathbb{N},$$

in latter case where it is possible that $q = \infty$.

It was shown by Nualart and Schoutens [34] that the set of pairwise strongly *orthogonal* martingales $\left(H^{(i)}\right)_{i=1}^{\infty}$ is a linear combination of the $Y_t^{(j)}$, $j = 1, ...$ defined by

$$H_t^{(i)} = Y_t^{(i)} + a_{i,i-1} Y_t^{(i-1)} + ... + a_{i,1} Y_t^{(1)}, \quad i \geq 1. \tag{1.7}$$

Then, we get

$$\left[H^{(i)}, Y^{(i)}\right]_t = L_t^{(i+j)} + a_{i,i-1} L_t^{(i+j-1)} + ... + a_{i,1} L_t^{(1+j)} + \sigma_0^2 t 1_{\{j=1\}},$$

and thus $\left[H^{(i)}, Y^{(i)}\right]$ is a martingale if and only if we have $\mathbb{E}\left[H^{(i)}, Y^{(i)}\right]_1 = 0$.

So we are looking for a set of pairwise strongly *orthonormal martingales* $\left(H^{(i)}\right)_{i=1}^{\infty}$. For this end, we consider two spaces:

(i) The first space S_1 is the space of all real polynomials on the positive real line. We endow this space with the scalar product $\langle ., . \rangle_1$, given by

$$\left\langle P(x), \hat{P}(x) \right\rangle_1 = \int_{-\infty}^{+\infty} P(x) \hat{P}(x) x^2 \nu(dx) + \sigma_0^2 P(0) \hat{P}(0).$$

According to hypothesis 1.3, this scalar product is well defined. Note that

$$\begin{aligned} \langle x^{i-1}, x^{j-1} \rangle_1 &= \int_{-\infty}^{+\infty} x^{i+j} \nu(dx) + \sigma_0^2 1_{\{i=j=1\}} \\ &= m_{i+j} + \sigma_0^2 1_{\{i=j=1\}}, \; i, \; j = 1, 2, ... \end{aligned}$$

(ii) The second space S_2 is the space of the process of the form

$$a_1 Y^{(1)} + a_2 Y^{(2)} + ... + a_n Y^{(n)},\ a_i \in \mathbb{R},\ n = 1, 2, ...$$

We endow this space with the scalar product $\langle ., . \rangle_2$, given by

$$\begin{aligned} \left\langle Y^{(i)}, Y^{(j)} \right\rangle_2 &= \mathbb{E}\left[\left[Y^{(i)}, Y^{(j)}\right]_1\right] \\ &= \mathbb{E}\left[L_1^{(i+j)}\right] + \sigma_0^2 1_{\{i=j=1\}}, \\ &= m_{i+j} + \sigma_0^2 1_{\{i=j=1\}},\ i,\ j = 1, 2, ... \end{aligned}$$

So it's clearly that $x^{i-1} \longleftrightarrow Y^{(i)}$ is an isometry between S_1 and S_2. It is therefore sufficient to orthogonalize the polynomials $\{1,\ x,\ x^2, ...\}$ in to obtain an orthogonalization of the martingale $\left\{Y^{(1)},\ Y^{(2)}, ...\right\}$ in S_2.
Then, we get this strong orthonormality is equivalent to the existence of coefficients $a_{i,k}$ correspond to the orthonormalization of the polynomials $1,\ x,\ x^2, ..$ with respect to the measure

$$\mu(dx) = x^2 \nu(dx) + \sigma_0^2 \delta_0(dx), \tag{1.8}$$

where $\delta_0(dx) = 1$ when $x = 0$ and zero otherwise, that is, the polynomials q_i defined by

$$q_{i-1}(x) = a_{i,i} x^{i-1} + a_{i,i-1} x^{i-2} + ... + a_{i,1},$$

then $\{q_i(x)\}$ is the system of orthonormalized polynomials such that $q_{i-1}(x)$ corresponds to $H^{(i)}(t)$. Also we define the polynomials

$$p_i(x) = x q_{i-1}(x) = a_{i,i} x^i + a_{i,i-1} x^{i-1} + ... + a_{i,1} x,$$

$$\tilde{p}_i(x) = x\left(q_{i-1}(x) - q_{i-1}(0)\right) = a_{i,i} x^i + a_{i,i-1} x^{i-1} + ... + a_{i,2} x^2.$$

We assume from now on that the family $\left(H^{(i)}\right)_{i=1}^{\infty}$ of strongly orthonormal martingales is the one thus obtained such that its *predictable quadratic variation process* (or *the compensator* of $\left[H^{(i)}, H^{(j)}\right]$) is $\left\langle H^{(i)}, H^{(j)}\right\rangle = \delta_{ij}t$ and $\left[H^{(i)}, H^{(j)}\right] - \left\langle H^{(i)}, H^{(j)}\right\rangle$ is a martingale.

Example 1.4 *(**Simple Lévy process**) The simple Lévy process is given by*

$$L_t = \sigma_0 W_t + \alpha_1 N_1(t) + ... + \alpha_k N_k(t), ...t \geq 0.$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion, $\{N_k(t), t \geq 0\}$ are independent Poisson processes (and independent of Brownian motion) of parameter $\lambda_1, ..., \lambda_k$, respectively, $\sigma_0 > 0$ and $\alpha_1, ..., \alpha_k$ are different non-null numbers. The Lévy measure of L is $\nu = \sum_{j=1}^{k} \lambda_j \delta_{\alpha_j}$ and satisfies the hypothesis 1.3.

However, in this context, in addition to the family $\left(H^{(i)}\right)_{i=1}^{k}$, we also have the set of martingales: $\{W_t,\ N_j(t) - \lambda_1 t, ...,\ N_k(t) - \lambda_k t\}$. It seems sensible to use the last family instead of the former. Indeed observe that

$$L_t^{(1)} = L_t = \sigma_0 W_t + \sum_{j=1}^{k} \alpha_j N_j(t),$$

and

$$L_t^{(i)} = \sum_{0<s\leq t} (\triangle L_s)^i = \sum_{j=1}^{k} \alpha_j^i N_j(t), \quad i \geq 2,$$

because two Poisson processes define in the same filtration are independent if and only if they do not jump at the same time (See Bertoin [5]*, p.5). Since*

$$Y_t^{(i)} = L_t^{(i)} - m_i t, \quad i \geq 1,$$

we have

$$Y_t^{(1)} = \sigma_0 W_t + \sum_{j=1}^{k} \alpha_j (N_j(t) - \lambda_j t),$$

and

$$Y_t^{(i)} = \sum_{j=1}^{k} \alpha_j^i \left(N_j\left(t\right) - \lambda_j t\right), \quad i \geq 2.$$

Then, the martingales $Y^{(i)}$, $i \geq 1$, *are linear combinations of* W_t, $N_1\left(t\right) - \lambda_1 t, ..., N_k\left(t\right) - \lambda_k t$. *Since the martingales* $H^{(i)}$ *are a linear combination of* $Y^{(1)}, ..., Y^{(i)}$ *it follows that* $H^{(i)}$ *are also a linear combination of* W_t, $N_1\left(t\right) - \lambda_1 t, ..., N_k\left(t\right) - \lambda_k t$.
Furthermore, we know that $H^{(i)} = 0$ *for all* $i \geq k+1$.

The following result gives the decomposition of Teugels martingale which used in the second chapter and in this case take $\sigma_0^2 = 1$.

Lemma 1.2 (i) *The process* $H_t^{(i)}$ *can be represented as follows:*

$$H_t^{(i)} = q_{i-1}\left(0\right) W_t + \int_{\mathbb{R}} p_i\left(x\right) \tilde{N}\left(t, dx\right),$$

where W_t *be a Brownian motion, and* $\tilde{N}\left(t, dx\right)$ *is the compensated Poisson random measure that corresponds to the pure jump part of* L_t *and the polynomials* $q_{i-1}\left(0\right)$ *and* $p_i\left(x\right)$ *associated to* L_t.
(ii) *The polynomials* p_i *and* q_j *are linked by the relation:*

$$\int_{\mathbb{R}} p_i\left(x\right) p_j\left(x\right) v\left(dx\right) = \delta_{ij} - q_{i-1}\left(0\right) q_{j-1}\left(0\right).$$

Proof. See Lemma 2.1 in Pereira and Shamarova [41]. ■
The main result in Nualart and Schoutens [34] is the predictable representation property (PRP).

Theorem 1.5 (***Predictable representation property***) *Every random variable* F *in* $L^2\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ *has representation of the form*

$$F = \mathbb{E}\left[F\right] + \sum_{i=1}^{\infty} \int_0^{+\infty} \varphi_s^{(i)} dH_s^{(i)},$$

where $\varphi_s^{(i)}$ *is predictable.*

Proof. See Theorem 2 in Nualart and Schoutens [34]. ■

1.4 Backward stochastic differential equations for Lévy processes (BSDE)

Let $(L_t)_{t\geq 0}$ be a Lévy process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration generated by (L_t) .

We consider the following BSDE:

$$\begin{cases} -dY_t & = f\left(t, w, Y_{t_-}, Z_t\right) dt - \sum\limits_{i=1}^{\infty} Z_t^{(i)} dH_t^{(i)}, \\ Y_T & = \xi, \end{cases} \tag{1.9}$$

where $H_t^{(i)}$ is the orthonormalized Teugels martingale of order i associated with the Lévy process L, ξ is some given $\mathcal{F}_T$-measurable real-valued with value in $\mathbb{R}$, and $(f(t, Y, Z))_{0\leq t\leq T}$ is a progressively measurable processes.

The map f is called *the generator* and ξ *the terminal datum.*

We consider the following assumptions:

$(\mathbf{H}_{1.1})$ there exist a constant C, such that $\forall t \in [0, T], \forall (Y, Z)$ and (Y', Z') in $\mathbb{R}\times \mathcal{P}^2(\mathbb{R})$

$$|f(t, Y, Z) - f(t, Y', Z')| \leq C\left(|Y - Y'| + \|Z - Z'\|_{\mathcal{P}^2(\mathbb{R})}\right);$$

$(\mathbf{H}_{1.2})$ the integrability condition:

$$\mathbb{E}\left[|\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds\right] < \infty.$$

If (f,ξ) satisfies the assumption $(\mathbf{H}_{1.1})$ and $(\mathbf{H}_{1.2})$, the pair (f,ξ) is said to be *standard data* for the BSDE (1.9).

Definition 1.11 *A solution of the BSDE* (1.9) *is a couple of processes* $\{(Y_t, Z_t), 0 \leq t \leq T\}$ *which belongs to space* $\mathbb{L}^2_{\mathcal{F}}(0,T,\mathbb{R}) \times l^2_{\mathcal{F}}(0,T,\mathbb{R})$ *and satisfies the following relation holds for all* $t \in [0,T]$:

$$Y_t = \xi + \int_t^T f\left(s, w, Y_{s_-}, Z_s\right) ds - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} dH_s^{(i)}. \tag{1.10}$$

Note that the progressive measurability of $\{(Y_t, Z_t), 0 \leq t \leq T\}$ implies that (Y_0, Z_0) is deterministic.

The following result was proved by Nualart and Schoutens [35].

Theorem 1.6 *Under assumptions* $(\mathbf{H}_{1.1})$ *and* $(\mathbf{H}_{1.2})$ *the BSDE* (1.10) *has a unique solution.*

Proof. See Theorem 1 in Nualart and Schoutens [35]. ■

The next two chapters, are devoted to study the solvability of fully coupled forward-backward stochastic differential equations driven by jumps Teugels martingales.

Chapter 2

Forward-backward stochastic differential equations driven by Teugels Martingales

2.1 Introduction

Let $(L_t)_{0 \leq t \leq T}$ be a $\mathbb{R}-$valued Lévy process defined on a complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P\right)$ satisfying the usual conditions.

We also assume that $\mathcal{F}_t = \mathcal{F}_0 \vee \sigma (L_s, s \leq t) \vee \mathcal{N}$.

The aim of this chapter is to prove an existence and uniqueness result of solutions of the following coupled forward-backward stochastic differential equation (FBSDE for short)

$$\begin{cases} X_t = & X_0 + \int\limits_0^t f(s, w, X_s, Y_s, Z_s)\, ds + \sum\limits_{i=1}^{\infty} \int\limits_0^t \sigma^i \left(s, w, X_{s_-}, y_{s_-}\right) dH_s^{(i)}, \\ Y_t = & \varphi(X_T) + \int\limits_t^T g(s, w, X_s, Y_s, Z_s)\, ds - \sum\limits_{i=1}^{\infty} \int\limits_t^T Z_s^i dH_s^{(i)}, \end{cases} \tag{2.1}$$

where $t \in [0, T]$, $H_t = (H_t^i)_{i=1}^{\infty}$ are pairwise strongly orthonormal Teugels martingales associated with the Lévy process L_t. For any $\mathbb{R}$-valued and $\mathcal{F}_0$-measurable random vector

x_0, satisfying $\mathbb{E}\left|X_0\right|^2 < \infty$, we are looking for an $\mathbb{R}\times\mathbb{R}\times\mathcal{P}^2\left(\mathbb{R}\right)$-valued solution (x_t, y_t, z_t) on an arbitrarily fixed large time duration, which is square-integrable and adapted with respect to the filtration $\mathcal{F}_t$ generated by L_t and $\mathcal{F}_0$ satisfying

$$\mathbb{E}\int_0^t \left(\left|X_t\right|^2 + \left|Y_t\right|^2 + \left\|Z_t\right\|^2_{\mathcal{P}^2(\mathbb{R})}\right) dt < \infty.$$

This type of equations has been introduced by R.S. Pereira and E. Shamarova [41], then by Baghery et al. [3].

Let us point out that our work extends the results of Jianfeng Zhang [53], to FBSDEs driven by general Lévy processes. We note that much of the technical difficulties coming from the Teugels martingales are due to the fact that the quadratic variation $[H^i, H^j]$ is not absolutely continuous, with respect to the Lebesgue measure. To overcome these difficulties, we use the fact that the predictable quadratic variation process $\left\langle H^i, H^j\right\rangle_t$ is equal to $\delta_{ij}t$ and that $\left[H^i, H^j\right]_t - \left\langle H^i, H^j\right\rangle_t$ is a martingale.

This chapter is organized as follows. In the second section, we formulate the problem of the existence and uniqueness of the solution to FBSDEs (2.1). The main idea of the proof is to construct a solution on small intervals, and then extend it piece by piece to the whole interval. The third section is devoted to prove stability and comparison theorems for the solutions to FBSDEs (2.1).

2.2 Existence and uniqueness of solutions

In this section, we will show that, under some assumptions on the derivatives of the coefficients, one can prove the existence and uniqueness of a solution of FBSDE (2.1).

Consider the FBSDEs :

$$\begin{cases} X_t = & X_0 + \int\limits_0^t f\left(s, X_s, Y_s, Z_s\right) ds + \int\limits_0^t \sigma\left(s, X_{s_-}, Y_{s_-}\right) dH_s, \\ Y_t = & \varphi\left(X_T\right) + \int\limits_t^T g\left(s, X_s, Y_s, Z_s\right) ds - \int\limits_t^T Z_s dH_s, \end{cases} \tag{2.2}$$

where

$$f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}^2\left(\mathbb{R}\right) \rightarrow \mathbb{R},$$

$$\sigma : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}^2\left(\mathbb{R}\right),$$

$$g : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}^2\left(\mathbb{R}\right) \rightarrow \mathbb{R},$$

$$\varphi : \Omega \times \mathbb{R} \rightarrow \mathbb{R},$$

are progressively measurable. Here, for notational simplicity, we shall denote

$$\int_0^t \sigma\left(s, w, X_{s_-}, Y_{s_-}\right) dH_s \quad \text{and} \quad \int_t^T Z_s dH_s$$

instead of

$$\sum_{i=1}^{\infty} \int_0^t \sigma^i\left(s, w, X_{s_-}, Y_{s_-}\right) dH_s^{(i)} \quad \text{and} \quad \sum_{i=1}^{\infty} \int_t^T Z_s^i dH_s^{(i)}$$

respectively, where $z_s = \{z_s^i\}_{i=1}^{\infty}$, $\sigma_s = \{\sigma_s^i\}_{i=1}^{\infty}$, $\sigma^i : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}^2\left(\mathbb{R}\right)$. We also use the following notation

$$M^2\left(0,T\right) = \mathcal{S}_{\mathcal{F}}^2\left(0,T,\mathbb{R}\right) \times \mathcal{S}_{\mathcal{F}}^2\left(0,T,\mathbb{R}\right) \times l_{\mathcal{F}}^2\left(0,T,\mathbb{R}\right).$$

Furthermore, we say that FBSDE (2.2) is *solvable* if it has an adapted solution. An FBSDE is said to be nonsolvable if it is *not solvable.*

The following assumptions will be considered in this chapter:

$(\mathbf{H}_{2.1})$ There exist λ, $\lambda_0 > 0$, such that $\forall t \in [0,T], \forall (x,y,z)$ and (x',y',z') in $\mathbb{R} \times \mathbb{R} \times$

$\mathcal{P}^2(\mathbb{R})$

$$\begin{aligned} \left| f(t,x,y,z) - f(t,x',y',z') \right| &\leq \lambda \left(|x-x'| + |y-y'| + \|z-z'\|_{\mathcal{P}^2(\mathbb{R})} \right), \\ \left| \sigma(t,x,y) - \sigma(t,x',y') \right|^2 &\leq \lambda^2 \left(|x-x'|^2 + |y-y'|^2 \right), \\ \left| g(t,x,y,z) - g(t,x',y',z') \right| &\leq \lambda \left(|x-x'| + |y-y'| + \|z-z'\|_{\mathcal{P}^2(\mathbb{R})} \right), \\ \left| \varphi(x) - \varphi(x') \right| &\leq \lambda_0 \left(|x-x'| \right). \end{aligned}$$

($\mathbf{H}_{2.2}$) The functions f, g, σ, φ are differentiable with respect to x, y, z with uniformly bounded derivatives such that

$$\sigma_y f_z = 0 \text{ and } f_y + \sigma_x f_z + \sigma_y g_z = 0. \tag{2.3}$$

($\mathbf{H}_{2.3}$) Assume that, for every $p \geq 1$,

$$\mathbf{V}_0^{2p} \triangleq \mathbf{E}\left\{ |X_0|^{2p} + |\varphi(0)|^{2p} + \int_0^T \left[|f(t,0,0,0)|^{2p} + \|\sigma(t,0,0)\|_{\mathcal{P}^2(\mathbb{R})}^{2p} \right. \right. \\ \left. \left. + |g(t,0,0,0)|^{2p} \right] dt \right\} < \infty.$$

Let us mention that assumption ($\mathbf{H}_{2.2}$) has been introduced for the first time by Zhang [53] in the case of FBSDEs without jumps.

2.2.1 Small time duration

In this subsection we try to adopt the method of *contraction mapping* to prove the solvability of FBSDE (2.2) in small time durations.

We shall start by giving and proving the following technical Lemma. Let us introduce the following semi-coupled FBSDE:

$$\begin{cases} \tilde{X}_t = & X_0 + \int_0^t f\left(s, \tilde{X}_s, Y_s, Z_s\right) ds + \int_0^t \sigma\left(s, \tilde{X}_{s-}, Y_{s-}\right) dH_s, \\ \tilde{Y}_t = & \varphi(X_T) + \int_t^T g\left(s, \tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s\right) ds - \int_t^T \tilde{Z}_s dH_s. \end{cases} \tag{2.4}$$

Lemma 2.1 *Assume that* $(\mathbf{H}_{2.1})$ *and* $(\mathbf{H}_{2.3})$ *are satisfied (for* $p=1$*). Let* $\left(\tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s\right)$ *and* $\left(\tilde{U}_t, \tilde{V}_t, \tilde{I}_s\right)$ *belong to* $M^2(0,T)$ *and satisfy the equation (2.4), then there exists three constants* c, c' *and* c'' *depending on* λ *and* λ_0*, such that the following estimates hold true*

$$\left(1-cT^{\frac{1}{2}}\right)\mathbb{E}\sup_{0\leq t\leq T}\left|\tilde{X}_s-\tilde{U}_t\right|^2\leq cT^{\frac{1}{2}}\left(\mathbb{E}\sup_{0\leq s\leq T}\left|Y_s-V_s\right|^2+\mathbb{E}\int_0^T\left\|Z_s-I_s\right\|^2_{\mathcal{P}^2(\mathbb{R})}ds\right), \tag{2.5}$$

$$\left(1-c''T\right)\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\tilde{Y}_t-\tilde{V}_t\right|^2\right)\leq c''\left(1+T\right)\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\tilde{X}_s-\tilde{U}_s\right|^2\right), \tag{2.6}$$

$$\mathbb{E}\left[\int_0^T\left\|\tilde{Z}_s-\tilde{I}_s\right\|^2_{\mathcal{P}^2(\mathbb{R})}ds\right]\leq c'\left((1+T)\,\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\tilde{X}_s-\tilde{U}_s\right|^2\right)+T\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\tilde{Y}_s-\tilde{V}_s\right|^2\right)\right). \tag{2.7}$$

Proof. Let us consider $(X_t,Y_t,Z_t)_{0\leq t\leq T}, \left(\tilde{X}_t,\tilde{Y}_t,\tilde{Z}_t\right)_{0\leq t\leq T}, (U_t,V_t,I_t)_{0\leq t\leq T}, \left(\tilde{U}_t,\tilde{V}_t,\tilde{I}_t\right)_{0\leq t\leq T}\in M^2(0,T)$.

First, we proceed to prove (2.5).

Applying Itô's formula to $\left|\tilde{X}_t-\tilde{U}_t\right|^2$, taking expectation and using assumption $(\mathbf{H}_{2.1})$, the fact that $[H^i,H^j]_t-\langle H^i,H^j\rangle_t$ is an $\mathcal{F}_t$-martingale and $\langle H^i,H^j\rangle_t=\delta_{ij}t$, then there exists a constant c, depending on λ such that

$$\begin{aligned}&\mathbb{E}\sup_{0\leq t\leq T}\left|\tilde{X}_t-\tilde{U}_t\right|^2\leq c\left[\mathbb{E}\int_0^T\left|\tilde{X}_s-\tilde{U}_s\right|\left(\left|\tilde{X}_s-\tilde{U}_s\right|+\left|Y_s-V_s\right|+\left\|Z_s-I_s\right\|^2_{\mathcal{P}^2(\mathbb{R})}\right)ds\right.\\&\left.+\mathbb{E}\int_0^T\left(\left|\tilde{X}_s-\tilde{U}_s\right|^2+\left|Y_s-V_s\right|^2\right)ds\right]\\&+2\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t\left(\tilde{X}_{s-}-\tilde{U}_{s-}\right)\sigma\left(s,\tilde{X}_{s-},Y_{s-}\right)-\sigma\left(s,\tilde{U}_{s-},V_{s-}\right)dH_s\right|.\end{aligned}$$

Burkholder-Davis-Gundy's inequality applied to the martingale

$$\int_0^t\left(\tilde{X}_s-\tilde{U}_s\right)\sigma\left(s,\tilde{X}_{s-},Y_{s-}\right)-\sigma\left(s,\tilde{U}_{s-},V_{s-}\right)dH_s,$$

yields the existence of a constant $C>0$, such that

$$\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t\left(\tilde{X}_s-\tilde{U}_s\right)\left(\sigma\left(s,\tilde{X}_{s-},Y_{s-}\right)-\sigma\left(s,\tilde{U}_{s-},V_{s-}\right)\right)dH_s\right|$$
$$\leq C\mathbb{E}\left(\left[\int_0^{\cdot}\left(\tilde{X}_{s-}-\tilde{U}_{s-}\right)\left(\sigma\left(s,\tilde{X}_{s-},Y_{s-}\right)-\sigma\left(s,\tilde{U}_{s-},V_{s-}\right)\right)dH_s\right]\right)^{\frac{1}{2}}.$$

Moreover, since $\langle H^i,H^i\rangle=\delta_{ij}t$ and $[M]_t=\langle M\rangle_t+\psi_t$, where ψ_t is a uniformly integrable martingale starting at 0, then

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t\left(\tilde{X}_s-\tilde{U}_s\right)\left(\sigma\left(s,\tilde{X}_{s-},Y_{s-}\right)-\sigma\left(s,\tilde{U}_{s-},V_{s-}\right)\right)dH_s\right|\right)^{1/2}\\&=C\mathbb{E}\left(\left\langle\int_0^{\cdot}\left(\tilde{X}_{s-}-\tilde{U}_{s-}\right)\sigma\left(s,\tilde{X}_{s-},Y_{s-}\right)-\sigma\left(s,\tilde{U}_{s-},V_{s-}\right)dH_s\right\rangle_t+\psi_t\right)^{1/2}\\&\leq C\mathbb{E}\left(\int_0^T\left|\tilde{X}_s-\tilde{U}_s\right|^2\left\|\sigma\left(s,\tilde{X}_s,Y_s\right)-\sigma\left(s,\tilde{U}_s,V_s\right)\right\|^2_{\mathcal{P}^2(\mathbb{R})}ds\right)^{1/2}.\end{aligned}$$

Then, modifying c if necessary, we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\tilde{X}_t-\tilde{U}_t\right|^2\right]\\&\leq cT^{1/2}\left(\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\tilde{X}_s-\tilde{U}_s\right|^2\right)+\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s-V_s\right|^2\right)+\mathbb{E}\left(\int_0^T\|Z_s-W_s\|^2_{\mathcal{P}^2(\mathbb{R})}ds\right)\right);\end{aligned}$$

which implies that,

$$\left(1-cT^{1/2}\right)\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\tilde{X}_t-\tilde{U}_t\right|^2\right)\leq cT^{1/2}\left(\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s-V_s\right|^2\right)+\mathbb{E}\left(\int_0^T\|Z_s-I_s\|^2_{\mathcal{P}^2(\mathbb{R})}ds\right)\right).$$

On the other hand, by applying Itô's formula to $\left|\tilde{Y}_t-\tilde{V}_t\right|^2$, we get

$$\begin{aligned}&\left|\tilde{Y}_t-\tilde{V}_t\right|^2+\int_t^T\left\|\tilde{Z}_s-\tilde{I}_s\right\|^2_{\mathcal{P}^2(\mathbb{R})}ds\\&=\left|\varphi\left(\tilde{X}_T\right)-\varphi\left(\tilde{U}_T\right)\right|^2+2\int_t^T\left(\tilde{Y}_s-\tilde{V}_s\right)\left(g\left(s,\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s\right)-g\left(s,\tilde{U}_s,\tilde{V}_s,\tilde{I}_s\right)\right)ds\\&-2\int_t^T\left(\tilde{Y}_s-\tilde{V}_s\right)\left(\tilde{Z}_s-\tilde{I}_s\right)dH_s-\sum_{i,j}\int_t^T\left(\tilde{Z}^i_s-\tilde{I}^i_s\right)\left(\tilde{Z}^j_s-\tilde{I}^j_s\right)d\left[H^i,H^j\right]_s.\end{aligned}\tag{2.8}$$

Thus, by taking expectations, invoking the assumption $(\mathbf{H}_{2.1})$ and using the fact that

$[H^i, H^j]_t - \langle H^i, H^j \rangle_t$ is an $\mathcal{F}_t$-martingale and $\langle H^i, H^j \rangle_t = \delta_{ij} t$, one can show that there exists a constant c', depending on λ and λ_0, such that

$$\mathbb{E}\int_0^T \left\|\tilde{Z}_s - \tilde{I}_s\right\|^2_{\mathcal{P}^2(\mathbb{R})} ds \leq c' \left[\mathbb{E}\left|\tilde{X}_T - \tilde{U}_T\right|^2 \right.$$
$$\left. + \mathbb{E}\int_0^T \left|\tilde{Y}_s - \tilde{V}_s\right| \left(\left|\tilde{X}_s - \tilde{U}_s\right| + \left|\tilde{Y}_s - \tilde{V}_s\right| + \left\|\tilde{Z}_s - \tilde{I}_s\right\|^2_{\mathcal{P}^2(\mathbb{R})}\right) ds\right].$$

Using the fact that $|ab| \leq \frac{1}{2}\left(|a|^2 + |b|^2\right)$ for any $a, b \in \mathbb{R}$, we have

$$\mathbb{E}\int_0^T \left|\tilde{Z}_s - \tilde{I}_s\right|^2 ds \leq c' \left[(1+T)\,\mathbb{E}\sup_{0\leq s\leq T}\left|\tilde{X}_s - \tilde{U}_s\right|^2 \right.$$
$$\left. + T\mathbb{E}\sup_{0\leq s\leq T}\left|\tilde{Y}_s - \tilde{V}_s\right|^2\right] + \frac{1}{2}\mathbb{E}\int_0^T \left\|\tilde{Z}_s - \tilde{I}_s\right\|^2_{\mathcal{P}^2(\mathbb{R})} ds.$$

By modifying c' if necessary, we obtain

$$\begin{aligned}&\mathbb{E}\int_0^T \left\|\tilde{Z}_s - \tilde{I}_s\right\|^2_{l^2(\mathbb{R})} ds \\ &\leq c'\left[(1+T)\,\mathbb{E}\sup_{0\leq s\leq T}\left|\tilde{X}_s - \tilde{U}_s\right|^2 + T\mathbb{E}\sup_{0\leq s\leq T}\left|\tilde{Y}_s - \tilde{V}_s\right|^2\right].\end{aligned} \tag{2.9}$$

Using equality (2.8) once again, and the Burkholder-Davis-Gundy inequality, we show that there exists a constant c'', only depending on λ and λ_0, such that

$$\mathbb{E}\sup_{0\leq t\leq T}\left|\tilde{Y}_t - \tilde{V}_t\right|^2 \leq c''\left[\mathbb{E}\left|\tilde{X}_T - \tilde{U}_T\right|^2 + \mathbb{E}\left(\int_0^T \left|\tilde{Y}_s - \tilde{V}_s\right|^2 \left\|\tilde{Z}_s - \tilde{I}_s\right\|^2_{l^2(\mathbb{R})} ds\right)^{1/2}\right.$$
$$\left. + \mathbb{E}\int_0^T \left|\tilde{Y}_s - \tilde{V}_s\right| \left(\left|\tilde{X}_s - \tilde{U}_s\right| + \left|\tilde{Y}_s - \tilde{V}_s\right| + \left\|\tilde{Z}_s - \tilde{I}_s\right\|^2_{\mathcal{P}^2(\mathbb{R})}\right) ds\right].$$

Then, Taking into account (2.9), using Young's inequality one more time, and modifying c'' if necessary, we get

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\tilde{Y}_t - \tilde{V}_t\right|^2\right) \leq c''\left[(1+T)\,\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\tilde{X}_s - \tilde{U}_s\right|^2\right) + T\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\tilde{Y}_s - \tilde{V}_s\right|^2\right)\right]$$
$$+\tfrac{1}{2}\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\tilde{Y}_t - \tilde{V}_t\right|^2\right)$$

Then, modifying c'' if necessary, we have

$$\left(1-c^{''}T\right)\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\tilde{Y}_t-\tilde{V}_t\right|^2\right)\leq c''\left(1+T\right)\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\tilde{X}_s-\tilde{U}_s\right|^2\right).$$

Lemma 2.1 is proved. ■

We now state and prove main result.

Theorem 2.1 *Suppose that* $(\mathbf{H}_{1.1})$ *and* $(\mathbf{H}_{1.3})$ *(for* $p=1$*) are satisfied. Then, for every* $\mathcal{F}_0$*-measurable random vector* X_0*, there exists a constant* δ *depending only on* λ *and* λ_0*, such that for* $T\leq\delta$*, equation* (2.2) *has a unique solution which belongs to* $M^2(0,T)$*.*

Proof. Let $(X_t,Y_t,Z_t)_{0\leq t\leq T}$ be a possible solution of FBSDE (2.2) and $\left(\tilde{X},\tilde{Y},\tilde{Z}\right)$ be defined as in Lemma 2.1. It is clear that the process $\tilde{X}$ is a solution of a Forward component of the SDE (2.4), whereas the couple $\left(\tilde{Y},\tilde{Z}\right)$ is a solution of a Backward component of the SDE (2.4) SDE. Then $\left(\tilde{X},\tilde{Y},\tilde{Z}\right)$ is a solution of the above semi-coupled Forward Backward SDE (2.4). To prove the existence and the uniqueness of the solution in $M^2(0,T)$, we use the fixed point method. Let us define a mapping Ψ from $M^2(0,T)$ into itself defined by

$$\Psi\left(X,Y,Z\right)=\left(\tilde{X},\tilde{Y},\tilde{Z}\right).$$

We want to prove that there exists a constant $\delta>0$, only depending on λ and λ_0, such that for $T\leq\delta$, Ψ is a contraction on $M^2(0,T)$ equipped with the norm

$$\|\Psi\left(X,Y,Z\right)\|^2_{M^2(0,T)}=\mathbb{E}\left\{\sup_{0\leq t\leq T}\left[|X_t|^2+|Y_t|^2\right]+\int_0^T\|Z_t\|^2_{\mathcal{P}^2(\mathbb{R})}\,dt\right\}.$$

In order to achieve this goal, we firstly assume that $T\leq 1$. Further, we set

$$\Psi\left(X,Y,Z\right)=\left(\tilde{X},\tilde{Y},\tilde{Z}\right),\ \Psi\left(U,V,I\right)=\left(\tilde{U},\tilde{V},\tilde{I}\right).$$

where$(X_t,Y_t,Z_t)_{0\leq t\leq T}\,,(U_t,V_t,I_t)_{0\leq t\leq T}$ be two elements of $M^2(0,T)$. Thus, by invoking and combining the results (2.5) , (2.6) and (2.7) of the Lemma 2.1, a simple computation shows that there exists a constant δ depending on λ and λ_0, such that for $T \leq \delta$, the following estimate holds true

$$\left\| \Psi\left(X, Y, Z\right) - \Psi\left(U, V, I\right) \right\|_{M^2(0,T)} \leq D \left\| (X, Y, Z) - (U, V, I) \right\|_{M^2(0,T)},$$

For some constant $0 < D < 1$.This proves that the map Ψ is contraction from $M^2\left(0, T\right)$ into itself. Furthermore, It follows immediately that this mapping has a unique fixed point (X_t, Y_t, Z_t) progressively measurable which is the unique solution of FBSDE (2.1). The proof is complete. ■

In the above proof, it is crucial that the time duration is small enough, besides condition $(\mathbf{H}_{2.1})$. Starting from the next chapter, we are going to use different methods to approach the solvability problem for the fully coupled FBSDE driven by Teugels martingales associated with some Lévy processes having moment of all orders and an independent Brownian motion in stopping time duration.

The following proposition gives a priori estimates, which shows in particular the continuous dependence of the solution upon the data.

Proposition 2.1 *Under the same assumptions of the Theorem* 2.1*, there exist* δ *and* C_0 *depending on* λ *and* λ_0*, such that for* $T \leq \delta$*, the following estimates hold true:*

i)

$$\|\Pi\| = \mathbb{E}\left(\sup_{0\leq t\leq T} \left[|X_t|^2 + |Y_t|^2 \right] + \int_0^T \|Z_t\|^2_{\mathcal{P}^2(\mathbb{R})}\, dt \right)^{\frac{1}{2}} \leq C_0 V_0.$$

ii)

$$\mathbb{E}\left\{ \sup_{0\leq t\leq T} \left[|X_t|^{2p} + |Y_t|^{2p} \right] + \left(\int_0^T \|Z_t\|^2_{\mathcal{P}^2(\mathbb{R})}\, dt \right)^p \right\} < \infty. \tag{2.10}$$

Proof. Arguing as in the proof of Lemma 2.1 and standard arguments of FBSDEs (see for example [1] for the Brownian case), one can prove (i) . ■

Now we proceed to prove (ii), we want to prove that there exists a constant $\delta > 0$, only depending on λ, λ_0 and p, such that for $T \leq \delta$, the process satisfy the inequality 2.10. For this end, we firstly assume that $T \leq 1$. Let us define the stopping time

$$R_k = \inf\{t : |X_t| + |Y_t| > k\} \text{ with } X_0 = 0.$$

Applying Itô's formula to $|X_t|^{2p}$, we obtain

$$\begin{aligned}|X_t|^{2p} = |X_0|^{2p} + 2p\int_0^{t\wedge R_k} X_s^{2p-1} f(s, X_s, Y_s, Z_s)\, ds + 2p\int_0^{t\wedge R_k} X_s^{2p-1}\sigma\left(s, X_{s_-}, Y_{s_-}\right) dH_s \\ +p(2p-1)\int_0^{t\wedge R_k} X_s^{2p-2} d[X,X]_s \\ + \sum_{0<s\leq t\wedge R_k}\left\{X_s^{2p} - X_{s_-}^{2p} - 2pX_s^{2p-1}\Delta X_s - p(2p-1)X_s^{2p-2}(\Delta X_s)^2\right\}.\end{aligned}$$

Then

$$\begin{aligned}&\sup_{0\leq t\leq T\wedge R_k}|X_t|^{2p} \leq |X_0|^{2p} + 2p\left|\int_0^{T\wedge R_k} X_s^{2p-1} f(s, X_s, Y_s, Z_s)\, ds\right| \\ &+2p\sup_{0\leq t\leq T\wedge R_k}\left|\int_0^{t\wedge R_k} X_s^{2p-1}\sigma\left(s, X_{s_-}, Y_{s_-}\right) dH_s\right| + p(2p-1)\int_0^{T\wedge R_k}|X_s|^{2p-2}|\sigma(s, X_s, Y_s)|^2 ds \\ &+p(2p-1)\sum_{i,j=1}^{\infty}\int_0^{T\wedge R_k}|X_s|^{2p-2}\left(\sigma^i(s, X_s, Y_s), \sigma^j(s, X_s, Y_s)\right) d\left([H^i, H^j]_t - \langle H^i, H^j\rangle_t\right) + \tilde{I}_1.\end{aligned}$$

Taking expectation, using Burkholder-Davis-Gundy inequality and the fact that $[H^i, H^j]_t - \langle H^i, H^j\rangle_t$ is an F_t-martingale and $\langle H^i, H^j\rangle_t = \delta_{ij}t$, we show that there exists a constant c such that

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T\wedge R_k}|X_t|^{2p} \leq \mathbb{E}|X_0|^{2p} + 2p\mathbb{E}\left(\int_0^{T\wedge R_k}|X_s|^{2p-1}|f(s, X_s, Y_s, Z_s)|\, ds\right) \\ +2pc\mathbb{E}\left(\int_0^{T\wedge R_k}|X_s|^{4p-2}|\sigma(s, X_s, Y_s)|^2 ds\right)^{\frac{1}{2}} \\ +p(2p-1)\mathbb{E}\left(\int_0^{T\wedge R_k}|X_s|^{2p-2}\left|\sigma\left(s, \tilde{X}_s, \tilde{Y}_s\right)\right|^2 ds\right) + \mathbb{E}\left|\tilde{I}_1\right|.\end{aligned}$$

Therefore, using assumption $(\mathbf{H}_1)$, there exists a constant $c'_{p,\lambda}$, only depending on p and

λ, we get

$$\mathbb{E}\sup_{0\le t\le T\wedge R_k}|X_t|^{2p}\le \mathbb{E}|X_0|^{2p}+\mathbb{E}\left|\tilde{I}_1\right|+c'_{p,\lambda}\Big[\mathbb{E}\int_0^{T\wedge R_k}|X_s|^{2p-1}\left(|X_s|+|Y_s|\right.$$
$$\left.+\|Z_s\|_{\mathcal{P}^2(\mathbb{R})}\right)+\mathbb{E}\int_0^{T\wedge R_k}|X_s|^{2p-1}|f(s,0,0,0)|\,ds\Big]++\mathbb{E}\left(\int_0^{T\wedge R_k}|X_s|^{4p-2}|\sigma(s,X_s,Y_s)|^2\,ds\right)^{\frac{1}{2}}$$
$$+\mathbb{E}\left(\int_0^{T\wedge R_k}|X_s|^{2p-2}|\sigma(s,X_s,Y_s)|^2\,ds\right).$$

We deduce, by using Young's estimate and modifying $c'_{p,\lambda}$ if necessary,

$$\mathbb{E}\sup_{0\le t\le T\wedge R_k}|X_t|^{2p}\le c'_{p,\lambda}\Big[\mathbb{E}|X_0|^{2p}+\mathbb{E}\left|\tilde{I}_1\right|+\mathbb{E}\left(\int_0^{T\wedge R_k}|f(s,0,0,0)|\,ds\right)^{2p}$$
$$+\mathbb{E}\left(\int_0^{T\wedge R_k}|\sigma(s,0,0)|^2\,ds\right)^{p}+\mathbb{E}\int_0^{T\wedge R_k}\left(|X_s|^{2p}+|Y_s|^{2p}\right)ds \tag{2.11}$$
$$+T^p\mathbb{E}\left(\int_0^{T\wedge R_k}||Z||^2_{\mathcal{P}^2(\mathbb{R})}\,ds\right)^{p}\Big).$$

On the other hand, applying Itô's formula to $|Y_s|^{2p}$, we have for every $t\in[0,T\wedge R_k]$

$$|Y_t|^{2p}=|\varphi(X_T)|^{2p}+2\left|\int_t^{T\wedge R_k}Y_s^{2p-1}g(s,X_s,Y_s,Z_s)\right|ds$$
$$+2p\left|\int_t^{T\wedge R_k}Y_s^{2p-1}Z_s dH_s\right|-p(2p-1)\sum_{i,j=1}^{\infty}\int_t^{T\wedge R_k}Y_s^{2p-2}\left(Z_s^i,Z_s^j\right)d\left[H^i,H^j\right]_s \tag{2.12}$$
$$+\sum_{0<s\le t\wedge R_k}\left\{Y_s^{2p}-Y_{s_-}^{2p}-2pY_s^{2p-1}\Delta Y_s-p(2p-1)Y_s^{2p-2}(\Delta Y_s)^2\right\}.$$

Taking expectation, and using the fact that $\left[H^i,H^j\right]_t-\left\langle H^i,H^j\right\rangle_t$ is an $\mathcal{F}_t$-martingale and $\left\langle H^i,H^j\right\rangle_t=\delta_{ij}t$, one can show that there exists a constant c such that

$$\mathbb{E}|Y_t|^{2p}+p(2p-1)\mathbb{E}\int_t^{T\wedge R_k}|Y_s|^{2p-2}||Z||^2_{\mathcal{P}^2(\mathbb{R})}\,ds\le\mathbb{E}|\varphi(X_T)|^{2p}$$
$$+2p\mathbb{E}\left(\int_t^{T\wedge R_k}|Y_s|^{2p-1}|g(s,X_s,Y_s,Z_s)|\,ds\right)+\mathbb{E}\left(\tilde{I}_2\right).$$

This leads to the following inequality,

$$\mathbb{E}\int_t^{T\wedge R_k}|Y_s|^{2p-2}||Z||^2_{\mathcal{P}^2(\mathbb{R})}\,ds\le\frac{1}{p(2p-1)}\left(\mathbb{E}|\varphi(X_T)|^{2p}+\mathbb{E}\left(\tilde{I}_2\right)\right)$$
$$+2p\mathbb{E}\left(\int_t^{T\wedge R_k}|Y_s|^{2p-1}|g(s,X_s,Y_s,Z_s)|\,ds\right)\Big) \tag{2.13}$$

Now, by invoking the inequality (2.12) one more time, using Burkholder-Davis-Gundy inequality, one can show that there exists a constant c_p,such that

$$\mathbb{E}\sup_{0\leq t\leq T\wedge R_k}|Y_t|^{2p}\leq c_p\left[\mathbb{E}\left|\varphi\left(X_T\right)\right|^{2p}+\mathbb{E}\left(\tilde{I}_2\right)+\mathbb{E}\left(\int_0^{T\wedge R_k}|Y_s|^{2p-1}\left|g\left(s,X_s,Y_s,Z_s\right)\right|ds\right)\right]$$
$$+\mathbb{E}\int_0^{T\wedge R_k}|Y_s|^{2p-2}\,||Z||^2_{\mathcal{P}^2(\mathbb{R})}\,ds\Big]$$

We substitute the inequality (2.13) into the previous one , we get

$$\mathbb{E}\sup_{0\leq t\leq T\wedge R_k}|Y_t|^{2p}\leq c_p\left[\mathbb{E}\left|\varphi\left(X_T\right)\right|^{2p}+\mathbb{E}\left(\tilde{I}_2\right)+\mathbb{E}\left(\int_0^{T\wedge R_k}|Y_s|^{2p-1}\left|g\left(s,X_s,Y_s,Z_s\right)\right|ds\right)\right].$$

Using assumption ($\mathbf{H}_{2.1}$) and Young's inequality, there exists a constant $c''_{p,\lambda,\lambda_0}$, depending upon p, λ and λ_0, such that

$$\mathbb{E}\sup_{0\leq t\leq T\wedge R_k}|Y_t|^{2p}+\mathbb{E}\int_0^{T\wedge R_k}|Y_s|^{2p-2}\,||Z||^2_{\mathcal{P}^2(\mathbb{R})}\,ds\leq c''_{p,\lambda,\lambda_0}\Big[\mathbb{E}\sup_{0\leq t\leq T\wedge R_k}|X_t|^{2p}+\mathbb{E}\left|\varphi\left(0\right)\right|^{2p}+\mathbb{E}\left(\tilde{I}_2\right)$$
$$+\mathbb{E}\int_0^{T\wedge R_k}|Y_s|^{2p-1}|Z_s|\,ds+\mathbb{E}\int_0^{T\wedge R_k}\left(|X_s|^{2p}+|Y_s|^{2p}\right)ds+\mathbb{E}\left(\int_0^{T\wedge R_k}\left|g\left(s,0,0,0\right)\right|ds\right)^{2p}\Big].$$

Hence, by modifying $c''_{p,\lambda,\lambda_0}$,

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T\wedge R_k}|Y_t|^{2p}\leq c''_{p,\lambda,\lambda_0}\Big[&\mathbb{E}\sup_{0\leq t\leq T\wedge R_k}|X_t|^{2p}+\mathbb{E}\left|\varphi\left(0\right)\right|^{2p}+\mathbb{E}\left(I_2\right)\\ &+\mathbb{E}\int_0^{T\wedge R_k}\left(|X_s|^{2p}+|Y_s|^{2p}\right)ds+\mathbb{E}\left(\int_0^{T\wedge R_k}\left|g\left(s,0,0,0\right)\right|ds\right)^{2p}\Big].\end{aligned}\tag{2.14}$$

In the other hand, by applying Itô's formula to $|Y_t|^2$, one can get

$$\int_t^{T\wedge R_k}\|Z_s\|^2_{\mathcal{P}^2(\mathbb{R})}\,ds+\sum_{i,j=1}^{\infty}\int_t^{T\wedge R_k}\left(Z_s^i,Z_s^j\right)d\left(\left[H^i,H^j\right]_s-\left\langle H^i,H^j\right\rangle_s\right)\leq\left|\varphi\left(X_T\right)\right|^2$$
$$+2\int_t^{T\wedge R_k}|Y_s|\left|g\left(s,X_s,Y_s,Z_s\right)\right|ds-2\int_t^{T\wedge R_k}|Y_s|\,\|Z_s\|_{\mathcal{P}^2(\mathbb{R})}\,dH_s,$$

Using assumption ($\mathbf{H}_{2.1}$) and Young's inequality, then there exists a constant $c''_{p,\lambda,\lambda_0}$, only depending on p, λ and λ_0, such that

$$\begin{aligned}\mathbb{E}\left(\int_0^{T\wedge R_k}\|Z_s\|^2_{\mathcal{P}^2(\mathbb{R})}ds\right)^p \le c''_{p,\lambda,\lambda_0}&\left[\mathbb{E}\left|\varphi(0)\right|^{2p}+\mathbb{E}\left(\int_0^{T\wedge R_k}|g(s,0,0,0)|\,ds\right)^{2p}\right.\\&\left.+\mathbb{E}\sup_{0\le s\le T}|X_s|^{2p}+\mathbb{E}\sup_{0\le s\le T}|Y_s|^{2p}\right]\end{aligned}\tag{2.15}$$

So, considering (2.11), (2.14) and (2.15), this proves that there exists two constants $\delta'>0$ and C_1 only depend on λ, λ_0 and p, such that for $T\le\delta'$,

$$\begin{aligned}&\mathbb{E}\sup_{0\le s\le T\wedge R_k}|X_s|^{2p}+\mathbb{E}\sup_{0\le s\le T\wedge R_k}|Y_s|^{2p}+\mathbb{E}\left(\int_0^{T\wedge R_k}\|Z_s\|^2_{\mathcal{P}^2(\mathbb{R})}ds\right)^p\le \mathbf{C}_1\mathbb{E}\left[|X_0|^{2p}+|\varphi(0)|^{2p}\right.\\&\left.+\mathbb{E}\left|\tilde{I}_1\right|+\mathbb{E}\left|\tilde{I}_2\right|+\left(\int_0^{T\wedge R_k}(|f(s,0,0,0)|+|g(s,0,0,0)|)\,ds\right)^{2p}+\left(\int_0^{T\wedge R_k}\|\sigma(s,0,0)\|^2_{\mathcal{P}^2(\mathbb{R})}ds\right)^p\right].\end{aligned}$$

Using Fatou's Lemma, we obtain

$$\begin{aligned}&\mathbb{E}\sup_{0\le s\le T}|X_s|^{2p}+\mathbb{E}\sup_{0\le s\le T}|Y_s|^{2p}+\mathbb{E}\left(\int_0^{T}\|Z_s\|^2_{\mathcal{P}^2(\mathbb{R})}ds\right)^p\le \mathbf{C}_1\mathbb{E}\left[|X_0|^{2p}+|\varphi(0)|^{2p}\right.\\&\left.+\mathbb{E}\left|I_1\right|+\mathbb{E}\left|I_2\right|+\left(\int_0^{T}(|f(s,0,0,0)|+|g(s,0,0,0)|)\,ds\right)^{2p}+\left(\int_0^{T}\|\sigma(s,0,0)\|^2_{\mathcal{P}^2(\mathbb{R})}ds\right)^p\right],\end{aligned}\tag{2.16}$$

where

$$I_1=\lim\tilde{I}_1,$$
$$I_2=\lim\tilde{I}_2.$$

For each k, denoting $\left(\tilde{X},\tilde{Y},\tilde{Z}\right)=\left(X1_{[0,R_k)},Y1_{[0,R_k)},Z1_{[0,R_k)}\right)$. It is clear that the stopped process $\tilde{X}=X1_{[0,R_k)}$ is bounded by k, and is a semimartingale as a product of two semimartingles, which is valid for $\tilde{Y}$ as well. Now, we proceed to prove that

$$\sum_{0<s\le t}\left\{\tilde{X}_s^{2p}-\tilde{X}_{s-}^{2p}-2p\tilde{X}_s^{2p-1}\Delta\tilde{X}_s-p(2p-1)\tilde{X}_s^{2p-2}\left(\Delta\tilde{X}_s\right)^2\right\}<C\left[\tilde{X},\tilde{X}\right]_t.$$

Since $\tilde{X}$ takes its values in intervals of the form $[-k,k]$, for $h(x)=x^{2p}$, it is easy to show that

$$\left|h(x) - h(y) - (y-x)h'(x) - (y-x)^2 h''(x)\right| \leq C(y-x)^2$$

Thus

$$\begin{aligned} I_1 &= \sum_{0<s\leq t} \left|\tilde{X}_s^{2p} - \tilde{X}_{s-}^{2p} - 2p\tilde{X}_s^{2p-1}\Delta\tilde{X}_s - p(2p-1)\tilde{X}_s^{2p-2}\left(\Delta\tilde{X}_s\right)^2\right| \\ &\leq C\sum_{0<s\leq t}\left(\Delta\tilde{X}_s\right)^2 \leq C\left[\tilde{X},\tilde{X}\right]_t < \infty, P-a.s. \end{aligned} \tag{2.17}$$

Arguing symmetrically, one can show,

$$\begin{aligned} I_2 &= \sum_{0<s\leq t} \left|\tilde{Y}_s^{2p} - \tilde{Y}_{s-}^{2p} - 2p\tilde{Y}_s^{2p-1}\Delta\tilde{Y}_s - p(2p-1)\tilde{Y}_s^{2p-2}\left(\Delta\tilde{Y}_s\right)^2\right| \\ &\leq C\sum_{0<s\leq t}\left(\Delta\tilde{Y}_s\right)^2 \leq C\left[\tilde{Y},\tilde{Y}\right]_t < \infty. P-a.s. \end{aligned} \tag{2.18}$$

Combining (2.17) and (2.18) with (2.16), we get

$$\mathbb{E}\sup_{0\leq s\leq T}\left|\tilde{X}_s\right|^{2p} + \mathbb{E}\sup_{0\leq s\leq T}\left|\tilde{Y}_s\right|^{2p} + \mathbb{E}\left(\int_0^T \left\|\tilde{Z}_s\right\|^2_{\mathcal{P}^2(\mathbb{R})} ds\right)^p < \infty. P-a.s.$$

Since the last inequality is valid for $\left(\tilde{X},\tilde{Y},\tilde{Z}\right)$ for each k, it also remains valid for (X,Y,Z) and this completes the proof.

2.2.2 Large time duration

In this subsection, under same assumption of Theorem 2.1 we extend the result in Theorem 2.1 to arbitrary large time duration.

The following lemma gives estimates of $\bar{\lambda}_0$ in terms of λ and λ_0. This estimation is the key step for the proof of the main result.

Lemma 2.2 *Consider the following linear FBSDE:*

$$\begin{cases} X_t = & 1 + \int_0^t \left(a_s^1 X_s + b_s^1 Y_s + c_s^1 Z_s\right) ds + \int_0^t \left(a_s^2 X_s + b_s^2 Y_s\right) dH_s, \\ Y_t = & F X_T + \int_t^T \left(a_s^3 X_s + b_s^3 Y_s + c_s^3 Z_s\right) ds - \int_t^T Z_s dH_s. \end{cases} \tag{2.19}$$

Assume $|a_t^i|, |b_t^i|, |c_t^i| \leq \lambda, i = 1, 2, 3$ *and* $|F| \leq \lambda_0$. *Let* δ *be as in Theorem* 2.1. *And assume further that*

$$b_t^2 c_t^1 = 0; \quad b_t^1 + a_t^2 c_t^1 + b_t^2 c_t^3 = 0. \tag{2.20}$$

Then for $T \leq \delta$,

i) The LFBSDE (2.19) *admits a unique solution.*

ii)

$$|Y_0| \leq \bar{\lambda}_0, \tag{2.21}$$

where

$$\bar{\lambda}_0 = c\left([\lambda_0 + 1] e^{\left(2\lambda + \lambda^2\right)T} - 1 \right). \tag{2.22}$$

Proof. First, we can easily check that LFBSDE (2.19) satisfy assumptions of Theorem 2.1, then it has a unique solution (X_t, Y_t, Z_t) which belongs to the space $M^2(0, T)$. This gives the proof of the assertion (i).

We shall prove the assertion (ii). We split the proof into two steps.

*Step*1. For any $t \in [0, T)$ and any $\xi \in L^2(\mathcal{F}_0)$, we put $\bar{\Pi}_s \triangleq (X_t \xi, Y_t \xi, Z_t \xi)$, $s \in [t, T]$. Then $\bar{\Pi}_s$ satisfies the following linear FBSDE

$$\begin{cases} \bar{X}_s = & X_t \xi + \int\limits_t^s \left[a_r^1 \bar{X}_r + b_r^1 \bar{Y}_r + c_r^1 \bar{Z}_r\right] dr + \int\limits_t^s \left[a_r^2 \bar{X}_r + b_r^2 \bar{Y}_r\right] dH_r, \\ \bar{Y}_s = & F \bar{X}_T + \int\limits_s^T \left[a_r^3 \bar{X}_r + b_r^3 \bar{Y}_r + c_r^3 \bar{Z}_r\right] dr - \int\limits_s^T \bar{Z}_r dH_r. \end{cases}$$

By assertion (i) of Proposition 2.1, we get

$$E\left\{ \left|\bar{Y}_t\right|^2 \right\} = E\left\{ |Y_t \xi|^2 \right\} \leq C_0^2 E\left\{ |X_t \xi|^2 \right\}.$$

Since ξ is arbitrary, we have $|Y_t| \leq C_0 |X_t|$, P-a.s.,$\forall t$.

*Step*2. We define

$$\varrho \triangleq \inf\{t > 0 : X_t = 0\} \wedge T; \quad \text{and } \varrho_n \triangleq \inf\left\{t > 0 : X_t = \frac{1}{n}\right\} \wedge T.$$

Then $\varrho_n \uparrow \varrho$ and $X_t > 0$ for $t \in [0, \varrho)$. We also define the pure jump process η, by the following formula

$$\eta_t = \prod_{0<s\leq t} \left(1 - (X_s)^{-1} \Delta X_s\right) \frac{(X_{s_-})^{-1}}{(X_s)^{-1}}$$

The above product is clearly càdlàg, adapted, converges and is of finite variation. We put for any $t \in [0, \varrho)$,

$$A_t = \eta_t (X_t)^{-1}.$$

It should be noted that when we apply Itô's formula to $(X_t)^{-1}$, a sum of discontinuous quantities appears. To eliminate this, we shall apply Itô's formula to $A_t = \eta_t (X_t)^{-1}$ instead of $(X_t)^{-1}$. Firstly, applying Itô's formula to A_t, we have

$$\begin{aligned} A_t = A_0 &- \textstyle\int_0^t \eta_{s_-} \left(X_{s_-}\right)^{-2} dX_s + \int_0^t \left(X_{s_-}\right)^{-1} d\eta_s + \int_0^t A_{s_-} (X_s)^{-2} d[X,X]_s^c \\ &+ \sum_{0<s\leq t} \Big(A_s - A_{s_-} + A_{s_-} \left(X_{s_-}\right)^{-1} (\Delta X_s) - \left(X_{s_-}\right)^{-1} \Delta\eta_s\Big), \end{aligned} \tag{2.23}$$

Note that η is a pure jump process. Hence $[\eta, X]^c = [\eta, \eta]^c = 0$ and

$$\int_0^t \left(\tilde{X}_{s_-}\right)^{-1} d\eta_s = \sum_{0<s\leq t} \left(\tilde{X}_{s_-}\right)^{-1} \Delta\eta_s.$$

Then (2.23) becomes

$$\begin{aligned} A_t = A_0 &- \textstyle\int_0^t \eta_{s_-} \left(X_{s_-}\right)^{-2} dX_s + \int_0^t A_{s_-} \left(X_{s_-}\right)^{-2} d[X,X]_s^c \\ &+ \sum_{0<s\leq t} \Big(A_s - A_{s_-} + A_{s_-} \left(X_{s_-}\right)^{-1} \Delta X_s\Big), \end{aligned}$$

The following equality is obvious, from the definition of the process A,

$$A_s = A_{s_-}\left(1-(X_t)^{-1}\Delta X_t\right).$$

Now by replacing the above equality into the previous one, one can get

$$\sum_{0<s\leq t}\left(A_s - A_{s_-} + A_{s_-}\left(X_{s_-}\right)^{-1}\Delta X_s\right) = 0.$$

Therefore,

$$A_t = A_0 - \int_0^t A_s\left(X_s\right)^{-1} dX_s + \int_0^t A_{s_-}\left(X_s\right)^{-2} d\left[X,X\right]_s^c,$$

with

$$d\left[X,X\right]_s^c = \sum_{i,j}\left(a_s^{2,i}X_s + b_s^{2,i}Y_s\right)\left(a_s^{2,j}X_s + b_s^{2,j}Y_s\right)q_{i-1}\left(0\right)q_{j-1}\left(0\right)ds.$$

Thanks to Lemma 1.2 in Chapter 1, we get

$$\begin{aligned} d\left[X,X\right]_s^c &= \left[\left(a_s^2X_s + b_s^2Y_s\right)^2 - \sum_{i,j}\left(a_s^{2,i}X_s + b_s^{2,i}Y_s\right)\left(a_s^{2,j}X_s + b_s^{2,j}Y_s\right)\int_{\mathbb{R}} p_i\left(x\right)p_j\left(x\right)v\left(dx\right)\right]ds. \\ &= \left[\left(a_s^2X_s + b_s^2Y_s\right)^2 - \Psi_s\right]ds. \end{aligned}$$

Hence

$$\begin{aligned} A_t &= A_0 - \textstyle\int_0^t \left[A_s\left(X_s\right)^{-1}\left(a_s^1X_s + b_s^1Y_s + c_s^1Z_s\right) - A_s\left(X_s\right)^{-2}\left(a_s^2X_s + b_s^2Y_s\right)^2\right]ds \\ &- \textstyle\int_0^t A_{s_-}\left(X_{s_-}\right)^{-1}\left(a_s^2X_{s_-} + b_s^2Y_{s_-}\right)dH_s - \int_0^t A_s\left(X_s\right)^{-2}\Psi_s ds. \end{aligned}$$

Let us define the following processes

$$\hat{Y}_t = Y_tA_t; \quad \hat{Z}_t \triangleq A_tZ_t - A_t\left(X_t\right)^{-1}Y_t\left(a_t^2X_t + b_t^2Y_t\right).$$

Then after the result of the Step 1, we have

$$\left|\hat{Y}_t\right| \leq C_0.$$

Now, applying Itô's formula to $\hat{Y}_t$, we obtain

$$\begin{aligned}
&d\hat{Y}_t = -A_t\left(a_t^3 X_t + b_t^3 Y_t + c_t^3 Z_t\right) dt \\
&- \left[Y_t A_t \left(X_t\right)^{-1}\left(a_t^1 X_t + b_t^1 Y_t + c_t^1 Z_t\right) dt - A_t \left(X_t\right)^{-2}\left(a_t^2 X_t + b_t^2 Y_t\right)^2\right] dt \\
&- \left[A_t \left(X_t\right)^{-1}\left(a_t^2 X_t + b_t^2 Y_t\right) Z_t\right] dt - \left[Y_t A_t \left(X_t\right)^{-2} \Psi_t\right] dt \\
&+ \left[A_{t_-} Z_t - Y_{t_-} A_{t_-}\left(X_{t_-}\right)^{-1}\left(a_t^2 X_{t_-} + b_t^2 Y_{t_-}\right)\right] dH_t + d\tilde{A}_t,
\end{aligned}$$

where we have denoted by $\tilde{A}_t = [A, Y]_t - \langle A, Y\rangle_t$. By using the definition of the processes $\left(\hat{Y}, \hat{Z}\right)$ it follows that

$$\begin{aligned}
&d\hat{Y}_t = \hat{Z}_t dH_t - \left[c_t^3 + c_t^1 \eta_t^{-1}\hat{Y}_t + a_t^2 + b_t^2 \eta_t^{-1}\hat{Y}_t\right]\hat{Z}_t dt \\
&- \left[c_t^1 b_t^2 \left(\eta_t^{-1}\right)^2 \hat{Y}_t^3 + \left(b_t^1 + a_t^2 c_t^1 + c_t^3 b_t^2\right)\eta_t^{-1}\hat{Y}_t^2\right] dt - \left[a_t^3 \eta_t + \left(b_t^3 + a_t^1 + c_t^3 a_t^2\right)\hat{Y}_t\right] dt \\
&- \left[Y_t A_t \left(X_t\right)^{-2}\Psi_t\right] dt + d\tilde{A}_t.
\end{aligned}$$

Thus, by taking into account (2.20),

$$\begin{aligned}
&d\hat{Y}_t = \hat{Z}_t dH_t - \left[c_t^3 + c_t^1 \eta_t^{-1}\hat{Y}_t + a_t^2 + b_t^2 \eta_t^{-1}\hat{Y}_t\right]\hat{Z}_t dt \\
&- \left[a_t^3 \eta_t + \left(b_t^3 + a_t^1 + c_t^3 a_t^2\right)\hat{Y}_t\right] dt - \left[Y_t A_t \left(X_t\right)^{-2}\Psi_t\right] dt + d\tilde{A}_t.
\end{aligned}$$

We put

$$\Gamma_t = 1 + \int_0^t \Gamma_s \left(X_s\right)^{-2}\Psi_s 1_{\{\varrho > s\}} ds.$$

$$M_t = 1 + \sum_{i=1}^{\infty}\int_0^t \left(q_{i-1}(0)\right)^{-1} M_s \left(\left(c_s^3 + a_s^2\right) + \left(c_s^1 + b_s^2\right)\eta_s^{-1}\hat{Y}_s\right) 1_{\{\varrho > s\}} dW_s;$$

$$N_t = 1 + \int_0^t N_s \left(a_s^1 + b_s^3 + a_s^2 c_s^3\right) 1_{\{\varrho > s\}} ds.$$

Applying Itô's formula to $\left(\Gamma_t N_t M_t \hat{Y}_t\right)$, we obtain

$$\begin{aligned}
& d\left(\Gamma_t N_t M_t \hat{Y}_t\right) = \Gamma_t N_t M_t \hat{Z}_t 1_{\{\varrho>t\}} dH_t \\
& + \sum_{i=1}^{\infty} \left(\Gamma_t \hat{Y}_t N_t M_t \left(q_{i-1}(0)\right)^{-1} \left(c_t^3 + a_t^2\right) + \left(c_t^1 + b_t^2\right) \eta_t^{-1} \hat{Y}_t\right) 1_{\{\varrho>t\}} dW_t \\
& - \eta_t \Gamma_t N_t M_t a_t^3 1_{\{\varrho>s\}} dt + \Gamma_t M_t N_t d\tilde{A}_t.
\end{aligned}$$

Taking expectations, we get

$$Y_0 = \mathbb{E}\left(\Gamma_{\varrho_n} N_{\varrho_n} M_{\varrho_n} \hat{Y}_{\varrho_n} + \int_0^{\varrho_n} \eta_t \Gamma_t N_t M_t a_t^3 dt\right). \tag{2.24}$$

Since $\left|\hat{Y}_t\right| \leq C_0$, M is an $\mathcal{F}_t$–martingale and $|N_t| \leq e^{\left(2\lambda+\lambda^2\right)t}$. Moreover, we observe that if $\tau = T$, $|Y_\tau| = |Y_T| = |FX_T| = |FX_\tau| \leq \lambda_0 |X_\tau|$.

If $\tau < T$, $X_\tau = 0$, and thus $|Y_\tau| \leq C_0 |X_\tau| = 0$.

Therefore, in both cases it holds that $|Y_\tau| \leq \lambda_0 |X_\tau|$.

Now, applying Ito's formula to $|Y_t|^2$ from $s = \varrho_n$ to $s = \varrho$, we obtain

$$\begin{aligned}
& |Y_{\varrho_n}|^2 + \mathbb{E}\left(\int_{\varrho_n}^{\varrho} \|Z_t\|^2_{\mathcal{P}^2(\mathbb{R})} dt \mid \mathcal{F}_{\varrho_n}\right) = \mathbb{E}\left(|Y_\tau|^2 + 2\int_{\varrho_n}^{\varrho} Y_t \left(a_t^3 X_t + b_t^3 Y_t + c_t^3 Z_t\right) dt \mid \mathcal{F}_{\varrho_n}\right) \\
& \leq \mathbb{E}\left(\lambda_0^2 |X_\varrho|^2 + C\int_{\varrho_n}^{\varrho} \left(|X_t|^2 + |Y_t|^2\right) dt + \frac{1}{2}\int_{\varrho_n}^{\varrho} \|Z_t\|^2_{\mathcal{P}^2(\mathbb{R})} dt \mid \mathcal{F}_{\varrho_n}\right).
\end{aligned}$$

Similarly, applying Ito's formula to $|X_t|^2$ from $s = \varrho_n$ to $s = \varrho$, we obtain,

$$\mathbb{E}\left(|X_\varrho|^2 \mid \mathcal{F}_{\varrho_n}\right) \leq \mathbb{E}\left(|X_\varrho|^2 + C\int_{\varrho_n}^{\varrho} \left(|X_t|^2 + |Y_t|^2\right) dt + \frac{1}{2\lambda_0^2}\int_{\varrho_n}^{\varrho} \|Z_t\|^2_{\mathcal{P}^2(\mathbb{R})} dt \mid \mathcal{F}_{\varrho_n}\right).$$

Thus

$$|Y_{\varrho_n}|^2 \leq \mathbb{E}\left(\lambda_0^2 |X_{\varrho_n}|^2 + C\int_{\varrho_n}^{\varrho} \left(|X_t|^2 + |Y_t|^2\right) dt \mid \mathcal{F}_{\varrho_n}\right).$$

Note that $|X_{\varrho_n}| \geq \frac{1}{n}$, then

$$\begin{aligned}\left|\hat{Y}_{\varrho_n}\right| &\leq \lambda_0 \left|\eta_{\varrho_n}\right| + C\mathbb{E}^{\frac{1}{2}} \left(\int_{\varrho_n}^{\varrho} \left(\left|\tilde{X}_t\right|^2 + \left|\tilde{Y}_t\right|^2 \right) dt \mid \mathcal{F}_{\varrho_n} \right) \\ &\leq \lambda_0 \left|\eta_{\varrho_n}\right| + C\mathbb{E}^{\frac{1}{2}} \left(\sup_{\varrho_n \leq t \leq \varrho} \left(\left|\tilde{X}_t\right|^2 + \left|\tilde{Y}_t\right|^2 \right) (\varrho - \varrho_n) \mid \mathcal{F}_{\varrho_n} \right),\end{aligned}$$

where

$$\tilde{X}_t \triangleq X_t \left|\eta_{\varrho_n}\right| (X_{\varrho_n})^{-1} \ ; \ \ \tilde{Y}_t \triangleq Y_t \left|\eta_{\varrho_n}\right| (X_{\varrho_n})^{-1}.$$

Now by (2.24), we get

$$\begin{aligned}&\left|\hat{Y}_0\right| \leq \lambda E\left(\Gamma_t M_t\right) \int_0^T \left|\eta_t\right| e^{\left(2\lambda+\lambda^2\right)t} dt \\ &+\mathbb{E}\left\{ e^{\left(2\lambda+\lambda^2\right)T} M_{\varrho_n} \Gamma_{\varrho_n} \left(\left|\eta_{\varrho_n}\right| \lambda_0 + C\mathbb{E}^{\frac{1}{2}} \left(\sup_{\varrho_n \leq t \leq \varrho} \left(\left|\tilde{X}_t\right|^2 + \left|\tilde{Y}_t\right|^2 \right) (\varrho - \varrho_n) \mid \mathcal{F}_{\varrho_n} \right) \right) \right\} \\ &\leq c' \left(e^{\left(2\lambda+\lambda^2\right)T} - 1 \right) + c'' \lambda_0 e^{\left(2\lambda+\lambda^2\right)T} \\ &+C\mathbb{E}\left\{ M_{\varrho_n} \Gamma_{\varrho_n} \mathbb{E}^{\frac{1}{2}} \left(\sup_{\varrho_n \leq t \leq \varrho} \left(\left|\tilde{X}_t\right|^2 + \left|\tilde{Y}_t\right|^2 \right) (\varrho - \varrho_n) \mid \mathcal{F}_{\varrho_n} \right) \right\} \\ &\leq \bar{\lambda}_0 + C\mathbb{E}^{\frac{1}{2}} \left(\left|M_{\varrho_n}\right|^2 \left|\Gamma_{\varrho_n}\right|^2 \right) \mathbb{E}^{\frac{1}{2}} \left(\sup_{\varrho_n \leq t \leq \varrho} \left(\left|\tilde{X}_t\right|^2 + \left|\tilde{Y}_t\right|^2 \right) (\varrho - \varrho_n) \right) \\ &\leq \bar{\lambda}_0 + C\mathbb{E}^{\frac{1}{4}} \left(\sup_{\varrho_n \leq t \leq \varrho} \left(\left|\tilde{X}_t\right|^4 + \left|\tilde{Y}_t\right|^4 \right) \right) \mathbb{E}^{\frac{1}{4}} \left(\left|\varrho - \varrho_n\right|^2 \right).\end{aligned}$$

Note that $\left(\tilde{X}_t, \tilde{Y}_t\right)$ satisfies the following LFBSDE:

$$\begin{cases} \tilde{X}_t = & 1 + \int_0^t \left[a_r^1 1_{\{\varrho_n \leq r\}} \tilde{X}_r + b_r^1 1_{\{\varrho_n \leq r\}} \tilde{Y}_r + c_r^1 1_{\{\varrho_n \leq r\}} \tilde{Z}_r \right] dr \\ & + \int_0^t \left[a_r^2 1_{\{\varrho_n \leq r\}} \tilde{X}_r + b_r^2 1_{\{\varrho_n \leq r\}} \tilde{Y}_r \right] dH_r, \\ \tilde{Y}_t = & F\tilde{X}_T + \int_t^T \left[a_r^3 1_{\{\varrho_n \leq r\}} \tilde{X}_r + b_r^3 1_{\{\varrho_n \leq r\}} \tilde{Y}_r + c_r^3 1_{\{\varrho_n \leq r\}} \tilde{Z}_r \right] dr - \int_t^T \tilde{Z}_r dH_r. \end{cases}$$

By (ii) of Proposition 2.1 (We choose $p = 2$), we have

$$\mathbb{E}\left(\sup_{\varrho_n \leq t \leq \varrho} \left(\left|\tilde{X}_t\right|^4 + \left|\tilde{Y}_t\right|^4 \right) \right) \leq \mathbb{E}\left(\sup_{0 \leq t \leq T} \left(\left|\tilde{X}_t\right|^4 + \left|\tilde{Y}_t\right|^4 \right) \right) \leq C_1.$$

Thus

$$\left|\hat{Y}_0\right| \leq \bar{\lambda}_0 + C\mathbb{E}^{\frac{1}{4}}\left(|\varrho - \varrho_n|^2\right).$$

Then for $n \to \infty$, we get $\left|\hat{Y}_0\right| \leq \bar{\lambda}_0$.That is, $|Y_0| \leq \bar{\lambda}_0 |X_0| |\eta_0| = \bar{\lambda}_0$.This complete the proof. ■

The following result is important.

Proposition 2.2 *Let* $\Pi^i, i = 0, 1,$ *be the solution to FBSDEs:*

$$\begin{cases} X_t^i = & x_i + \int_0^t f\left(s, \Pi_s^i\right) ds + \int_0^t \sigma\left(s, X_{s-}^i, Y_{s-}^i\right) dH_s, \\ Y_t^i = & \varphi\left(X_T^i\right) + \int_t^T g\left(s, \Pi_s^i\right) ds - \int_t^T Z_s^i dH_s. \end{cases}$$

Assume that $(\mathbf{H}_{2.1})$ *and* $(\mathbf{H}_{2.3})$ *(for* $p = 1$*) are satisfied. Then*

$$\left|Y_0^1 - Y_0^0\right| \leq \bar{\lambda}_0 \left|x_1 - x_0\right|,$$

where $\bar{\lambda}_0$ *is defined by* (2.22).

Proof. The proof is the same as in Corollary 1 in [53], by replacing the Brownian part by the Teugels martingales and using the above lemma. ■

Now we are able to state and prove of our main result by using similar arguments introduced in [53] consisting in solving the system iteratively in small intervals having fixed length.

Theorem 2.2 *Assume* $(\mathbf{H}_{2.2})$ *and* $(\mathbf{H}_{2.3})$ *(for* $p = 1$*) are satisfied. Then*

i) Equation (2.1) *has a unique solution* $\Pi \in M^2(0, T)$.

ii) The following estimate holds

$$\|\Pi\|^2 \leq CV_0^2.$$

Proof. First we prove (i). Let λ and λ_0 be as in Theorem 2.1, and $\bar{\lambda}_0$ is a constant defined as in (2.22). Let δ be a constant as in Theorem 2.1, but corresponding to λ and $\bar{\lambda}_0$ instead of λ and λ_0. For some integer n, we assume $(n-1)\,\delta < T \leq n\delta$ and consider a partition of $[0,T]$, with $T_i \triangleq \frac{iT}{n}, i = 0, ..., n$.

We consider the mapping:

$$\begin{aligned} G_n : \quad \Omega \times \mathbb{R} \rightarrow & \quad \mathbb{R} \\ \omega \times x \mapsto & \quad \varphi(\omega, x) \end{aligned}$$

Let us consider the following FBSDE over the small interval $[T_{n-1}, T_n]$,

$$\begin{cases} X_t^n = & x + \int_{T_{n-1}}^t f(s, \Pi_s^n)\, ds + \int_{T_{n-1}}^t \sigma\left(s, X_{s_-}^n, Y_{s_-}^n\right) dH_s, \\ Y_t^n = & G_n\left(X_{T_n}^n\right) + \int_t^{T_n} g(s, \Pi_s^n)\, ds - \int_t^{T_n} Z_s^n dH_s. \end{cases} \tag{2.25}$$

Let L_{G_n} denotes the Lipschitz constant of the mapping G_n. Then, by Theorem 2.1 the required solution of FBSDE (2.25) exists and is unique. Define $G_{n-1}(x) \triangleq Y_{T_{n-1}}^n$, then for fixed x, $G_{n-1}(x) \in \mathcal{F}_{T_{n-1}}$. Further, in view of the Proposition 2.2, it's straightforward to verify that

$$L_{G_{n-1}} \leq \lambda_1 \triangleq c\left(\left[\lambda_0 + 1\right] e^{\left(2\lambda+\lambda^2\right)\left(T_n - T_{n-1}\right)} - 1\right) \leq \bar{\lambda}_0.$$

Next, for $t \in [T_{n-2}, T_{n-1}]$, we consider the following FBSDE:

$$\begin{cases} X_t^{n-1} = & x + \int_{T_{n-2}}^t f(s, \Pi_s^{n-1})\, ds + \int_{T_{n-2}}^t \sigma\left(s, X_{s_-}^{n-1}, Y_{s_-}^{n-1}\right) dH_s, \\ Y_t^{n-1} = & G_{n-1}\left(X_{T_{n-1}}^{n-1}\right) + \int_t^{T_{n-1}} g(s, \Pi_s^{n-1})\, ds - \int_t^{T_{n-1}} Z_s^{n-1} dH_s. \end{cases} \tag{2.26}$$

Once again, since $L_{G_{n-1}} \leq \bar{\lambda}_0$, by Theorem 2.1, the FBSDE (2.26) has a unique solution. Then as well, we may define $G_{n-2}(x)$, such that

$$\begin{aligned} L_{G_{n-2}} \leq \lambda_2 \triangleq & \; c\left(\left[\lambda_1 + 1\right] e^{\left(2\lambda+\lambda^2\right)\left(T_{n-1} - T_{n-2}\right)} - 1\right) \\ = & \; c\left(\left[\lambda_0 + 1\right] e^{\left(2\lambda+\lambda^2\right)\left(T_n - T_{n-2}\right)} - 1\right) \leq \bar{\lambda}_0. \end{aligned}$$

Repeating this procedure backwardly for $i = n, ..., 1$, we may define G_i such that

$$L_{G_i} \leq \lambda_{n-i} \triangleq c\left(\left[\lambda_0 + 1\right] e^{\left(2\lambda+\lambda^2\right)\left(T_n - T_i\right)} - 1\right) \leq \bar{\lambda}_0.$$

As a conclusion, one can repeat the above construction and, after a finite number of steps, we obtain the required unique solution in each subinterval of the type $[T_{i-1}, T_i]$ for $i = 0, ..., n$.

Now, we prove by induction that FBSDE (2.1) has a unique solution in the whole time interval $[0, T]$. Our starting point is the small time interval $[0, T_i]$ with $i = 1$, we know from the first stage that the following FBSDE has a unique solution in the small time interval $[0, T_1]$ for any $X_0 \in L^2(\mathcal{F}_0)$ and $t \in [0, T_1]$

$$\begin{cases} X_t = & X_0 + \int_0^t f(s, \Pi_s)\, ds + \int_{T_{i-1}}^t \sigma(s, X_{s-}, Y_{s-})\, dH_s, \\ Y_t = & G_1(X_{T_1}) + \int_t^{T_1} g(s, \Pi_s)\, ds - \int_t^{T_1} Z_s dH_s. \end{cases}$$

Once again, for $i = 2$, one can easily check that the following FBSDE

$$\begin{cases} X_t = & X_{T_1} + \int_{T_1}^t f(s, \Pi_s)\, ds + \int_{T_1}^t \sigma(s, X_{s-}, Y_{s-})\, dH_s, \\ Y_t = & G_2(X_{T_2}) + \int_t^{T_2} g(s, \Pi_s)\, ds - \int_t^{T_2} Z_s dH_s. \end{cases} \quad t \in [T_1, T_2]$$

has a unique solution in the time interval $[0, T_2]$.

Repeating this procedure forwardly for $i = 2, ..., n$ and for any $X_0 \in L^2(\mathcal{F}_0)$, by extending the time interval piece by piece, we construct a solution for the following FBSDE

$$\begin{cases} X_t = & X_{T_{i-1}} + \int_{T_{i-1}}^t f(s, \Pi_s)\, ds + \int_{T_{i-1}}^t \sigma(s, X_{s-}, Y_{s-})\, dH_s, \\ Y_t = & G_i(X_{T_i}) + \int_t^{T_i} g(s, \Pi_s)\, ds - \int_t^{T_i} Z_s dH_s. \end{cases} \quad t \in [T_{i-1}, T_i]$$

Clearly this provides a solution to the FBSDE (2.1). From the construction and the uniqueness of each step, it is clear that this solution is unique.

Now, we turn out to prove (ii). Denote

$$V_t^2 = |f(t, 0, 0, 0)|^2 + \|\sigma(t, 0, 0)\|^2_{\mathcal{P}^2(\mathbb{R})} + |g(t, 0, 0, 0)|^2.$$

From Theorem 2.1 and by the definition of G_i, we get

$$E\left\{\left|G_{i-1}(0)\right|^2\right\} \leq C_0 E\left\{\left|G_i(0)\right|^2 + \int_{T_{i-1}}^{T_i} V_t^2 dt\right\}.$$

By induction one can easily prove that

$$\begin{aligned}&\max_{0\leq i\leq n} E\left\{\left|G_i(0)\right|^2\right\} \leq C_0^n E\left\{\left|\varphi(0)\right|^2 + \textstyle\int_0^T V_t^2 dt\right\}\\ &= CE\left\{\left|\varphi(0)\right|^2 + \textstyle\int_0^T V_t^2 dt\right\}.\end{aligned}$$

Set $n \leq \frac{T}{\delta} + 1$ is a fixed constant depending only on λ, λ_0 and T, then so is C. Now for $t \in [T_0, T_1]$, by using (ii) of Theorem 2.1, we get

$$\begin{aligned}&E\left\{\sup_{0\leq t\leq T}|X_t|^2 + \sup_{0\leq t\leq T}|Y_t|^2\right\} \leq CE\left\{|X_0|^2 + |G_1(0)|^2 + \textstyle\int_{T_0}^{T_1} V_t^2 dt\right\}\\ &\leq CE\left\{|X_0|^2 + |\varphi(0)|^2 + \textstyle\int_0^T V_t^2 dt\right\}.\end{aligned}$$

Then by induction one can prove

$$E\left\{\sup_{0\leq t\leq T}|X_t|^2 + \sup_{0\leq t\leq T}|Y_t|^2\right\} \leq CE\left\{|X_0|^2 + |\varphi(0)| + \int_0^T V_t^2 dt\right\}. \tag{2.27}$$

On the other hand, applying Ito's formula to Y_t , we obtain

$$\begin{aligned}&E\left\{|Y_0|^2 + \textstyle\int_0^T |Z_t|^2 dt\right\} = E\left\{|Y_T|^2 + 2\textstyle\int_0^T Y_t g(t, \Pi_t) dt\right\}\\ &\leq E\left\{|Y_T|^2 + C\textstyle\int_0^T \left[|g(t,0,0,0)|^2 + |X_t|^2 + |Y_t|^2\right] dt + \frac{1}{2}\textstyle\int_0^T \|Z_t\|^2_{\mathcal{P}^2(\mathbb{R})} dt\right\}.\end{aligned}$$

Therefore

$$E\left\{\int_0^T |Z|_t^2 dt\right\} \leq CE\left\{|X_0|^2 + |\varphi(0)| + \int_0^T V_t^2 dt\right\}. \tag{2.28}$$

Finally, combining (2.27) and (2.28) leads to $\|\Pi\|^2 \leq CV_0^2$, which achieves the proof. ■

2.3 Proprieties of solutions

In this section, we establish some further properties of the solution to the FBSDE (2.1). These will include a stability result and a comparison theorem for FBSDE.

2.3.1 Stability theorem

The following results state the stability of the solution of FBSDE (2.1) with respect to the initial condition and the data. This means that the solution of equation (2.1) does not change too much under small perturbations of the data. In other words, the trajectories which are close to each other at specific instant should therefore remain close to each other at all subsequent instants. To state the next theorem and its corollary, let us consider $\Pi^i, i = 0, 1$ the solutions of (2.1) corresponding to $(f^i, \sigma^i, g^i, \varphi^i)$. We shall consider the following notations, $\Delta\Pi \triangleq \Pi^1 - \Pi^0$ and for any function $h \triangleq f, \sigma, g, \varphi$, we set $\Delta h \triangleq h^1 - h^0$. We now give the stability of the solutions.

Theorem 2.3 *Assume that* $(f^i, \sigma^i, g^i, \varphi^i, X_0^i)\,, i = 0, 1,$ *satisfy the same conditions of Theorem 2.2. Then*

$$\|\Delta\Pi\|^2 \leq C\mathbb{E}\left\{|\Delta X_0|^2 + \left|\Delta\varphi\left(X_T^1\right)\right|^2 + \int_0^T \left[|\Delta f|^2 + \|\Delta\sigma\|^2_{\mathcal{P}^2(\mathbb{R})} + |\Delta g|^2\right]\left(t, \Pi_t^1\right) dt\right\}.$$

Proof. For $0 \leq \varepsilon \leq 1$, let Π^ε be the solution to the following FBSDE:

$$\begin{cases} X_t^\varepsilon = & X_0 + \varepsilon\Delta X_0 + \int_0^t \left(f^0\left(s, \Pi_s^\varepsilon\right) + \varepsilon\Delta f\left(s, \Pi_s^1\right)\right) ds \\ & + \int_0^t \left(\sigma^0\left(s, X_{s_-}^\varepsilon, Y_{s_-}^\varepsilon\right) + \varepsilon\Delta\sigma\left(s, X_{s_-}^1, Y_{s_-}^1\right)\right) dH_s; \\ Y_t^\varepsilon = & \left(\varphi^0\left(X_T^\varepsilon\right) + \varepsilon\Delta\varphi\left(X_T^1\right)\right) + \int_t^T \left(g^0\left(s, \Pi_s^\varepsilon\right) + \varepsilon\Delta g\left(s, \Pi_s^1\right)\right) ds - \int_t^T Z_s^\varepsilon dH_s. \end{cases}$$

and $\nabla\Pi^{\varepsilon}$ be the solution of the following variational linear FBSDE

$$\begin{cases} \nabla X_t^{\varepsilon} = & \Delta X_0 + \int_0^t \left(f_x^0\left(s,\Pi_s^{\varepsilon}\right)\nabla X_s^{\varepsilon} + f_y^0\left(s,\Pi_s^{\varepsilon}\right)\nabla Y_s^{\varepsilon} + f_z^0\left(s,\Pi_s^{\varepsilon}\right)\nabla Z_s^{\varepsilon} + \Delta f\left(s,\Pi_s^1\right)\right)ds \\ & + \int_0^t \left(\sigma_x^0\left(s,X_{s_-}^{\varepsilon},Y_{s_-}^{\varepsilon}\right)\nabla X_s^{\varepsilon} + \sigma_y^0\left(s,X_{s_-}^{\varepsilon},Y_{s_-}^{\varepsilon}\right)\nabla Y_s^{\varepsilon} + \Delta\sigma\left(s,\Pi_s^1\right)\right)dH_s; \\ \nabla Y_t^{\varepsilon} = & \varphi_x^0\left(X_T^{\varepsilon}\right) + \Delta\varphi\left(X_T^1\right) + \int_t^T \left(g_x^0\left(s,\Pi_s^{\varepsilon}\right)\nabla X_s^{\varepsilon} + g_y^0\left(s,\Pi_s^{\varepsilon}\right)\nabla Y_s^{\varepsilon} + g_z^0\left(s,\Pi_s^{\varepsilon}\right)\nabla Z_s^{\varepsilon}\right. \\ & \left.+\Delta g\left(s,\Pi_s^1\right)\right)ds - \int_t^T \nabla Z_s^{\varepsilon}dH_s; \end{cases}$$

Then by Theorem 2.1, the above FBSDEs has a unique solution. Moreover, a simple calculation shows that

$$\Delta\Pi_t = \int_0^1 \frac{d}{d\varepsilon}\Pi_t^{\varepsilon}d\varepsilon = \int_0^1 \nabla\Pi_t^{\varepsilon}d\varepsilon.$$

since (f^0,σ^0,g^0) satisfies (2.20), by Lemma 2.2, we obtain

$$\|\Delta\Pi^{\varepsilon}\|^2 \leq CE\left\{|\Delta X_0|^2 + \left|\Delta\varphi\left(X_T^1\right)\right|^2 + \int_0^T \left[|\Delta f|^2 + |\Delta\sigma|^2 + |\Delta g|^2\right]\left(t,\Pi_t^1\right)dt\right\},$$

which implies the desired result. ■

Corollary 2.1 *Suppose that* $(f^n,\sigma^n,\varphi^n,g^n,X_0^n)$, *for* $n=0,1...$ *satisfy the same conditions of Theorem* 2.2*. Moreover assume that:*

i) $X_0^n \to X_0^0$ *in* L^2.

ii) For $h \triangleq f,\sigma,\varphi,g$, $h^n(t,\Pi) \to h^0(t,\Pi)$ *as* $n\to\infty$.

iii) $\mathbb{E}\left\{|X_0^n - X_0^0|^2 + |\varphi^n-\varphi^0|^2(0) + \int_0^T \left[|f^n-f^0|^2 + \|\sigma^n-\sigma^0\|_{\mathcal{P}^2(\mathbb{R})}^2 + |g^n-g^0|^2\right](t,0,0,0)\,dt\right\} \to 0$

Then if Π^n *(resp.*Π*) denotes the solution of* (2.1) *corresponding to* $(f^n,\sigma^n,\varphi^n,g^n,X_0^n)$ *(resp.* $(f,\sigma,\varphi,g,X_0^0)$*), we obtain*

$$\left\|\Pi^n - \Pi^0\right\| \to 0 \text{ as } n \longrightarrow +\infty.$$

Proof. Using Theorem 2.3 we have

$$\|\Pi^n - \Pi^n\|^2 \leq CE\left\{\left|X_0^n - X_0^0\right|^2 + \left|\varphi^n - \varphi^0\right|^2 (X_T^0)\right.$$
$$\left. + \int_0^T \left[\left|f^n - f^0\right|^2 + \|\sigma^n - \sigma^0\|^2_{\mathcal{P}^2(\mathbb{R})} + \left|g^n - g^0\right|^2\right](t, \Pi_t^0)\, dt\right\}.$$

Thus, the desired result follows immediately, by letting n tend to 0, and using the dominated convergence theorem. ■

2.3.2 Comparison theorem

In what follows we provide, under the same assumptions as for the existence and uniqueness results, another important result, which is the comparison theorem. Let (X, Y, Z) be the solution to the following LFBSDE:

$$\begin{cases} X_t = & \int_0^t (a_s^1 X_s + b_s^1 Y_s + c_s^1 Z_s)\, ds + \int_0^t (a_s^2 X_s + b_s^2 Y_s)\, dH_s, \\ Y_t = & P X_T + \alpha + \int_t^T (a_s^3 X_s + b_s^3 Y_s + c_s^3 Z_s + \beta_s)\, ds - \int_t^T Z_s dH_s. \end{cases} \tag{2.29}$$

Firstly, we state and show the following proposition,, which is the linear version of the comparison theorem.

Proposition 2.3 *Assume* $|a_t^i|, |b_t^i|, |c_t^i| \leq \lambda$, $|P| \leq \lambda_0$ *and* $(\mathbf{H}_{2.2})$ *holds true. Assume further that* $\alpha \geq 0$ *and* $\beta_s \geq 0$*. Then*

$$Y_0 \geq 0.$$

In order to prove Proposition 2.3, we need the following two Lemmas. Let us introduce the following linear FBSDE

$$\begin{cases} X_t = & \int_0^t \left(\bar{a}_s^1 X_s + \bar{b}_s^1 \bar{Y}_s + \bar{c}_s^1 \bar{Z}_s\right) ds + \int_0^t \left(\bar{a}_s^2 X_s + \bar{b}_s^2 \bar{Y}_s\right) dH_s, \\ \bar{Y}_t = & \int_t^T \left(\bar{a}_s^3 X_s + \bar{b}_s^3 \bar{Y}_s + \bar{c}_s^3 \bar{Z}_s\right) ds - \int_t^T \bar{Z}_s dH_s. \end{cases} \tag{2.30}$$

Here, $\bar{Y}_t \triangleq Y_t - P_t X_t$, $\bar{Z}_t \triangleq Z_t - P_t\left(a_t^2 X_t + b_t^2 Y_t\right) - g_t X_t$, where $P = E(P) + \int_0^T p_t dH_t$, $P_t \triangleq E(P) + \int_0^t p_t dH_t$;and

$$\begin{cases} \bar{a}_t^1 \triangleq a_t^1 + P_t b_t^1 + P_t a_t^2 c_t^1 + \left|P_t\right|^2 b_t^2 c_t^1 + p_t c_t^1; \\ \bar{b}_t^1 \triangleq b_t^1 + P_t b_t^2 c_t^1 = b_t^1; \\ \bar{c}_t^1 \triangleq c_t^1; \\ \bar{a}_t^2 \triangleq a_t^2 + P_t b_t^2; \\ \bar{b}_t^2 \triangleq b_t^2; \\ \bar{a}_t^3 \triangleq a_t^3 + p_t a_t^2 + P_t a_t^1 + \left(b_t^3 + p_t b_t^2 + P_t b_t^1\right) P_t \\ \qquad + \left(c_t^3 + P_t c_t^1\right)\left(p_t + P_t a_t^2 + \left|P_t\right|^2 b_t^2\right); \\ \bar{b}_t^3 \triangleq b_t^3 + p_t b_t^2 + P_t b_t^1 + P_t b_t^2 c_t^3 + \left|P_t\right|^2 b_t^2 c_t^1; \\ \bar{c}_t^3 \triangleq c_t^3 + P_t c_t^1. \end{cases}$$

Lemma 2.3 *Let (X, Y, Z) be the solution of LFBSDE (2.29), assume $\beta = 0$ and $p \leq C$ .Then $\left(X, \tilde{Y}, \tilde{Z}\right)$ is the solution of the linear FBSDE* (2.30)*.*

Proof. By the definition of P_t, $\bar{Y}_t$ and $\bar{Z}_t$, we get

$$\begin{aligned} dX_t &= \left(a_t^1 X_t + b_t^1\left(\bar{Y}_t + P_t X_t\right) + c_t^1\left(\bar{Z}_t + P_t a_t^2 X_t + P_t b_t^2\left(\bar{Y}_t + P_t X_t\right) + p_t X_t\right)\right) dt \\ &+ \left(a_t^2 X_t + b_t^2\left(\bar{Y}_t + P_t X_t\right)\right) dH_t \\ &= \left(\bar{a}_t^1 X_t + \bar{b}_t^1 \bar{Y}_t + \bar{c}_t^1 \bar{Z}_t\right) dt + \left(\bar{a}_t^2 X_t + \bar{b}_t^2 \bar{Y}_t\right) dH_t, \end{aligned}$$

and

$$\begin{aligned} d\bar{Y}_t &= -\left(a_t^3 X_t + b_t^3 Y_t + c_t^3 Z_t\right) dt + Z_t dH_t - p_t\left(a_t^2 X_t + b_t^2 Y_t\right) dt \\ &-P_t\left(a_t^1 X_t + b_t^1 Y_t + c_t^1 Z_t\right) dt - P_t\left(a_t^2 X_t + b_t^2 Y_t\right) dH_t - p_t X_t dH_t \\ &= \bar{Z}_t dH_t - \left[\left(a_t^3 + p_t a_t^2 + P_t a_t^1\right) X_t + \left(b_t^3 + p_t b_t^2 + G_t b_t^1\right)\left(\bar{Y}_t + p_t X_t\right)\right. \\ &\left. + \left(c_t^3 + P_t c_t^1\right)\left(\bar{Z}_t + \left(p_t + P_t a_t^2\right) X_t + P_t b_t^2\left(\bar{Y}_t + P_t X_t\right)\right) dt\right] \\ &= -\left(\bar{a}_t^3 X_t + \bar{b}_t^3 \bar{Y}_t + \bar{c}_t^3 \bar{Z}_t\right) dt + \bar{Z}_t dH_t, \end{aligned}$$

Is easy to prove that $\bar{a}_t^i, \bar{b}_t^i, \bar{c}_t^i$ are bounded and still satisfy the assumptions (2.20). Then this gives the desired result. ■

Lemma 2.4 *Assume $\alpha = 0$, $c_t^3 = 0$, for some integer m, we assume $\frac{1}{m} \leq \kappa_2 \leq m$. Then there exist small constants δ and C depending on λ and λ_0, such that $T \leq \delta$, and that for some $\varepsilon > 0$,*

$$\left| E\left(PX_t + \int_0^T \left(a_t^3 X_t + b_t^3 Y_t\right) dt\right)\right| \leq Cm\sqrt{\varepsilon}T.$$

Proof. By standard arguments and using Young's inequality, for every $\varepsilon > 0$, there exist constant C depending only on λ, λ_0, that

$$\sup_{0\leq t\leq T} E\left(|X_t|^2 + |Y_t|^2\right) + E\left(\int_0^T \|Z_t\|^2_{\mathcal{P}^2(\mathbb{R})}\, dt\right) \leq C\varepsilon^{-1} E\left(\int_0^T \left(|X_t|^2 + |Y_t|^2\right) dt\right) + \frac{\varepsilon}{2} E\left(\int_0^T |\beta_t|^2\, dt\right)$$
$$\leq C\varepsilon^{-1} T \sup\nolimits_{0\leq t\leq T} E\left(|X_t|^2 + |Y_t|^2\right) + \frac{\varepsilon}{2} m^2 T.$$

If we choose the constant $\delta = \dfrac{\epsilon}{2C}$ and will specify ε later. Then for $T \leq \delta$, we get

$$\sup_{0\leq t\leq T} E\left(|X_t|^2 + |Y_t|^2\right) + E\left(\int_0^T \|Z_t\|^2_{\mathcal{P}^2(\mathbb{R})}\, dt\right) \leq m^2 \varepsilon T.$$

And

$$E\left(|X_t|^2\right) \leq CE\left(\left|\int_0^T \left(a_t^1 X_t + b_t^1 Y_t + c_t^1 Z_t\right) dt\right|^2 + \left|\int_0^T \left(a_t^2 X_t + b_t^2 Y_t\right) dH_t\right|^2\right)$$
$$\leq CE\left(T\int_0^T \left(|X_t|^2 + |Y_t|^2 + \|Z_t\|^2_{\mathcal{P}^2(\mathbb{R})}\right) dt + \int_0^T \left(|X_t|^2 + |Y_t|^2\right) dt\right)$$
$$\leq Cm^2 \varepsilon T^2.$$

Thus

$$\left|E\left(PX_t\right) + \int_0^T \left(a_t^3 X_t + b_t^3 Y_t\right) dt\right|$$
$$\leq CE^{\frac{1}{2}}\left(|X_T|^2\right) + CT \sup_{0\leq t\leq T} E^{\frac{1}{2}}\left(|X_t|^2 + |Y_t|^2\right) \leq Cm\sqrt{\varepsilon}T.$$

This ends the proof. ■

Proof of Proposition.2.3. The proof of this proposition will be divided into several steps.

Step 1. Assume that $P = 0$ and $\beta = 0$. If $Y_0 < 0$, let us define the following stopping time

$$\tau \triangleq \inf\{t : Y_t = 0\} \wedge T.$$

Since $Y_T = \alpha \geq 0$, we get $Y\tau = 0$. Define

$$\hat{a}_t^i \triangleq a_t^i 1_{\{\tau > t\}}; \hat{b}_t^i \triangleq b_t^i 1_{\{\tau > t\}}; \hat{c}_t^i \triangleq c_t^i 1_{\{\tau > t\}}$$

$$\hat{X}_t \triangleq X_{\tau \wedge t}; \quad \hat{Y}_t \triangleq Y_{\tau \wedge t}; \quad \hat{Z}_t \triangleq Z_{\tau \wedge t}$$

In view of Lemma 2.2, the following LFBSDE:

$$\begin{cases} \hat{X}_t = & \int_0^t \left(\hat{a}_s^1 \hat{X}_s + \hat{b}_s^1 \hat{Y}_s + \hat{c}_s^1 \hat{Z}_s \right) ds + \int_0^t \left(\hat{a}_s^2 \hat{X}_s + \hat{b}_s^2 \hat{Y}_s \right) dH_s, \\ \hat{Y}_t = & \int_t^T \left(\hat{a}_s^3 \hat{X}_s + \hat{b}_s^3 \hat{Y}_s + \hat{c}_s^3 \hat{Z}_s \right) ds - \int_t^T \hat{Z}_s dH_s, \end{cases}$$

has a unique solution, with $\hat{Y}_T = 0$.That is to say $Y_0 = \hat{Y}_0 = 0$, obviously this leads to a contradiction. In other words, we have proved that $Y_0 \geq 0$.

Step 2. Assume that all the conditions in Lemma 2.3 are fulfilled, then $\bar{Y}_T = \alpha \geq 0$. Applying *Step* 1 we get $Y_0 = \hat{Y}_0 \geq 0$.

Step 3. Assume $\beta = 0$. One can find P_n satisfying the condition in Lemma 2.3 such that $P_n \to P$ a.s. and $|P_n| \leq \lambda$. Let (X^n, Y^n, Z^n) denotes the solution corresponding to G_n. Apply the result of *Step*2 to conclude that $Y_0^n \geq 0$. Then from Corollary 2.1, we get $Y_0 = \lim_{n\to\infty} Y_0^n \geq 0$.

Step 4. Assume all the conditions in Lemma 2.4 are in force. Then

$$\begin{aligned} & Y_0 = E\left(PX_T + \int_0^T \left(a_t^3 X_t + b_t^3 Y_t + \beta_t\right) dt\right) \\ & \geq m^{-1}T - \left|E\left(PX_T + \int_0^T \left(a_t^3 X_t + b_t^3 Y_t\right) dt\right)\right| \\ & \geq m^{-1}T - Cm\sqrt{\varepsilon}T. \end{aligned}$$

Now choose $\varepsilon = C^{-2}m^{-4}$, we get $Y_0 \geq 0$.

Step 5. Assume $\frac{1}{m} \leq \beta \leq m$ and $T \leq \delta$, where δ is the same as in Lemma 2.4. Denote

$$\begin{cases} X'_t = & \int_0^t (a_s^1 X'_s + b_s^1 Y'_s + c_s^1 Z'_s)\, ds + \int_0^t (a_s^2 X'_s + b_s^2 Y'_s)\, dH_s, \\ Y'_t = & P X'_T + \alpha + \int_t^T (a_s^3 X'_s + b_s^3 Y'_s + c_s^3 Z'_s)\, ds - \int_t^T Z'_s dH_s, \end{cases}$$

and

$$\begin{cases} X''_t = & \int_0^t (a_s^1 X''_s + b_s^1 Y''_s + c_s^1 Z''_s)\, ds + \int_0^t (a_s^2 X''_s + b_s^2 Y''_s)\, dH_s, \\ Y''_t = & L X''_T + \int_t^T (a_s^3 X''_s + b_s^3 Y''_s + c_s^3 Z''_s + \beta_s)\, ds - \int_t^T Z''_s dH_s, \end{cases}$$

By Step 3, $Y'_0 \geq 0$, and by Step 4, $Y''_0 \geq 0$. Then, $Y_0 = Y'_0 + Y''_0 \geq 0$.

Step 6. Assume $\frac{1}{m} \leq \beta \leq m$. Let δ be as in Lemma 2.4 but corresponding to $\left(\lambda, \bar{\lambda}_0, m\right)$ instead of (λ, λ_0, m), and assume $(n-1)\,\delta < T < n\delta$. Denote $T_i \triangleq \frac{iT}{n}$, $L_n \triangleq L$ and $\alpha_n \triangleq \alpha$. For $t \in [T_{n-1}, T_n]$, let

$$\begin{cases} X_t^{n,1} = & 1 + \int_{T_{n-1}}^t (a_s^1 X_s^{n,1} + b_s^1 Y_s^{n,1} + c_s^1 Z_s^{n,1})\, ds + \int_{T_{n-1}}^t (a_s^2 X_s^{n,1} + b_s^2 Y_s^{n,1})\, dH_s, \\ Y_t^{n,1} = & P_n X_T^{n,1} + \int_t^{T_n} (a_s^3 X_s^{n,1} + b_s^3 Y_s^{n,1} + c_s^3 Z_s^{n,1})\, ds - \int_t^{T_n} Z_s^{n,1} dH_s, \end{cases}$$

and

$$\begin{cases} X_t^{n,2} = & 1 + \int_{T_{m-1}}^t (a_s^1 X_s^{n,2} + b_s^1 Y_s^{n,2} + c_s^1 Z_s^{n,2})\, ds + \int_{T_{n-1}}^t (a_s^2 X_s^{n,2} + b_s^2 Y_s^{n,2})\, dH_s, \\ Y_t^{n,2} = & P_n X_T^{n} + \alpha_n + \int_t^{T_n} (a_s^3 X_s^{n,2} + b_s^3 Y_s^{n,2} + c_s^3 Z_s^{n,2} + \beta_s)\, ds - \int_t^{T_n} Z_s^{n,2} dH_s, \end{cases}$$

Denote

$$P_{n-1} \triangleq Y_{T_{n-1}}^{n,1}, \alpha_{n-1} \triangleq Y_{T_{n-1}}^{n,2}.$$

By the proof of Theorem 2.2, we know that $|P_{n-1}| \leq \lambda_1 \leq \bar{\lambda}_0$. Apply the result of *Step* 5, we get $\alpha_{n-1} \geq 0$. We note that, for $t \in [0, T_{n-1}]$, (X, Y, Z) satisfies

$$\begin{cases} X_t = & \int_0^t (a_s^1 X_s + b_s^1 Y_s + c_s^1 Z_s)\, ds + \int_0^t (a_s^2 X_s + b_s^2 Y_s)\, dH_s, \\ Y_t = & P_{n-1} X_{T_{n-1}} + \alpha_{n-1} + \int_t^{T_{n-1}} (a_s^3 X_s + b_s^3 Y_s + c_s^3 Z_s + \beta_s)\, ds - \int_t^{T_{n-1}} Z_s dH_s. \end{cases}$$

Repeating the same arguments, we may define L_1 and $\alpha_1 \geq 0$, and it holds that

$$\begin{cases} X_t = & \int_0^t \left(a_s^1 X_s + b_s^1 Y_s + c_s^1 Z_s\right) ds + \int_0^t \left(a_s^2 X_s + b_s^2 Y_s\right) dH_s, \\ Y_t = & P_1 X_{T_1} + \alpha_1 + \int_t^{T_1} \left(a_s^3 X_s + b_s^3 Y_s + c_s^3 Z_s + \beta_s\right) ds - \int_t^{T_1} Z_s dH_s. \end{cases}$$

By step 5, we have $Y_0 \geq 0$.

Step 7. In the general case, we put $\beta^m \triangleq (\beta \wedge m) \vee \frac{1}{m}$ and let (X^m, Y^m, Z^m) denote the solution corresponding to β^m. We know by Step 6, that $Y_0^m \geq 0$. Then by Corollary 2.1, $Y_0 = \lim\limits_{m \to \infty} Y_0^m \geq 0$. This gives the result.■

We are now in position to give the comparison theorem. Let $\Pi^i, i = 0, 1$, be the solution of the following FBSDE:

$$\begin{cases} X_t^i = & X_0 + \int_0^t f\left(s, \Pi_s^i\right) ds + \int_0^t \sigma\left(s, X_{s_-}^i, Y_{s_-}^i\right) dH_s, \\ Y_t^i = & \varphi^i\left(X_T^i\right) + \int_t^T g^i\left(s, \Pi_s^i\right) ds - \int_t^T Z_s^i dH_s, i = 0, 1 \end{cases} \tag{2.31}$$

Theorem 2.4 *Let* $\Pi^i, i = 0, 1,$ *be the solutions of the FBSDEs (2.1). If*

i) $\left(f, \sigma, g^i, \varphi^i\right), i = 0, 1$ *satisfy all the conditions in Theorem* 2.2.

ii) For any $(t, \Pi), \varphi^0(X) \leq \varphi^1(X)$ *and* $g^0(t, \Pi) \leq g^1(t, \Pi)$. *Then*

$$Y_0^0 \leq Y_0^1.$$

Proof. For $0 \leq \varepsilon \leq 1$, let Π^ε and $\nabla \Pi^\varepsilon$ be as in the prove of Theorem 2.3. Then, we get $\Delta X_0 = 0, \Delta f = 0, \Delta \sigma = 0, \Delta g \geq 0, \ \Delta \varphi \geq 0$. From Proposition 2.3, we have $\nabla Y_0^\varepsilon \geq 0$. This proves the theorem. ■

We would like to mention that the above comparison theorem holds true only at time $t = 0$. We cannot get the result in the whole interval $[0, T]$, even in the Brownian case. See for instance, the counterexample which is given in [50].

Chapter 3

Forward-backward SDEs driven by Lévy process in stopping time duration

3.1 Introduction

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ be a complete probability space. Let $(W_t)_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}^d$ and $(L_t)_{t\geq 0}$ is an $\mathbb{R}-$valued Lévy process independent of $(W_t)_{t\geq 0}$.
Let $\tau \geq 0$ be a given $\mathcal{F}_t$-stopping time with value in $[0, \infty]$. For any $t \geq 0$, we assume that

$$\mathcal{F}_t = \mathcal{F}_0 \vee \sigma\left(L_s, s \leq t\right) \vee \sigma\left(W_s, s \leq t\right) \vee \mathcal{N}.$$

The purpose of the current chapter is to discuss the problem of existence, uniqueness for a class of FBSDEs driven by Teugels martingales associated with some Lévy processes and an independent Brownian motion in stopping time duration of the type

$$\begin{cases} x_t = & x_0 + \int_0^{t\wedge\tau} f\left(s, w, x_s, y_s, z_s, k_s\right) ds + \int_0^{t\wedge\tau} \sigma\left(s, w, x_s, y_s, z_s, k_s\right) dW_s \\ & + \sum_{i=1}^{\infty} \int_0^{t\wedge\tau} \theta^i \left(s, w, x_{s_-}, y_{s_-}, z_s, k_s\right) dH_s^{(i)}, \\ y_t = & \varphi\left(x_\tau\right) + \int_{t\wedge\tau}^{\tau} g\left(s, w, x_s, y_s, z_s, k_s\right) ds - \int_{t\wedge\tau}^{\tau} z_s dW_s - \sum_{i=1}^{\infty} \int_{t\wedge\tau}^{\tau} k_s^i dH_s^{(i)}, \end{cases} \tag{3.1}$$

where $t > 0$, B_t is a d-dimensional Brownian motion and $H_t = (H_t^i)_{i=1}^{\infty}$ are pairwise strongly orthonormal Teugels martingales associated with a Lévy process L_t such that $\langle H^i, H^j \rangle_t = \delta_{ij}.t$ and that $[H^i, H^j] - \langle H^i, H^j \rangle$ is an $\mathcal{F}_t$-martingale.

In this chapter, we will use a completely different method to approach the solvability of (3.1) in stopping time duration. Such a method is called *the method of continuation.* The main assumptions for this method is the so called monotonicity conditions, and the proofs are based on point fix theorem.

The rest of this chapter is organized as follows. In section 2, we give the existence and uniqueness result for backward stochastic differential equation driven by Teugels martingales associated with some Lévy processes and an independent Brownian motion in stopping time duration. In section 3, under a monotonicity condition, we prove an existence and uniqueness result for fully coupled FBSDE driven by Teugels martingales associated with some Lévy processes and an independent Brownian motion on a stopping time duration. As an application, we deal in Section 4 a model of hedging options for a large investor in a Lévy-type market. Finally, section 5 concludes the chapter.

3.2 BSDE in stopping time duration

Let us first present the existence and uniqueness result of the following backward SDE driven by a family of Teugels martingales and an independent Brownian motion in stopping time duration:

$$y_t = \xi + \int_{t\wedge\tau}^{\tau} g\left(s, y_s, z_s, k_s\right) ds - \int_{t\wedge\tau}^{\tau} z_s dW_s - \int_{t\wedge\tau}^{\tau} k_s dH_s, \tag{3.2}$$

where

$$g : [0,\infty] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}^2\left(\mathbb{R}^m\right) \rightarrow \mathbb{R}^m,$$

is progressively measurable. Here, we have used the following shorthand notational, $\int_{t\wedge\tau}^{\tau} k_s dH_s := \sum_{i=1}^{\infty} \int_{t\wedge\tau}^{\tau} k_s^i dH_s^i$, where $k_s = \{k_s^i\}_{i=1}^{\infty}$. We introduce the following basic assumption on coefficients:

$(\mathbf{H}_{3.1})$ $\xi \in L^2\left(\Omega, \mathcal{F}, P, \mathbb{R}^m\right).$

$(\mathbf{H}_{3.2})$ $\mathbb{E}\left(\int_0^{\infty} \left|g\left(s,0,0,0\right)\right| ds\right)^2 < \infty.$

$(\mathbf{H}_{3.3})$ $\forall \left(y, z, k\right) \in \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}^2\left(\mathbb{R}^m\right)$ and $\forall \left(y', z', k'\right) \in \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}^2\left(\mathbb{R}^m\right)$, there exists three positive deterministic functions $u_1\left(t\right)$, $u_2\left(t\right)$ and $u_3\left(t\right)$ satisfying

$$\int_0^{\infty} u_1\left(t\right) dt < \infty, \int_0^{\infty} u_2^2\left(t\right) dt < \infty, \int_0^{\infty} u_3^2\left(t\right) dt < \infty,$$

such that

$$\left|g\left(t, y, z, k\right) - g\left(t, y', z', k'\right)\right| \leq u_1\left(t\right)\left|y - y'\right| + u_2\left(t\right)\left\|z - z'\right\| + u_3\left(t\right)\left\|k - k'\right\|_{\mathcal{P}^2(\mathbb{R}^m)}, t \geq 0.$$

We also denote by

$$\mathcal{B}_\tau^2 = \mathcal{S}_{\mathcal{F}}^2\left(0, \tau, \mathbb{R}^m\right) \times \mathcal{L}_{\mathcal{F}}^2\left(0, \tau, \mathbb{R}^{m\times d}\right) \times l_{\mathcal{F}}^2\left(0, \tau, \mathbb{R}^m\right),$$

In the following theorem, we show an existence and uniqueness result for the equation (3.2) on $[s, \tau]$. That is a triplet $(y_., z_., k_.)$ of adapted $\mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}^2\left(\mathbb{R}^m\right)$-valued processes belong to $\mathcal{B}^2$ and satisfying BSDE (3.2).

Theorem 3.1 *Assume that* $(\mathbf{H}_{3.1}), (\mathbf{H}_{3.2})$ *and* $(\mathbf{H}_{3.3})$ *are in force.*

Then equation (3.2) *has a unique solution* $\pi = (y, z, k) \in \mathcal{B}_\tau^2$.

Proof. (A) The uniqueness part.

In this part, we will adopt the proof in [35] to our case, with suitable changes. This will be done by replacing the fixed time duration T by stopping time and the Lipschitz constants by $u_1(t)$, $u_2(t)$ and $u_3(t), t \geq 0$.

First, we assume that $\pi = (y, z, k)$ and $\pi' = (y', z', k')$ are two solution of equation (3.2). Setting $\hat{y} = y - y'$, $\hat{z} = z - z'$ and $\hat{k} = k - k'$. Then, by applying Itô's formula to $|\hat{y}_t|^2$, owing the fact that $[H^i, H^j]_t - \langle H^i, H^j\rangle_t$ is an $\mathcal{F}_t$-martingale, $\langle H^i, H^j\rangle_t = \delta_{ij}t$, using assumption $(\mathbf{H}_{3.3})$, we get

$$\mathbb{E}\left|\hat{y}_t\right|^2 + \mathbb{E}\int_t^\tau \left\|\hat{z}_s\right\|^2 ds + \mathbb{E}\int_t^\tau \left\|\hat{k}_s\right\|^2_{\mathcal{P}^2(\mathbb{R}^m)} ds \leq c\mathbb{E}\int_t^\tau \left|\hat{y}_t\right|^2 ds.$$

The uniqueness result follows immediately, by using Gronwall's lemma.

(B) The existence part.

In this part we shall use the fix point argument to prove the existence part of theorem 3.1. Noting that the stopping time duration is unbounded and can be infinite, so we shall divide the proof into two steps.

Step1. Firstly, we assume that the following condition holds true,

$$\left(\int_0^\infty u_1(t)\, dt\right)^2 + \int_0^\infty u_2^2(t)\, dt + \int_0^\infty u_3^2(t)\, dt < \frac{1}{15}.$$

For each $(y., z., k.) \in \mathcal{B}_{\tau}^{2}$, we set $M_t = \mathbb{E}\left(\xi + \int_0^{\tau} g(t, y_t, z_t, k_t)\, dt \mid \mathcal{F}_{t\wedge\tau}\right)$. Then $M.$ is a square integrable martingale. Indeed

$$\begin{aligned}
&\mathbb{E}\left[\xi + \int_0^{\tau} g(t, y_t, z_t, k_t)\, dt\right]^2 \\
&\leq \mathbb{E}\left[|\xi| + \int_0^{\tau}\left(|g(t,0,0,0)| + u_1(t)|y_t| + u_2(t)\|z_t\| + u_3(t)\|k_t\|_{\mathcal{P}^2(\mathbb{R}^m)}\right) dt\right]^2 \\
&= c\left[\mathbb{E}|\xi|^2 + \mathbb{E}\left(\int_0^{\infty}|g(s,0,0,0)|\, ds\right)^2 + \left(\int_0^{\tau} u_1(t)\, dt\right)^2 \mathbb{E}\sup_{0\leq t\leq\tau}|y_t|^2\right. \\
&\left.\left(\int_0^{\tau} u_2^2(t)\, dt\right)\mathbb{E}\int_0^{\tau}\|z_t\|^2 dt + \left(\int_0^{\tau} u_3^2(t)\, dt\right)\mathbb{E}\int_0^{\tau}\|k_t\|^2_{\mathcal{P}^2(\mathbb{R}^m)} dt\right] < \infty.
\end{aligned}$$

It can be shown by using the martingale representation theorem, that there exists $(Z., K.) \in \mathcal{L}_{\mathcal{F}}^{2}\left(\mathbb{R}^{m\times d}\right) \times \mathcal{P}^2(\mathbb{R}^m)$ satisfying

$M_t = \mathbb{E}\left[\xi + \int_0^{\tau} g(t, y_t, z_t, k_t)\, dt\right] + \int_0^{t\wedge\tau} Z_s dB_s + \int_0^{t\wedge\tau} K_s dH_s.$

Again we set $Y_{t\wedge\tau} = \mathbb{E}\left[\xi + \int_{t\wedge\tau}^{\tau} g(t, y_t, z_t, k_t)\, dt/\mathcal{F}_{t\wedge\tau}\right]$, hence $Y. \in \mathcal{S}_{\mathcal{F}}^{2}$ and $(Y., Z., K.)$ is the solution of the following BSDE

$$Y_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} g(s, y_s, z_s, k_s)\, ds - \int_{t\wedge\tau}^{\tau} Z_s dW_s - \int_{t\wedge\tau}^{\tau} K_s dH_s. \tag{3.3}$$

Next, we define the mapping $\Phi : \mathcal{B}_{\tau}^{2} \to \mathcal{B}_{\tau}^{2}$ such that $\Phi(\pi_t) = \Phi(y, z, k) = (Y, Z, K)$.

We want to prove that the mapping Φ is contract on $\mathcal{B}_{\tau}^{2}$. Suppose that $(y, z, k), (y', z', k')$ are two elements of $\mathcal{B}_{\tau}^{2}$, such that $\Phi(y, z, k) = (Y, Z, K)$ and $\Phi(y', z', k') = (Y, Z, K)$. Denote $\left(\hat{y}, \hat{z}, \hat{k}\right) = (y - y', z - z', k - k')$, $\left(\hat{Y}, \hat{Z}, \hat{K}\right) = (Y - Y', Z - Z', K - K')$ and $\hat{g}_s = g(s, y_s, z_s, k_s) - g(s, y'_s, z'_s, k'_s)$. An application of Doob's and Jensen inequalities leads to,

$$\begin{aligned}
\left\|\hat{Y}.\right\|^2_{\mathcal{S}_{\mathcal{F}}^{2}(\mathbb{R}^m)} &= \mathbb{E}\sup_{0\leq t\leq\tau}\left|\mathbb{E}\left(\int_t^{\tau}\hat{g}_s ds \mid \mathcal{F}_t\right)\right|^2 \leq 4\mathbb{E}\left(\int_0^{\tau}|\hat{g}_s|\, ds\right)^2, \\
\left\|\hat{Z}.\right\|^2_{\mathcal{L}_{\mathcal{F}}^{2}\left(\mathbb{R}^{m\times d}\right)} + \left\|\hat{K}_t\right\|^2_{l_{\mathcal{F}}^{2}(\mathbb{R}^m)} &= \mathbb{E}\left|\int_0^{\tau}\hat{g}_s ds\right|^2 - \left|\mathbb{E}\int_0^{\tau}\hat{g}_s ds\right|^2 \leq \mathbb{E}\left(\int_0^{\tau}|\hat{g}_s|\, ds\right)^2.
\end{aligned}$$

So

$$
\begin{aligned}
&\left\|\Phi(y,z,k)-\Phi(y',z',k')\right\|_{\mathcal{B}_\tau^2} \\
&=\left\|\hat{Y}.\right\|^2_{\mathcal{S}^2_{\mathcal{F}}(\mathbb{R}^m)}+\left\|\hat{Z}.\right\|^2_{\mathcal{L}^2_{\mathcal{F}}\left(\mathbb{R}^{m\times d}\right)}+\left\|\hat{K}.\right\|^2_{l^2_{\mathcal{F}}(\mathbb{R}^m)}\leq 5\mathbb{E}\left(\int_0^\tau |\hat{g}_s|\,ds\right)^2 \\
&\leq 15\left[\left(\int_0^\infty u_1(t)\,dt\right)^2+\int_0^\infty u_2^2(t)\,dt+\int_0^\infty u_3^2(t)\,dt\right] \\
&\times\left[\|\hat{y}.\|^2_{\mathcal{S}^2_{\mathcal{F}}(\mathbb{R}^m)}+\|\hat{z}.\|^2_{\mathcal{L}^2_{\mathcal{F}}\left(\mathbb{R}^{m\times d}\right)}+\left\|\hat{k}.\right\|^2_{l^2_{\mathcal{F}}(\mathbb{R}^m)}\right].
\end{aligned}
$$

Since $\left(\int_0^\infty u_1(t)\,dt\right)^2+\int_0^\infty u_2^2(t)\,dt+\int_0^\infty u_3^2(t)\,dt\leq\frac{1}{15}$, the above inequality has it that the mapping Φ is a contraction, and therefore it has a unique fixed point which is the unique solution of BSDE (3.2).

Step 2. Assume $\left(\int_0^\infty u_1(t)\,dt\right)^2+\int_0^\infty u_2^2(t)\,dt+\int_0^\infty u_3^2(t)\,dt<\infty$. Then there exists $T>0$, such that $\left(\int_T^\infty u_1(t)\,dt\right)^2+\int_T^\infty u_2^2(t)\,dt+\int_T^\infty u_3^2(t)\,dt<\frac{1}{15}$. Denoting $g_1(t,y_t,z_t,k_t)=1_{[T,\infty]}g(t,y_t,z_t,k_t)$, $\bar{u}_1(t)=1_{[T,\infty]}u_1(t)$, $\bar{u}_2(t)=1_{[T,\infty]}u_2(t)$, $\bar{u}_3(t)=1_{[T,\infty]}u_3(t)$, we see that g_1 satisfies $(\mathbf{H}_{3.3})$ with $\bar{u}_1(t),\bar{u}_2(t)$ and $\bar{u}_3(t)$ such that

$$
\left(\int_0^\infty \bar{u}_1(t)\,dt\right)^2+\int_0^\infty \bar{u}_2^2(t)\,dt+\int_0^\infty \bar{u}_3^2(t)\,dt<\frac{1}{15}.
$$

By the first step, we know that, for $t\geq T$ there exists a solution $\left(\tilde{y}.,\tilde{z}.,\tilde{k}.\right)$ satisfying

$$
\tilde{y}_{t\wedge\tau}=\xi+\int_{t\wedge\tau}^{\tau} g_1\left(s,\tilde{y}_s,\tilde{z}_s,\tilde{k}_s\right)ds-\int_{t\wedge\tau}^{\tau}\tilde{z}_s dB_s-\int_{t\wedge\tau}^{\tau}\tilde{k}_s dH_s.
$$

Then for $t\in[0,T\wedge\tau]$, we consider the following BSDE

$$
\bar{y}_t=\tilde{y}_{T\wedge\tau}+\int_t^{T\wedge\tau} g\left(s,\bar{y}_s,\bar{z}_s,\bar{k}_s\right)ds-\int_t^{T\wedge\tau}\bar{z}_s dB_s-\int_t^{T\wedge\tau}\bar{k}_s dH_s.
$$

Arguing as in the proof of the Theorem 3.1 in [3] or Theorem 1 in [35], one can prove that there exists a unique solution $\left(\bar{y}_., \bar{z}_., \bar{k}_.\right) \in \mathcal{B}^2$. Putting $y_t = 1_{[0,T\wedge\tau]}\bar{y}_t + 1_{(T\wedge\tau,\tau]}\tilde{y}_t$, $z_t = 1_{[0,T\wedge\tau]}\bar{z}_t + 1_{(T\wedge\tau,\tau]}\tilde{z}_t$, $k_t = 1_{[0,T\wedge\tau]}\bar{k}_t + 1_{(T\wedge\tau,\tau]}\tilde{k}_t$, it follows that the process $y_.$ is a solution of BSDE (3.2) and this complete the proof. ■

3.3 Fully coupled FBSDE in stopping time duration

In this section, we establish the existence and uniqueness theorem to the fully coupled FBSDE driven by a family of Teugels martingales and independent Brownian motion in stopping time duration. Let us, consider FBSDE (3.1) with the following coefficients

$$\begin{aligned}
&f : [0,\infty] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}^2\left(\mathbb{R}^m\right) \to \mathbb{R}^n,\\
&\sigma : [0,\infty] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}^2\left(\mathbb{R}^m\right) \to \mathbb{R}^{n\times d},\\
&\theta : [0,\infty] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}^2\left(\mathbb{R}^m\right) \to \mathcal{P}^2\left(\mathbb{R}^n\right),\\
&g : [0,\infty] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}^2\left(\mathbb{R}^m\right) \to \mathbb{R}^m,\\
&\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m.
\end{aligned}$$

We also denote by

$$\mathcal{M}_\tau^2 = \mathcal{S}_{\mathcal{F}}^2\left(0,\tau,\mathbb{R}^n\right) \times \mathcal{S}_{\mathcal{F}}^2\left(0,\tau,\mathbb{R}^m\right) \times \mathcal{L}_{\mathcal{F}}^2\left(0,\tau,\mathbb{R}^{m\times d}\right) \times l_{\mathcal{F}}^2\left(0,\tau,\mathbb{R}^m\right).$$

Let us recall the definition of a solution of FBSDE (3.1).

Definition 3.1 *A solution of the FBSDE* (3.1) *is an adapted process* $(x_., y_., z_., k_.)$ *which belongs to the space* $\mathcal{M}_\tau^2$ *and satisfies* (3.1).

Let us define

$$\lambda = \begin{pmatrix} x \\ y \\ z \\ k \end{pmatrix}, \quad A(t,\lambda) = \begin{pmatrix} -G^{*}g \\ Gf \\ G\sigma \\ G\theta \end{pmatrix}(t,\lambda),$$

where G is given $m \times n$ full-rank matrix, $G\sigma = \left(G\sigma^{1}, ..., G\sigma^{d}\right)$ and $G\theta = (G\theta^{1}, ..., G\theta^{n}, ...)$. We use the usual inner product $\langle \cdot, \cdot \rangle$ and Euclidean norm $|.|$ in $\mathbb{R}^{n}, \mathbb{R}^{m}$ and $\mathbb{R}^{m\times d}$. And, we consider the following conditions:

($\mathbf{H}_{3.4}$) For every $(x, y, z, k) \in \mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{R}^{m\times d} \times \mathcal{P}^{2}(\mathbb{R}^{m}), \varphi(x) \in L^{2}, f, \sigma, g$ and θ are progressively measurable and

$$\begin{aligned} &\mathbb{E}\left(\int_0^\infty |f(s,0,0,0,0)|\,ds\right)^2 + \mathbb{E}\left(\int_0^\infty |g(s,0,0,0,0)|\,ds\right)^2 \\ &+\mathbb{E}\left(\int_0^\infty \left(\|\sigma(s,0,0,0,0)\|^2 + \|\theta(s,0,0,0,0)\|^2_{\mathcal{P}^2(\mathbb{R}^m)}\right) ds\right) < \infty. \end{aligned}$$

($\mathbf{H}_{3.5}$) $\forall (x, y, z, k) \in \mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{R}^{m\times d} \times \mathcal{P}^{2}(\mathbb{R}^{m})$ and $\forall (x', y', z', k') \in \mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathbb{R}^{m\times d} \times \mathcal{P}^{2}(\mathbb{R}^{m})$, there exists a positive deterministic bounded function $u_1(t)$ satisfying

$$\int_0^\infty u_1(t)\,dt < \infty, \int_0^\infty u_1^2(t)\,dt < \infty,$$

such that

$$\begin{aligned} &|\rho(t,x,y,z,k) - \rho(t,x',y',z',k')| \\ &\leq u_1(t)\left[|x-x'| + |y-y'| + \|z-z'\| + \|k-k'\|_{\mathcal{P}^2(\mathbb{R}^m)}\right], t \geq 0 \end{aligned}$$

where $\rho = f, \sigma, g, \theta$ resp. Besides, $\forall x, x' \in \mathbb{R}^{n} \times \mathbb{R}^{n}$, there exists a constant $C > 0$, such that

$$|\varphi(x) - \varphi(x')| \leq C\,|x - x'|.$$

($\mathbf{H}_{3.6}$) For every $\lambda = (x, y, z, k)$, $\lambda' = (x', y', z', k')$ and $\hat{\lambda} = \left(\hat{x}, \hat{y}, \hat{z}, \hat{k}\right) = (x - x', y - y', z - z', k - k')$,

$$\begin{cases} \left\langle A(t,\lambda) - A(t,\lambda'), \hat{\lambda} \right\rangle \leq -\beta_1 u_1(t) |G\hat{x}|^2 - \beta_2 u_1(t) \left(|G^*\hat{y}|^2 + \|G^*\hat{z}\|^2 + \left\|G^*\hat{k}\right\|^2_{\mathcal{P}^2(\mathbb{R}^m)} \right), \\ \langle \varphi(x) - \varphi(x'), G(x - x') \rangle \geq \beta_3 |G\hat{x}|^2, \end{cases}$$

where β_1, β_2 and β_3 are given nonnegative constants with $\beta_1 + \beta_2 > 0$, $\beta_2 + \beta_3 > 0$. Moreover, we have $\beta_1 > 0$, $\beta_3 > 0$ (resp. $\beta_2 > 0$), where $m \geq n$ (resp. $m < n$).

($\mathbf{H}_{3.7}$) For every $\lambda = (x, y, z, k)$, $\lambda' = (x', y', z', k')$ and $\hat{\lambda} = \left(\hat{x}, \hat{y}, \hat{z}, \hat{k}\right) = (x - x', y - y', z - z', k - k')$,

$$\begin{cases} \left\langle A(t,\lambda) - A(t,\lambda'), \hat{\lambda} \right\rangle \geq \beta_1 u_1(t) |G\hat{x}|^2 + \beta_2 u_1(t) \left(|G^*\hat{y}|^2 + \|G^*\hat{z}\|^2 + \left\|G^*\hat{k}\right\|^2_{\mathcal{P}^2(\mathbb{R}^m)} \right), \\ \langle \varphi(x) - \varphi(x'), G(x - x') \rangle \leq -\beta_3 |G\hat{x}|^2, \end{cases}$$

where β_1, β_2 and β_3 are given nonnegative constants with $\beta_1 + \beta_2 > 0$, $\beta_2 + \beta_3 > 0$. Moreover, we have $\beta_1 > 0$, $\beta_3 > 0$ (resp. $\beta_2 > 0$), where $m \geq n$ (resp. $m < n$).

We note that, to simplify the notations, we take the same function $u_1(t)$ in ($\mathbf{H}_{3.5}$), ($\mathbf{H}_{3.6}$) and ($\mathbf{H}_{3.7}$), and we assume that the state variables $x_.$ and $y_.$ have the same dimensional $n = m$, in that case the matrix $G \equiv I_n$.

We also mention that, in what follows, we shall handle two different cases according to the signs of β_1, β_2 and β_3.

Case1. $\beta_1 > 0, \beta_3 > 0$ and $\beta_2 \geq 0$. We introduce the following family of FBSDE, parameterized by $\alpha \in [0, 1]$,

$$\begin{cases} x_t^\alpha = & x_0 + \int_0^{t\wedge\tau} \left[\alpha f(s, \lambda_s^\alpha) + \phi_s^1\right] ds + \int_0^{t\wedge\tau} \left[\alpha\sigma(s, \lambda_s^\alpha) + \phi_s^2\right] dW_s \\ & + \int_0^{t\wedge\tau} \left[\alpha\theta\left(s, \lambda_{s_-}^\alpha\right) + \phi_{s_-}^3\right] dH_s, \\ y_t^\alpha = & \alpha\varphi(x_\tau^\alpha) + (1-\alpha) x_\tau^\alpha + \xi + \int_{t\wedge\tau}^\tau \left[(1-\alpha)\beta_1 u_1(s) x_s^\alpha + \alpha g(s, \lambda_s^\alpha) + \phi_s^4\right] ds \\ & - \int_{t\wedge\tau}^\tau z_s^\alpha dW_s - \int_{t\wedge\tau}^\tau k_s^\alpha dH_s, \end{cases} \tag{3.4}$$

where $\xi \in L^2$, x_0 is in $\mathbb{R}^n$, ϕ^1, ϕ^2, ϕ^3 and ϕ^4 are given processes with values in $\mathbb{R}^n, \mathbb{R}^{n\times d}, \mathcal{P}^2(\mathbb{R}^n)$ and $\mathbb{R}^n$ respectively, such that

$$\mathbb{E}\left(\int_0^\tau \left|\phi_s^1\right| ds\right)^2 + \mathbb{E}\left(\int_0^\tau \left|\phi_s^2\right| ds\right)^2 + \mathbb{E}\int_0^\tau \left\|\phi_s^3\right\|^2_{\mathcal{P}^2(\mathbb{R}^n)} ds + \mathbb{E}\int_0^\tau \left|\phi_s^4\right|^2 ds < \infty.$$

Obviously, when $\alpha = 0$, we can easily check that equation (3.4) has a unique solution. Furthermore, when $\alpha = 1$, the existence of solution of equation (3.4) implies that of FBSDE (3.1).

Lemma 3.1 *We assume* $(\mathbf{H}_{3.4})$, $(\mathbf{H}_{3.5})$ *and* $(\mathbf{H}_{3.6})$, *If for an* $\alpha_0 \in [0,1)$ *equation* (3.4) *has a unique solution* $\lambda^{\alpha_0}_{\cdot} = (x^{\alpha_0}_{\cdot}, y^{\alpha_0}_{\cdot}, z^{\alpha_0}_{\cdot}, k^{\alpha_0}_{\cdot}) \in \mathcal{M}^2_\tau$, *then there exists a positive constant* ε_0, *such that for each* $\varepsilon \in [0, \varepsilon_0]$, *there exists a unique solution* $\lambda^{\alpha_0+\varepsilon}_{\cdot} = (x^{\alpha_0+\varepsilon}_{\cdot}, y^{\alpha_0+\varepsilon}_{\cdot}, z^{\alpha_0+\varepsilon}_{\cdot}, k^{\alpha_0+\varepsilon}_{\cdot})$ *of FBSDE* (3.4).

Proof. Since for each $\phi^1(t), \phi^2(t), \phi^3(t)$ and $\phi^4(t)$ and $\alpha \in [0,1)$, there exists a unique solution of (3.4), then for each $\lambda_s = (x_s, y_s, z_s, k_s) \in \mathcal{M}^2_\tau$ we can define the following FBSDE:

$$\begin{cases} X_t = & x_0 + \int_0^{t\wedge\tau} \left[\alpha_0 f(s,\Lambda_s) + \varepsilon f(s,\lambda_s) + \phi_s^1\right] ds + \int_0^{t\wedge\tau} \left[\alpha_0 \sigma(s,\Lambda_s) + \varepsilon\sigma(s,\lambda_s) + \phi_s^2\right] dW_s \\ & + \int_0^{t\wedge\tau} \left[\alpha_0 \theta\left(s,\Lambda_{s_-}\right) + \varepsilon\theta\left(s,\lambda_{s_-}\right) + \phi_{s_-}^3\right] dH_s, \\ Y_t = & \alpha_0\varphi(X_\tau) + (1-\alpha_0) X_\tau + \varepsilon(\varphi(x_\tau) - x_\tau) + \xi \\ & + \int_{t\wedge\tau}^\tau \left[(1-\alpha_0)\beta_1 u_1(s) X_s + \alpha_0 g(s,\Lambda_s) + \varepsilon(-\beta_1 u_1(s) x_s) + g(s,\lambda_s) + \phi^4(s)\right] ds \\ & - \int_{t\wedge\tau}^\tau Z_s dW_s - \int_{t\wedge\tau}^\tau K_s dH_s, \end{cases}$$

where $\Lambda_t = (X_t, Y_t, Z_t, K_t)$. Define the mapping $\Phi_{\alpha_0+\varepsilon}$ from $\mathcal{M}^2 \times L^2$ into itself, as follows: for $(\lambda, x_\tau) \in \mathcal{M}^2 \times L^2$, $\Phi_{\alpha_0+\varepsilon}(\lambda, x_\tau) = \Lambda \times X_\tau$. We want to prove that the mapping $\Phi_{\alpha_0+\varepsilon}$ is a contraction if $\mathcal{M}^2 \times L^2$ is endowed with the norm $\left[\mathbb{E}\int\limits_0^\tau |\lambda_s|^2 ds + \mathbb{E}|x_\tau|^2\right]^{1/2}$. Indeed, let $\lambda_t = (x_t, y_t, z_t, k_t)$ and $\lambda' = (x'_t, y'_t, z'_t, k'_t)$ be two elements of $\mathcal{M}^2$ such that $\Phi_{\alpha_0+\varepsilon}(\lambda_t, x_\tau) = (\Lambda_t, X_\tau)$ and $\Phi_{\alpha_0+\varepsilon}(\lambda'_t, x'_\tau) = (\Lambda'_t, X'_\tau)$. Putting $\hat{\lambda} = \left(\hat{x}, \hat{y}, \hat{z}, \hat{k}\right) = (x_t - x'_t, y_t - y'_t, z_t - z'_t, k_t - k'_t)$, and $\hat{\Lambda} = \left(\hat{X}, \hat{Y}, \hat{Z}, \hat{K}\right) = (X - X', Y - Y', Z - Z', K - K')$.

Applying Itô's formula to $\left\langle \hat{X}_s, \hat{Y}_s \right\rangle$, using the fact that $[H^i, H^j]_t - \langle H^i, H^j \rangle_t$ is an $\mathcal{F}_t$-martingale and $\langle H^i, H^j \rangle_t = \delta_{ij} t$, one can obtain

$$\begin{aligned}
&\alpha_0 \mathbb{E} \left\langle \varphi(X_\tau) - \varphi(X'_\tau), \hat{X}_\tau \right\rangle + (1-\alpha_0) \mathbb{E} \left| \hat{X}_\tau \right|^2 + \varepsilon \mathbb{E} \left\langle \varphi(x_\tau) - \varphi(x'_\tau) - \hat{x}_\tau, \hat{x}_\tau \right\rangle \\
&= \mathbb{E} \int_0^\tau \left\langle \alpha_0 \left(A(s, \lambda_s) - A(s, \lambda'_s) \right), \hat{\lambda}_s \right\rangle ds - \beta_1 (1-\alpha_0) \mathbb{E} \int_0^\tau u_1(s) \left| \hat{X}_\tau \right|^2 ds \\
&+ \varepsilon \mathbb{E} \int_0^\tau \left[\beta_3 u_1(s) \left\langle \hat{X}_s, \hat{x}_s \right\rangle + \left\langle \hat{X}_s, -g_s \right\rangle + \left\langle \hat{Y}_s, \hat{f}_s \right\rangle + \left\langle \hat{Z}_s, \hat{\sigma}_s \right\rangle + \sum_{i=1}^\infty \left\langle \hat{K}^i, \hat{\theta}^i_s \right\rangle \right] ds,
\end{aligned}$$

where $\hat{\rho}_s = \rho(s, \lambda_s) - \rho(s, \lambda'_s)$ for $\rho = g, f, \sigma$ and θ. Then by invoking the assumptions $(\mathbf{H}_{3.4}), (\mathbf{H}_{3.5})$ and $(\mathbf{H}_{3.6})$, seeing that $\beta_1 > 0$, $\beta_3 > 0$, one can show that there exists a constant C_1 depends on β_1, β_3 and C, such that

$$\begin{aligned}
&[\alpha_0 \beta_3 + (1-\alpha_0)] \mathbb{E} \left| \hat{X}_\tau \right|^2 + \beta_1 \mathbb{E} \int_0^\tau u_1(s) \left| \hat{X}_s \right|^2 ds \\
&\leq \varepsilon C_1 \left[\int_0^\infty u_1^2(s) ds + \left(\int_0^\infty u_1(s) ds \right)^2 \right] \mathbb{E} \int_0^\tau \left[\left| \hat{\Lambda}_s \right|^2 + \left| \hat{\lambda}_s \right|^2 \right] ds \\
&+ \varepsilon C_1 \left(\mathbb{E} \left| \hat{x}_\tau \right|^2 + \mathbb{E} \left| \hat{X}_\tau \right|^2 \right).
\end{aligned}$$

Again, Itô's formula applied to $\left| \hat{Y}_s \right|^2$ together with Gronwall's Lemma and Burkholder-Davis-Gundy inequality, lead to

$$\begin{aligned}
&\mathbb{E} \int_0^\tau \left(\left| \hat{Y} \right|^2 + \left\| \hat{Z} \right\|^2 + \left\| \hat{K} \right\|^2_{\mathcal{P}^2(\mathbb{R}^n)} \right) ds \\
&\leq C_2 \left[\mathbb{E} \int_0^\tau u_1(s) \left| \hat{X}_s \right|^2 ds + \mathbb{E} \left| \hat{X}_\tau \right|^2 \right] + \varepsilon C_2 \mathbb{E} \left| \hat{x}_\tau \right|^2 \\
&+ \varepsilon C_2 \left[\int_0^\infty u_1^2(s) ds + \left(\int_0^\infty u_1(s) ds \right)^2 \right] \mathbb{E} \int_0^\tau \left| \hat{\lambda}_s \right|^2 ds,
\end{aligned}$$

here the constant C_2 depends on β_1, β_3 and C. Now, combining the two above inequalities, taking under consideration that, $\beta_1 > 0, \beta_3 > 0$, one can indicate the existence of a constant C_3 only depending on β_1, β_3 and C, such that

$$\mathbb{E} \int_0^\tau \left(\left| \hat{\Lambda}_s \right|^2 \right) ds + \mathbb{E} \left| \hat{X}_\tau \right|^2 \leq \varepsilon C_3 \left[\mathbb{E} \int_0^\tau \left(\left| \hat{\lambda}_s \right|^2 \right) ds + \mathbb{E} \left| \hat{x}_\tau \right|^2 \right].$$

Clearly, by taking $\varepsilon_0 = \frac{1}{2C_3}$, we get, for each fixed $\varepsilon \in [0, \varepsilon_0]$,

$$\left\|\hat{\Lambda}\right\|^2_{\mathcal{M}^2_\tau} + \left|\hat{X}_\tau\right|^2_{L^2} \leq \frac{1}{2}\left[\left\|\hat{\lambda}\right\|^2_{\mathcal{M}^2_\tau} + \left|\hat{x}_\tau\right|^2_{L^2}\right].$$

Hence, the mapping $\Phi_{\alpha_0+\varepsilon}$ is a contraction and has a unique fixed point $\Lambda^{\alpha+\varepsilon}_\cdot = (X^{\alpha+\varepsilon}_\cdot, Y^{\alpha+\varepsilon}_\cdot, Z^{\alpha+\varepsilon}_\cdot, K^{\alpha+\varepsilon}_\cdot)$, which is the unique solution of FBSDE (3.4). The proof of Lemma is complete. ■

Case 2. $\beta_2 > 0, \beta_1 \geq 0$ and $\beta_3 \geq 0$.

We consider the following FBSDE for each $\alpha \in [0, 1]$:

$$\begin{cases} x^\alpha_t = & x_0 + \int_0^{t\wedge\tau} \left[(1-\alpha)\beta_2\left(-u_1(s)\, y^\alpha_s\right) + \alpha f\left(s, \lambda^\alpha_s\right) + \phi^1_s\right] ds \\ & + \int_0^{t\wedge\tau} \left[(1-\alpha)\beta_2\left(-u_1(s)\, z^\alpha_s\right) + \alpha\sigma\left(s, \lambda^\alpha_s\right) + \phi^2_s\right] dW_s \\ & + \int_0^{t\wedge\tau} \left[(1-\alpha)\beta_2\left(-u_1(s)\, k^\alpha_{s_-}\right) + \alpha\theta\left(s, \lambda^\alpha_{s_-}\right) + \phi^4_{s_-}\right] dH_s, \\ y^\alpha_t = & \alpha\varphi\left(x^\alpha_\tau\right) + \xi + \int_{t\wedge\tau}^\tau \left(\alpha g\left(s, \lambda^\alpha_s\right) + \phi^4_s\right) ds - \int_{t\wedge\tau}^\tau z^\alpha_s dW_s - \int_{t\wedge\tau}^\tau k^\alpha_s dH_s, \end{cases} \quad (3.5)$$

Here $\phi^1, \phi^2, \phi^3, \phi^4$ and ξ satisfy the same assumptions as those in case 1.

Lemma 3.2 *Assume that hypotheses* $(\mathbf{H}_{3.4}), (\mathbf{H}_{3.5})$ *and* $(\mathbf{H}_{3.6})$ *hold. If for an* $\alpha_0 \in [0, 1)$ *equation* (3.5) *has a unique solution* $\lambda^{\alpha_0}_\cdot = (x^{\alpha_0}_\cdot, y^{\alpha_0}_\cdot, z^{\alpha_0}_\cdot, k^{\alpha_0}_\cdot) \in \mathcal{M}^2_\tau$, *then there exists a positive constant* ε_0, *such that for each* $\varepsilon \in [0, \varepsilon_0]$, *there exists a unique solution* $\lambda^{\alpha_0+\varepsilon}_\cdot = (x^{\alpha_0+\varepsilon}_\cdot, y^{\alpha_0+\varepsilon}_\cdot, z^{\alpha_0+\varepsilon}_\cdot, k^{\alpha_0+\varepsilon}_\cdot)$ *of FBSDE* (3.5).

Proof. For each $\lambda_s = (x_s, y_s, z_s, k_s)$ there exists a unique process $\Lambda_s = (X_s, Y_s, Z_s, K_s)$ satisfying the following FBSDE:

$$\begin{cases} X_t = & x_0 + \int_0^{t\wedge\tau} \left[(1-\alpha_0)\beta_2\left(-u_1(s)\, Y_s\right) + \alpha_0 f\left(s, \Lambda_s\right) + \varepsilon f\left(s, \lambda_s\right) + \phi^1(s) + \varepsilon\beta_2 u_1(s)\, y_s\right] ds \\ & + \int_0^{t\wedge\tau} \left[(1-\alpha_0)\beta_2\left(-u_1(t)\, Z_s\right) + \alpha_0\sigma\left(s, \Lambda_s\right) + \varepsilon\sigma\left(s, \lambda_s\right) + \phi^1(s) + \varepsilon\beta_2 u_1(t)\, z_s\right] dW_s \\ & + \int_0^{t\wedge\tau} \left[(1-\alpha_0)\beta_2\left(-u_1(s)\, k_s\right) + \alpha_0\theta\left(s, \Lambda_{s_-}\right) + \varepsilon\theta\left(s, \lambda_{s_-}\right) + \phi^3(s_-) + \varepsilon\beta_2 u_1(s)\, k_s\right] dH_s, \\ Y_t = & \alpha_0\varphi\left(X_\tau\right) + \varepsilon\varphi\left(x_\tau\right) + \xi + \int_{t\wedge\tau}^\tau \left[\alpha_0 g\left(s, \Lambda_s\right) + \varepsilon g\left(s, \lambda_s\right) + \phi^4(s)\right] ds \\ & - \int_{t\wedge\tau}^\tau z_s dW_s - \int_{t\wedge\tau}^\tau k_s dH_s. \end{cases}$$

Arguing as in the proof of Lemma 3.1. First, let $\Phi_{\alpha_0+\varepsilon}$ be described as in the above lemma. Then, we are going to prove that the mapping $\Phi_{\alpha_0+\varepsilon}$ is a contraction on $\mathcal{M}^2 \times L^2$. Applying Itô's to $\left\langle \hat{X}_s, \hat{Y}_s \right\rangle$, using the assumptions $(\mathbf{H}_{3.4})$, $(\mathbf{H}_{3.5})$ and $(\mathbf{H}_{3.6})$, and taking under consideration that $\beta_2 > 0$, one can get

$$\begin{aligned}
&\beta_2 \mathbb{E} \int_0^\tau u_1(s) \left[\left| \hat{Y}_s \right|^2 + \left\| \hat{Z}_s \right\|^2 + \left\| \hat{K}_s \right\|_{\mathcal{P}^2(\mathbb{R}^n)} \right] ds \\
&\leq \varepsilon C_1 \left(\int_0^T u_1^2(s)\, ds + \left(\int_0^T u_1(s)\, ds \right)^2 \right) \mathbb{E} \int_0^\tau \left[\left| \hat{\Lambda}_s \right|^2 + \left| \hat{\lambda}_s \right|^2 \right] ds \\
&+ \varepsilon C_1 E \left| \hat{X}_\tau \right|^2 + \varepsilon C_1 E \left| \hat{x}_\tau \right|^2 .
\end{aligned}$$

Then, we obtain, by using a standard arguments to the forward part,

$$\begin{aligned}
\sup_{0 \leq s \leq \tau} \mathbb{E} \left| \hat{X}(s) \right|^2 \leq \varepsilon C_1 \left(\int_0^\infty u_1^2(s)\, ds + \left(\int_0^\infty u_1(s)\, ds \right)^2 \right) \mathbb{E} \int_0^\tau \left| \hat{\lambda}_s \right|^2 ds \\
+ C_1 \mathbb{E} \int_0^\tau u_1(s) \left(\left| \hat{Y}_s \right|^2 + \left\| \hat{Z}_s \right\| + \left\| \hat{K}_s \right\|_{\mathcal{P}^2(\mathbb{R}^n)} \right) ds
\end{aligned}$$

Here the constant C_1 depends on the Lipschitz constant C as well as β_2. Now, let us combine the above two inequalities, to get

$$\mathbb{E} \int_0^\tau \left[\left| \hat{\Lambda}_s \right|^2 \right] ds + E \left| \hat{X}_\tau \right|^2 \leq \varepsilon C_2 \left[\mathbb{E} \int_0^\tau \left| \hat{\Lambda}_s \right|^2 ds + E \left| \hat{x}_\tau \right|^2 \right].$$

Here the constant C_2 depending only on β_2, and the Lipschitz constant C. We now pick up $\varepsilon_0 = \frac{1}{2C_2}$, and obtain for each fixed $\varepsilon \in [0, \varepsilon_0]$,

$$\left\| \hat{\Lambda} \right\|_{\mathcal{M}^2}^2 + \left| \hat{X}_\tau \right|_{L^2}^2 \leq \frac{1}{2} \left[\left\| \hat{\lambda} \right\|_{\mathcal{M}^2}^2 + \left| x_\tau \right|_{L^2}^2 \right].$$

So that the mapping $\Phi_{\alpha_0+\varepsilon}$ is a contraction and consequently, there exists a unique fixed point $\hat{\Lambda}_\cdot^{\alpha_0+\varepsilon} = \left(\hat{X}_\cdot^{\alpha_0+\varepsilon}, \hat{Y}_\cdot^{\alpha_0+\varepsilon}, \hat{Z}_\cdot^{\alpha_0+\varepsilon}, \hat{K}_\cdot^{\alpha_0+\varepsilon} \right)$ which is the required unique solution to the

FBDSDE (3.5). The proof is complete. ■

We are now in a position to state your main result in this section.

Theorem 3.2 *Under the conditions* $(\mathbf{H}_{3.4}), (\mathbf{H}_{3.5})$ *and* $(\mathbf{H}_{3.6})$ *(or* $(\mathbf{H}_{3.7})$*), there exists a unique solution* $(x., y., z., k.) \in \mathcal{M}_\tau^2$ *of the FBSDE* (3.1).

Proof. **(A) Proof of the uniqueness part.** Let $\lambda_s = (x_s, y_s, z_s, k_s)$ and $\lambda'_s = (x'_s, y'_s, z'_s, k'_s)$ are two solutions of (3.1). We set $\hat{\lambda}_s = (x_s - x'_s, y_s - y'_s, z_s - z'_s, k_s - k'_s) = \left(\hat{x}_s, \hat{y}_s, \hat{z}_s, \hat{k}_s\right)$.

Using Itô's formula applied to $\langle \hat{x}_s, \hat{y}_s \rangle$ from t to τ and using the fact that $[H^i, H^j]_t - \langle H^i, H^j \rangle_t$ is an $\mathcal{F}_t$-martingale and $\langle H^i, H^j \rangle_t = \delta_{ij} t$, we get

$$\begin{aligned} &\mathbb{E}\left\langle \varphi(x_\tau) - \varphi(x'_\tau), \hat{x}_\tau \right\rangle - \mathbb{E}\left\langle \hat{x}_\tau, \hat{y}_\tau \right\rangle = \mathbb{E}\int_t^\tau \left\langle A(s, \lambda_s) - A(s, \lambda'_s), \hat{\lambda}_s \right\rangle ds \\ &\leq -\beta_1 \mathbb{E}\int_t^\tau u_1(s) |\hat{x}_s|^2 ds - \beta_2 \mathbb{E}\int_t^\tau u_1(s) \left[|\hat{y}_s|^2 + \|\hat{z}_s\|^2 + \left\|\hat{k}_s\right\|^2_{\mathcal{P}^2(\mathbb{R}^n)} \right] ds. \end{aligned}$$

This together with the monotonicity conditions of φ and A imply,

$$\beta_1 \mathbb{E}\int_0^\tau u_1(s) |\hat{x}_s|^2 ds + \beta_2 \mathbb{E}\int_0^\tau u_1(s) \left[|\hat{y}_s|^2 + \|\hat{z}_s\|^2 + \left\|\hat{k}_s\right\|^2_{\mathcal{P}^2(\mathbb{R}^n)} \right] ds \leq 0.$$

Case 1. In this case $\beta_1 > 0$. Thus $x_s = x'_s$. In particular, $\varphi(x_\tau) = \varphi(x'_\tau)$. Thus from Theorem **??**, it follows that $y_s = y'_s$, $z_s = z'_s$ and $k_s = k'_s$.

Case 2. In this case $\beta_2 > 0$, then $y_s = y'_s$, $z_s = z'_s$ and $k_s = k'_s$, $s \in [0, \tau]$. Returning to the forward part of (3.1), similarly as in the proof of the uniqueness result to SDEs driven by Teugels martingales, for fixed time duration T, see for instance [**?**], one can get $x_s = x'_s$.

(A) Proof of the existence part.

First case: When $\beta_1 > 0, \beta_3 > 0$ and $\beta_2 \geq 0$, we know that equation (3.4) has a unique solution for $\alpha = 0$. It then follows from Lemma 3.1 that there exists a positive constant ε_0 such that, for each $\varepsilon \in [0, \varepsilon_0]$, for $\alpha = \alpha_0 + \varepsilon$ equation (3.4) has a unique solution.

We can repeat this process for N-time white $1 \leq N\varepsilon_0 < 1 + \varepsilon_0$. It then follows that, in particular, for $\alpha = 1$ with $\phi^1 = 0, \phi^2 = 0, \phi^3 = 0, \phi^4 = 0$ and $\xi = 0$, FBSDE (3.1) has a unique solution.

Second case: When $\beta_1 \geq 0, \beta_3 \geq 0$ and $\beta_2 > 0$, for $\alpha = 0$, FBSDE (3.5) has a unique solution. Using Lemma 3.2, and repeating the same process as in the first case, one can get the desired conclusion. This achieve the proof. ∎

We note that we will give the proof of the above theorem only under the assumptions $(\mathbf{H}_{3.4})$, $(\mathbf{H}_{3.5})$ and $(\mathbf{H}_{3.6})$. The proof under $(\mathbf{H}_{3.4})$, $(\mathbf{H}_{3.5})$ and $(\mathbf{H}_{3.7})$ is similar.

Remark 3.1 *We can replace $u_1(t)$ by a constant. Then if the stopping time verify this condition $\tau \leq T < \infty$, one can consider the existence and uniqueness result of FBSDE in bounded time duration as a special case of our Theorem* 3.2.

3.4 Application

In this section, we consider an option pricing problem with a Large Investor motivated by a portfolio selection in Lévy-type market, that is a market whose price processes are semimartingales with the martingale parts represented as a sum of stochastic integrals with respect to a Lévy process. Note that the Brownian continuous case has been treated by Cvitanic and Ma [10], where they developed a model for hedging options in the presence of a large investor in a Brownian market. See also [41] for other approaches in the context of Lévy jump processes. In our approach, we shall use the existence and uniqueness result established in the preceding sections.

Let $\tau \geq 0$ be a given $\mathcal{F}_t$-stopping time with value in $[0, \infty]$. We assume that in the financial market, there are two kind of assets:

i) A non risky asset, also called a money market account, whose price process $S_0(t)$ at time t is given by the following forward stochastic differential equation:

$$\begin{cases} dS_t^0 &= S_t^0 r(t, X_t, Z_t, K_t)\, dt, 0 \leq t \leq \tau \\ S_0^0 &= 1, \end{cases} \tag{3.6}$$

where $X_.$ is the wealth process, $Z_.$ and $K_.$ are the portfolio processes they will be determined in the sequel.

ii) A d-dimensional risky asset, which called the stock, whose price process $S_. = \{S_.^i\}_{i=1}^d$at time t, evolves according to the following stochastic differential equation:

$$\begin{cases} dS_t^i &= S_t^i f_i(t, S_t, X_t, Z_t, K_t)\, dt \\ & \quad +S_i(t)\, \sigma_i(t, S_t, X_t, Z_t, K_t)\, dW(t) \\ & \quad +\sum_{j=1}^{\infty} S_t^i \theta_j^i \left(t, S_{t_-}, X_{t_-}, Z_t, K_t\right) dH^{(j)}(t), \\ S_0^i &= p^i, \quad p^i \geq 0, 1 \leq i \leq d, t \in [0, \tau]. \end{cases} \tag{3.7}$$

Our starting point is to derive the BSDE for the wealth process $X_.$ with the final condition $X_\tau = h(S_\tau)$, Arguing as in [10], by using the following formula,

$$dX_t = \sum_{i=1}^d \alpha_t^i dS_t^i + \frac{X_t - \sum_{i=1}^d \alpha_t^i S_t^i}{S_t^0} dS_t^0$$

where $\alpha_. = (\alpha_.^i)_{i=1}^{i=d}$ is the portfolio process. Now, due to the definition of (3.6) and (3.7), one can easily derive the following BSDE:

$$\begin{aligned} dX_t &= \sum_{i=1}^d \alpha_t^i \{S_t^i f_i(t, S_t, X_t, Z_t, K_t)\, dt \\ & \quad +S_t^i \sigma_i(t, S_t, X_t, Z_t, K_t)\, dW_t \\ & \quad +S_t^i \theta^i \left(t, S_{t_-}, X_{t_-}, Z_t, K_t\right) dH_t\} \\ & \quad + \left(X_t - \sum_{i=1}^d \alpha_t^i dS_t^i\right) r(t, X_t, Z_t, K_t)\, dt. \end{aligned}$$

Consequently,

$$dX(t) = g(t, S(t), X(t), Z_t, K_t, \alpha_t) dt + Z_t dB_t + K_t dH_t, \tag{3.8}$$

where we have used the following notations,

$$g(t, \pi, x, z, k, a) = \sum_{i=1}^{d} a^i \pi^i f^i (t, \pi, x, z, k, a) + \left(x - \sum_{i=1}^{d} q^i \pi^i \right) r(t, x, z, k),$$

$$a = \left\{ a^i \right\}_{i=1}^{d}, \pi = \left\{ \pi^i \right\}_{i=1}^{d},$$

$$Z_t = \sum_{i=1}^{d} \alpha_t^i \sigma^i (t, S_t, X_t, Z_t, K_t), \tag{3.9}$$

$$K_t = \sum_{i=1}^{d} \alpha_t^i \theta_j^i (t, S_t, X_t, Z_t, K_t) \;\; i = 1, 2, ... \tag{3.10}$$

Noting that, From the relations (3.9) and (3.10), one can rule out the dependence on $\alpha(.)$ in (3.8). In addition, the BSDE (3.8) takes the following form:

$$\begin{aligned} dX_t &= h(S_T) + g(t, S_t, X_t, Z_t, K_t, \alpha_t) dt \\ &\quad + Z_t dW_t + K_t dH_t. \end{aligned} \tag{3.11}$$

Combining (3.7) and (3.11), to obtain

$$\begin{cases} dS_t &= p + \int_0^t S_r f(r, S_r, X_r, Z_r, K_r) dt \\ &\quad + \int_0^t S_r \sigma(r, S_r, X_r, Z_r, K_r) dW_r \\ &\quad + \int_0^t S_r \sigma\left(r, S_{r_-}, X_{r_-}, Z_r, K_r\right) dH_r, \\ dX_t &= h(S_\tau) + \int_t^\tau g(r, S_r, X_r, Z_r, \alpha_r) dr \\ &\quad + \int_t^\tau Z_r dW_r + \int_t^\tau K_r dH_r, \end{cases} \tag{3.12}$$

where $p = \{p^i\}_{i=1}^{i=d}$. Finally, we deduce the following theorem:

Theorem 3.3 *Let $\alpha_. = (\alpha_.^i)_{i=1}^{i=d}$ be an $\mathbb{R}^d$-valued stochastic process such that $\alpha_t^i \geq 0$. Assume that the relations* (3.9) *and* (3.10) *hold true on top of the assumptions of Theorem*

3.2. *Then, FBSDEs* (3.12) *has a unique solution* (S, X, Z, K) *such that the pair* (S, X) *is càdlàg and* α. *is a replicating portfolio.*

3.5 Conclusion

Through this previous chapter, we have proved some existence and uniqueness theorems for BSDE and fully coupled FBSDE driven by Teugels martingales associated to some Lévy processes and an independent Brownian motion in the case where the final time T is allowed to be random. Our main results could be perceived as an extension of the results in [3] and [35] to more general cases. Indeed, in one hand, if $\nu = 0$, then, $H_t^{(1)}$ is a standard Brownian motion and $H_t^{(i)} = 0$, for $i \geq 2$, this case has been studied by many authors, see for instance [11], [23], [30], [39], [40]. In the other hand, if we assume that μ only has mass at 1, then $H_t^{(1)}$ is the compensated Poisson process and also $H_t^{(i)} = 0$, for $i \geq 2$, for this case we refer the reader to [49] and [50] for more details in this respect. Motivated by our theoretical results, we have applied a Lévy-FBSDE approach to select a replicating portfolio for a large investor in a Lévy-type market.

Chapter 4

Optimal Control Strategies for Premium Policy of an Insurance Firm with Jump Diffusion Assets and Stochastic Interest Rate

4.1 Introduction

It is well known that an insurance is a contract, represented by a policy, used as a method of protection against losses, whether big or small. This means that the insured can receive or reimburse some financial amount to offset his or her losses from an insurance company. Due the fact that an insurance premium is the amount of money that a person or a company ought to pay for an insurance policy, one can perceive it in two different ways. In one side, it can be considered as an income by the insurance company. In the other side it can be also considered as a liability in that the insurer must provide coverage for claims being made against the policy. In the current paper we focus on the first case where the policy maker look forward to maximize the terminal wealth of its firm's cash-balance under a demand law.

The main problem in optimal control theory is to characterize an optimal control process. There are two main approaches, the Pontryagin's maximum principle and Bellman's dynamic programming. We note that optimal control theory has been used for example in [21] in order to determine an optimal dynamic unlimited excess of loss reinsurance strategy to minimize infinite time ruin probability, see also [22], where the authors applied a proportional reinsurance policy for diffusion models in order to find a policy that maximizes a given return function before the time of ruin. In [6], Cairns study the optimization problem of stochastic pension fund models in continuous time. The mean-variance portfolio management for an insurance company is studied by Josa-Fombellida and Rincon-Zapatero [26] by using dynamic programming techniques and also by Xie et al. [51] using the general stochastic linear quadratic control technique. In [32], Moore and Young solve an optimal dynamic consumption, investment, and insurance strategies, using the dynamic programming principle and a Markov chain approximation method. Ngwira and Gerrad in [33] show that the optimal contribution and asset allocation policies have similar forms as in the pure diffusion case, but with a modification due to the effect of jumps. In [24], Huang et al. explicitly derived the insurance company's optimal premium strategy and the associated optimal cost function.

Motivated by the above results, in this chapter, we solve an optimal premium policy problem of an insurance firm. The main tool used in proving our main results is Pontryagin maximum principle. More precisely, the sufficient condition of optimality. Noting that, in pretty much all of the previous papers, the authors dealt with the problem of optimal insurance in continuous-time models, we impose here to work with a quite general semimartingale framework assuming that the liability process is driven by both a Brownian motion and a family of pairwise orthogonal martingales associated with a Lévy process. This kind of models comes naturally from the fact that in many real cases the continuity of trajectories condition cannot be satisfied. Indeed, the empirical distribution of cash balance process tend to deviate from normal distributions, either due to inspected dusters or huge profits, many successive incidents or even because of the lack of continuity in the real world of applications.

4.2 Problem statement and description of the model

Let be $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{F}, P)$ a complete filtered probability space supporting a standard Brownian motions W and a Lévy process L with triplet $(\sigma_0^2, \nu, \alpha)$, defined on $[0, T]$, and all of them independent each other.
We also assume that $\mathbb{F}$ is the complete right-continuous natural filtration generated by processes W and L.
Throughout this chapter we address an optimal premium policy problem of an insurance firm under stochastic interest rate when the liability process, also called the payment function, B_t, is modeled by the stochastic differential equation:

$$-dB_t = (b_t + v_t)\, dt + \sigma_t dW_t + \sum_{i=1}^{\infty} \pi_t^i dH_t^i,$$

where b denotes the liability rate, that is, the expected liability per unit time due to premium loading, v is the premium rate (premium policy) and σ and $\{\pi^i\}_{i=1}^{\infty}$ are the volatility rates measuring the liability risks belonging respectively to the Brownian and Teugels martingale components. Assume moreover that the cash balance process of the insurer X_t is described by the formula,

$$X_t = e^{\Delta_t} \left(X_0 - \int_0^t e^{-\Delta_s} dB_s \right), \tag{4.1}$$

where $X_0 = x \geq 0$ represents the initial reserve and

$$d\Delta(t) = \delta(t)\, dt + \alpha(t)\, dW(t), \; t \in [0, T], \; \Delta(0) = \Delta_0 \tag{4.2}$$

is a stochastic process which represents the interest rate. Note that X_t is the difference between the initial capital and the net expenses up to time t.

Now the Itô's formula applied to the process $X_.$ leads to the following controlled SDE

$$\begin{cases} dX_t = f\left(t, X_t, v_t\right) dt + \sigma\left(t, X_t, v_t\right) dW_t + \pi_t dH_t, \\ X_0 = x, \end{cases} \tag{4.3}$$

where

$$f\left(t, X_t, v_t\right) = \left(\delta_t + \frac{1}{2}\alpha_t^2\right) X_t + \alpha_t \sigma_t + b_t + v_t,$$

and

$$\sigma\left(t, X_t, v_t\right) = X_t \alpha_t + \sigma_t.$$

Herein, the process $v_.$ stands for the control variable. We require that process $v_.$ is adapted, with càdlàg trajectories and taking values in U, a non-empty convex subset of $\mathbb{R}$, such that the following fourth-power condition $\mathbb{E} \int_0^T |v_t|^4 \, dt < \infty$ holds true. Furthermore, we assume there exists a positive constant c_0 such that the SDE (4.3) has a unique solution satisfying the terminal constraint

$$\mathbb{E}\left[X_T\right] = c_0. \tag{4.4}$$

This last equality means that the insurance firm is looking for some regulatory requirement c_0 described by the average value of its cash balance process at the terminal time T. Herein, a control variable is said to be an admissible control if and only if satisfies all the above three conditions.

Let us point out that the aim of the policy maker is to minimize simultaneously the deviation between the firm's cash-balance process and its dynamic benchmark over the set of all admissible controls which will be denoted by $\mathcal{U}$, the cost of the premium policy over the whole time interval $[0,T]$, and the terminal variance of the cash-balance process under some given constraint. Therefore, it is quite natural that the cost functional takes the following form

$$J(v.) = \mathbb{E}\left[\int_0^T e^{-\beta t} g_1(t, X_t, v_t)\,dt + e^{-\beta T}\varphi(X_T)\right], \tag{4.5}$$

with

$$g_1(t,x,v) = \frac{1}{2}\left(R_t(x - A_t)^2 + N_t v^2\right), \tag{4.6}$$

and

$$\varphi(x) = \frac{1}{2} M (x - c_0)^2. \tag{4.7}$$

Here β is a discounting factor, A_t is some dynamic pre-set target, representing the dynamic benchmark of X. Processes R, N and constant M are the weighting factors which make the cost functional (4.5) more general and flexible to control the preference of the policy-maker. Furthermore, we suppose that process A converges to c_0 as t goes to T.

Now we can formulate the firm's optimal premium problem as

Problem A: To find $\hat{v} \in \mathcal{U}$ such that $\hat{v}$ minimizes the cost function (4.5) subject to (4.3) and the state constraint (4.4).

To deal with this problem, we need to impose the following assumptions on the previous coefficients.

Assumption ($\mathbf{H}_{4.1}$): Functions $R_. \geq 0$, $N_. > 0$, $N_.^{-1}$, $\Delta_.$, $b_.$, $\sigma_.$ and $A_.$ are all deterministic and uniformly bounded on the time interval $[0,T]$. Moreover, $M \geq 0$ and $\beta > 0$.

Let us now reformulate the above control problem $(4.3) - (4.5)$ as a generalized stochastic recursive optimal control with state constraint by introducing the following backward stochastic differential equation

$$\begin{cases} -dY_t = & [g_1(t, X_t, v(t)) - \beta Y_t] \, dt - Z_t dW_t - \sum_{i=1}^{\infty} K_t^i dH_t^{(i)}, \\ Y_T = & \varphi(X_T). \end{cases} \tag{4.8}$$

To conclude this subsection let us notice that (4.3) together with (4.8) form a semi-coupled FBSDE driven by both the Teugels martingales and an independent Brownian motion. Then Lemma 2.1 in [31] shows that under $(\mathbf{H}_{4.1})$, (4.3) admits a unique solution X_t for each $v \in \mathcal{U}$ and satisfies

$$\sup_{0 \leq t \leq T} \mathbb{E}\left(X_t^4\right) dt < +\infty.$$

For proof of the above estimate, we use the same technique as in Proposition 2.1 in Chapter 2.

As consequence, the terminal condition of (4.8) is square-integrable. Then, for the foregoing v_t and X_t, thanks to Theorem 3.1 in [4], the BSDE (4.8) admits a unique solution pair (Y_t, Z_t) under $(\mathbf{H}_{4.1})$. That is, for any $v \in \mathcal{U}$ semi-coupled FBSDE consisting of (4.3) with (4.8) admits a unique solution $(X_., Y_., Z_., K_.)$.

Obviously, by using the dual technique to the BSDE (4.8) one can get $Y_0 = J(v.)$, then we can reformulate Problem A in the following way:

Problem B: To find $\hat{v} \in \mathcal{U}$ such that

$$J(\hat{v}.) = \mathbb{E}(\hat{Y}_0) \tag{4.9}$$

subject to (4.3), (4.4), and (4.8).

In the next section we are going to prove a sufficient condition of optimality for the above problem but in a more general form by assuming that coefficients are not necessarily linear with respect to the state variables.

4.3 Sufficient condition of optimality

4.3.1 Problem formulation

Motivated by the above optimal premium policy of an insurance firm, we are now going to focus on the following control problem where the state process is described by the following controlled stochastic differential equation driven by both a Brownian motion and a family of Teugels martingales,

$$\begin{cases} dX_t = & f\left(t, X_t, v_t\right) dt + \sigma\left(t, X_t, v_t\right) dW_t + \sum_{i=1}^{\infty} \pi^i \left(t, X_{t_-}, v_t\right) dH_t^{(i)}, \\ X_0 = & x, \end{cases} \tag{4.10}$$

and the general stochastic differential utility is given by the BSDE

$$\begin{cases} -dY_t = & g\left(t, \mathcal{X}_t, v_t\right) dt - Z_t dW_t - \sum_{i=1}^{\infty} K_t^i dH_t^{(i)}, \\ Y_T = & \varphi\left(X_T\right) \end{cases} \tag{4.11}$$

where $\mathcal{X}_t := \left(X_t, Y_t, Z_t, K_t\right)$, and

$$\begin{aligned} &f : [0,T] \times \Omega \times \mathbb{R} \times \mathcal{U} \rightarrow \mathbb{R}, \\ &\sigma : [0,T] \times \Omega \times \mathbb{R} \times \mathcal{U} \rightarrow \mathbb{R}, \\ &\pi : [0,T] \times \Omega \times \mathbb{R} \times \mathcal{U} \rightarrow \mathcal{P}^2\left(\mathbb{R}\right), \\ &g : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}^2\left(\mathbb{R}\right) \times \mathcal{U} \rightarrow \mathbb{R}, \\ &\varphi : [0,T] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}, \end{aligned}$$

are progressively measurable functions.

In the sequel, for notational simplicity, we shall use the shorthand notation

$$\pi\left(t, X_{t_-}, v_t\right) dH_t \quad \text{and} \quad K_t dH_t,$$

instead of

$$\sum_{i=1}^{\infty} \pi^i \left(t, X_{t_-}, v_t\right) dH^{(i)}(t) \quad \text{and} \quad \sum_{i=1}^{\infty} K_t^i dH_t^{(i)},$$

respectively, where $K_t = \{K_t^i\}_{i=1}^{\infty}$, $\pi_t = \{\pi_t^i\}_{i=1}^{\infty}$ and $\pi^i : [0,T] \times \Omega \times \mathbb{R} \times \mathcal{U} \to \mathcal{P}^2(\mathbb{R})$ is a progressively measurable function for any i

Let us now introduce the following basic assumption on coefficients which will be needed in the sequel.

Assumption $(\mathbf{H}_{4.2})$**:**

i) The function g is $\mathcal{F}_t$-progressively measurable for all $(y, z, k) \in \mathbb{R} \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R})$ and for any $v_t \in \mathcal{U}$,

$$g(t, \mathcal{X}_t, v_t) = G(t, Y_t, Z_t, K_t) + R_t (X_t - A_t)^2 + N_t v_t^2,$$

with

$$\mathbb{E}\left(\int_0^T |g(s, 0, 0, 0, 0, v_s)|^2 ds\right) < \infty.$$

ii) For every $x \in \mathbb{R}$, $\varphi \in L^2$, f, σ and π are progressively measurable and

$$\mathbb{E}\left(\int_0^T |f(s, 0, v_s)|^4 ds\right) + \mathbb{E}\left(\int_0^T \left(|\sigma(s, 0, v_s)|^4 + \|\pi(s, 0, v_s)\|_{\mathcal{P}^2(\mathbb{R})}^4\right) ds\right) < \infty.$$

iii) The functions G, f, σ and π are continuous and continuously differentiable with respect to $x, y, z, k,$ and v. Moreover, their derivatives are bounded.

Noting that, Lemma 2.1 in [31] shows that under the assumptions $(\mathbf{H}_{4.2})$, the SDE (4.10) admits a unique solution belongs to $\mathcal{S}_{\mathcal{F}}^2(0, T, \mathbb{R})$. On the other hand, since the function

g is uniformly Lipschitz with respect to y, z and k, by using the assumptions $(\mathbf{H}_{4.2})$ one can easily check that the BSDE (4.11), satisfies all the conditions in Theorem 3.1 in [4], and hence it has a unique solution belongs to $\mathcal{S}^2_{\mathcal{F}}(0,T,\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R}) \times l^2_{\mathcal{F}}(0,T,\mathbb{R})$.

4.3.2 Sufficient maximum principle

In this subsection, we study Problem **B** with more general state process. We establish a sufficient stochastic maximum principle for stochastic control of forward-backward SDEs driven by Brownian motion and Teugels martingales where the control domain is assumed to be convex.

First, by combining (4.10) and (4.11) we get the following controlled semi-coupled FBSDE driven by Brownian motion and Teugels martingales:

$$\begin{cases} X_t = & X_0 + \int_0^t f(s, X_s, v_s)\, ds + \int_0^t \sigma(s, X_s, v_s)\, dW_s \\ & + \int_0^t \pi(s, X_s, v_s)\, dH(s), \\ Y_t = & \varphi(X_T) + \int_t^T g(s, \chi_s, v_s)\, dt - \int_t^T Z_s dW_s - \int_t^T K_s dH_s. \end{cases} \tag{4.12}$$

We introduce the following cost functional

$$J(v) = \mathbb{E}\left[\int_0^T g(s, \chi_s, v_s)\, ds + \varphi(X_T)\right], \tag{4.13}$$

The optimal control problem is to minimize the cost functional $J(\cdot)$ over the set of all admissible controls.

To deal with the above control problem, We first define the Hamiltonian function

$$\mathcal{H} : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \to \mathbb{R}$$

by

$$\begin{aligned}\mathcal{H}(t,x,y,z,k,v,p,q,\lambda,\rho) = g(t,x,y,z,k,v)\,\lambda + f(t,x,v)\,p \\ +\sigma(t,x,v)\,q + \pi(t,x,v)\,\rho.\end{aligned} \tag{4.14}$$

Using the shorthand notation

$$\mathcal{H}_y(t) = \frac{\partial \mathcal{H}}{\partial y}(t,x,y,z,k,v,p,q,\lambda,\rho),$$

and similarly, with $\mathcal{H}_z(t)$, $\mathcal{H}_k(t)$, $\mathcal{H}_v(t)$,$\mathcal{H}_x(t)$, and similar notations are made for $\mathcal{H}(t)$ and $l(t)$ where $l = f, \sigma, \pi$ and g.

The adjoint equations are described by the following stochastic Hamiltonian systems,

$$\begin{cases} d\lambda_t = \mathcal{H}_y(t)\,dt + \mathcal{H}_z(t)\,dW_t + \mathcal{H}_k(t)\,dH_t, \\ dp_t = -\mathcal{H}_x(t)\,dt + q_t dW_t + \rho_t dH_t, \\ \lambda_0 = 1, \\ p_T = \lambda_T \varphi'(X_T),\ q_T = \rho_T = 0. \end{cases} \tag{4.15}$$

In addition to $(\mathbf{H}_{4.2})$ we need the following assumptions:

Assumption $(\mathbf{H}_{4.3})$**:** We assume,

i) Functions $x \to \varphi(x)$ and $(t,x,v,y,z,k) \to \mathcal{H}\left(t,x,v,y,z,k,\hat{p}_t,\hat{q}_t,\hat{\lambda}_t,\hat{\rho}_t\right)$ are convex.

ii) Function $\mathcal{H}$ satisfies

$$\begin{aligned}&\mathcal{H}\left(t,\hat{X}_t,\hat{Y}_t,\hat{Z}_t,\hat{K}_t,v_t,\hat{p}_t,\hat{q}_t,\hat{\lambda}_t,\hat{\rho}_t\right) \\ &-\hat{\mathcal{H}}(t,\hat{X}_t,\hat{Y}_t,\hat{Z}_t,\hat{K}_t,\hat{v}_t,\hat{p}_t,\hat{q}_t,\hat{\lambda}_t,\hat{\rho}_t) \geq 0,\end{aligned} \tag{4.16}$$

for $v_\bullet \in \mathcal{U}$ and any for almost all $(t,w) \in [0,T] \times \Omega$.

Then we have the following sufficient condition for an optimal control of Problem **B**.

Theorem 4.1 ***(Sufficient maximum principle)*** *Assume* $(\mathbf{H}_{4.3})$. *Let* $\hat{v} \in \mathcal{U}$ *with corresponding solutions* $\hat{X}_.$, $\left(\hat{Y}_., \hat{Z}_., \hat{K}_.\right)$ *and* $\left(\hat{\lambda}_., \hat{p}_., \hat{q}_., \hat{\rho}_.\right)$ *of* (4.10), (4.11) *and* (4.15) *respectively. Then,* $\hat{v}_.$ *is an optimal control for Problem* ***B***.

Proof. Let $(X^v_., Y^v_., Z^v_., K^v_., v_.)$ be an admissible solution of (4.12), it follows from the definition of the cost functional (4.13) that

$$J(\hat{v}_.) - J(v_.) = \mathbb{E}\left[\hat{Y}_0 - Y^v_0\right].$$

From a forward component of SDE (4.15), the right hand side of the above equality can be rewritten

$$\mathbb{E}\left[\hat{Y}_0 - Y^v_0\right] = \mathbb{E}\left[\left(\hat{Y}_0 - Y^v_0\right)\hat{\lambda}_0\right].$$

Applying Itô's formula to $(\hat{Y}_t - Y^v_t)\lambda^v_t$ from $t = 0$ to $t = T$ and using the fact that $\left\langle H^{(i)}, H^{(j)}\right\rangle_t = \delta_{ij} t$ and $\left[H^{(i)}, H^{(j)}\right] - \left\langle H^{(i)}, H^{(j)}\right\rangle$ is a martingale, to obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\hat{Y}_0 - Y^v_0\right)\hat{\lambda}_0\right] = \mathbb{E}\left[\left(\varphi\left(\hat{X}_T\right) - \varphi\left(X^v_T\right)\right)\hat{\lambda}_T\right] \\
&-\mathbb{E}\left[\int_0^T \left(\hat{Y}_t - Y^v_t\right) d\hat{\lambda}_t\right] - \mathbb{E}\left[\int_0^T \hat{\lambda}_t d\left(\hat{Y}_t - Y^v_t\right)\right] \\
&-\mathbb{E}\left[\int_0^T \mathcal{H}_z(t)\left(\hat{Z}_t - Z^v_t\right) dt\right] - \mathbb{E}\left[\int_0^T \mathcal{H}_k(t)\left(\hat{K}_t - K^v_t\right) dt\right].
\end{aligned}$$

Since φ is convex, one can get

$$\begin{aligned}
&\mathbb{E}\left[\left(\varphi\left(\hat{X}_T\right) - \varphi\left(X^v_T\right)\right)\hat{\lambda}_T\right] \\
&\leq \mathbb{E}\left[\left(\hat{X}_T - X^v_T\right)\varphi'\left(X^v_T\right)\hat{\lambda}_T\right].
\end{aligned}$$

We remark that $\hat{p}_T = \varphi'\left(X^v_T\right)\hat{\lambda}_T$, then

$$\begin{aligned}
&\mathbb{E}\left[\left(\hat{Y}_0 - Y^v_0\right)\hat{\lambda}_0\right] \leq \mathbb{E}\left[\left(\hat{X}_T - X^v_T\right)\hat{p}_T\right] \\
&-\mathbb{E}\left[\int_0^T \mathcal{H}_y(t)\left(\hat{Y}_t - Y^v_t\right) dt\right] + \mathbb{E}\left[\int_0^T (\hat{\lambda}_t\left(\hat{g}_t - g^v_t\right))dt\right] \\
&-\mathbb{E}\left[\int_0^T \mathcal{H}_z(t)\left(\hat{Z}_t - Z^v_t\right) dt\right] - \mathbb{E}[\mathcal{H}_k(t)\left(\hat{K}_t - K^v_t\right) dt],
\end{aligned} \tag{4.17}$$

on the other hand, Itô's formula applied to $\left(\hat{X}_t - X_t\right)\hat{p}_t$, gives us

$$\begin{gathered}\mathbb{E}\left[\left(\hat{X}_T - X_T^v\right)\hat{p}_T\right] = \mathbb{E}\left[\int_0^T \left(\hat{X}_t - X_t^v\right) d\hat{p}_t\right] \\ +\mathbb{E}\left[\int_0^T \hat{p}_t d\left(\hat{X}_t - X_t^v\right)\right] + \mathbb{E}\left[\int_0^T \left(\hat{\sigma}_t - \sigma_t\right)\hat{q}_t dt\right] \\ +\mathbb{E}\left[\int_0^T \left(\hat{\pi}_t - \pi_t\right)\hat{\rho}_t dt\right],\end{gathered} \tag{4.18}$$

Substituting (4.18) into (4.17), it follows immediately that,

$$\begin{aligned}&\mathbb{E}\left[\left(\hat{Y}_0 - Y_0^v\right)\right] \leq \mathbb{E}\left[\int_0^T -\mathcal{H}_x(t)\left(\hat{X}_t - X_t^v\right) dt\right. \\ &+ \int_0^T \left\{\left(\hat{f}_t - f_t\right)\hat{p}_t + \left(\hat{\sigma}_t - \sigma_t\right)\hat{q}_t + \left(\hat{\pi}_t - \pi_t\right)\hat{\rho}_t\right. \\ &+\hat{\lambda}_t\left(\hat{g}_t - g_t\right) - \mathcal{H}_y(t)\left(\hat{Y}_t - Y_t^v\right) \\ &\left.\left.-\mathcal{H}_z(t)\left(\hat{Z}_t - Z_t^v\right) - \mathcal{H}_k(t)\left(\hat{K}_t - K_t^v\right)\right\} dt\right].\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}\left[\left(\hat{Y}_0 - Y_0^v\right)\right] \leq \mathbb{E}\left[\int_0^T \left\{\left(\hat{\mathcal{H}}(t) - \mathcal{H}_t\right) - \mathcal{H}_x(t)\left(\hat{X}_t - X_t^v\right)\right.\right. \\ &-\mathcal{H}_y(t)\left(\hat{Y}_t - Y_t^v\right) - \mathcal{H}_z(t)\left(\hat{Z}_t - Z_t^v\right) \\ &\left.\left.-\mathcal{H}_k(t)\left(\hat{K}_t - K_t^v\right)\right\} dt\right].\end{aligned}$$

By virtue of the convexity property of the Hamiltonian $\mathcal{H}$ with respect to (x, y, z, k) for almost all $(t, w) \in [0, T] \times \Omega$, one can get

$$\begin{aligned}&\mathbb{E}\left[\left(\hat{Y}_0 - Y_0^v\right)\right] \leq \mathbb{E}\left[\int_0^T \left\{\mathcal{H}_x(t)\left(\hat{X}_t - X_t^v\right)\right.\right. \\ &+\mathcal{H}_y(t)\left(\hat{Y}_t - Y_t^v\right) + \mathcal{H}_z(t)\left(\hat{Z}_t - Z_t^v\right) \\ &+\mathcal{H}_k(t)\left(\hat{K}_t - K_t^v\right) + \mathcal{H}_v(t)\left(\hat{v}_t - v_t\right) \\ &-\mathcal{H}_x(t)\left(\hat{X}_t - X_t^v\right) - \mathcal{H}_y(t)\left(\hat{Y}_t - Y_t^v\right) \\ &\left.\left.-\mathcal{H}_z(t)\left(\hat{Z}_t - Z_t^v\right) - \mathcal{H}_k\left(\hat{K}_t - K_t^v\right)\right\} dt\right] \\ &= \mathbb{E}\left[\int_0^T \mathcal{H}_v(t)\left(\hat{v}_t - v_t\right) dt\right].\end{aligned}$$

By invoking the necessary condition of optimality (4.16), we conclude that

$$\mathbb{E}\left[\left(\hat{Y}_0 - Y_0^v\right)\right] \leq 0,$$

which implies,

$$J(\hat{v}_\cdot) - J(v_\cdot) \leq 0, \quad \forall v_\cdot \in \mathcal{U}.$$

This achieves the proof. ■

4.4 Applications

4.4.1 Optimal premium problem

In this subsection, firstly, we will use the Lagrangian method to treat the terminal state constraint and after that we apply a sufficient maximum principle to deal with the resulting unconstrained optimization problem. Throughout this subsection, we assume that $\alpha = 0$ in (4.2) which means that the insurance firm only invests in a money account with compounded interest rate δ_t, and hence, $\Delta_t = \int_0^t \delta_s ds$. We recall that this kind of problem in the continuous Brownian case was solved in [24].

Then, SDE (4.3) becomes

$$\begin{cases} dX_t = (\delta_t X_t + b_t + v_t)\, dt + \sigma_t dW_t + \pi_t dH_t, \\ X_0 = x. \end{cases} \tag{4.19}$$

By using the Lagrangian multiplier method the cost function (4.5) becomes

$$J(v) = \mathbb{E}\left[\int_0^T e^{-\beta t} g_1(t, X_t, v_t)\, dt + e^{-\beta T}\varphi(X_T) + \theta\left((X_T - c_0)\right)\right], \tag{4.20}$$

where θ, the Lagrange multiplier, is some constant to be determined. Then we can reformulate the problem **A**, as the following:

We look forward to find $\hat{v} \in \mathcal{U}$ which minimize the cost function (4.20) subject to (4.19) and (4.4).

For this end, let us firstly define the Hamiltonian function

$$\mathcal{H} : [0,T] \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}^2 (\mathbb{R}) \rightarrow \mathbb{R},$$

by

$$\begin{aligned} \mathcal{H}(t, X, v, p, q, \rho) &= (\delta_t X + b_t + v)\, p + \sigma_t q + \pi_t \rho \\ &\quad + \frac{1}{2} e^{-\beta t} \left[R_t (X - A_t)^2 + N_t v^2 \right]. \end{aligned} \tag{4.21}$$

Then, the adjoint equation can be rewritten in a Hamiltonian form:

$$\begin{cases} -dp_t = & \left[\delta_t p_t + (X_t - A_t) R_t e^{-\beta t} \right] dt - q_t dW_t - \rho_t dH_t, \\ p_T = & \theta + M e^{-\beta T} (X_T - c_0), \end{cases} \tag{4.22}$$

where $X_.$ stands for the optimal process under the optimal premium policy v and satisfies the equation (4.19).

Let ϕ and ψ the solutions of

$$\begin{cases} \phi_t' + 2\delta_t \phi_t - N_t^{-1} e^{\beta t} \phi_t^2 + R_t e^{-\beta t} = 0, \\ \phi_T = M e^{-\beta T}, \end{cases} \tag{4.23}$$

and

$$\begin{cases} \psi_t' + \left(\delta_t - N_t^{-1} e^{\beta t} \phi_t \right) \psi_t + b_t \phi_t - A_t R_t e^{-\beta t} = 0, \\ \psi_T = \theta - c_0 M e^{-\beta T}. \end{cases} \tag{4.24}$$

We now in position to stat and prove the main result of this section.

Theorem 4.2 *Under Assumption* $(\mathbf{H}_{4.1})$*, the optimal premium policy is given by*

$$\hat{v}_t = -N_t^{-1} e^{\beta t} \left(\phi_t X_t + \psi_t\right),$$

where $X_.$ *satisfies* (4.19) *and* ϕ_t *and* ψ_t *are the solution of* (4.23) *and* (4.24) *respectively. Moreover, the optimal cost functional is given by*

$$\begin{aligned} J\left(\hat{v}\right) = & \frac{1}{2}\left(\int_0^T e^{-\beta t} R_t A_t^2 dt + M e^{-\beta T} c_0^2\right) + \frac{1}{2}\phi_0 x^2 + \psi_0 x - c_0\theta \\ & + \frac{1}{2}\int_0^T \left[\phi_t \sigma_t^2 + \phi_t \pi_t^2 + \psi_t \left(2b_t - N_t^{-1} e^{\beta t}\psi_t\right)\right] dt. \end{aligned} \tag{4.25}$$

Proof. We shall divide the proof into several steps.

Step 1: We start by proving the existence of optimal premium policy. Since for each t, (4.5) is quadratic with respect to X_t, v_t, X_T, and the weight N_t of v_t^2 is larger than 0, then there exists an optimal premium policy $\hat{v}_.$ which solves Problem **A**. Indeed, from the maximum principle, the optimal premium policy $\hat{v}$ satisfy

$$\begin{aligned} 0 &= \frac{\partial \mathcal{H}}{\partial v_t}\left(t, X_t, v_t, p_t, q_t, \rho_t\right) \\ &= \frac{\partial}{\partial v_t}\left\{\left(\delta_t X_t + b_t + v_t\right) p_t + \sigma_t q_t + \pi_t \rho_t \right. \\ &\left. + \frac{1}{2} e^{-\beta t}\left[R_t \left(X_t - A_t\right)^2 + N_t v_t^2\right]\right\} \\ &= p_t + e^{-\beta t} N_t v_t, \end{aligned}$$

and is given by

$$\hat{v}_t = -N_t^{-1} e^{\beta t} p_t. \tag{4.26}$$

Substituting (4.26) into (4.19) and combining it with (4.22), we get the generalized Hamiltonian system

$$\begin{cases} dX_t = & \left(\delta_t X_t - N_t^{-1} e^{\beta t} p_t + b_t\right) dt + \sigma_t dW_t + \pi_t dH_t, \\ -dp\left(t\right) = & \left[\delta_t p_t + \left(X_t - A_t\right) R_t e^{-\beta t}\right] dt - q_t dW_t - \rho_t dH_t, \\ X_0 = x, & p_T = \theta + M e^{-\beta T}\left(X_T - c_0\right), \end{cases} \tag{4.27}$$

which is a coupled FBSDE system. Using si a similar argument as the ones used in Baghery and al. [3], we can verify that FBSDE (4.27) admits a unique solution under ($\mathbf{H}_{4.1}$). Note that the optimal premium policy exists and is unique. Furthermore the relation (4.26) implies that it is linear with respect to p.

Step 2: In this step, we are going to prove that the optimal premium policy $\hat{v}$ is in fact a linear feedback of the optimal process $X_.$. First of all, according to the terminal condition of p in (4.27), it is quite natural to suggest that

$$p_t = \phi_t X_t + \psi_t, \tag{4.28}$$

with $\phi_T = Me^{-\beta T}$ and $\psi_T = \theta - c_0 Me^{-\beta T}$. Then, we apply Itô's formula to p to obtain

$$\begin{aligned} dp_t = & \left[\left(\phi_t' + \delta_t \phi_t\right) X_t + \psi_t' + \left(b_t - N_t^{-1} e^{\beta t} p_t\right)\phi_t\right] dt \\ & + \phi_t \delta_t dW_t + \phi_t \pi_t dH_t. \end{aligned}$$

Comparing their generator terms with those of the BSDE in (4.27) we get (4.23) and (4.24).

It is well known that (4.23) is a standard Riccati differential equation which admits a unique solution under ($\mathbf{H}_{4.1}$), so does (4.24) and

$$\psi_t = \int_t^T \left(b_s \phi_s - A_s R_s e^{-\beta s}\right) \Lambda_t\left(s\right) ds + \left(\theta - c_0 M e^{-\beta T}\right) \Lambda_t\left(T\right), \tag{4.29}$$

where

$$\Lambda_t(s) = \exp\{\int_t^s \left(\delta_r - N_r^{-1} e^{\beta r}\phi_r\right) dr\}.$$

Step 3: In this step we are going to find the value of θ. Combining (4.26) and (4.28), we get

$$\begin{cases} dX_t = & \left[\left(\delta_t - N_t^{-1} e^{\beta t}\phi_t\right) X_t + b_t - N_t^{-1} e^{\beta t}\psi_t\right] dt \\ & +\sigma_t dW_t + \pi_t dH_t, \\ X_0 = & x. \end{cases}$$

Define $\tilde{X}_t = \mathbb{E}[X_t]$, then

$$\begin{cases} \tilde{X}_t' = & \left(\delta_t - N_t^{-1} e^{\beta t}\phi_t\right) \tilde{X}_t + b_t - N_t^{-1} e^{\beta t}\psi_t, \\ \tilde{X}_0 = & x \end{cases} \tag{4.30}$$

Solving (4.30) and keeping in mind the terminal constraint (4.4), we easily derive

$$c_0 = x\Lambda_0(T) + \int_0^T \left(b_t - N_t^{-1} e^{\beta t}\psi_t\right) \Lambda_t(T)\, dt, \tag{4.31}$$

where

$$\Lambda_t(s) = \exp\{\int_t^s \left(\delta_r - N_r^{-1} e^{\beta r}\phi_r\right) dr\}.$$

Note that ϕ in (4.23) does not depend on θ and ψ in (4.24) is linear with respect to θ. Inserting (4.24) into (4.31), we get the equation

$$\theta = \frac{\int_0^T \left\{b_t - N_t^{-1}\left[F_t - Q_t\right]\Lambda_t(T)\right\} dt + x\Lambda_0(T) - c_0}{\int_0^T N_t^{-1} e^{\beta t}\Lambda_t^2(T)\, dt}$$

where

$$F_t = \int_t^T \left(e^{\beta t} b_s\phi_s - A_s R_s e^{\beta(t-s)}\right) \Lambda_t(s)\, ds,$$

$$Q_t = c_0 M \exp\{\int_t^T \left(\delta_s - N_s^{-1} e^{\beta s}\phi_s - \beta\right) ds\}$$

and

$$\Lambda_t(s) = \exp\{\int_t^s \left(\delta_r - N_r^{-1} e^{\beta r} \phi_r\right) dr\}.$$

Step 4: We now proceed to determine the optimal cost functional. Substituting (4.26) into (4.5) we have

$$\begin{aligned} J(\hat{v}) = \ & \frac{1}{2}\mathbb{E}\left\{\int_0^T e^{-\beta t}\left[\left(R_t + N_t^{-1} e^{2\beta t}\phi_t^2\right) X_t^2 - 2\left(A_t R_t\right.\right.\right. \\ & \left. -N_t^{-1} e^{2\beta t}\phi_t\psi_t\right) X_t + N_t^{-1} e^{2\beta t}\psi_t^2\Big] dt \\ & +Me^{-\beta T} X_T^2 - 2c_0 M e^{-\beta T} X_T\Big\} \\ & +\frac{1}{2}\left(\int_0^T e^{-\beta t} R_t A_t^2 dt + Me^{-\beta T} c_0^2\right). \end{aligned} \tag{4.32}$$

From Itô's formula,

$$\begin{aligned} d\left(\phi_t X_t^2 + 2\psi_t X_t\right) = \ & \left[\phi_t \sigma_t^2 + 2\psi_t\left(b_t - N_t^{-1} e^{\beta t}\psi_t\right)\right. \\ & +2e^{-\beta t}\left(A_t R_t - N_t^{-1} e^{2\beta t}\phi_t\psi_t\right) X_t \\ & \left. -e^{-\beta t}\left(R_t + N_t^{-1} e^{2\beta t}\phi_t^2\right) X_t^2\right] dt \\ & +2\sigma_t\left(\phi_t X_t + \psi_t\right) dW_t \\ & +2\pi_t\left(\phi_t X_t + \psi_t\right) dH_t + \phi_t \sum_{i,j} \pi_s^i \pi_s^j d\left[H^i, H^j\right]_s. \end{aligned}$$

Integrating from 0 to T, taking expectations on both sides of the above equality and using the fact that $\left[H^i, H^j\right]_t - \left\langle H^i, H^j\right\rangle_t$ is an $\mathcal{F}_t$-martingale and $\left\langle H^i, H^j\right\rangle_t = \delta_{ij} t$, we get

$$\begin{aligned} & \mathbb{E}\left(\phi_T X_T^2 + 2\psi_T X_T\right) \\ & = \int_0^T \left[\phi_t\left(\sigma_t^2 + \pi_t^2\right) + 2\psi_t\left(b_t - N_t^{-1} e^{\beta t}\psi_t\right)\right] dt + \phi_0 x^2 + 2\psi_0 x \\ & +2\mathbb{E}\int_0^T e^{-\beta t}\left(A_t R_t - N_t^{-1} e^{2\beta t}\phi_t\psi_t\right) X_t dt \\ & -\mathbb{E}\int_0^T e^{-\beta t}\left(R_t + N_t^{-1} e^{2\beta t}\phi_t^2\right) X_t^2 dt. \end{aligned}$$

On the other hand, it follows from (4.23) and (4.24) that

$$\mathbb{E}\left[\phi_T X_T^2 + 2\psi_T X_T\right] = \mathbb{E}\left[Me^{-\beta T} X_T^2 - 2c_0 Me^{-\beta T} X_T + 2\theta X_T\right].$$

Inserting the above two equalities into (4.32) we get the optimal cost functional (4.25). This gives the desired result. ■

Specifically, in case we rule out the terminal constraint (4.4), that is, $\theta = 0$, Theorem 4.2 solves Problem **A** without constraint (4.4).

Note also that ϕ, ψ and θ don't depend on σ and π. This leads to the fact that the optimal cost function $J(v)$ increases with respect to σ and π (describing the liability risk). This implies that more uncertainty in the liability process, the higher the operation costs of our optimal premium policy.

4.4.2 Optimal premium problem under stochastic interest rate.

In the current subsection we want to discuss the optimal premium problem (Problem **B**), in the case where the interest rate is allowed to be stochastic. More precisely, we shall consider two different cases, where in the first case we assume that the payment function and the stochastic interest rate are given by the same Brownian motion and in the second case, we assume that they are given by a different and independent Brownian motions.

First case.

In this case we proceed to solve problem **B**, assuming that the utility function is that of (4.9) and the cash-balance process satisfies (4.3) where the stochastic interest rate is given by (4.2). We also point out that we also solve a mean variance problem in this section as a particular case of the problem **B.**

Since the coefficients in (4.3), (4.6) and (4.7) are linear, we shall assume

$$\begin{cases} dX_t = & (a_1(t) X_t + a_2(t) v_t + a_3(t)) dt \\ & + (b_1(t) X_t + b_2(t) v_t + b_3(t)) dW(t) + \pi_t dH(t), \\ X_0 = & x > 0, \end{cases} \tag{4.33}$$

and

$$\begin{cases} dY_t = & \left(\frac{1}{2}(c_1(t) X_t^2 + c_2(t) X_t + c_3(t) v_t^2 + c_4(t)) - \beta Y_t\right) dt \\ & -Z_t dW_t - K_t dH_t \\ Y_T = & \frac{1}{2}(X_T - a)^2, \end{cases} \tag{4.34}$$

where a is real constant and $a_1, a_2, a_3, b_1, b_2 \neq 0, b_3, c_1, c_2, c_3, c_4$ are deterministic functions satisfying some properties. Note that, a particular choices of the coefficients of (4.33) and (4.34) generate the SDE (4.10) and the BSDE (4.11).

We shall derive the solution of Problem **B** where $Y.$ satisfies (4.34). Here, the Hamiltonian (4.14) gets the form

$$\begin{aligned} \mathcal{H} = & \left[\frac{1}{2}(c_1(t) X_t^2 + c_2(t) X_t + c_3(t) v_t^2 + c_4(t)) - \beta Y_t\right] \lambda_t \\ & + (a_1(t) X(t) + a_2(t) v(t) + a_3(t)) p_t \\ & + (b_1(t) X_t + b_2(t) v_t + b_3(t)) q_t + \pi_t \rho_t, \end{aligned}$$

$\lambda_t = e^{-\beta t}$ and $(p., q., \rho.)$ satisfies the following adjoint backward equation

$$\begin{cases} dp_t = & -\left(c_1(t) X_t \lambda_t + a_1(t) p_t + b_1(t) q_t + \frac{1}{2} c_2(t) \lambda_t\right) dt \\ & + q_t dW_t + \rho_t dH_t, \\ p_T = & \theta + \lambda_t \varphi'(X_T). \end{cases} \tag{4.35}$$

By using the sufficient condition (4.16) given in Section 3, we get

$$\lambda(t) c_3(t) v_t + a_2(t) p_t + b_2(t) q_t = 0. \tag{4.36}$$

To solve (4.35), we try a process p of the form

$$p_t = \phi_t X_t + \psi_t, \tag{4.37}$$

where ϕ and ψ are deterministic C^1 functions.

Applying Itô's formula to (4.37) and using (4.33), we get

$$\begin{aligned} dp_t &= \left[\left(\phi_t a_1(t) + \phi_t'\right) X_t + \phi_t a_2(t) v_t + \phi_t a_3(t) + \psi_t'\right] dt \\ &+ \phi_t \left[b_1(t) X_t + b_2(t) v_t + b_3(t)\right] dW_t + \phi_t \pi_t dH_t \end{aligned} \tag{4.38}$$

Comparing with (4.35) we get

$$\rho_t = \phi_t \pi_t,$$

$$q_t = \phi_t \left[b_1(t) X_t + b_2(t) v_t + b_3(t)\right], \tag{4.39}$$

$$\begin{aligned} &-\left(c_1(t) X_t \lambda_t + a_1(t) p_t + b_1(t) q_t + \frac{1}{2} c_2(t) \lambda_t\right) \\ &= \left(\phi_t a_1(t) + \phi_t'\right) X_t + \phi_t a_2(t) v_t + \phi_t a_3(t) + \psi_t'. \end{aligned} \tag{4.40}$$

Substituting (4.39) into (4.36), we get

$$\hat{v}_t = -\frac{\left(a_2(t) + b_1(t) b_2(t)\right) \phi_t \hat{X}_t + a_2(t) \psi_t + b_2(t) b_3(t) \phi_t}{\lambda_t c_3(t) + b_2^2(t) \phi_t}. \tag{4.41}$$

On the other hand, from (4.40) we have

$$\begin{aligned} \hat{v}_t = &-\frac{\left(c_1(t) \lambda_t + 2a_1(t) \phi_t + b_1^2(t) \phi_t + \phi_t'\right) \hat{X}_t}{\left(a_2(t) + b_1(t) b_2(t)\right) \phi_t} \\ &- \frac{a_1(t) \psi_t + b_1(t) b_3(t) \phi_t + \frac{1}{2} c_2(t) \lambda_t + a_3(t) \phi_t + \psi_t'}{\left(a_2(t) + b_1(t) b_2(t)\right) \phi_t}. \end{aligned} \tag{4.42}$$

Combining (4.41) and (4.42) , we get

$$\begin{aligned}\phi_t^2\left[\left(a_2(t)+b_1(t)b_2(t)\right)^2-b_2^2(t)\left(2a_1(t)+b_1^2(t)\right)\right]-\phi_t\phi_t'b_2^2(t)\\-\left[c_3(t)\left(2a_1(t)+b_1^2(t)\right)+c_1(t)b_2^2(t)\right]\phi_t\lambda_t\\-c_3(t)\phi_t'\lambda_t=c_1(t)c_3(t)\lambda^2(t),\ \phi_T=\lambda_T.\end{aligned}\tag{4.43}$$

and

$$\begin{aligned}&\psi_t\left[\phi_t\left(a_2(t)\left(a_2(t)+b_1(t)b_2(t)\right)-a_1(t)b_2^2(t)\right)-c_3(t)a_1(t)\lambda_t\right]\\&-\psi'(t)\left(c_3(t)\lambda_t+\phi_tb_2^2(t)\right)\\&=-\phi_t^2\left[\left(a_2(t)+b_1(t)b_2(t)\right)b_2(t)b_3(t)-b_2^2(t)\left(b_1(t)b_3(t)+a_3(t)\right)\right]\\&+c_3(t)\left(b_1(t)b_3(t)+a_3(t)\right)\phi_t\lambda_t\\&+\frac{1}{2}\left(c_2(t)b_2^2(t)\phi_t+c_3(t)c_2(t)\lambda_t\right)\lambda_t,\ \psi_T=\theta-\lambda_Tc_0.\end{aligned}\tag{4.44}$$

It is well-known that (4.43) is a standard Riccati differential equation which admits a unique solution. Equation (4.44) has and explicit solution given by

$$\psi_t=\left(\theta-\lambda_Tc_0\right)\exp\left(-\Lambda_t(T)\right)-\exp\Lambda_t(T)\int_t^T e^{-\Lambda_t(s)}G_t(s)\,ds,$$

with

$$\begin{aligned}\Lambda_t(T)=\int_t^T\frac{\phi_s\left[\left(a_2(s)+b_1(s)b_2(s)\right)a_2(s)-a_1(s)b_2^2(s)\right]}{c_3(s)\lambda(s)+b_2^2(s)\phi_s}ds\\-\int_t^T\frac{c_3(s)a_1(s)\lambda_s}{c_3(s)\lambda_s+\phi_sb_2^2(s)}ds,\end{aligned}$$

and

$$\begin{aligned}G_t(s)=-\int_t^T\frac{\phi_s^2b_2(s)b_3(s)\left(a_2(s)+b_1(s)b_2(s)\right)}{c_3(s)\lambda_s+\phi_sb_2^2(s)}ds\\+\int_t^T\frac{\phi_s^2b_2^2(s)\left(b_1(s)b_3(s)+a_3(s)\right)}{c_3(s)\lambda_s+\phi_sb_2^2(s)}ds\\+\int_t^T\frac{c_3(s)\left(b_1(s)b_3(s)+a_3(s)\right)\lambda_s\phi_s}{c_3(s)\lambda_s+\phi_sb_2^2(s)}ds.\\+\int_t^T\frac{\frac{1}{2}\left(c_2(s)b_2^2(s)\phi_s+c_3(s)c_2(s)\lambda_s\right)\lambda_s}{c_3(s)\lambda_s+\phi_sb_2^2(s)}ds\end{aligned}$$

Thus, we have the following theorem.

Theorem 4.3 *Let X_t^v be the cash balance satisfying* (4.33)*, the optimal premium policy of Problem **B** is*

$$\hat{v}_t = -\frac{(a_2(t) + b_1(t)\, b_2(t))\, \phi_t \hat{X}_t + a_2(t)\, \psi_t + b_2(t)\, b_3(t)\, \phi_t}{c_3(t)\, \lambda_t + b_2^2(t)\, \phi_t},$$

where $\lambda = e^{-\beta t}$ and ϕ and ψ are the solutions of (4.43) *and* (4.44)*, respectively.*

This is clearly the unique solution under Assumption ($\mathbf{H}_{4.1}$). Notice that we can get explicit solutions in some particular cases. For example, if $g_1 = 0$ we can see the problem as a mean-variance one, where our objective is to find $v(t)$ such that it minimizes

$$var(X_T) = \mathbb{E}\left[(X_T - \mathbb{E}(X_T))^2\right],$$

under the terminal constraint condition (4.4).

By the Lagrangian multiplier method the problem can be reduced to minimize the following equivalent problem

$$J(v.) = \mathbb{E}\left[\frac{1}{2}(X_T^v - c_0)^2 + \theta (X_T^v - c_0)\right]. \tag{4.45}$$

In this case, the Hamiltonian (4.14) gets the form

$$\mathcal{H} = ((a_1(t)\, X_t + a_2(t)\, v_t + a_3(t)))\, p_t + ((b_1(t)\, X_t + b_2(t)\, v_t + b_3(t)))\, q_t$$
$$+ \pi(t)\, \rho_t.$$

Hence the adjoint equation takes the following form

$$\begin{cases} dp_t = -(a_1(t)\, p_t + b_1(t)\, q_t)\, dt + q_t dW_t + \rho_t dH_t \\ p_T = \theta + (X_T - c_0). \end{cases}$$

Note that in this case $\lambda_t = 1, \forall t \in [0, T]$.

In the following corollary, we solve problem (4.45) assuming that X_t^v satisfies (4.33).

Corollary 4.1 *Let X_t^v be the cash balance satisfying (4.33). Then, the optimal premium policy of Problem (4.45) is*

$$\hat{v}_t = -\frac{(a_2(t) + b_1(t) b_2(t)) \phi_t \hat{X}_t + a_2(t) \psi_t + b_2(t) b_3(t) \phi_t}{b_2^2(t) \phi_t},$$

where ϕ and ψ are given respectively by

$$\phi_t = \exp\left[-\int_t^T \frac{(a_2(s) + b_1(s) b_2(s))^2 - (2a_1(s) + b_1^2(s)) b_2^2(s)}{b_2^2(s)} ds\right]$$

and

$$\psi_t = (\theta - c_0) \exp(\Lambda_t(T))$$
$$- \exp(\Lambda_t(T)) \int_t^T \exp(-\Lambda_t(s)) \phi_s (a_3(s) b_2(s) - a_2(s) b_3(s)) b_2(s) ds,$$

with

$$\Lambda_t(T) = -\int_t^T \frac{(a_2(s)(a_2(s) + b_1(s) b_2(s))) - a_1(s) b_2^2(s)}{b_2^2(s)} ds.$$

Second case

In this case, we assume the liability of the surplus process and the interest rate are given by two different Brownian motions. Let $\bar{V}_\cdot = (\bar{V}_t)_{t\geq 0}$ be an other Brownian motion defined in $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that $\bar{V}_\cdot$ is independent of $W_\cdot$ and $L_\cdot$. In this subsection, we also assume that

$$\mathcal{F}_t = \sigma\left(W_s, \bar{V}_s, 0 \leq s \leq t\right) \vee \sigma(L_s, s \leq t) \vee \mathcal{N}.$$

We consider that the interest rate is given by the following stochastic differential equation:

$$d\Delta_t = \delta_t dt + \alpha_t dV_t,\ t \in [0, T],\ \Delta(0) = \Delta_0. \tag{4.46}$$

Itô's formula applied to the process $X.$ described by (4.1) in section 2, leads to

$$\begin{cases} dX_t = & \left(X_t\left(\delta_t + \frac{1}{2}\alpha_t^2\right) + b_t + v_t\right)dt + X_t\alpha_t d\bar{V}_t \\ & +\sigma_t dW_t + \pi_t dH_t, \\ X_0 = & x. \end{cases} \tag{4.47}$$

Define a process V such that

$$X_t\alpha_t d\bar{V}_t + \sigma_t dW_t = \left(X_t^2\alpha_t^2 + \sigma_t^2\right)^{\frac{1}{2}} dV_t,$$

that is,

$$V_t = \int_0^t \left(X_t^2\alpha_t^2 + \sigma_t^2\right)^{-\frac{1}{2}} \left(X_t\alpha_t d\bar{V}_t + \sigma_t dW_t\right).$$

Clearly, the process V is a continuous martingale with quadratic variation $\langle V\rangle_t = t$, and so, it must be a standard Brownian motion, see for example Theorem 6.1 in [8]. Thus, we can write equation (4.47) by

$$\begin{cases} dX_t = & \left(X_t\left(\delta_t + \frac{1}{2}\alpha_t^2\right) + b_t + v_t\right)dt + (X^2\alpha_t^2 + \sigma_t^2)^{\frac{1}{2}} dV_t \\ & +\pi(t)\, dH_t, \\ X_0 = & x. \end{cases} \tag{4.48}$$

We shall derive the solution of Problem (4.5). Here, the Hamiltonian (4.14) gets the form

$$\begin{aligned} \mathcal{H} = & \left(X_t\left(\delta_t + \frac{1}{2}\alpha_t^2\right) + b_t + v_t\right)p + (X_t^2\alpha_t^2 + \sigma_t^2)^{\frac{1}{2}} q \\ & +\pi_t\rho + \frac{1}{2}e^{-\beta t}\left[R_t\left(X_t - A_t\right)^2 + N_t v_t^2\right]. \end{aligned}$$

The adjoint process $(p(t), q(t), \rho(t))$ satisfies the following BSDE:

$$\begin{cases} -dp_t = & \left(\left(\delta_t + \frac{1}{2}\alpha_t^2 \right) p_t + \frac{\alpha_t^2 X_t q_t}{\sqrt{X_t^2 \alpha_t^2 + \sigma_t^2}} + R_t \left(X_t - A_t \right) e^{-\beta t} \right) dt \\ & -q_t dV_t - \rho_t dH_t, \\ p_T = & \theta + M e^{-\beta T} \left(X_T - c_0 \right). \end{cases}$$

Note in this case $\lambda_t = 1, \forall t \in [0, T]$.

By the same technique used in Subsection 4.1, we infer that the optimal premium policy is given by

$$\hat{v}_t = -N_t^{-1} e^{\beta t} \left(\phi_t X_t + \psi_t \right),$$

where $X.$ satisfies (4.48) and ϕ and ψ are respectively the solutions of the equations

$$\begin{cases} \phi_t' + 2 \left(\delta_t + \alpha_t^2 \right) \phi_t - N_t^{-1} e^{\beta t} \phi_t^2 + R_t e^{-\beta t} = 0 \\ \phi_T = M e^{-\beta T}, \end{cases}$$

and

$$\begin{cases} \psi_t' + \left(\delta_t + \frac{1}{2}\alpha_t^2 - N_t^{-1} e^{\beta t} \phi_t \right) \psi_t + b_t \phi_t - A_t R_t e^{-\beta t} = 0, \\ \psi_T = \theta - c_0 M e^{-\beta T}. \end{cases}$$

Moreover, the optimal cost functional is given by

$$\begin{aligned} J\left(\hat{v}_{.}\right) = & \frac{1}{2} \left(\int_0^T e^{-\beta t} R_t A_t^2 dt + M e^{-\beta T} c_0^2 \right) + \frac{1}{2} \phi_0 x^2 + \psi_0 x - c_0 \theta \\ & + \frac{1}{2} \int_0^T \left[\phi_t \sigma_t^2 + \phi_t \alpha_t^2 + \psi_t \left(2 b_t - N_t^{-1} e^{\beta t} \psi_t \right) \right] dt. \end{aligned}$$

Conclusion

This study aimed to highlight some novel existence and uniqueness results to fully coupled forward-backward stochastic differentials equations driven by a family of Teugels martingales associated to some Lévy processes. For this purpose, we have extended some knowledge results for FBSDE driven by a Brownian motions to case of FBSDEs with jumps. Such jumps come out naturally, for instance, in models with default risk which happen in default time. This type of equations turned out to be useful in many fields, namely in the stochastic control problems, mathematical finance and partial differential equations.

The most important results and the main contributions of this thesis are summarized as follows:

1) Under some monotonicity conditions, we have provided an existence and uniqueness solution to fully coupled FBSDEs associated to Lévy process, where the final time is allowed to be random. As an intermediate result, we also have treated one kind of BSDEs with jumps in stopping time duration.

2) The well-posedness of FBSDEs driven by Teugels martingales on an arbitrarily fixed large time duration, is also investigated under some assumption on the derivatives of the coefficient.

3) We have proved a verification theorem for an admissible control to be optima for a semi-coupled FBSDEs driven by Teugels martingales, which is illustrated by various examples.

Finally, we provide a few prospects for future studies of FBSDEs driven by Lévy process as follows:

i) Applications to stochastic control problems.

ii) Generalization of the results provided in chapter two to the multi-dimensionnal case.

iii) State and prove a comparison theorem, under some suitable condition, for the system that we have dealt with in chapter one.

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