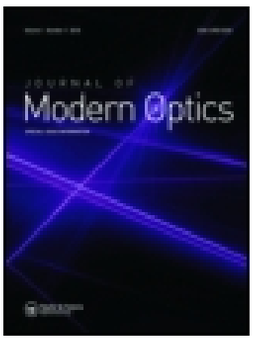

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A semi-inverse variational method for generating the bound state energy eigenvalues in a quantum system: the Dirac Coulomb type-equation

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ABSTRACT

Exact eigenspectra and eigenfunctions of the Dirac quantum equation are established using the semi-inverse variational method. This method improves of a considerable manner the efficiency and accuracy of results compared with the other usual methods much argued in the literature. Some applications for different state configurations are proposed to concretize the method.

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1. Introduction

Over past decades, considerable attention has been devoted to the investigation of new physical models and numerical approaches to evaluate the solutions of wave equations for certain potentials of physical interest (*1–9*). There are relatively few quantum mechanical problems for which the Schrödinger equation and the Dirac equation are exactly solvable (*10–15*).

A great deal of approximate schemas and numerical approaches have appeared to calculate the quantum quantities of the Schrödinger and Dirac equations for numerous potential functions (*16–22*). The scope of this area remains until now a more active field of diverse problems.

The main part of this work serves to the construction of a specific functional in the framework of spherically symmetric potentials using the semi-inverse variational method (*23–31*). In the application, a special attention is focused on the Coulomb potential. This last is chosen among other forms because it has been the subject of intense study and plays a central role in quantum physics.

The organization of the present work is as follows. In Section 2, we present the necessary arguments which underlie the variational formulation in connection with the problems of quantum nature. In Section 3 , some applications are proposed to concretize the method and conclusion and trends for future work are made in Section 4.

2. Semi-inverse method

In recent years, an alternative schema known as the semi-inverse variational method has been introduced for constructing a specific functional of a given variational problem. This method is developed for solving solutions of the coupled first-order differential system for any spherically symmetric potential $V(r)$, i.e. a particle moving under the influence of a central potential, and can be written as (*32–34*)

$$\begin{pmatrix} V(r)-mc^2 & \hbar c(\frac{\mathrm{d}}{\mathrm{d}r}-\kappa/r) \\ -\hbar c(\frac{\mathrm{d}}{\mathrm{d}r}+\kappa/r) & V(r)+mc^2 \end{pmatrix} S(r) = ES(r) \quad (1)$$

where

$$S(r) = \begin{pmatrix} F(r) \\ G(r) \end{pmatrix} \quad (2)$$

$F(r)$ and $G(r)$ denote the large and small components of the radial solution $S(r) = \begin{pmatrix} F(r) \\ G(r) \end{pmatrix}$ and are real radial square-integrable functions, κ is the angular momentum quantum number (also called the spin–orbit quantum number) $\kappa = \pm\left(j+\frac{1}{2}\right), j+\frac{1}{2} \leq n, l = j \pm \frac{1}{2}, l \leq n-1, n = n' + \left(j+\frac{1}{2}\right)$. For $n' = 0$, $\kappa > 0$ and for $n' > 0$, all positive and negative integer values are allowed for κ, n is the principal quantum number, n' is the radial quantum number, c is the speed of light, m (E) is the masse (energy) of the particle and $\hbar$ being the Planck constant. In next section, for the examples,

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$V(r)$ is taken to be of Coulomb potential type as $V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}$, where Z is the atomic number and e is the charge of the particle. The non-relativistic case has been treated in (35) for the Schrödinger equation. Here we treat the states of an electron in a hydrogen-like ion with nuclear charge Z.

The Equation (1), therefore, can be re-written in the form

$$\begin{aligned} M(r;F;F',G,G) &= F' + \frac{\kappa}{r}F \\ &\quad - \frac{1}{\hbar c}\left(mc^2 - E - \frac{\beta}{r}\right)G = 0, \\ N(r;F;F',G,G) &= G' - \frac{\kappa}{r}G \\ &\quad - \frac{1}{\hbar c}\left(mc^2 + E + \frac{\beta}{r}\right)F = 0, \end{aligned} \tag{3}$$

where the superscript $'$ denotes the derivative with respect to r and $\beta = Z\alpha\hbar c$. For the practical requirement, we must put

$$\begin{aligned} \epsilon_\pm &= \frac{1}{\hbar c}\left(mc^2 \pm E\right) \\ \gamma &= Z\alpha \\ \rho &= \sqrt{\epsilon_+\epsilon_-}\,r, \end{aligned} \tag{4}$$

where the fine-structure constant is given by $\alpha = \frac{e^2}{(4\pi\varepsilon_0)\hbar c}$, and after solving the coupled Equation (3) must became

$$\begin{aligned} M(\rho;F;F',G,G') &= F' + \frac{\kappa}{\rho}F - \left(\sqrt{\frac{\epsilon_-}{\epsilon_+}} - \frac{\gamma}{\rho}\right)G = 0, \\ N(\rho;F;F',G,G') &= G' - \frac{\kappa}{\rho}G - \left(\sqrt{\frac{\epsilon_+}{\epsilon_-}} + \frac{\gamma}{\rho}\right)F = 0. \end{aligned} \tag{5}$$

The consistency conditions for the existence of a functional integral are described in (36–38). For this system, we have

$$\begin{aligned} \frac{\partial N}{\partial G} &= \frac{\partial M}{\partial F} - \frac{\partial}{\partial \rho}\left(\frac{\partial M}{\partial F'}\right), \\ \frac{\partial N}{\partial G'} &= -\frac{\partial M}{\partial F'}. \end{aligned} \tag{6}$$

From the consistency conditions (6), we show that the condition of existence of a functional integral for the system (5) is not satisfied. Consequently, we use two auxiliary factors $h(\rho)$ and $l(\rho)$ and rewrite the Equation (5) as

$$\begin{aligned} A(\rho;F;F',G,G') &\equiv h(\rho)M(\rho;F;F',G,G') = 0 \\ B(\rho;F;F',G,G') &\equiv l(\rho)N(\rho;F;F',G,G') = 0 \end{aligned} \tag{7}$$

and in the Equation (6), M and N are replaced by A and B, respectively. Now the consistency conditions are satisfied for (7) provided that $h(\rho) = -l(\rho) = a$, where a is a non-zero constant and can be taken equal to 1. In other words, if we merely multiply the second equation of the system (5) by -1, it is easy to verify that, the consistency conditions (6) are naturally satisfied.

We note that, the problem of variational calculus requires the construction of a functional J such that the general derivative $\delta J = 0$, leads to the Euler–Lagrange equations. Finally, the system (1) is derived, in all cases, directly from the variational integral including a specific functional. Here we show that, the semi-inverse variational method is distinguished by simplicity and elegance and provides a practical way to construct the variational problem. The basic idea of the semi-inverse variational method is illustrated in (23–31).

Our aim hereby is to search for a variational principle whose stationary conditions satisfy the system (1). We begin with the following general trial functional for Equation (1) using the semi-inverse method as

$$J(F,G) = \int_0^{+\infty} \mathcal{L}\,\mathrm{d}\rho \tag{8}$$

in which $\mathcal{L}$ is a trial-Lagrange function, which reads

$$\mathcal{L} = G\frac{\mathrm{d}F}{\mathrm{d}\rho} + \frac{\kappa}{\rho}FG - \left(\sqrt{\frac{\epsilon_-}{\epsilon_+}} - \frac{\gamma}{\rho}\right)G^2 + W, \tag{9}$$

where W is an unknown function of F, G and/or its derivatives. In order to identify the unknown function W, requiring to the trial-functional (9), the stationary condition with respect to F yields the following trial-Euler equation

$$\frac{\kappa}{\rho}G - \frac{\mathrm{d}G}{\mathrm{d}\rho} + \frac{\delta W}{\delta F} = 0. \tag{10}$$

we refer to $\delta W/\delta F$ as the variational derivative of W with respect to F (23, 39, 40), expressed by

$$\begin{aligned} \frac{\delta}{\delta F} &= \sum_{k=0}^{n}(-1)^k\frac{\mathrm{d}^k}{\mathrm{d}\rho^k}\left(\frac{\partial}{\partial\varphi_k}\right), \\ \varphi_k &= \frac{\partial^k F}{\partial\rho^k}. \end{aligned} \tag{11}$$

We search for such an F so that the Equation (10) turns out to be one of two original equations, for example, the second equation of (7). Accordingly, the unknown W can be identified as follows

$$W = \frac{1}{2}\left(\sqrt{\frac{\epsilon_+}{\epsilon_-}} + \frac{\gamma}{\rho}\right)F^2 + W_1, \tag{12}$$

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where W_1 is a newly unknown function of G and/or its derivatives.

Now, we can reconstruct the trial-Lagrangian with unknown function W_1 as

$$\mathcal{L} = G\frac{\mathrm{d}F}{\mathrm{d}\rho} + \frac{\kappa}{\rho}FG - \left(\sqrt{\frac{\epsilon_-}{\epsilon_+}} - \frac{\gamma}{\rho}\right)G^2 + \frac{1}{2}\left(\sqrt{\frac{\epsilon_+}{\epsilon_-}} + \frac{\gamma}{\rho}\right)F^2 + W_1. \tag{13}$$

Making the updated trial-Lagrangian (13), stationary with respect to G results in the following Euler–Lagrange equation

$$\frac{\kappa}{\rho}F + \frac{\mathrm{d}F}{\mathrm{d}\rho} - 2\left(\sqrt{\frac{\epsilon_-}{\epsilon_+}} - \frac{\gamma}{\rho}\right)G + \frac{\delta W_1}{\delta G} = 0. \tag{14}$$

Comparison of (14) and the first equation of (7) leads to the following results

$$\frac{\delta W_1}{\delta G} = \left(\sqrt{\frac{\epsilon_-}{\epsilon_+}} - \frac{\gamma}{\rho}\right)G \tag{15}$$

We, therefore, identify the unknown W_1 as follows

$$W_1 = \frac{1}{2}\left(\sqrt{\frac{\epsilon_-}{\epsilon_+}} - \frac{\gamma}{\rho}\right)G^2. \tag{16}$$

We, therefore, obtain the final variational principle of the system (1), which reads

$$J(F,G) = \int_0^{+\infty}\left\{G\frac{\mathrm{d}F}{\mathrm{d}\rho} + \frac{\kappa}{\rho}FG - \frac{1}{2}\left(\sqrt{\frac{\epsilon_-}{\epsilon_+}} - \frac{\gamma}{\rho}\right)G^2 + \frac{1}{2}\left(\sqrt{\frac{\epsilon_+}{\epsilon_-}} + \frac{\gamma}{\rho}\right)F^2\right\}\mathrm{d}\rho. \tag{17}$$

It is easy to verify that, the Lagrangian in (17), when substituted in the Euler–Lagrange equations, reproduces the Dirac Equation (7). In the next section, we illustrate the above theory by treating some specific examples.

3. Illustrative applications

In this section, we explore the method with three examples. To be more precise, we use therefore the case of the angular momentum $l = 0$ (s-wave setting), $l = 1$ (p-wave setting) and $l = 2$ (d-wave setting). In this application, we consider an electron bound to the atomic nucleus by a coulomb potential in the central force. Because the wave functions vanish at the origin and at $r \longrightarrow \infty$, it is desirable to guess the solutions by making the following trial radial wave functions such that

$$\begin{pmatrix} F(\rho) \\ G(\rho) \end{pmatrix} = e^{-\rho}\rho^s \begin{pmatrix} \sum_{k=0}^{M} a_k\rho^k \\ \sum_{k=0}^{N} b_k\rho^k \end{pmatrix}, \tag{18}$$

where $e^{-\rho}$ is the asymptotic solution at infinity and $s = \sqrt{\kappa^2 - \gamma^2}$ gives the formal asymptotic solution ρ^s at origin, while a_j and b_j are constant quantities to be determined and are used with the energy E as free variational parameters of the problem under consideration. These variational parameters are inferred from the variational functional $J(F,G)$ (Equation (17)) subject to the minimization condition

$$\delta J(F,G) = 0 \tag{19}$$

with respect to specific variational parameters. The optimization condition (19) yields sets of algebraic equations in terms of the variational parameters. In most cases, these parameters must suffice to describe the bound state solutions of the system.

Usually, for bound states, we can find many eigenfunction solutions for a given set of three quantum numbers (n, l, j) where n is the principal quantum number. To distinguish the different configurations, we must label the solutions with these three quantum numbers as $F^j_{n,l}(\rho)$ and $G^j_{n,l}(\rho)$, while the energy is labelled with $E_{n,j}$. We can also label F and G as $F_{n,\kappa}(\rho)$ and $G_{n,\kappa}(\rho)$, respectively.

In the sequel, unless otherwise specified we adopt our chosen units in which $\hbar = \frac{e^2}{4\pi\varepsilon_0} = 1$ and $c = 137$.

3.1. First test example: case $l = 0$, $\kappa = 1$ and $M = N = 1$

We begin with the candidate bound state wave function in the s -configuration. The solution we are looking for is expressed in the form:

$$\begin{pmatrix} F(\rho) \\ G(\rho) \end{pmatrix} = e^{-\rho}\rho^s \begin{pmatrix} (a_0 + a_1\rho) \\ (b_0 + b_1\rho) \end{pmatrix} \tag{20}$$

substituting (20) into (17), then using the minimization condition (19) with respect to a_0, b_0, a_1, b_1 and with the help of symbolic software packages in *Mathematica* , we can get a system of algebraic equations for a_0, b_0, a_1, b_1 and $p = E$

$$\begin{aligned}
&2b_1\left(274\sqrt{1173}g\gamma - 137\sqrt{1173} - 9384\right) \\
&\quad - 137b_0\left(4\sqrt{1173} - 137g\gamma\right) + 137\Big(137a_0 \\
&\quad + 6\sqrt{1173}a_1\Big)g = 0, \\
&18769b_0 + 274\sqrt{1173}b_1 + 548\sqrt{1173}a_0g \\
&\quad + 274\sqrt{1173}a_1g + 18768a_1g \\
&\quad + 18769a_0\gamma + 548\sqrt{1173}a_1\gamma = 0, \\
&548\left(411\sqrt{1173}b_0 + \left(9384 + 137\sqrt{1173}\right)b_1\right.
\end{aligned}$$

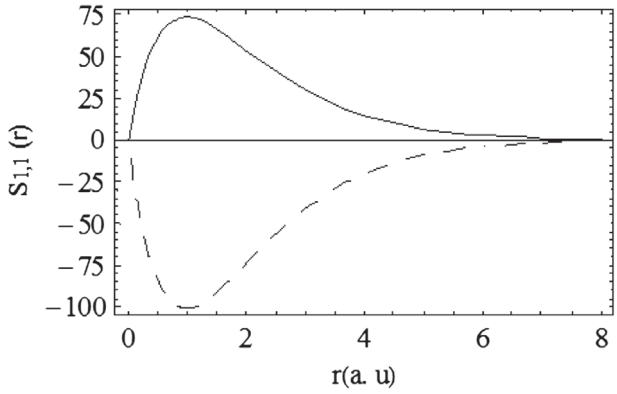

Figure 1. Representation of the unnormalized wavefunction $S_{1,1}(r)$ for the configuration $1S_{1/2}$ in set 1, obtained with the present method for the Coulomb potential. Solid curve (large component $F_{1,1}(r)$), Dash curve (small component $G_{1,1}(r)$).

$$
\begin{aligned}
&+a_0\left(\left(9384+137\sqrt{1173}\right)g+274\sqrt{1173}\gamma\right)\Big)\\
&+4a_1\left(\left(1928412+56305\sqrt{1173}\right)g\right.\\
&\left.+137\left(9384+137\sqrt{1173}\right)\gamma\right)=0,\\
&137\left(137\sqrt{1173}a_0+\left(9384+137\sqrt{1173}\right)a_1g\right.\\
&\left.+137b_0274\sqrt{1173}g\gamma-137\sqrt{1173}-9384\right)\\
&+b_1\Big(137\left(9384+137\sqrt{1173}\right)g\gamma\\
&-56305\sqrt{1173}-1928412\Big)=0,\\
&g=\sqrt{\frac{mc^2+p}{mc^2-p}}
\end{aligned}
\tag{21}
$$

for which, we distinguish with the aid of *Mathematica*, two sets of solutions

- Set 1

$$
\begin{cases}
b_0=-273.99635031634966a_0,\\
b_1=0,\\
a_1=0,\\
E=0.9999733599733551mc^2.
\end{cases}
\tag{22}
$$

This variant corresponds to the configuration $1S_{1/2}$, and the eigenstates associated are indicated by $S_{1,1}(\rho)=\begin{pmatrix}F_{1,1}(\rho)\\G_{1,1}(\rho)\end{pmatrix}F_{1,1}(\rho)$ and $G_{1,1}(\rho)$. The energy is labelled as $E_{1,1/2}$.

- Set 2

$$
\begin{cases}
b_0=-273.99635031634966a_0,\\
b_1=274.0018248722094a_0,\\
a_1=-0.5000083251539453a_0,\\
E=0.9999933399711608mc^2.
\end{cases}
\tag{23}
$$

This variant corresponds to the configuration $2S_{1/2}$, and the eigenstates associated are indicated by $S_{2,1}(\rho)=\begin{pmatrix}F_{2,1}(\rho)\\G_{2,1}(\rho)\end{pmatrix}F_{2,1}(\rho)$ and $G_{2,1}(\rho)$. The energy is labelled as $E_{2,1/2}$.

The coefficient a_0 is easily evaluated from the following normalization condition $\int_0^\infty\left(F_{n,\kappa}^2(r)+G_{n,\kappa}^2(r)\right)\mathrm{d}r=1$. The graphical illustration of the wave functions $S_{1,1}(r)$ is depicted in Figure 1 for the configuration $1S_{1/2}$ in set 1.

3.2. Second test example: case $l=1, \kappa=2$ and $M=N=1$

We could get a better estimate by introduction some parameters in our trial wave function as

$$
\begin{pmatrix}F(\rho)\\G(\rho)\end{pmatrix}=e^{-\rho}\rho^s\begin{pmatrix}(a_0+a_1\rho)\\(b_0+b_1\rho)\end{pmatrix}.
\tag{24}
$$

Similarly, substituting (24) into (17), and making J stationary with respect to a_0, b_0, a_1, b_1 results in

$$
\begin{aligned}
&137\left(548a_0+25\sqrt{3003}a_1\right)g\\
&+5b_1\left(274\sqrt{3003}g\gamma-137\sqrt{3003}-30030\right)\\
&-274b_0\left(5\sqrt{3003}-137g\gamma\right)=0,\\
&75076b_0+2055\sqrt{3003}b_1+1370\sqrt{3003}a_0g\\
&+685\sqrt{3003}a_1g+150150a_1g+37538a_0\gamma\\
&+1370\sqrt{3003}a_1\gamma=0,\\
&93845\sqrt{3003}b_0+274\left(30030+137\sqrt{3003}\right)b_1\\
&+18769\sqrt{3003}a_0g+4114110a_0g\\
&+168919\sqrt{3003}a_1g+6171165a_1g+37538\gamma\\
&\times\sqrt{3003}a_0+18769\sqrt{3003}a_1\gamma+4114110a_1\gamma=0,\\
&137\left(\left(411\sqrt{3003}a_0+274\sqrt{3003}a_1+60060a_1\right)g\right.\\
&\left.+b_0\left(274\sqrt{3003}g\gamma-137\sqrt{3003}-30030\right)\right)\\
&+b_1\left(137\left(30030+137\sqrt{3003}\right)g\gamma\right.\\
&\left.-168919\sqrt{3003}-6171165\right)=0,\\
&g=\sqrt{\frac{mc^2+p}{mc^2-p}}.
\end{aligned}
\tag{25}
$$

Solving the set of Equation (25) with respect to b_0, a_1, b_1 and $p=E$ with the help of *Mathematica*, we obtain two sets of parameters given by

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- Set 1

$$\begin{cases} b_0 = -547.9981751764051a_0, \\ b_1 = 0, \\ a_1 = 0, \\ E_{2,3/2}{=}0.9999933400598727mc^2. \end{cases} \tag{26}$$

- Set 2

$$\begin{cases} b_0 = -547.9981751764051a_0, \\ b_1 = 274.0006082776825a_0, \\ a_1 = -0.3333350599937004a_0, \\ E_{3,3/2} = 0.9999970400233246mc^2. \end{cases} \tag{27}$$

Similarly, from these results, we identify the following configuration states in spectroscopic language: $2P_{3/2}$ and $3P_{3/2}$ for the set 1 and the set 2 respectively and the associated eigenstates are $S_{2,2}(\rho) = \begin{pmatrix} F_{2,2}(\rho) \\ G_{2,2}(\rho) \end{pmatrix}$ and $S_{3,2}(\rho) = \begin{pmatrix} F_{3,2}(\rho) \\ G_{3,2}(\rho) \end{pmatrix}$, respectively.

Indeed, it may be noted that the eigenenergies and the wave functions obtained for this potential are exact compared with those obtained by the expansion method reported in the literature (*41*). As illustration, we sketch the variation of the p-wave states vs. r to represent the two non-normalized wave functions $F_{3,2}(r)$ and $G_{3,2}(r)$ in Figure 2 for the set 2 (with ρ essentially replaced by $\sqrt{\epsilon_+\epsilon_-}r$). We can finally underline that the behaviour of wave functions is very well outlined and preserved for all the states considered.

3.3. Third test example: case $l = 2$, $\kappa = 3$ and $M = N = 0$

We search for a wave function in the form

$$\begin{pmatrix} F(\rho) \\ G(\rho) \end{pmatrix} = e^{-\rho}\rho^s \begin{pmatrix} a_0 \\ b_0. \end{pmatrix} \tag{28}$$

Substituting Equation (28) into Equation (17) and making J stationary with respect to a_0 and b_0 results in

$$\begin{aligned} 3a_0 + b_0\gamma - b_0\sqrt{\frac{c^2m-p}{mc^2+p}}\sqrt{9-\gamma^2} = 0, \\ \gamma a_0 + \sqrt{\frac{mc^2+p}{c^2m-p}}\sqrt{9-\gamma^2}a_0 + 3b_0 = 0. \end{aligned} \tag{29}$$

Solving the set of Equation (29) with respect to b_0 and $p = E$, we obtain one set of parameters given by

$$b_0 = -\frac{1}{3}a_0\left(\gamma + \sqrt{9-\gamma^2}\right.$$

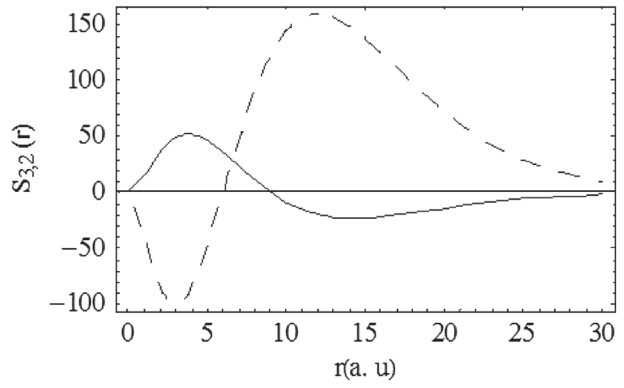

Figure 2. Representation of the unnormalized wavefunction $S_{3,2}(r)$ for the configuration $3S_{3/2}$ in set 2, obtained with the present method for the Coulomb potential. Solid curve(large component $F_{3,2}(r)$), Dash curve (small component $G_{3,2}(r)$).

$$\left.\times\sqrt{-\frac{3m\left(\gamma^2-9\right)c^2+\sqrt{\gamma^2-9}\sqrt{-c^4m^2\left(\gamma^2-9\right)^2}}{\sqrt{\gamma^2-9}\sqrt{-c^4m^2\left(\gamma^2-9\right)^2}-3c^2m\left(\gamma^2-9\right)}}\right),$$

$$p = \frac{mc^2\sqrt{\left(9-\gamma^2\right)}}{3}. \tag{30}$$

Again we invoke Equation (30) and set $\gamma = 1/137$, we obtain

$$\begin{aligned} b_0 &= -821.9987834458900a_0, \\ E_{3,5/2} &= 0.999997040032086mc^2. \end{aligned} \tag{31}$$

From these results, we identify the following configuration state in spectroscopic notation: $3D_{5/2}$ and the associated eigenstate is $S_{3,3}(\rho) = \begin{pmatrix} F_{3,3}(\rho) \\ G_{3,3}(\rho) \end{pmatrix}$

We thus see that, the negative κ is associated with the larger of the two values of l belonging to the particular j considered in the same state n. Often, some authors adopt the opposite of this convention.

The Table 1 shows numerically the comparison of eigenenergies for the Dirac hydrogen atom obtained from the semi-inverse variational method and the direct calculations using the following exact formula (*41*)

$$E_{n,j} = mc^2\left[1+\left(\frac{\gamma}{n-\left(j+\frac{1}{2}\right)+\sqrt{\left(j+\frac{1}{2}\right)^2-\left(\gamma\right)^2}}\right)^2\right]^{-1/2}. \tag{32}$$

From these results, we notice that, the form given in (18) is a good candidate and an excellent approximation

Table 1. Energy eigenvalues obtained with the present work and the exact formula for the Dirac hydrogen atom in Hartree atomic units.

n	j	l	Terms nL_j	$E_{n,j}/mc^2$: our calculations	$E_{n,j}/mc^2$: exact formula
1	1/2	0	$1S_{1/2}$	0.9999733599733551	0.999 973 359 973 36
2	1/2	0	$2S_{1/2}$	0.9999933399711608	0.999 993 339 971 16
2	3/2	1	$2P_{3/2}$	0.9999933400598727	0.999 993 340 059 87
3	3/2	1	$3P_{3/2}$	0.9999970400233246	0.999 997 040 023 32
3	5/2	2	$3D_{5/2}$	0.999997040032086	0.999 997 040 032 09

to the correct eigenfunction for all configurations found by this method.

It is important to remark that the degenerate levels have been omitted in this study. In particular, if we choose $M = N = 1, \kappa = -1, l = 1$, then we find the state $2P_{1/2}$ is exactly degenerate with the state $2S_{1/2}$ (with the same energy level), while the ground state energy $1S_{1/2}$ corresponding to the minimum eigenvalue $E_{1,1/2}$ (example 1, set 1) is always non-degenerate. The configuration $2P_{1/2}$ is represented by a set of realizations (33)

$$\begin{cases} b_0 = 0.0036496836503374652a_0, \\ b_1 = 182.66686942844495a_0, \\ a_1 = -0.3333370333761329a_0, \\ E_{2,1/2} = p = 0.9999933399711608mc^2. \end{cases} \tag{33}$$

4. Conclusion

In this work, we have seen that the semi-inverse method works well and can provide an excellent tool for exploring solutions in a quantum environment. The relativistic bound state energy spectrum and the corresponding wave functions have been obtained by the semi-inverse variational method. Using some illustrative examples, it was shown that the semi-inverse variational method is a powerful and straightforward solution method to find closed-form for Dirac-type equations.

In this paper, we have shown a possible connection between the semi-inverse variational method and the evaluation of solutions of a given quantum system. Specially, we have introduced the potential of Coulomb as a reference to obtain simultaneously some bound energies and the associated quantum states. Furthermore, we have proposed some trial forms for the solutions for several configurations, and we have demonstrated that the trial Lagrangian build by this approach has provided an exact aspect of solutions that are the same as those obtained by the expansion method (*41*). We feel that the semi-inverse variational method as treated here will be useful to study more complex systems such that the Klein–Gordon Hamiltonian. A significant development must be provided in this direction in order to obtain the appropriate solutions. The details will be given elsewhere and will be reported in our future research work.

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Disclosure statement

No potential conflict of interest was reported by the authors.

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