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Contribution on Backward Doubly Stochastic Differential Equations.

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Dedication

In all letters can not find the right words, not all words express love, respect, gratitude to my dear parents. I dedicate this modest work:

To the memory of my **father.** To the person who surrounded me with love and taught me how to be patient, and taught me and directed me on the right path, to my lovely mother in the universe. I hope that Allah will keep her for us, inchàallah.

To my **brothers,** especially **Mohamed El-Hachemi**, my **sisters** and all members of the **Saouli family** and to all those who have watched over my success during the years of study.

To my Ph.D supervisor **DR: Mansouri Baderddine** and my colleagues without exception.

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I would like to thank all those who taught me during my studies, especially our professors from University of Biskra and to the members of the Applied Mathematics Laboratory (**LMA**).

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Resumé

Dans cette thèse, nous étudions une classe des equations différentielles doublement stochastiques retrogrades (EDDSR). Dans le première partie, notre contribution consiste à établir l'existence et l'unicité lorsque le coefficient f est faiblement monotone et a une croissance générale et que la condition terminale ξ et de carré intégrable et aussi donnons une application à des équations différentielles partielles stochastique (EDPS). Nos démonstrations sont basées sur des techniques d'approximation.

Dans le même esprit mais avec des techniques différentes, nous prouvons des nouveaux résultats d'existence dans deux autres directions. Tout d'abord, nous prouvons le résultat d'existence d'une solution minimale au EDDSR avec un barrière continu et dirigée par le sauts de poisson lorsque le coefficient est continu dans (Y, Z, U) et a une croissance linéaire. Nous étudions également ce type d'équation sous la condition de croissance linéaire et de la continuité a gauche en y sur le générateur. Deuxièmement, nous prouvons aussi l'existence et unicité des solutions aux équations différentielles doublement stochastiques rétrogrades réfléchies anticipées dirigées par une famille de martingales de teughels, nous montrons également le théorème de comparaison pour une classe spéciale équations différentielles doublement stochastiques rétrogrades réfléchies anticipées dans des conditions légèrement plus fortes. De plus, nous obtenons un résultat d'existence et d'unicité de la solution de l'équation précédente lorsque, $S = -\infty$ i.e., $K \equiv 0$. La nouveauté de notre résultat réside dans le fait que nous permettons à l'intervalle de temps d'être infini.

Phrases-clé: Equations différentielles doublement stochastiques retrogrades; Equations différentielles doublement stochastiques retrograde réfléchie; Equation différentielle partielle stochastique; Solution faible de sobolev; Inégalité de Bihari, Mesure aléatoire de Poisson, Théorème de comparaison.

Abstract

In this thesis, we study a class of baclward doubly stochastic differential equations (BDSDEs in short). In a first part, our contribution is to establish existence and uniqueness when the coefficient f is is weakly monotonous and has general growth and the terminal condition ξ is only square integrable and give application to the homogenization of stochastic partial differential equations (SPDE's). Our demonstrations are based on approximation techniques.

In the same spirit but with different techniques we prove the new existence results in two other directions. First, we prove the existence result of minimal solution to the RBDSDE with poisson jumps when the coefficient is continuous in (Y, Z, U) and has linear growth. Also, we study this type of equation under the condition of linear growth and the continuity left inand the continuity left in y on the generator. Second, existence and uniqueness of solutions to the reflected anticipated backward doubly stochastic differential equation equations driven by teughles martingales (RABDSDEs in short), we also show the comparison theorem for a special class of reflected ABDSDEs under some slight stronger conditions. Furthermore we get a existence and uniqueness result of the solution to the previous equation when, $S = -\infty$ i.e., $K \equiv 0$. The novelty of our result lies in the fact that we allow the time interval to be infinite.

Key-phrases: Backward doubly stochastic differential equations; Reflected backward doubly stochastic differential equations; Stochastic partial differential equation; Sobolev weak solution; Bihari inequality, Random Poisson measure, Comparison theorem.

Contents

Index of notations

The different symbols and abbreviations used in this thesis.

$(\Omega, \mathcal{F}, P)$	:	Probability space.
$\{W_t, 0 \leq t \leq T\}$	:	Brownian motion.
$\sigma(B)$	:	$\sigma -$ algebre generated by B.
$\mathcal{F}_t^\eta$	:	$\sigma -$ algebre generated by η.
$\mathcal{F}_{t,T}^B$	:	$\sigma -$ fields generated by $\{B_s - B_t; t \leq s \leq T\}$
$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$	:	$\sigma -$ fields generated by $\mathcal{F}_t^W \cup \mathcal{F}_{t,T}^B$.
$\mathcal{G}_t := \mathcal{F}_t^W \vee \mathcal{F}_T^B,$	:	The collection $(\mathcal{G}_t)_{t\in[0,T]}$ is a filtration.
$a.e$	:	Means almost everywhere with respect to the Lebesgue measure.
$a.s$	:	Means almost surely with respect to the probability measure.
C_b^k	:	Set of function of class C^k, whose partial derivatives of order less then or egal to k are bounded.
$J\left(\hat{X}_s^{t,x}\right)$	:	The determinan to the Jacobian matrix of $\hat{X}_s^{t,x}$.
$\mathbb{L}^1([0,T])$	:	Is the space of the functions whose absolute value is integrable.
$\mathbb{R}^d$	:	$d -$ dimensional real Euclidean space.
$\mathbb{R}^{k\times d}$	:	The set of all $k \times d$ real matrixes.
$\mathbb{L}^2\left(\mathbb{R}^d, \pi(x)\,dx\right)$	:	Be the weight $\mathbb{L}^2$ space with weight $\pi(x)$ endowed with the following norm, $\|\|u\|\|_\pi^2 = \int_{\mathbb{R}^d} \|u(x)\|^2 \pi(x)\,dx$.
$\mathcal{M}^2\left(0,T,\mathbb{R}^d\right)$	:	The set of $d -$ dimensional, $\mathcal{F}_t -$ measurable processes $\{\varphi_t; t\in[0,T]\}$, such that $\mathbb{E}\int_0^T \|\varphi_t\|^2 dt < \infty$.
$S^2\left(0,T,\mathbb{R}^d\right)$	:	The set of continuous $\mathcal{F}_t -$ measurable processes $\{\varphi_t; t\in[0,T]\}$, which satisfy $\mathbb{E}(\sup_{0\leq t\leq T}\|\varphi_t\|^2) < \infty$.
l^2	:	Be the space of real valued sequences $(x_n)_{n\geq 0}$ suchthat $\sum_{i=1}^{i=\infty} x_i^2 < \infty$, and $\|\|x\|\|_{l^2}^2 = \sum_{i=1}^{i=\infty} x_i^2$.
$\mathcal{M}_{\mathcal{H}}^2([0,T];l^2)$ and $\mathcal{S}_{\mathcal{H}}^2([0,T];l^2)$	:	Are the corresponding spaces of l^2-valued processes equipped with the norm $\|\|\varphi\|\|_{l^2}^2 = \mathbb{E}\int_0^T \sum_{i=1}^{i=\infty}\left\|\varphi_t^{(i)}\right\|^2 dt < \infty$ associated to the $\mathcal{H}_t -$ measurable processes.

$\mathcal{L}^2\left(0,T,\tilde{\mu},\mathbb{R}^d\right)$	:	The space of mappings $U:\Omega\times[0,T]\times E\to\mathbb{R}^d$ which are $\mathcal{P}\otimes\mathcal{E}$ measurable such that, $\|\|U_t\|\|^2_{\mathcal{L}^2\left(0,T,\tilde{\mu},\mathbb{R}^d\right)}=\mathbb{E}\int_0^T\|\|U_t\|\|^2_{L^2\left(E,\mathcal{E},\lambda,\mathbb{R}^d\right)}dt<\infty$, where $\mathcal{P}\otimes\mathcal{E}$ denoted the σ-algebra of $\mathcal{F}_t$-predectable sets of $\Omega\times[0,T]$, and $\|\|U_t\|\|^2_{L^2\left(E,\mathcal{E},\lambda,\mathbb{R}^d\right)}=\int_E\|U_t(e)\|^2\lambda(de)$.
$\mathbb{M}^2\left(0,T,\mathbb{R}^d\right)$	:	The set of $d-$ dimensional, $\mathcal{F}_t-$ measurable processes $\{\varphi_t;t\in[0,T]\}$, such that $\mathbb{E}\left(\int_0^T\|\varphi_t\|dt\right)^2<\infty$.
$\mathcal{A}^2$	:	Set of continuous, increasing, $\mathcal{F}_t$-measurable process K such that $K:[0,T]\times\Omega\to[0,+\infty($ with $K_0=0$, $\mathbb{E}\left(K_T\right)^2<+\infty$.
$\mathbb{L}^2,\ \mathbb{L}^2\left(\mathcal{H}_T\right)$	:	Set of $\mathcal{F}_T-$ (resp $\mathcal{H}_T$) measurable random variables $\xi:\Omega\to\mathbb{R}^k$ with $\mathbb{E}\|\xi\|^2<+\infty$.
$\mathbf{E}(X),\ \mathbf{E}(\cdot\|\mathcal{F})$	:	Expectation at X and conditional expectation.
$SDEs$	:	Stochastic differential equation.
$BSDEs$	:	Backward stochastic differential equation.
$BDSDEs$	:	Backward doubly stochastic differential equation.
$RBDSDEs$	:	Reflected backward doubly stochastic differential equation.
$RBDSDEJs$	:	Reflected backward doubly stochastic differential equation with poisson jump.
$RABDSDEs$	:	Reflected anticipated backward doubly stochastic differential equation.
$SPDEs$	:	Stochastic partial differential equation.

Introduction

0.1 Historical of Backward Doubly Stochastic Differential Equations.

It was mainly during the last decade that the theory of backward doubly stochastic differential equations (BDSDE for short) took shape as a distinct mathematical discipline. This theory has found a wide field of applications as in stochastic optimal problems, see Han, Peng and Wu [11], Zhang and Shi [31] and stochastic partial differential equations (SPDEs) see Z. Wu, F. Zhang [29] and Zhu, Q., Shi, Y [34], we are especially concerned in this thesis with the last connection. The nonlinear Backward doubly stochastic differential equation are equations of the following type:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s, Z_s)\, d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \qquad 0 \le t \le T \quad \left(E^{\xi,f,g}\right)$$

with two different directions of stochastic integrals, i.e., the equation involves both a standard (forward) stochastic integral dW_t and a backward stochastic integral dB_t. Was firstly initiated by Pardoux and Peng [24] they have proved the existence and uniqueness under uniformly Lipschitz conditions and they give probabilistic interpretation for the solutions of a class of semilinear SPDEs where the coefficients are smooth enough, the idea is to connect the following BDSDEs system

$$\begin{cases} Y_s^{t,x} &= h\left(X_T^{t,x}\right) + \int_s^T f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr + \int_s^T g\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) d\overleftarrow{B}_r - \int_s^T Z_r^{t,x} dW_r, \\ X_s^{t,x} &= x + \int_t^s b\left(X_r^{t,x}\right) dr + \int_t^s \sigma\left(X_r^{t,x}\right) dW_r, \end{cases}$$

with the following semilinear SPDE,

$$\begin{aligned} u(s,x) &= h(x) + \int_s^T \Big(\mathcal{L}u(r,x) + f(r,x,u(r,x),\sigma^*\nabla u(r,x))\Big)dr \\ &+ \int_s^T g(r,x,u(r,x),\sigma^*\nabla u(r,x))\, d\overleftarrow{B}_r, \quad t \leq s \leq T, \end{aligned}$$

where

$$\mathcal{L} := \frac{1}{2}\sum_{i,j}(a_{ij})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}, \qquad with\ \ (a_{ij}) := \sigma\sigma^*.$$

The result of Pardoux and Peng [24], several works have attempted to relax the Lipschitz condition and the growth of the generator function; see Bahlali et all [7] have provide the existence and uniqueness of a solution for BDSDE with superlinear growth generators, Z. Wu, F. Zhang [29] gave the existence and uniqueness result of BDSDEs with locally monotone assumptions, in which the coefficient f is assumed to be locally monotone in the variable y and locally Lipschitz in the variable z.

In addition, Bahlali et all [5] prove the existence and uniqueness of solutions with uniformly Lipschitz coefficients to the following reflected backward doubly stochastic differential equations (RBDSDEs for schort)

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)d\overleftarrow{B}_s + \int_t^T dK_s - \int_t^T Z_s dW_s,\ 0 \leq t \leq T, \quad \left(\bar{E}^{\xi,f,g}\right)$$

The role of the nondecreasing continuous process $(K_t)_{t\in[0,T]}$ is to puch upward the process Y in order to keep it above S, it satisfies the skorohod condition

$$\int_0^T (Y_s - S_s)\, dK_s = 0.$$

The existence of a maximal and a minimal solution for RBDSDEs with continuous generator is also established.

Note that when $g = 0$ and $S = -\infty$ i.e., $K \equiv 0$ the previous backward doubly stochastic differential equations becomes a classical backward stochastic differential equation (BSDE) and can be related to semilinear and quasi linear partial differential equations (PDEs).

0.2 Our results

In this thesis, we present three new results in the theory of BDSDEs.

1. We establish existence and uniqueness results for the previous type of multidimensional backward doubly stochastic differential equations $\left(E^{\xi,f,g}\right)$, for the case where the generator f is weak monotonicity and general growth with respect to (Y,Z). Also we establish the existence and uniqueness of probabilistic solutions to some stochastic PDEs by using the solution of BDSDE with weak monotonicity and general growth generator. See [18] (submitted).

Mansouri, Badreddine, Saouli, M, A, ouahab. (2019). Backward Doubly SDEs and SPDEs with weak Monotonicity and General Growth Generators.

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2. We prove the existence of a minimal and a maximal solution to the following reflected backward doubly stochastic differential equations with poisson jumps (RBDSDEPs in short)

$$\begin{cases} Y_t = \xi + \int_t^T f(s,Y_s,Z_s,U_s)ds + \int_t^T g(s,Y_s,Z_s,U_s)d\overleftarrow{B}_s + \int_t^T dK_s \\ -\int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\,\tilde{\mu}(ds,de)\,,\ 0 \le t \le T, \qquad \left(EP^{\xi,f,g}\right). \end{cases}$$

when the generator has a linear growth condition and left continuity in y on the generator, the case where the generator is continuous in (Y,Z,U) and has a linear growth is also study. We state a new version of a comparison principle which allows us to compare the solutions to RBDSDEs. See [20]

Mansouri, Badreddine, Saouli, M, A, ouahab. (2018). Reflected Discontinuous Backward Doubly Stochastic Differential Equation With Poisson Jumps. Journal of Numerical Mathematics and Stochastics, 10 (1) : 73-93.

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3. Motivated by the above results and by the result introduced by Xiaoming Xu [30], we establish the existence and uniqueness of the solution to the reflected ABDSDE (RABDSDEs) driven by teugels martingales associated with a Lévy process where the coefficient of this BDSDE depend on the future and present value of the solution (Y,Z).

We also show the comparison theorem for a special class of reflected ABDSDEs under some slight stronger conditions. Furthermore we get a existence and uniqueness result of the solution to the previous equation when, $S = -\infty$ i.e., $K \equiv 0$. The novelty of our result lies in the fact that we allow the time interval to be infinite. See [17] (Submitted; February, 05, 2019; In Filomat journal).

Mansouri, Badreddine, Saouli, M, A, ouahab. (2019). Reflected solutions of Anticipated Backward Doubly SDEs driven by Teugels Martingales.
arXiv preprint arXiv:1703.09105.

0.3 Outline of the thesis.

The organization of this thesis is as follows: **In Chapter 1**, we present, under classical assumptions and by means of a fixed point, an existence and uniqueness theorem for solutions of BDSDE's. In particular, we obtain a result for BDSDE's with Lipschitz coefficient. Then, we state BDSDE's with continuous coefficient. A comparison theorem for BDSDE's is also presented.

Chapter 2, is devoted to the study of existence and uniqueness results for backward doubly SDE with superlinear growth generators.

In Chapter 3, we prove existence and uniqueness results of solution to the multidimensional backward doubly stochastic differential equation. Our contribution in this topic is to weaken the Lipschitz assumption on the data (ξ, f, g), see [18]. This is done with weak monotonicity and general growth coefficient f and an only square integrable terminal condition ξ i.e. f and g satisfying the following assumptions:

- $dP \times dt$-a.e., $z \in \mathbb{R}^{k\times d}$ $y \to\ f(w,t,y,z)$ is continuous.
- f satisfies the weak monotonicity condition in y, i.e., there exist a nondecreasing and concave function $k(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ with $k(u) > 0$ for $u > 0$, $k(0) = 0$ and $\int_{0^+} k^{-1}(u)du = +\infty$ such that $dP \times dt$-a.e.,$\forall\,(y_1, y_2) \in \mathbb{R}^{2k}, z \in \mathbb{R}^{k\times d}$,

$$\langle y_1 - y_2, f(t, \omega, y_1, z) - f(t, \omega, y_2, z)\rangle \le k\left(\left|y_1 - y_2\right|^2\right).$$

- f is lipschitz in z, uniformly with respect to (w,t,y) i.e., there exists a constant $c>0$ such that $dP\times dt$-a.e.,

$$\left|f\left(w,t,y,z\right)-f\left(w,t,y,z'\right)\right|\leq c\left|z-z'\right|.$$

- There exists a constant $c>0$ and a constant $0<\alpha\leq\frac{1}{4}$ such that $dP\times dt$-a.e.,

$$\left|g\left(w,t,y,z\right)-g\left(w,t,y^{'},z^{'}\right)\right|\leq c\left|y-y^{'}\right|+\alpha\left|z-z'\right|.$$

- f has a general growth with respect to y, i.e., $dP\times dt$-a.e., $\forall y\in\mathbb{R}^k$

$$\left|f\left(t,\omega,y,0\right)\right|\leq\left|f\left(t,\omega,0,0\right)\right|+\varphi\left(\left|y\right|\right),$$

where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is increasing continuous function.

- and $f\left(t,\omega,0,0\right)\in\mathcal{M}^2\left(0,T,\mathbb{R}^k\right),\qquad g\left(t,\omega,0,0\right)\in\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times l}\right).$

More precisely, let (f_n) be a sequence of processes which converges to f locally uniformly and (ξ_n) a sequence of random variable which converge to ξ in $\mathbb{L}^2\left(\Omega\right)$, then the solutions Y_n of BDSDE (ξ_n,f_n) converges to Y the solution of (ξ,f). Also we prove the existence and uniqueness of Sobolev solution for some SPDEs by constructing it with the help of some BDSDE with weak monotonicity and general growth generator.

In Chapter 4, we present the existence and uniqueness results of solution to the refleccted backward doubly stochastic differential equation. In particular, we obtain a result for reflected BDSDE's with Lipschitz coefficient and continuous coefficient.

In Chapter 5, we prove the existence result of RBDSDE with poisson jumps $\left(EP^{\,\xi,f,g}\right)$. More generally, our results in this part focus essentially in two directions, see [20].

First, we study the existence of a minimal and a maximal solution to the reflected backward doubly stochastic differential equation with poisson jumps (RBDSDEPs in short) where the coefficient is continuous in the variables Y, Z and U and has linear growth i.e. f and g satisfying the following assumptions:

- There exists $C > 0$ $s.t.$ for all $(t, \omega, y, z, u) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbb{R})$, $\left(t, \omega, y', z', u'\right) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbb{R})$

$$\begin{cases} |f(t, \omega, y, z, u)| \leq C(1 + |y| + |z| + |u|), \\ \left|g(t, \omega, y, z, u) - g(t, \omega, y', z', u')\right|^2 \leq C\left|y - y'\right|^2 + \alpha\left\{\left|z - z'\right|^2 + \left|u - u'\right|^2\right\}. \end{cases}$$

- For fixed ω and t, $f(t, \omega, \cdot, \cdot, \cdot)$ is continuous.

Also, we study the existence of a minimal and a maximal solution for RBDSDEPs under a linear growth condition and left continuity in y on the generator i.e.f and g satisfying the following assumptions:

- There exists a positive process $f_t \in \mathcal{M}^2(0, T, \mathbb{R})$ such that$\forall (t, y, z, u) \in [0,T] \times \mathcal{B}^2(\mathbb{R})$,

$$|f(t, y, z, u)| \leq f_t(\omega) + C(|y| + |z| + |u|).$$

- $f(t, \cdot, z, u) : \mathbb{R} \to \mathbb{R}$ is a left continuous and $f(t, y, \cdot, \cdot)$ is a continuous.

- There exists a continuous fonction $\pi : [0,T] \times \mathcal{B}^2(\mathbb{R})$ satisfying for $y \geq y'$, $\left(z, z'\right) \in \mathbb{R}^{2d}$, $\left(u, u'\right) \in \left(L^2(E, \mathcal{E}, \lambda, \mathbb{R})\right)^2$

$$\begin{cases} |\pi(t, y, z, u)| \leq C(|y| + |z| + |u|), \\ f(t, \omega, y, z, u) - f\left(t, \omega, y', z', u'\right) \geq \pi\left(t, y - y', z - z', u - u'\right). \end{cases}$$

- There exist constant $C \geq 0$ and a constant $0 < \alpha < 1$ such that for every $(\omega, t) \in \Omega \times [0,T]$ and $\left(y, y'\right) \in \mathbb{R}^2$, $\left(z, z'\right) \in \left(\mathbb{R}^d\right)^2$, $\left(u, u'\right) \in \left(L^2(E, \mathcal{E}, \lambda, \mathbb{R})\right)^2$

$$\left|g(t, \omega, y, z, u) - g(t, \omega, y', z', u')\right|^2 \leq C\left|y - y'\right|^2 + \alpha\left\{\left|z - z'\right|^2 + \left|u - u'\right|^2\right\}.$$

In Chapter 6, motivated by the above results we prove the existence and uniqueness of solutions to the following anticipated BDSDE driven by teughles martingales associated by lévy process (ABDSDE in short),

$$\begin{cases} Y_t = \xi + \int_t^T f(s,\Lambda_s,\Lambda_s^{\phi,\psi})ds + \int_t^T g(s,\Lambda_s,\Lambda_s^{\phi,\psi})d\overleftarrow{B}_s + \int_t^T dK_s - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)}dH_s^{(i)}, \ t\in[0,T], \\ (Y_t,Z_t) = (\eta_t,\vartheta_t), \qquad t\in[T,T+\rho], \end{cases}$$

and $Y_t \geq S_t$ a.s. for any $t \in [0,T+\rho]$ where $\Lambda_s = (Y_s, Z_s)$, $\Lambda_s^{\phi,\psi} = \left(Y_{s+\phi(s)}, Z_{s+\psi(s)}\right)$, and $\phi : [0,T] \to \mathbb{R}_+^*$, and $\psi : [0,T] \to \mathbb{R}_+^*$ are continuous functions satisfying:

- There exists a constant $\rho \geq 0$ such that for all $t \in [0,T]$,

$$t + \phi(t) \leq T + \rho, \qquad t + \psi(t) \leq T + \rho.$$

- There exists a constant $M \geq 0$ such that for each $t \in [0,T]$ and for all nonnegative integrable functions $h(\cdot)$,

$$\begin{cases} \int_t^T h(s+\phi(s))ds \leq M\int_t^{T+\rho} h(s)ds, \\ \int_t^T h(s+\psi(s))ds \leq M\int_t^{T+\rho} h(s)ds, \end{cases}$$

by means of the fixed-point theorem where the coefficients of these BDSDEs depend on the future and present value of the solution (Y,Z). We also show the comparison theorem for a special class of reflected ABDSDEs under some slight stronger conditions. Furthermore we get a existence and uniqueness result of the solution to the previous equation when, $S = -\infty$ i.e., $K \equiv 0$. The novelty of our result lies in the fact that we allow the time interval to be infinite see [17].

Part one:

Backward Doubly Stochastic Differential Equation

This part is intended to give a thorough description of BDSDE's and then we present in Chapter 1, existence and uniqueness results under classical Lipshitz conditions see Pardoux. E, Peng. S, [24], also we present a comparison theorem and the existence result of BDSDE under continuous coefficient, see [27]. The existence and uniqueness of solution to BDSDE with superlinear growth generators is presented in Chapter 2, for more detail see [7]. In Chapter 3, we present our contribution in this part, see [18] which is the existence and uniqueness of the solution for multidimensional backward doubly stochastic differential equation whose coefficient f has a waek monotonicity and general growth. We establish also the existence and uniqueness of probabilistic solutions to some semilinear stochastic partial differential equations (SPDEs) under the same assumptions. By probabilistic solution, we mean a solution which is representable throughout a BDSDEs

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Chapter 1

A background on Backward Doubly SDEs.

This Chapter is organized as follow:

- **In section one**, we present a backward doubly stochastic differential equations (BDSDEs) with a Lipschtiz coefficien and a square integrable terminal datum.
- **In section two,** we state the comparison theorem which allows us to compare the solutions of BDSDEs.
- **In section three,** we study BDSDE with continuous coefficient.

1.1 Backward Doubly SDEs with Lipschtiz coefficient.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. For $T > 0$, let $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two independent standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively.

Let $\mathcal{F}_t^W := \sigma(W_s; 0 \leq s \leq t)$ and $\mathcal{F}_{t,T}^B := \sigma(B_s - B_t; t \leq s \leq T)$, completed with P-null sets. We put,

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B.$$

It should be noted that $(\mathcal{F}_t)$ is not an increasing family of sub $\sigma-$fields, and hence it is not a filtration.

Let $f:\Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^{d\times r}\longmapsto\mathbb{R}^d,\ g:\Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^{d\times r}\longmapsto\mathbb{R}^{d\times l}$ be measurable functions such that, for every $(y,\,z)\in\mathbb{R}^d\times\mathbb{R}^{d\times r}$, $f(.,y,z)\in M^2(0,T,\mathbb{R}^d)$ and $g(.,y,z)\in M^2(0,T,\mathbb{R}^{d\times l})$.

The following hypotheses are satisfied for some strictly positive finite constant C and $0<\alpha<1$ such that for any $(y_1;z_1),(y_2;z_2)\in\mathbb{R}^d\times\mathbb{R}^{d\times r}$:

$$(H.1)\qquad \begin{cases} |f(t,\omega,y_1,z_1)-f(t,\omega,y_2,z_2)|^2\leq C\left[|y_1-y_2|^2+||z_1-z_2||^2\right], \\ |g(t,\omega,y_1,z_1)-g(t,\omega,y_2,z_2)|^2\leq C\,|y_1-y_2|^2+\alpha\,||z_1-z_2||^2. \end{cases}$$

Throughout this paper, $\langle\cdot\,;\,\cdot\rangle$ will denote the scalar product on $\mathbb{R}^d$, i.e $\langle x;y\rangle:=\sum_{i=0}^{i=d}x_iy_i$, for all $(x;y)\in\mathbb{R}^{2d}$: Sometimes, we will also use the notation $x*y$ to designate $\langle x;y\rangle$.

We point out that by C we always denote a finite constant whose value may change from one line to the next, and which usually is (strictly) positive.

1.1.1 Existence and uniqueness theorem.

Suppose that we are given a terminal condition $\xi\in L^2\left(\Omega,\mathcal{F}_T,P\right)$.

Definition 1.1 *A solution of equation* $\left(E^{\xi,f,g}\right)$ *is a couple* (Y,Z) *which belongs to the space* $\mathcal{S}^2\left([0,\,T],\mathbb{R}^d\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{d\times r}\right)$ *and satisfies* $\left(E^{\xi,f,g}\right)$.

Theorem 1.1 *Let ξ be a square integrable random variable. Assume that* ***(H.1)*** *are satisfied. Then equation* $(E^{f,g,\xi})$ *has a unique solution.*

Let us first establish the result in Theorem for BDSDEs, where the coefficients f, g do not depend on $(Y;Z)$. More precisely, let $f:\Omega\times[0,T]\longmapsto\mathbb{R}^d,\ g:\Omega\times[0,T]\longmapsto\mathbb{R}^{d\times l}$ satisfy **(H.1)**, and let ξ be as before. We consider the following BDSDE,

$$Y_t=\xi+\int_t^T f(s)ds+\int_t^T g(s)d\overleftarrow{B}_s-\int_t^T Z_s dW_s,\qquad 0\leq t\leq T. \tag{E.1}$$

Then we have the following result.

Proposition 1.1 *Assume that* ***(H.1)*** *are satisfied. Then equation* $(E.1)$ *has a unique solution.*

Proof. Existance: To show the existence, we consider the filtration $\mathcal{G}_t := \mathcal{F}_t^W \vee \mathcal{F}_T^B$ and the martingale

$$M_t = \mathbf{E}\left(\xi + \int_0^T f(s)ds + \int_0^T g(s)d\overleftarrow{B}_s \middle| \mathcal{G}_t\right), \tag{1.1}$$

which is clearly a square integrable martingale by (H.1). An extension of Itô's martingale representation theorem yields the existence of a $\mathcal{G}_t$-progressively measurable process Z_t with values in $\mathbb{R}^{d\times r}$ such that

$$\mathbf{E}\int_0^T ||Z_s||\, ds < +\infty \qquad \text{and} \qquad M_T = M_t + \int_t^T Z_s dW_s. \tag{1.2}$$

We subtract the quantity $\int_0^t f(s)ds + \int_0^t g(s)d\overleftarrow{B}_s$ from both sides of the martingale in (1.1) and we employ the martingale representation in (1.2) to obtain

$$Y_t = \xi + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B}_s - \int_t^T Z_s dW_s,$$

where

$$Y_t = \mathbf{E}\left(\xi + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B}_s \middle| \mathcal{G}_t\right).$$

It remains to show that Y_t and Z_t are in fact $\mathcal{F}_t$-adapted. For Y_t, this is obvious since for each t,

$$Y_t = \mathbf{E}\left(\Gamma | \, \mathcal{F}_t \vee \mathcal{F}_t^B\right)$$

where

$$\Gamma = \xi_T + \int_0^T f(s)ds + \int_0^T g(s)d\overleftarrow{B}_s,$$

is $\mathcal{F}_t \vee \mathcal{F}_t^B$-mesurable. Using the fact that $\mathcal{F}_t^B$ is independent of $\mathcal{F}_t \vee \sigma(\Gamma)$, we deduce that $Y_t = \mathbb{E}(\Gamma | \mathcal{F}_t)$. Moreover, we have

$$\int_t^T Z_s dW_s = \xi + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B}_s - Y_t,$$

and the right-hand side is $\mathcal{F}_T^W \vee \mathcal{F}_{t,T}^B$-mesurable. Hence, from Itô's martingale representation theorem, $Z_s, t < s < T$ is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$ adapted. Consequently Z_s is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$ measurable, for any $t < s$, so it is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$ measurable.

Uniqueness. Let (Y, Z) and $\left(\tilde{Y}, \tilde{Z}\right)$ be two solution of $(E.1)$ and define $\theta \in \{Y, Z\}$, $\Delta\theta = \theta - \tilde{\theta}$. Then the triplet $(\Delta Y, \Delta Z)$ solves the equation

$$\Delta Y_t + \int_t^T \Delta Z_s dW_s = 0, \qquad t \in [0, T].$$

Itô's formula implies

$$\mathbb{E}\left|\Delta Y_t\right|^2 + \mathbb{E}\int_t^T \left|\Delta Z_s\right|^2 dW_s = 0, \qquad t \in [0, T].$$

The proof of Proposition 1.1 is complete. ■

We will also need the following Itô-formula.

Lemma 1.1 *Let* $\alpha \in \mathcal{S}^2([0,T], \mathbb{R}^n)$, $\beta \in \mathcal{M}^2([0,T], \mathbb{R}^n)$, $\gamma \in \mathcal{M}^2\left([0,T], \mathbb{R}^{n\times d}\right)$, $\delta \in \mathcal{M}^2\left([0,T], \mathbb{R}^{n\times d}\right)$ *de such that*

$$\alpha_t = \alpha_0 + \int_0^T \beta_s ds + \int_0^T \gamma_s d\overleftarrow{B}_s - \int_0^T \delta_s dW_s, \qquad 0 \le t \le T$$

Then, for any function $\phi \in C^2(\mathbb{R}^n)$, *we have*

$$\phi(\alpha_t) = \phi(\alpha_0) + \int_0^T \langle \nabla\phi(\alpha_s), \beta_s\rangle ds + \int_0^T \langle \nabla\phi(\alpha_s), \gamma_s\rangle d\overleftarrow{B}_s + \int_0^T \langle \nabla\phi(\alpha_s), \delta_s\rangle dW_s$$
$$- \frac{1}{2}\int_0^T Tr\left[\phi''(\alpha_s)\gamma_s\gamma_s^*\right] ds + \frac{1}{2}\int_0^T Tr\left[\phi''(\alpha_s)\delta_s\delta_s^*\right] ds$$

In particular,

$$\begin{aligned}|\alpha_t| &= |\alpha_0| + 2\int_0^T \alpha_s\beta_s ds + 2\int_0^T \langle \alpha_s, \gamma_s d\overleftarrow{B}_s\rangle + 2\int_0^T \langle \alpha_s, \delta_s dW_s\rangle \\ &- \int_0^T ||\gamma_s||^2 ds + \int_0^T ||\delta_s||^2 ds.\end{aligned}$$

Proof. See E, Pardoux; S, Peng [24]. ■

We are now in a position to give the proof of Theorem 1.1.

Proof. It remains to show the existence which will be obtained via a fixed point of the contraction of the function Φ defined as follows

$$\Phi: \ \mathcal{D} \ \rightarrow \ \mathcal{D}$$

where $\mathcal{D}$ the space of couple process $(Y_\cdot, Z_\cdot) \in \mathcal{S}^2\left([0,T];\mathbb{R}^d\right) \times \mathcal{M}^2\left([0,T];\mathbb{R}^{d\times r}\right)$, endowed with the norm

$$||(Y,Z)||_\beta = \left(\mathbb{E}\left[\int_0^T e^{\beta s}\left(|Y_s|^2 ds + \int_t^T ||Z_s||^2\right) ds\right]\right)^{\frac{1}{2}}.$$

Let Φ be the map from $\mathcal{D}$ into itself which to (Y,Z) associates $\Phi(Y,Z) = \left(\tilde{Y},\tilde{Z}\right)$ where the couple $(Y_t,Z_t)_{0\le t\le T} \in \mathcal{D}$ and satisfies the equation $\left(E^{\xi,f,g}\right)$. Thanks to Proposition (1.1), the mapping Φ is well defined. Let $\left(\tilde{Y},\tilde{Z}\right)$ and $\left(\tilde{Y}',\tilde{Z}'\right)$ be two elements of $\mathcal{D}$ such that

$$(Y,Z) = \Phi\left(\tilde{Y},\tilde{Z}\right), \qquad \left(\acute{Y},\acute{Z}\right) = \Phi\left(\tilde{Y}',\tilde{Z}'\right),$$

where $\left(\tilde{Y},\tilde{Z}\right)$ and $\left(\tilde{Y}',\tilde{Z}'\right)$ is the solution of the BDSDE $\left(E^{\xi,f,g}\right)$ associated with $\left(\xi, f(s,\tilde{Y}_s,\tilde{Z}_s), g(s,\tilde{Y}_s,\tilde{Z}_s)\right)$ and $\left(\xi, f(s,\tilde{Y}_s',\tilde{Z}_s'), g(s,\tilde{Y}_s',\tilde{Z}_s')\right)$. We use the following notation $\Delta\tilde{\Psi}_s = \tilde{\Psi}_s - \tilde{\Psi}_s'$ and $\Delta\Psi_s = \Psi_s - \Psi_s'$.

Then, we get

$$\mathbb{E}\int_t^T e^{\beta s}\left(|\Delta Y_s|^2 + ||\Delta Z_s||^2\right) ds \le \gamma\mathbb{E}\int_t^T e^{\beta s}\left(\left|\Delta\tilde{Y}_s\right|^2 + \left|\left|\Delta\tilde{Z}_s\right|\right|^2\right) ds,$$

where $0 < \gamma < 1$. Thus, the mapping Φ is a strict contraction on $\mathcal{D}$ and it has a unique fixed point $(Y_., Z_.) \in \mathcal{D}$. Consequently, $(Y_., Z_.) \in \mathcal{S}^2\left([0,T]\,;\mathbb{R}^d\right) \times \mathcal{M}^2\left([0,T]\,;\mathbb{R}^{d\times r}\right)$ is the unique solution of BDSDE $\left(E^{\xi,f,g}\right)$. Finally we complete the proof of **Theorem 1.1.** ■

1.2 Comparison principle.

In this section our objective is to present a comparison result for the following equations for $j = 1, 2$

$$Y_t^j = \xi^j + \int_t^T f^j(s, Y_s^j, Z_s^j)ds + \int_t^T g(s, Y_s^j, Z_s^j)d\overleftarrow{B}_s - \int_t^T Z_s^j dW_s, \qquad t \in [0,T]\,. \tag{1.3}$$

Theorem 1.2 *Assume that the BDSDE associated with dates* (ξ^1, f^1, g, T), $\left(resp\ \left(\xi^2, f^2, g, T\right)\right)$ *has a solution* $(Y_t^1, Z_t^1)_{t\in[0,T)]}$, (*resp* $(Y_t^2, Z_t^2)_{t\in[0,T]}$). *Each one satisfying the assumption **(H.1)**, assume moreover that:*

$$\begin{cases} \xi^1 \le \xi^2, \\ \forall t \le T,\ S_t^1 \le S_t^2, \\ f^1(t, Y_t, Z_t) \le f^2(t, Y_t, Z_t)\,. \end{cases}$$

Then we have $\mathbb{P} - a.s.$, $Y_t^1 \le Y_t^2$.

Proof. Let us show that $\left(Y_t^1 - Y_t^2\right)^+ = 0$, using the equations (1.3), by the notation $\bar{\delta}_. = \delta_.^1 - \delta_.^2$, we get

$$\bar{Y}_t = \bar{\xi} + \int_t^T \left(f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)\right) ds + \int_t^T \left(g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)\right) d\overleftarrow{B}_s - \int_t^T \bar{Z}_s dW_s.$$

Applying Tanaka-Itô's formula and taking expectation, we get

$$\begin{aligned}\mathbb{E}\left|\left(\bar{Y}_t\right)^+\right|^2 + \mathbb{E}\int_t^T 1_{\{\bar{Y}_s>0\}}\left\|\bar{Z}_s\right\|^2 ds \le \mathbb{E}\left|\left(\bar{\xi}\right)^+\right|^2 + 2\mathbb{E}\int_t^T \left(\bar{Y}_s\right)^+ \left(f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)\right) ds \\ + \mathbb{E}\int_t^T 1_{\{\bar{Y}_s>0\}}\left\|g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2\right\|^2 ds.\end{aligned}$$

Since $\int_t^T (\bar{Y}_s)^+ (g(s,Y_s^1,Z_s^1)-g(s,Y_s^2,Z_s^2))\, d\overleftarrow{B}_s$ and $\int_t^T (\bar{Y}_s)^+ \bar{Z}_s dW_s$ are a uniformly integrable martingale, we get

$$\mathbb{E}\left\{\left|(\bar{Y}_t)^+\right|^2+\int_t^T 1_{\{\bar{Y}_s>0\}}\left|\left|\bar{Z}_s\right|\right|^2 ds\right\} \leq 2\mathbb{E}\int_t^T (\bar{Y}_s)^+\left(f^1(s,Y_s^1,Z_s^1)-f^2(s,Y_s^2,Z_s^2)\right) ds$$
$$+\mathbb{E}\int_t^T 1_{\{\bar{Y}_s>0\}}\left|\left|g(s,Y_s^1,Z_s^1)-g(s,Y_s^2,Z_s^2)\right|\right|^2 ds,$$

since $(\xi^1-\xi^2)^+=0$. We obtain, by hypothesis **(H.1),** and Young's inequality the following inequality

$$\begin{aligned} I &= 2\mathbb{E}\int_t^T (\bar{Y}_s)^+\left(f^1(s,Y_s^1,Z_s^1)-f^2(s,Y_s^2,Z_s^2)\right) ds \\ &= 2\mathbb{E}\int_t^T \left[(\bar{Y}_s)^+\left(f^1(s,Y_s^1,Z_s^1)-f^1(s,Y_s^2,Z_s^2)\right)+(\bar{Y}_s)^+\left(f^1(s,Y_s^2,Z_s^2)-f^2(s,Y_s^2,Z_s^2)\right)\right] ds \\ : &= I_1+I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= 2\mathbb{E}\int_t^T (\bar{Y}_s)^+\left(f^1(s,Y_s^1,Z_s^1)-f^1(s,Y_s^2,Z_s^2)\right) ds, \\ I_2 &= 2\mathbb{E}\int_t^T (\bar{Y}_s)^+\left(f^1(s,Y_s^2,Z_s^2)-f^2(s,Y_s^2,Z_s^2)\right) ds \leq 0. \end{aligned}$$

From **(H.1)** and Young's inequality, it follows that

$$\begin{aligned} I \quad \leq I_1 \quad &\leq 2C\mathbb{E}\int_t^T (\bar{Y}_s)^+\left(|Y_1-Y_2|+||Z_1-Z_2||\right) ds, \\ &\leq \left(2C+\tfrac{C^2}{1-\alpha}\right)\mathbb{E}\int_t^T \left|(\bar{Y}_s)^+\right|^2 ds+(1-\alpha)\,\mathbb{E}\int_t^T 1_{\{\bar{Y}_s>0\}}\left|\bar{Y}_s\right|^2 ds \end{aligned}$$

again we applying the assumption **(H.1)** for g, we get

$$\mathbb{E}\left|(\bar{Y}_t)^+\right|^2 \leq C\mathbb{E}\int_t^T \left|\bar{Y}_s^+\right|^2 ds,$$

By Gronwall's inequality, it follows that $\mathbb{E}\left[\left|(\bar{Y}_t)^+\right|^2\right]=0$, finally, we have $Y_t^1 \leq Y_t^2$. ■

1.3 Backward Doubly SDEs with continuous coefficient.

In this section we are interested in weakening the conditions on f. We assume that f and g satisfy the following assumptions:

(C1.1) Let $f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^{d}\longmapsto\mathbb{R},\ g:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^{d}\longmapsto\mathbb{R}$ be measurable functions such that, for every $(y,\, z)\in\mathbb{R}\times\mathbb{R}^{d}$, $f(.,y,z)\in M^{2}(0,T,\mathbb{R})$ and $g(.,y,z)\in M^{2}(0,T,\mathbb{R})$

(C1.2) There exists $C>0$ $s.t.$ for all $(t,\omega,y,z)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^{d}$, $\left(t,\omega,y',z'\right)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^{d}$

$$\begin{cases}|f(t,\omega,y,z)|\leq C\left(1+|y|+|z|\right),\\ \left|g(t,\omega,y,z)-g(t,\omega,y',z')\right|^{2}\leq C\left|y-y'\right|^{2}+\alpha\left|\left|z-z'\right|\right|^{2}.\end{cases}$$

(C1.3) For fixed ω and t, $f(t,\omega,\cdot,\cdot)$ is continuous.

Theorem 1.3 *[see **Theorem 4.1 in [27]**] Assume that $(C1.1)-(C1.3)$ holds. Then Eq $\left(E^{\xi,f,g}\right)$ admits a solution $(Y,Z)\in\mathcal{D}^{2}\left(\mathbb{R}\right)$. Moreover there is a minimal solution (Y^{*},Z^{*}) of BDSDE $\left(E^{\xi,f,g}\right)$ in the sense that for any other solution (Y,Z) of Eq. $\left(E^{\xi,f,g}\right)$, we have $Y^{*}\leq Y$.*

We still assume that $l=d=1$. Before giving the proof of Theorem 1.3, we define, as the classical approximation can be proved by adapting the proof given in J. J. Alibert and K. Bahlali [2], the sequence $f_{n}\left(t,\omega,y,z\right)$ associated to f,

$$f_{n}\left(t,\omega,y,z\right)=\inf_{\left(y',z'\right)\in\mathbb{R}\times\mathbb{R}^{d}}\left[f\left(t,\omega,y',z'\right)+n\left(\left|y-y'\right|+|z-z'|\right)\right]$$

then for $n\geq K$, f_{n} is jointly measurable and uniformly linear growth in $y;z$ with constant K.

Given $\xi\in\mathbb{L}^{2}$, by Theorem 1.1, there exist two pair of processes $(Y_{\cdot}^{n},Z_{\cdot}^{n})$ and $(U_{\cdot},V_{\cdot})$, which are the solutions to the following BDSDEs, respectively,

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n)ds + \int_t^T g(s, Y_s^n, Z_s^n)d\overleftarrow{B}_s \int_t^T Z_s^n dW_s, \ 0 \leq t \leq T,$$
$$U_t = \xi + \int_t^T F(s, U_s, V_s)ds + \int_t^T g(s, U_s, V_s)d\overleftarrow{B}_s - \int_t^T V_s dW_s, \ 0 \leq t \leq T,$$

where $F(s, \omega, U, V) = K(1 + |U| + |V|)$. From **Theorem 1.2** and **lemma 1** of [15], we get for all, t and $\forall n \leq m$,

$$Y_t^n \leq Y_t^m \leq U_t. \tag{1.4}$$

Lemma 1.2 ***[see Lemma 4.2 in [27]]*** *Assume that* $(C1.1)-(C1.3)$ *is in force. Then there exists a constant* $A > 0$ *depending only on* K, C, α, ξ *and* T *such that:*

$$||Y^n||_{\mathcal{S}^2} \leq A, \qquad ||Z^n||_{\mathcal{M}^2} \leq A, \qquad ||U||_{\mathcal{S}^2} \leq A, \qquad ||V||_{\mathcal{M}^2} \leq A.$$

Proof. First of all, we prove that $\|U\|$ and $\|V\|$ are all bounded. Clearly, from (1.4) there exist a constant B depending only on K, C, α, T and ξ, such that

$$(E\int_0^T |Y_s^n|^2 ds)^{1/2} \leq B, \ (E\int_0^T |U_s|^2 ds)^{1/2} \leq B, \ ||V||_{\mathcal{M}^2} \leq B.$$

Applying Itô's formula to $|U_s|^2$, we have

$$\begin{aligned} |U_t|^2 = |\xi|^2 &+ 2\int_t^T U_s F(s, U_s, V_s)ds + 2\int_t^T U_s g(s, U_s, V_s)dB_s \\ &- 2\int_t^T U_s V_s dW_s + \int_t^T |g(s, U_s, V_s)|^2 ds - \int_t^T |V_s|^2 ds \end{aligned} \tag{1.5}$$

From **(C1.2)**, for all $\alpha < \alpha' < 1$, there exists a constant $C(\alpha') > 0$ such that

$$|g(t, u, v)|^2 \leq C(\alpha')(|u|^2 + |g(t, 0, 0)|^2) + \alpha'|v|^2 \tag{1.6}$$

From (1.5) and (1.6), it follows that

$$\begin{aligned}
|U_t|^2+\int_t^T|V_s|^2ds &\leq |\xi|^2+2K\int_t^T|U_s|(1+|U_s|+|V_s|)ds+2\int_t^TU_sg(s,U_s,V_s)dB_s\\
&-2\int_t^TU_sV_sdW_s+C(\alpha^{'})\int_t^T(|U_s|^2+|g(t,0,0)|^2)ds+\alpha^{'}\int_t^T|V_s|^2ds\\
&\leq |\xi|^2+K^2(T-t)+C(\alpha^{'})\int_t^T|g(t,0,0)|^2ds+\frac{1+\alpha^{'}}{2}\int_t^T|V_s|^2ds\\
&+(1+2K+C(\alpha^{'})+\frac{2K^2}{1-\alpha^{'}})\int_t^T|U_s|^2ds\\
&+2\int_t^TU_sg(s,U_s,V_s)dB_s-2\int_t^TU_sV_sdW_s.
\end{aligned}$$

Taking expectation, we get by Young's inequality,

$$\begin{aligned}
\|U_t\|^2+\frac{1-\alpha^{'}}{2}\int_t^T\|V_s\|^2ds &\leq E(|\xi|^2+K^2T+C(\alpha^{'})\int_t^T|g(t,0,0)|^2ds)\\
&+(1+2K+C(\alpha^{'})+\frac{2K^2}{1-\alpha^{'}})E\int_t^T|U_s|^2ds\\
&+2E\left(\sup_{0\leq t\leq T}|\int_t^TU_sg(s,U_s,V_s)dB_s|\right)+2E\left(\sup_{0\leq t\leq T}|\int_t^TU_sV_sdW_s|\right).
\end{aligned}\tag{1.7}$$

By **B-D-G**'s inequality, we deduce

$$\begin{aligned}
E\left(\sup_{0\leq t\leq T}|\int_t^TU_sg(s,U_s,V_s)dB_s|\right) &\leq C_pE\left(\int_0^T|U_s|^2\,|g(s,U_s,V_s)|^2\,ds\right)^{\frac{1}{2}},\\
&\leq C_pE\left[\left(\sup_{0\leq t\leq T}|U_s|^2\right)^{\frac{1}{2}}\left(\int_0^T|g(s,U_s,V_s)|^2\,ds\right)^{\frac{1}{2}}\right],\\
&\leq 2C_p^{,2}C(\alpha^{'})E\left(\int_0^T|U_s|^2+|g(s,0,0)|^2\,ds\right)^{\frac{1}{2}}\\
&+\frac{1}{8}\,||U||_{\mathcal{S}^2}+2C_p^{,2}\alpha^{'}\,||V||_{\mathcal{M}^2}\,,
\end{aligned}\tag{1.8}$$

In the same, way, we have

$$E\left(\sup_{0\leq t\leq T}|\int_t^TU_sV_sdW_s|\right)\leq\frac{1}{8}\,||U||_{\mathcal{S}^2}+2C_p^{,2}\,||V||_{\mathcal{M}^2}\,.\tag{1.9}$$

From Eqs. (1.7), (1.8) and (1.9), it follows that

$$\begin{aligned}\|U_t\|^2 + \frac{1-\alpha'}{2}\int_t^T \|V_s\|^2 ds &\leq E\left[|\xi|^2 + K^2T + C(\alpha')\left(1+4C_p^{,2}\right)\int_t^T |g(t,0,0)|^2 ds\right] \\ &\quad + 2\left(1+2K+C(\alpha')\left(1+4C_p^{,2}\right)+4C_p^{,2}\left(1+\alpha'\right)+\frac{2K^2}{1-\alpha'}\right)A^2, \\ &: = \frac{1-\alpha}{2}\bar{A}^2,\end{aligned}$$

that is $||U||_{\mathcal{S}^2} \leq \bar{A}, \qquad ||V||_{\mathcal{M}^2} \leq \bar{A}$. From Eq. (1.4), it easily follows that $||Y^n||_{\mathcal{S}^2} \leq \bar{A}$. Next, we prove that boundedness of $||Z^n||_{\mathcal{M}^2}$. Applying Itô's formula to $|Y_t^n|^2$ and taking expectation, it follows that

$$\mathbb{E}\left|Y_t^n\right|^2 + \mathbb{E}\int_t^T \left|Z_s^n\right|^2 ds \leq \mathbb{E}\left|\xi\right|^2 + 2\mathbb{E}\int_t^T Y_s^n f_n(s,Y_s^n,Z_s^n)ds + \mathbb{E}\int_t^T \left|\left|g(s,Y_s^n,Z_s^n)\right|\right|^2 ds.$$

From the well-known Young's inequality, it follows that

$$\begin{aligned}&\mathbb{E}\left(\left|Y_t^n\right|^2 + \textstyle\int_t^T \left|Z_s^n\right|^2 ds\right) \leq \mathbb{E}\left|\xi\right|^2 + \mathbb{E}\left(\textstyle\int_0^T \left(C'\left|Y_s^n\right|^2 + \frac{1-\alpha'}{2}\int_t^T \left|Z_s^n\right|^2\right) ds\right) \\ &+\mathbb{E}\left(\textstyle\int_0^T \left(C(\alpha')\left|\left|g(s,0,0,0)\right|\right|^2 + \alpha'\left|Z_s^n\right|^2\right) ds\right) + K\left(T-t\right).\end{aligned}$$

where $C' = 1 + 2K + C(\alpha') + \frac{2K}{1-\alpha'}$, and we know $0 < \alpha' < 1$ from Eq. (1.5). Then

$$||Z^n||_{\mathcal{M}^2}^2 \leq A^2,$$

where $A^2 : = \frac{2}{1-\alpha'}\left(C'T\bar{A}^2 + K^2T + \mathbb{E}\left|\xi\right|^2 + C(\alpha')\mathbb{E}\left(\int_0^T \left|\left|g(s,0,0,0)\right|\right|^2 ds\right)\right)$. The prove is complete. ■

Lemma 1.3 ***[see Lemma 4.3 in [27]]*** *Assume that* $(C1.1)-(C1.3)$ *is in force. Then the sequence* (Y^n, Z^n) *converges a.s. in* $\mathcal{S}^2(0,T,\mathbb{R}) \times \mathcal{M}^2(0,T,\mathbb{R})$.

Proof. Let $n_0 \geq K$. Since Y^n is increasing and bounded in $\mathcal{S}^2(0,T,\mathbb{R})$ we deduce from the dominated convergence theorem that Y^n converges in $\mathcal{S}^2(0,T,\mathbb{R})$. We shall denote by Y the limit of Y^n. Applying Itô's formula to $|Y_t^n - Y_t^m|^2$, we get for n, $m \geq n_0$,

$$\mathbb{E}\left(|Y_t^n - Y_t^m|^2 + \int_t^T |Z_s^n - Z_s^m|^2\, ds\right) \leq 2\mathbb{E}\int_t^T (Y_s^n - Y_s^m)\left(f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)\right) ds$$
$$+\,\mathbb{E}\int_t^T ||g(s, Y_s^n, Z_s^n) - g(s, Y_s^m, Z_s^m)||^2\, ds.$$

we deduce that

$$\begin{aligned}
&\mathbb{E}\left(|Y_t^n - Y_t^m|^2 + \int_t^T |Z_s^n - Z_s^m|^2\, ds\right)\\
&\leq 2\left(\mathbb{E}\int_t^T (Y_s^n - Y_s^m)^2\, ds\right)^{\frac{1}{2}}\left(\mathbb{E}\int_t^T \left(f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)\right)^2 ds\right)^{\frac{1}{2}}\\
&+\mathbb{E}\int_t^T \left(C\,|Y_s^n - Y_s^m| + \alpha\,|Z_s^n - Z_s^m|^2\right) ds.
\end{aligned}$$

Since f_n and f_m are uniformly linear growth and (Y^n, Z^n) is bounded, similarly to **Lemma 1.2**, there exists a constant $\bar{K} > 0$ depending only on K, C, T and ξ, such that

$$\mathbb{E}\left(|Y_0^n - Y_0^m|^2 + \int_t^T |Z_s^n - Z_s^m|^2\, ds\right) \leq \mathbb{E}\int_t^T \left(\bar{K}\,|Y_s^n - Y_s^m| + \alpha\,|Z_s^n - Z_s^m|^2\right) ds.$$

So

$$\mathbb{E}\int_t^T |Z_s^n - Z_s^m|^2\, ds \leq \frac{\bar{K}T}{1-\alpha}\mathbb{E}\left(\sup_{s\in[0,T]} |Y_s^n - Y_s^m|\right),$$

thus Z^n is a Cauchy sequence in $\mathcal{M}^2(0, T, \mathbb{R})$, from which the result follows. ■

Proof. of Theorem 1.3 [see pages 107 and 108 in [27]]. ■

Chapter 2

Backward Doubly SDEs and SPDEs with superlinear growth generators.

In this Chapter we present a multidimensional backward doubly stochastic differential equations (BDSDEs) with a superlinear growth generator and a square integrable terminal datum. As application, we establish the existence and uniqueness of probabilistic solutions to some semilinear stochastic partial differential equations (SPDEs) with superlinear growth gernerator. By probabilistic solution, we mean a solution which is representable throughout a BDSDEs.

Definition 2.1 *A solution of equation* $\left(E^{\xi,f,g}\right)$ *is a couple* (Y, Z) *which belongs to the space* $\mathcal{S}^2\left([0,\, T], \mathbb{R}^k\right) \times \mathcal{M}^2\left(0, T, \mathbb{R}^{k\times d}\right)$ *and satisfies* $\left(E^{\xi,f,g}\right)$.

We consider the following assumptions:

(H2.1) f is continuous in (y, z) for $a.e.$ (t, ω).

(H2.2) There exist $K > 0$, $M > 0$, and $\eta \in \mathbb{L}^1\left(\Omega; \mathbb{L}^1([0, T])\right)$ such that,

$$\langle y, f(t, \omega, y, z)\rangle \leq \eta_t + M|y|^2 + K|y||z| \quad P - a.s.,\, a.e.\, t \in [0, T].$$

(H2.3) g is continuous in $(., y, z)$ and there exist $L > 0$, $0 < \lambda < 1$, $0 < \alpha_1 < 1$, and η'_t, $0 \leq t \leq T$ verify $E \int_0^T |\eta'_s|^{\frac{2}{\alpha_1}} ds < \infty$ such that,

(i) $$|g(t, y, z) - g(t, y', z')|^2 \leq L|y - y'|^2 - \lambda|z - z'|^2.$$

(ii) $$|g(t, y, z)| \leq \eta'_t + L|y|^{\alpha_1} + \lambda|z|^{\alpha_1}$$

(H2.4) There exist $M_1 > 0$, $0 \leq \alpha < 2$, $\alpha' > 1$ and $\bar{\eta} \in \mathbb{L}^{\alpha'}([0, T] \times \Omega)$ such that:

$$|f(t, \omega, y, z)| \leq \bar{\eta}_t + M_1(|y|^\alpha + |z|^\alpha).$$

(H2.5) There exists $v \in \mathbb{L}^2(\Omega; \mathbb{L}^2([0, T]))$, a real valued sequence $(A_N)_{N>1}$ and constants $M_2 > 1$, $r > 0$ such that:

(i) $\forall N > 1, \quad 1 < A_N \leq N^r$.

(ii) $\lim_{N\to\infty} A_N = \infty$.

(iii) For every $N \in \mathbb{N}^*$ and every $y, y' z, z'$ such that $|y|, |y'|, |z|, |z'| \leq N$, we have

$$\langle y-y', f(t, y, z)-f(t, y', z')\rangle \mathbf{1}_{\{v_s(\omega)\leq N\}} \leq M_2|y-y'|^2 \log A_N + M_2|y-y'||z-z'|\sqrt{\log A_N} + M_2 A_N^{-1}.$$

For $n \in \mathbb{N}$, we define $\rho_n(f) := \mathbb{E}\int_0^T \sup_{|y|,|z|\leq n} |f(s, y, z)|ds.$

Let us give some remarks about the previous assumptions.

1. In assumptions **(H2.2)** and **(H2.3)**, the conditions $\gamma < \frac{1}{4}$ and $\lambda < \frac{1}{2}$ can be replaced by the condition : $2\gamma + \lambda < 1$.

2. The parameter α_1 appearing in assumption **(H2.3)** has a role in the construction of solution. More precisely, it allows to identify the backward stochastic integral driven by B.

3. Assumption **(H2.2)** shows expresses the fact that the generator f can have a superlinear growth on y and z.

4. The term $\mathbf{1}_{\{v_s(\omega)\leq N\}}$ appearing in assumption **(H2.5)** allows to cover generators with stochastic Lipschitz condition.

2.1 Existence and uniqueness of solutions.

Theorem 2.1 *Let ξ be a square integrable random variable. Assume that **(H2.1)**–**(H2.5)** are satisfied. Then equation $(E^{f,g,\xi})$ has a unique solution.*

Proof. See Bahlai et all [7]. ■

Let us recall the following approximation lemma which will be useful in the sequel.

Lemma 2.1 *Let f satisfy **(H2.1)**– **(H2.5)**. Then there exists a sequence (f_n) such that,*

(a) For each n, f_n is bounded and globally Lipschitz in (y,z) a.e. t and P-a.s.ω.

There exists $M'>0$, such that:

(b) $\sup_n |f_n(t,\omega,y,z)| \leq \bar{\eta} + M' + M_1(|y|^\alpha + |z|^\alpha)$, *for a.e.* (t,ω).

(c) $\sup\limits_n < y, f_n(t,\omega,y,z) > \leq \eta_t + M' + M|y|^2 + K|y||z|$, *for a.e.* (t,ω).

(d) *For every N, $\rho_N(f_n - f) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Proof. Let $\bar{\rho}_n : \mathbb{R}^d \times \mathbb{R}^{d\times r} \longrightarrow \mathbb{R}_+$ be a sequence of smooth functions with compact support which approximate the Dirac measure at 0 and which satisfy $\int \bar{\rho}_n(u)du = 1$. Let $\varphi_n : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be a sequence of smooth functions such that $0 \leq \varphi_n \leq 1$, $\varphi_n(u) = 1$ for $|u| \leq n$ and $\varphi_n(u) = 0$ for $|u| \geq n+1$. Likewise we define the sequence ψ_n from $\mathbb{R}^{d\times r}$ to $\mathbb{R}_+$. We put, $f_{q,n}(t,y,z) = \mathbb{1}_{\{\bar{\eta}\leq q\}} \int f(t,(y,z)-u)\bar{\rho}_q(u)du\varphi_n(y)\psi_n(z)$. For $n \in \mathbb{N}^*$, let $q(n)$ be an integer such that $q(n) \geq n + n^\alpha$. It is not difficult to see that the sequence $f_n := f_{q(n),n}$ satisfy all the assertions (a)-(d).

■

2.2 Stability of solutions.

Let (f_n) be a sequence of processes which are $\mathcal{F}_t$-progressively measurable for each n. Let (ξ_n) be a sequence of random variables which are $\mathcal{F}_T$-measurable for each n and such that

$E(|\xi_n|^2) < \infty$. We will assume that for each n, the BDSDE (E^{f_n,g,ξ_n}) corresponding to the data (f_n, g, ξ_n) has a (not necessarily unique) solution. Each solution to equation (E^{f_n,g,ξ_n}) will be denoted by (Y^{f_n}, Z^{f_n}). Let $(Y,\, Z)$ be the unique solution of the BDSDE $E^{(f,g,\xi)}$. We also assume that :

(H2.6) For every N, $\rho_N(f_n - f) \longrightarrow 0$ as $n \to \infty$.

(H2.7) $E(|\xi_n - \xi|^2) \longrightarrow 0$ as $n \to \infty$.

(H2.8) There exist $K > 0$, $M > 0$ and $\eta \in \mathbb{L}^1(\Omega; \mathbb{L}^1([0,T]))$ such that,

$$\sup_n \langle y, f_n(t,\omega,y,z)\rangle \leq \eta_t + M|y|^2 + K|y||z| \quad P-a.s.,\ a.e.\, t \in [0,T].$$

(H2.9) There exist $M_1 > 0$, $0 \leq \alpha < 2$, $\alpha' > 1$ and $\bar{\eta} \in \mathbb{L}^{\alpha'}([0,T] \times \Omega)$ such that:

$$\sup_n |f_n(t,\omega,y,z)| \leq \bar{\eta}_t + M_1(|y|^\alpha + |z|^\alpha).$$

Theorem 2.2 *Let f, g and ξ be as in Theorem 2.1. Assume that* ***(H2.1)-(H2.9)*** *are satisfied. Then, for all $q < 2$ we have*

$$\lim_{n\to+\infty} \left(\mathbb{E} \sup_{0\leq t\leq T} |Y_t^{f_n} - Y_t|^q + \mathbb{E}\int_0^T |Z_s^{f_n} - Z_s|^q ds \right) = 0.$$

Proof. See Bahlai et all [7]. ■

2.3 Application to Sobolev solutions of SPDEs

Let σ and b be two functions which satisfy

$$\begin{cases} b \in \mathcal{C}_b^2\left(\mathbb{R}^k, \mathbb{R}^k\right) \quad and \quad \sigma \in \mathcal{C}_b^3\left(\mathbb{R}^k, \mathbb{R}^{k\times r}\right), \\ \text{and} \\ \mathcal{L} := \dfrac{1}{2}\displaystyle\sum_{i,j}(a_{ij})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}, \qquad with\ (a_{ij}) := \sigma\sigma^*. \end{cases}$$

Let $0 < q < 2$ be fixed. Let $\mathcal{H}$ be the set of random fields $u(t,x), 0 \leq t \leq T, x \in \mathbb{R}^k$ such that, for every (t,x), $u(t,x)$ is $\mathcal{F}^B_{t,T}$-measurable and

$$\|u\|^q_{\mathcal{H}} = E[\int_{\mathbb{R}^k}\int_0^T (|u(r,x)|^q + |(\sigma^*\nabla u)(r,x)|^q) dr e^{-\delta|x|} dx] < \infty.$$

The couple $(\mathcal{H},\ \|.\|_{\mathcal{H}})$ is a Banach space.

The SPDE under consideration is,

$$(\mathcal{P}^{(f,g)}) \qquad \begin{cases} u(s,x) = h\left(x\right) + \displaystyle\int_s^T \left\{\mathcal{L}u(r,x) + f(r,x,u(r,x),\sigma^*\nabla u(r,x)\right\} dr \\ \qquad + \displaystyle\int_s^T g\left(r,x,u(r,x),\sigma^*\nabla u(r,x)\right) d\overleftarrow{B}_r, \quad t \leq s \leq T, \end{cases}$$

Definition 2.2 *We say that u is a Sobolev solution to SPDE $(\mathcal{P}^{(f,g)})$, if $u \in \mathcal{H}$ and for any $\varphi \in \mathcal{C}^{1,\infty}_c([0,T] \times \mathbb{R}^d)$,*

$$\begin{aligned} &\int_{\mathbb{R}^k}\int_s^T f(r,x,u(r,x),\sigma^*\nabla u(r,x))\varphi(r,x) dr dx + \int_{\mathbb{R}^k}\int_s^T g(r,x,u(r,x),\sigma^*\nabla u(r,x))\varphi(r,x) d\overleftarrow{B}_r dx \\ &= \int_{\mathbb{R}^k}\int_s^T u(r,x)\frac{\partial\varphi(r,x)}{\partial r}(r,x) dr dx + \int_{\mathbb{R}} u(r,x)\varphi(r,x) dx - \int_{\mathbb{R}^k} h(x)\varphi(T,x) dx \qquad (2.1) \\ &- \frac{1}{2}\int_{\mathbb{R}^k}\int_s^T \sigma^* u(r,x)\sigma^*\varphi(r,x) dr dx - \int_{\mathbb{R}^k}\int_s^T u div\big[(b-A)\varphi\big](r,x) dr dx, \end{aligned}$$

where A is a d-vector whose coordinates are defined by $A_j := \frac{1}{2}\sum_{i=1}^d \frac{\partial a_{ij}}{\partial x_i}$.

This subsection is devoted to the study of the existence and uniqueness of Sobolev solutions to SPDE $(\mathcal{P}^{(f,g)})$ by using a decoupled system of SDE-BDSDEs. To this end, we will connect the SPDE $(\mathcal{P}^{(f,g)})$ with the following system of SDE-BDSDE.

$$X^{t,x}_s = x + \int_t^s b\left(X^{t,x}_r\right) dr + \int_t^s \sigma\left(X^{t,x}_r\right) dW_r, \qquad (2.2)$$

$$Y^{t,x}_s = h\left(X^{t,x}_T\right) + \int_s^T f\left(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r\right) dr + \int_s^T g\left(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r\right) d\overleftarrow{B}_r - \int_s^T Z^{t,x}_r dW_r. \qquad (2.3)$$

Our goal consists to establish the existence and uniqueness of solutions u to SPDE $(\mathcal{P}^{(f,g)})$ such that $u(t,X_s^{t,x}) = Y_s^{t,x}$ and $\nabla u(s,X_s^{t,x}) = Z_s^{t,x}$.

Assumptions.

We assume that there exist $\delta \geq 0$ such that

(H2.10) h belongs to $\mathbb{L}^2(\mathbb{R}^k, e^{-\delta|x|}dx;\ \mathbb{R}^d)$, that is $\int_{\mathbb{R}^d} |h(x)|^2 e^{-\delta|x|}dx < \infty$.

(H2.11) $f(t,x,.,.)$ is continuous for a.e. (t,x)

(H2.12) There exist $M > 0$, $K > 0$ and $\eta \in \mathbb{L}^1([0,T]\times\mathbb{R}^k, e^{-\delta|x|}dtdx;\ \mathbb{R}_+)$ such that,

$$\langle y, f(t,x,y,z)\rangle \leq \eta(t,x) + M|y|^2 + K|y||z| \qquad \mathbb{P}-a.s., a.e. t \in [0,T].$$

(H2.13) $\int_{\mathbb{R}^k}\int_0^T |g(t,x,0,0)|^2 e^{-\delta|x|}dtdx < \infty$ and there existe $L > 0$, $0 < \lambda < 1$ $0 < \alpha_1 < 1$, and $\eta \in \mathbb{L}^{\frac{2}{\alpha_1}}([0,T]\times\mathbb{R}^k, e^{-\delta|x|}dtdx;\ \mathbb{R}_+)$, such that,

(i) $$|g(t,x,y,z) - g(t,x,y',z')|^2 \leq L|y-y'|^2 - \lambda|z-z'|^2.$$

(ii) $$|g(t,x,y,z)| \leq \eta'(t,x) + L|y|^{\alpha_1} + \lambda|z|^{\alpha_1}$$

(H2.14) There exists $M_1 > 0$, $0 \leq \alpha < 2$, $\alpha' > 1$ and $\bar{\eta} \in \mathbb{L}^{\alpha'}([0,T]\times\mathbb{R}^k, e^{-\delta|x|}dtdx;\mathbb{R}_+)$ such that

$$|f(t,x,y,z)| \leq \bar{\eta}(t,x) + M_1(|y|^\alpha + |z|^\alpha).$$

(H2.15) There exist $M_2 > 0$ such that, for every $N \in \mathbb{N}, \forall\, y, y', z, z'$ such that $|y|, |y'|, |z|, |z'| \leq N$, we have

$$\langle y-y', f(t,x,y,z) - f(t,x,y',z')\rangle \leq M_2 \log N\left(\frac{1}{N} + |y-y'^2\right) + \sqrt{M_2\log N}|y-y'||z-z'|.$$

The proof of the following lemma can be found for instance in [13, 14] and in [8].

Lemma 2.2 *There exist a constant $K_{\delta,T} > 1$, such that for any $t \leq s \leq T$ and $\Phi \in L^1\left(\Omega \times \mathbb{R}^k,\, \mathbb{P}\otimes e^{-\delta|x|}dx\right)$*

$$K_{\delta,T}^{-1}\left[\int_{\mathbb{R}^k} |\Phi(x)|\, e^{-\delta|x|}dx\right] \leq E\left[\int_{\mathbb{R}^k} \left|\Phi\left(X_s^{t,x}\right)\right| e^{-\delta|x|}dx\right] \leq K_{\delta,T}\left[\int_{\mathbb{R}^k} |\Phi(x)|\, e^{-\delta|x|}dx\right].$$

Moreover for any $\Psi \in L^1\left(\Omega \times [0,\ T] \times \mathbb{R}^k \times\ \mathbb{P}\otimes dt \otimes e^{-\delta|x|}dx\right)$

$$\begin{aligned} K_{\delta,T}^{-1}\left[\int_{\mathbb{R}^k}\int_t^T |\Psi(s,x)|\, ds\ e^{-\delta|x|}dx\right] &\leq E\left[\int_{\mathbb{R}^k}\int_t^T \left|\Psi\left(s, X_s^{t,x}\right)\right| ds\ e^{-\delta|x|}dx\right], \\ &\leq K_{\delta,T}\left[\int_{\mathbb{R}^k}\int_t^T |\Psi(s,x)|\, ds\ e^{-\delta|x|}dx\right]. \end{aligned}$$

Theorem 2.3 *Under assumptions **(H2.10)-(H2.15)**, the SPDE $(\mathcal{P}^{(f,g)})$ admits a unique Sobolev solution u such that for every $t \in [0,\ T]$*

$$u(s, X_s^{t,x}) = Y_s^{t,x} \quad \textit{and} \quad \sigma^*\nabla u(s, X_s^{t,x}) = Z_s^{t,x} \qquad \textit{for a.e. } (s,\omega,x) \textit{ in } [t,\ T]\times\Omega\times\mathbb{R}^k$$

where $\{(X_s^{t,x},\, Y_s^{t,x},\, Z_s^{t,x}),\, t \leq s \leq T\}$ is the unique solution of the SDE-BDSDE (2.2)-(2.3).

The following lemma can be proved by arguing as in Theorem 2.1 and Theorem 2.2.

Lemma 2.3 *Assume **(H2.10)-(H2.15)** be satisfied. Let $(X^{t,x})$ be the unique solution of SDE (2.2) and $(Y^{t,x},\, Z^{t,x})$ be the unique solution of BDSDE (2.3). Let f^n be a sequence of functions we construct as in Lemma 2.1. For a fixed $n \in \mathbb{N}^*$, let $(Y^{n,t,x},\, Z^{n,t,x})$ be the unique solution of the BDSDE*

$$\begin{aligned} Y_s^{n,t,x} &= h(X_T^{t,x}) + \int_s^T f^n(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x})dr \\ &\quad + \int_s^T g(r, X_r^{t,x}, u^n(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x})d\overleftarrow{B}_r - \int_t^T Z_r^{n,t,x}dW_r. \end{aligned}$$

Then,

(i) there exists $K(T,t,x) \in \mathbb{L}^1(e^{-\delta|x|}dx)$ such that:

$$\sup_n \mathbb{E}\left[\sup_{s\leq T}|Y_s^{n,t,x}|^2 + \sup_{s\leq T}|Y_s^{t,x}|^2 + \int_s^T |Z_s^{n,t,x}|^2 ds + \int_s^T |Z_s^{n,t,x}|^2 ds\right] \leq K(T,t,x),$$

(ii) for every $q < 2$,

$$\lim_{n\to+\infty}\left(\mathbb{E}\sup_{0\leq s\leq T}|Y_s^{n,t,x} - Y_s^{t,x}|^q + \mathbb{E}\int_0^T |Z_s^{n,t,x} - Z_s^{t,x}|^q ds\right) = 0.$$

Proof of Theorem 2.3. The uniqueness of solutions follows from the uniqueness of BDSDE (2.3). We shall prove the existence, for detail of the demonstration see Bahlai et all [7]. The prove is from the following path:

Step 1. *Approximation of the problem* $(P^{(f,g)})$.

Step 2. *Convergence of the problem* $(P^{(f^n,g)})$.

Step 3. $u\left(s, X_s^{t,x}\right) = Y_s^{t,x}$ *and* $\sigma^*\nabla u(s, X_s^{t,x}) = Z_s^{t,x}$.

Step 4. *u is a Sobolev solution to the problem* $(P^{(f,g)})$.

Chapter 3

Backward Doubly SDEs and SPDEs with weak Monotonicity and General Growth Generators.

In this Chapter we deal with multidimensional backward doubly stochastic differential equations (BDSDEs) with a weak monotonicity and general growth generators and a square integrable terminal datum. We show the existence and uniqueness of solutions. As application, we establish the existence and uniqueness of probabilistic solutions to some semilinear stochastic partial differential equations (SPDEs) with a weak monotonicity and general growth generators. By probabilistic solution, we mean a solution which is representable throughout a BDSDEs.

Assumptions.

We consider the following assumptions:

(H3.1) $dP \times dt$-a.e., $z \in \mathbb{R}^{k\times d}$ $y \to \ f(w,t,y,z)$ is continuous.

(H3.2) f satisfies the weak monotonicity condition in y, i.e., there exist a nondecreasing and concave function $k(\cdot):\mathbb{R}^{+}\to\mathbb{R}^{+}$ with $k\left(u\right)>0$ for $u>0$, $k\left(0\right)=0$ and $\int_{0^{+}} k^{-1}(u)du=+\infty$ such that $dP \times dt$-a.e.,$\forall\left(y_{1},y_{2}\right)\in\mathbb{R}^{2k}$, $z\in\mathbb{R}^{k\times d}$,

$$\langle y_{1}-y_{2},f\left(t,\omega,y_{1},z\right)-f\left(t,\omega,y_{2},z\right)\rangle\leq k\left(\left|y_{1}-y_{2}\right|^{2}\right).$$

(H3.3) **i)** f is lipschitz in z, uniformly with respect to (w,t,y) i.e., there exists a constant $c>0$ such that $dP \times dt$-a.e.,

$$\left|f\left(w,t,y,z\right)-f\left(w,t,y,z'\right)\right|\leq c\left|z-z'\right|.$$

ii) There exists a constant $c>0$ and a constant $0<\alpha\leq\frac{1}{4}$ such that $dP \times dt$-a.e.,

$$\left|g\left(w,t,y,z\right)-g\left(w,t,y^{'},z^{'}\right)\right|\leq c\left|y-y^{'}\right|+\alpha\left|z-z'\right|.$$

(H3.4) f has a general growth with respect to y, i.e., $dP \times dt$-a.e., $\forall y\in\mathbb{R}^{k}$

$$\left|f\left(t,\omega,y,0\right)\right|\leq\left|f\left(t,\omega,0,0\right)\right|+\varphi\left(\left|y\right|\right),$$

where $\varphi:\mathbb{R}^{+}\to\mathbb{R}^{+}$ is increasing continuous function.

(H3.5)

$$\begin{cases} f\left(t,\omega,0,0\right)\in\mathcal{M}^{2}\left(0,T,\mathbb{R}^{k}\right), \\ g\left(t,\omega,0,0\right)\in\mathcal{M}^{2}\left(0,T,\mathbb{R}^{k\times l}\right). \end{cases}$$

3.1 The main results.

Theorem 3.1 *Let $\xi \in \mathbb{L}^2$, assume that* ***(H3.1)–(H3.5)*** *are satisfied. Then equation $\left(E^{f,g,\xi}\right)$ has a unique solution.*

3.1.1 Estimate for the solutions of BDSDE $\left(E^{\xi,f,g}\right)$.

We will use the following assumption on f and g.

(H3.6) $dP \times dt$-a.e., $\forall (y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$

$$\langle y, f(t,\omega,y,z)\rangle \leq \psi\left(|y|^2\right) + \lambda |y||z| + |y|\sigma_t,$$

where λ is a positive constante, σ_t is a positive and $(\mathcal{F}_t)$ progressively measurable processus with $E\int_0^T |\sigma_t|^2 dt < \infty$ and $\psi(\cdot)$ is a nondecreasing concave function from $\mathbb{R}^+$ to itself with $\psi(0) = 0$.

(H3.7) $dP \times dt$-a.e., $\forall (y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$

$$\left|g(t,\omega,y,z)\right|^2 \leq \lambda |y|^2 + \gamma |z|^2 + \eta_t,$$

with λ is a positive constant such that $\gamma \leq \frac{1}{4}$ and η_t is a positive and $(\mathcal{F}_t)$ progressively measurable processes with $E\int_0^T \eta_t dt < \infty$.

Proposition 3.1 *Let f and g satisfy* ***(H3.6)*** *and* ***(H3.7)****, let $(Y_t, Z_t)_{t\in[0,T]}$ be a solution to the BDSDE with parameters (ξ, T, f, g). Then for each $\delta > 0$ there exists a constants $K > 0$ depending only on δ, λ and γ such that*

(i) *for each $0 \leq t \leq T$:*

$$\mathbb{E}(\sup_{0\leq s\leq T} |Y_s|^2) + \mathbb{E}\left(\int_t^T |Z_s|^2 ds\right) \leq \Big(\mathbb{E}|\xi|^2 + 2\int_t^T \psi\left(\mathbb{E}|Y_s|^2\right) ds + \frac{1}{\delta}\mathbb{E}\int_t^T |\sigma_s|^2 ds + \mathbb{E}\int_t^T \eta_s ds\Big) K \exp\left(K(T-t)\right).$$

(ii) *Moreover for each $\delta > 0$ there exists a constants $\bar{K} > 0$ and depending only on δ, λ and*

γ *such that for* $0 \leq r \leq t \leq T$:

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\bigg|\mathcal{F}_r\right)+\mathbb{E}\left(\int_t^T|Z_s|^2ds\bigg|\mathcal{F}_r\right)\\
&\leq\left(\mathbb{E}\left(|\xi|^2\big|\mathcal{F}_r\right)+2\int_t^T\psi\left(\mathbb{E}\left(|Y_s|^2\big|\mathcal{F}_r\right)\right)ds\right.\\
&\left.+\frac{1}{\delta}\mathbb{E}\left(\int_t^T|\sigma_s|^2ds\bigg|\mathcal{F}_r\right)+2\mathbb{E}\left(\int_t^T\eta_sds\bigg|\mathcal{F}_r\right)\right)\bar{K}\exp\left(\bar{K}T\right).
\end{aligned}$$

Proof. For the first part, applying Itô's formula to $|Y_t|^2$ yields that, for each $0 \leq t \leq T$, we have

$$\begin{aligned}
|Y_t|^2+\int_t^T|Z_s|^2ds&=|\xi|^2+2\int_t^T\langle Y_s,f\left(s,Y_s,Z_s\right)\rangle ds+2\int_t^T\langle Y_s,g\left(s,Y_s,Z_s\right)\rangle d\overleftarrow{B}_s\\
&-2\int_t^T\langle Y_s,Z_s\rangle dW_s+\int_t^T|g(s,Y_s,Z_s)|^2ds,
\end{aligned}$$

taking expectation, we get

$$\begin{aligned}
\mathbb{E}|Y_t|^2+\mathbb{E}\int_t^T|Z_s|^2ds&=\mathbb{E}|\xi|^2+2\mathbb{E}\int_t^T\langle Y_s,f\left(s,Y_s,Z_s\right)\rangle ds+2\mathbb{E}\int_t^T\langle Y_s,g\left(s,Y_s,Z_s\right)\rangle d\overleftarrow{B}_s\\
&-2\mathbb{E}\int_t^T\langle Y_s,Z_s\rangle dW_s+\mathbb{E}\int_t^T|g(s,Y_s,Z_s)|^2ds.
\end{aligned}$$

Now, by $(H3.6)$ and Young's inequality, we have

$$\begin{aligned}
2\int_t^T\langle Y_s,f(s,Y_s,Z_s)\rangle ds&\leq2\int_t^T\Big(\psi(|Y_s|^2)+\lambda|Y_s||Z_s|+|Y_s|\sigma_s\Big)ds,\\
&\leq2\int_t^T\psi\left(|Y_s|^2\right)ds+\left(2\lambda^2+\delta\right)\int_t^T|Y_s|^2ds\\
&+\int_t^T\frac{|\sigma_s|^2}{\delta}ds+\int_t^T\frac{|Z_s|^2}{2}ds,
\end{aligned}$$

Then by $(H3.7)$, we have

$$\begin{aligned}
\mathbb{E}|Y_t|^2+\left(\frac{1}{2}-\gamma\right)\mathbb{E}\int_t^T|Z_s|^2ds&\leq\mathbb{E}|\xi|^2+2\mathbb{E}\int_t^T\psi\left(|Y_s|^2\right)ds+\left(2\lambda^2+\lambda+\delta\right)\mathbb{E}\int_t^T|Y_s|^2ds\\
&+\frac{1}{\delta}\mathbb{E}\int_t^T|\sigma_s|^2ds+\mathbb{E}\int_t^T\eta_sds.
\end{aligned}$$

Since $\int_0^t \langle Y_s, Z_s\rangle dW_s$ and $\int_0^t \langle Y_s, g(s,Y_s,Z_s)\rangle dB_s$ are a uniformly integrable martingale. For each $0 \leq t \leq T$, we have the following inequality

$$\left(\frac{1}{2}-\gamma\right)\mathbb{E}\int_t^T |Z_s|^2\,ds \leq \mathbb{E}\left(\Delta_t\right), \tag{3.1}$$

where,

$$\Delta_t = |\xi|^2 + 2\int_t^T \psi(|Y_s|^2)ds + \left(2\lambda^2+\lambda+\delta\right)\int_t^T |Y_s|^2\,ds + \frac{1}{\delta}\int_t^T |\sigma_s|^2\,ds + \int_t^T \eta_s ds.$$

Furthermore, it follows from the Burkhölder-Davis-Gundy and Young's inequality, we have

$$\begin{cases} 2\mathbb{E}\left(\sup_{t\leq u\leq T}\left|\int_u^T \langle Y_s, Z_s\rangle dW_s\right|\right) \leq 2C_p\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|\sqrt{\int_t^T |Z_s|^2\,ds}\right), \\ \leq \frac{1+2\gamma}{2}\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\right) + \frac{2C_p^2}{1+2\gamma}\mathbb{E}\left(\int_t^T |Z_s|^2\,ds\right), \\ < \infty, \\ \text{and} \\ 2\mathbb{E}\left(\sup_{t\leq u\leq T}\left|\int_u^T \langle Y_s, g\left(s,Y_s,Z_s\right)\rangle d\overleftarrow{B}_s\right|\right) \leq \frac{1}{\epsilon}\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\right) + \epsilon C_p^2\mathbb{E}\int_0^T |g\left(s,Y_s,Z_s\right)|^2\,ds, \\ < \left(\frac{1}{\epsilon}+\lambda\epsilon C_p^2\right)\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\right) + \gamma\epsilon C_p^2\mathbb{E}\int_0^T |Z_s|^2\,ds + \epsilon C_p^2\mathbb{E}\int_0^T |\eta_s|^2\,ds, \\ \infty. \end{cases} \tag{3.2}$$

By assumptions $(H3.6)$, $(H3.7)$ and using $(3.1)-(3.2)$, we have for $\tilde{C}>0$ the following inequality,

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_t|^2\right) + \mathbb{E}\int_t^T |Z_s|^2\,ds \leq \tilde{C}\mathbb{E}\left(\Delta_t\right),$$

Gronwall's Lemma, Fubini's theorem and Jensen's inequality, in view of the concavity condition of $\psi(\cdot)$, then there exists a constant $K>0$ such that $t\in[0,T]$

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s|^2\right) + \mathbb{E}\int_t^T |Z_s|^2 ds \leq \left(K\mathbb{E}|\xi|^2 + 2K\int_t^T \psi(\mathbb{E}|Y_s|^2)ds + \frac{K}{\delta}\mathbb{E}\int_t^T |\sigma_s|^2 ds \right.$$
$$\left. + K\mathbb{E}\int_t^T \eta_s ds\right)\exp\left(K\left(T-t\right)\right).$$

For the second part, we use the conditional expectation with respect to $\mathcal{F}_r$ instead of using the mathematical expectation. Using the Burkhölder-Davis-Gundy, $2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$ inequalities and assumption $(H3.7)$, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\leq u\leq T}\left|\int_u^T \langle Y_s, g(s,Y_s,Z_s)\rangle d\overleftarrow{B}_s\right|\middle|\mathcal{F}_r\right) \leq C_p\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|\sqrt{\int_t^T |g(s,Y_s,Z_s)|^2\,ds}\middle|\mathcal{F}_r\right),\\
&\leq \frac{1}{2\epsilon}\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\middle|\mathcal{F}_r\right) + \frac{\epsilon C_p^2}{2}\mathbb{E}\left(\int_t^T |g(s,Y_s,Z_s)|^2\,ds\middle|\mathcal{F}_r\right), \qquad (3.3)\\
&\leq \left(\frac{1}{2\epsilon} + \frac{\epsilon\lambda C_p^2}{2}\right)\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\middle|\mathcal{F}_r\right) + \frac{\epsilon\gamma C_p^2}{2}\mathbb{E}\left(\int_t^T |Z_s|^2\,ds\middle|\mathcal{F}_r\right) + \frac{\epsilon C_p^2}{2}\mathbb{E}\left(\int_t^T \eta_s ds\middle|\mathcal{F}_r\right),\\
&< \infty.
\end{aligned}$$

Applying Itô's formula to $|Y_t|^2$, $\forall t \in [0,T]$, and we using $(H3.6)$, $(H3.7)$, (3.3), $\mathbb{E}\left(\int_t^T \langle Y_s, Z_s\rangle dW_s\middle|\mathcal{F}_r\right) = 0$ and

$$\begin{aligned}
2\mathbb{E}\left(\sup_{t\leq u\leq T}\int_u^T \langle Y_s, Z_s\rangle dW_s\middle|\mathcal{F}_r\right) &\leq 2C_p\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|\left(\int_t^T |Z_s|^2\,ds\right)^{\frac{1}{2}}\middle|\mathcal{F}_r\right)\\
&\leq \frac{2}{\epsilon}\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\middle|\mathcal{F}_r\right) + \frac{\epsilon}{2}C_p^2\mathbb{E}\left(\int_t^T |Z_s|^2\,ds\middle|\mathcal{F}_r\right),
\end{aligned}$$

we have for any $0 \leq r \leq t \leq T$

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\middle|\mathcal{F}_r\right) + \mathbb{E}\left(\int_t^T |Z_s|^2\,ds\middle|\mathcal{F}_r\right)\\
&\leq \mathbb{E}\left((\Delta_t)\middle|\mathcal{F}_r\right) + \left(\frac{3}{\epsilon} + \epsilon\lambda C_p^2\right)\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2\middle|\mathcal{F}_r\right)\\
&+ \left(\frac{\epsilon}{2}C_p^2 + \frac{1}{2} + \left(1+\epsilon C_p^2\right)\gamma\right)\mathbb{E}\left(\int_t^T |Z_s|^2\,ds\middle|\mathcal{F}_r\right) + \epsilon C_p^2\mathbb{E}\left(\int_t^T \eta_s ds\middle|\mathcal{F}_r\right).
\end{aligned}$$

Since $0 \leq \gamma \leq \frac{1}{4}$ it is enough to take $C_p^2 = \frac{1}{\epsilon^2}$ and $\epsilon = \frac{4\lambda+9}{3}$, we get

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2 + \int_t^T |Z_s|^2\,ds\middle|\mathcal{F}_r\right) &\leq \frac{3\lambda+9}{4\lambda+9}\mathbb{E}\left(\sup_{t\leq u\leq T}|Y_u|^2 + \int_t^T |Z_s|^2\,ds\middle|\mathcal{F}_r\right)\\
&+ \mathbb{E}\left((\Delta_t)\middle|\mathcal{F}_r\right) + \frac{3}{4\lambda+9}\mathbb{E}\left(\int_t^T \eta_s ds\middle|\mathcal{F}_r\right),
\end{aligned}$$

since $0 < \frac{3\lambda+9}{4\lambda+9} < 1$, we obtain

$$\mathbb{E}\left(\sup_{t\leq u\leq T} |Y_u|^2 + \int_t^T |Z_s|^2 ds \middle| \mathcal{F}_r \right) \leq \frac{4\lambda+9}{\lambda}\left(\mathbb{E}\left((\Delta_t) \middle| \mathcal{F}_r \right) + \frac{3}{4\lambda+9}\mathbb{E}\left(\int_t^T \eta_s ds \middle| \mathcal{F}_r \right) \right),$$

from which together with Gronwall's Lemma, Fubini's theorem and Jensen's inequality, in view of the concavity condition of $\psi(\cdot)$ then there exists a constants $\bar{K} > 0$ such that for $0 \leq r \leq t \leq T$

$$\mathbb{E}\left(\sup_{t\leq u\leq T} |Y_u|^2 \middle| \mathcal{F}_r \right) + \mathbb{E}\left(\int_t^T |Z_s|^2 ds \middle| \mathcal{F}_r \right) \leq \left(\mathbb{E}\left(|\xi|^2 \middle| \mathcal{F}_r \right) + 2\int_t^T \psi\left(\mathbb{E}\left(|Y_s|^2 \middle| \mathcal{F}_r \right) \right) ds + \frac{1}{\delta}\mathbb{E}\left(\int_t^T |\sigma_s| ds \middle| \mathcal{F}_r \right) + 2\mathbb{E}\left(\int_t^T \eta_s ds \middle| \mathcal{F}_r \right) \right) \bar{K} \exp\left(\bar{K}T\right)$$

Hence the required result. ■

3.1.2 Existence and uniqueness result.

Now we can give proof of Theorem 3.1, let us start with studying the uniqueness part.

Proof of uniqueness.

Proof. Suppose that f and g satisfies the assumption $(H3.1) - (H3.5)$. Let (Y_t^1, Z_t^1) and (Y_t^2, Z_t^2) be two solutions of the BDSDE with parameters (ξ, T, f, g). Then $\left(\bar{Y}_t, \bar{Z}_t\right) = (Y_t^1 - Y_t^2, Z_t^1 - Z_t^2)$ is a solution to the following BDSDE

$$\bar{Y}_t = \int_t^T \bar{f}\left(s, \bar{Y}_s, \bar{Z}_s\right) ds + \int_t^T \bar{g}\left(s, \bar{Y}_s, \bar{Z}_s\right) dB_s - \int_t^T \bar{Z}_s dW_s, \qquad t \in [0, T],$$

where for each $\left(\bar{Y}_t, \bar{Z}_t\right) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$

$$\begin{cases} \bar{f}\left(t, \bar{Y}_t, \bar{Z}_t\right) = f\left(t, \bar{Y}_t + Y_t^2, \bar{Z}_t + Z_t^2\right) - f\left(t, Y_t^2, Z_t^2\right), \\ \bar{g}\left(t, \bar{Y}_t, \bar{Z}_t\right) = g\left(t, \bar{Y}_t + Y_t^2, \bar{Z}_t + Z_t^2\right) - g\left(t, Y_t^2, Z_t^2\right). \end{cases}$$

It follows from $(H3.2)$ and $(H3.3)\,(i)$ that $dP \times dt - a.e.$,

$$\begin{aligned} \langle \bar{Y}, \bar{f}\left(t, \bar{Y}, \bar{Z}\right)\rangle &= \langle \bar{Y}, f\left(t, \bar{Y} + Y^2, \bar{Z} + Z^2\right) - f\left(t, Y^2, Z^2\right)\rangle \\ &\le k\left(\left|\bar{Y}\right|^2\right) + c\left|\bar{Y}\right|\left|\bar{Z}\right|, \end{aligned}$$

then the assumption $(H3.6)$ is satisfied for the generator $\bar{f}\left(s, \bar{Y}_s, \bar{Z}_s\right)$ of BDSDE with $\psi\left(u\right) = k\left(u\right),\ \lambda = c,\ \sigma_t = 0.$

It follows from $(H3.3)\,(ii)$ that $dP \times dt - a.e.$,

$$\left|\bar{g}\left(t, \bar{Y}, \bar{Z}\right)\right|^2 \le 2c^2\left|\bar{Y}\right|^2 + 2\alpha^2\left|\bar{Z}\right|^2,$$

then the assumption $(H3.7)$ is satisfied for the generator $\bar{g}\left(s, \bar{Y}_s, \bar{Z}_s\right)$ of BDSDE with $\gamma = 2\alpha^2$ and $\eta_t = 0.$

Thus, it follow from Proposition 3.1 (i) that there exists a constant $K > 0$ depending only on δ, λ and γ such that, for each $0 \le t \le T$, we have

$$\mathbb{E}\left(\sup_{0\le s\le T}\left|\bar{Y}_s\right|^2\right) + \mathbb{E}\left(\int_t^T \left|\bar{Z}_s\right|^2 ds\right) \le C\int_t^T \left(k\left(\mathbb{E}\sup_{s\le u\le T}\left|\bar{Y}_u\right|^2\right)\right) ds,$$

where $C = 2K\exp\left(KT\right)$ in view of $\int_{0^+} k^{-1}\left(u\right)du = \infty$, Bihari's inequality yields that, $\forall t \in [0, T]$

$$\mathbb{E}\left(\sup_{0\le s\le T}\left|\bar{Y}_s\right|^2 + \int_t^T \left|\bar{Z}_s\right|^2 ds\right) = 0.$$

The proof of the uniqueness part of Theorem 3.1 is then complete. ∎

Proof of Existence.

Let ϕ be a function of $C^{\infty}\left(\mathbb{R}^k,\mathbb{R}^+\right)$ with the closed unit as compact support, and satisfies $\int_{\mathbb{R}^k}\phi(v)\,dv=1$. For each $n\geq 1$ and each $(\omega,t,Y)\in\Omega\times[0,T]\times\mathbb{R}^k$, we set

$$\begin{aligned} f_n(t,Y_t,V_t) &= n^k f(t,Y_t,V_t)*\phi(nY_t)\,, \\ &= n^k\int_{\mathbb{R}^k} f(t,v,V_t)\,\phi\left(n\left(Y_t-v\right)\right)dv. \end{aligned} \tag{3.4}$$

Then f_n is an $(\mathcal{F}_t)$-progressively measurable process for each $Y\in\mathbb{R}^k$ and

$$\begin{aligned} f_n(t,Y_t,V_t) &= \int_{\mathbb{R}^k} f\left(t,Y_t-\frac{v}{n},V_t\right)\phi(v)\,dv, \\ &= \int_{\{v:|v|\leq 1\}} f\left(t,Y_t-\frac{v}{n},V_t\right)\phi(v)\,dv. \end{aligned} \tag{3.5}$$

Let us turn to the existence part. The proof will be split into three lemmas and after the proof of Theorem 3.1.

Lemma 3.1 *Let f and g satisfies the hypothesis **(H3.1)–(H3.5)**, $V\in\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$ and $\xi\in\mathbb{L}^2\left(\mathcal{F}_T,\mathbb{R}^k\right)$, if there exists a positive constant β such that*

$$dP-a.s.,\left|\xi\right|\leq\beta\quad dP\times dt-a.e.,\left|g(t,\omega,0,0)\right|\leq\beta\quad\left|f(t,\omega,0,0)\right|\leq\beta\quad and\quad\left|V_t\right|\leq\beta. \tag{3.6}$$

Then there exists a unique solution to the following BDSDE:

$$Y_t=\xi+\int_t^T f(s,Y_s,V_s)\,ds+\int_t^T g(s,Y_s,V_s)\,d\overleftarrow{B}_s-\int_t^T Z_s dW_s\quad t\in[0,T]\,. \tag{3.7}$$

Proof. It follows from $(H3.3)\,(i)$, $(H3.4)$ and (3.6) that, for each $Y\in\mathbb{R}^k$ $dP\times dt-a.e.$,

$$\left|f(s,Y_s,V_s\right|\leq c\beta+\beta+\varphi\left(|Y_s|\right). \tag{3.8}$$

Thus, checked from (3.4) that for each $n\geq 1$, $f_n(t,Y_t,V_t)$ is locally lipschitz in Y uniformly with respect to (t,ω). Furthermore, for each $n\geq 1$ and $Y\in\mathbb{R}^k$, it follows from (3.5) and

(3.8) that $dP\times dt-a.e.$,

$$\begin{aligned}\left|f_n(t,Y_t,V_t)\right| &= \Big|\int_{\{v:|v|\leq 1\}} f(t,Y_t-\frac{v}{n},V_t)\phi\left(v\right)dv\Big|,\\ &\leq \left(c\beta+\beta+\varphi\left(|Y_t|+1\right)\right)\int_{\{v:|v|\leq 1\}}\phi\left(v\right)dv = c\beta+\beta+\varphi\left(|Y_t|+1\right). \end{aligned} \tag{3.9}$$

Now, for some large enough integer $u>0$ which will be chosen later, let ρ_u be a smooth function such that $0\leq\rho_u\leq 1$, $\rho_u(Y_t)=1$ for $|Y_t|\leq u$ and $\rho_u(Y_t)=0$ as soon as $|Y_t|\geq u+1$. Then for each $n\geq 1$, the function $\rho_u\left(Y_t\right)f_n\left(t,Y_t,V_t\right)$ is globally lipschitz in Y, uniformly with respect to (t,ω).

Thus, from Pardoux-Peng [24], we know that for each $n\geq 1$, the following BDSDE has a unique solution $(Y_t^n,Z_t^n)_{t\in[0,T]}$:

$$Y_t^n=\xi+\int_t^T\rho_u\left(Y_s^n\right)f_n\left(s,Y_s^n,V_s\right)ds+\int_t^T g\left(s,Y_s^n,V_s\right)d\overleftarrow{B}_s-\int_t^T Z_s^n dW_s,\quad 0\leq t\leq T. \tag{3.10}$$

It follows from $(H3.2)$ and (3.5) that for each $n\geq 1$ and $(Y_t^1,Y_t^2)\in\mathbb{R}^{2k}$, $dP\times dt-a.e.$,

$$\langle Y_t^1-Y_t^2,f_n\left(t,Y_t^1,V_t\right)-f_n\left(t,Y_t^2,V_t\right)\rangle\leq\int_{\{v:|v|\leq 1\}}k\left(\left|Y_t^1-Y_t^2\right|^2\right)\phi\left(v\right)dv=k\left(\left|Y_t^1-Y_t^2\right|^2\right). \tag{3.11}$$

For each $n\geq 1$ and $Y_t\in\mathbb{R}^k$, combing (3.9) and (3.11) yields that $dP\times dt-a.e.$,

$$\begin{aligned}\langle Y_t,\rho_u\left(Y_t\right)f_n\left(t,Y_t,V_t\right)\rangle &= \rho_u\left(Y_t\right)\langle Y_t,f_n\left(t,Y_t,V_t\right)\rangle,\\ &\leq k\left(|Y_t|^2\right)+|Y_t|\left(c\beta+\beta+\varphi\left(1\right)\right),\end{aligned}$$

Then the assumption $(H3.6)$ is satisfied for the generator $\rho_u\left(Y_t^n\right)f_n\left(t,Y_t^n,V_t\right)$ of BDSDE (3.10) with $\psi\left(u\right)=k\left(u\right)$, $\lambda=0$, $\sigma_t=c\beta+\beta+\varphi\left(1\right)$.

It follows from $(H3.3)\,(ii)$ that $dP\times dt-a.e.$,

$$\begin{aligned}\left|g\left(t,Y_t^n,V_t\right)\right|^2 &\leq 2\left|g\left(t,Y_t^n,V_t\right)-g\left(t,0,0\right)\right|^2+2\left|g\left(t,0,0\right)\right|^2,\\ &\leq 4c^2\left|Y_t^n\right|^2+4\alpha^2\left|V_t\right|^2+2\left|g\left(t,0,0\right)\right|^2.\end{aligned}$$

Then the assumption $(H3.7)$ is satisfied for the generator $g\left(t, Y_t^n, V_t\right)$ of BDSDE (3.10) with $\lambda = 4c^2$, $\gamma = 4\alpha^2$ and $\eta_t = 2|g\left(t,\omega,0,0\right)|^2$.

Thus, it follow from Proposition 3.1 (ii) with $\delta = 1$ that there exists a constant $\bar{K} > 0$ depending only on δ, λ and γ such that, for each $0 \leq r \leq t \leq T$, we have

$$
\begin{aligned}
&\mathbb{E}\left(\left|Y_t^n\right|^2 \middle| \mathcal{F}_r\right) + \mathbb{E}\left(\int_t^T \left|Z_s^n\right|^2 ds \middle| \mathcal{F}_r\right) \\
&\leq \left(\mathbb{E}\left(|\xi|^2 \middle| \mathcal{F}_r\right) + 2\int_t^T k\left(\mathbb{E}\left(\left|Y_s^n\right|^2 \middle| \mathcal{F}_r\right)\right) ds + \left(c\beta + \beta + \varphi\left(1\right)\right)^2 T \right.\\
&\left. + 4\mathbb{E}\left(\int_t^T |g(s,\omega,0,0)|^2 ds \middle| \mathcal{F}_r\right)\right) \bar{K} \exp\left(\bar{K}T\right).
\end{aligned}
$$

Note $\bar{\theta} = \bar{K}\exp\left(\bar{K}T\right)$ and using the (3.6), we get

$$
\begin{aligned}
&\mathbb{E}\left(\left|Y_t^n\right|^2 \middle| \mathcal{F}_r\right) + \mathbb{E}\left(\int_t^T \left|Z_s^n\right|^2 ds \middle| \mathcal{F}_r\right) \\
&\leq \bar{\theta}\beta^2 + 2\bar{\theta}\int_t^T k\left(\mathbb{E}\left(\left|Y_s^n\right|^2 \middle| \mathcal{F}_r\right)\right) ds + \bar{\theta}\left(c\beta + \beta + \varphi\left(1\right)\right)^2 T + 4\bar{\theta}\beta^2 T.
\end{aligned}
$$

Furthermore, since $k\left(\cdot\right)$ is a nondecreasing and concave function with $k\left(0\right) = 0$ it increases at most linearly, i.e., there exists $A > 0$ such that $k\left(x\right) \leq A(x+1)$ for each $x \geq 0$, yields that

$$
\begin{aligned}
\mathbb{E}\left(\left|Y_t^n\right|^2 \middle| \mathcal{F}_r\right) + \mathbb{E}\left(\int_t^T \left|Z_s^n\right|^2 ds \middle| \mathcal{F}_r\right) &\leq \bar{\theta}\beta^2\left(4T+1\right) + 2\bar{\theta}A + \bar{\theta}\left(c\beta + \beta + \varphi\left(1\right)\right)^2 T \\
&+ 2\bar{\theta}A\int_t^T \mathbb{E}\left(\left|Y_s^n\right|^2 \middle| \mathcal{F}_r\right) ds.
\end{aligned}
$$

By Gronwall's lemma and with $r = t$, yields that

$$
\left|Y_t^n\right|^2 + \mathbb{E}\left(\int_t^T \left|Z_s^n\right|^2 ds\right) \leq u^2.
$$

where $u^2 = \left(\bar{\theta}\beta^2\left(4T+1\right) + 2A\bar{\theta} + \bar{\theta}\left(c\beta + \beta + \varphi\left(1\right)\right)^2 T\right)\exp\left(2A\bar{\theta}T\right)$. By the previous inequality, yields that for each $n \geq 1$ and $\forall t \in [0,T]$

$$\begin{cases} |Y_t^n|^2 \leq u^2, \\ \mathbb{E}\left(\int_0^T |Z_s^n|^2 \, ds\right) \leq u^2. \end{cases} \tag{3.12}$$

By (3.10) and (3.12), we can conclude that $(Y_t^n, Z_t^n)_{t\in[0,T]}$ solves the following BDSDE:

$$Y_t^n = \xi + \int_t^T f_n\left(s, Y_s^n, V_s\right) ds + \int_t^T g\left(s, Y_s^n, V_s\right) d\overleftarrow{B}_s - \int_t^T Z_s^n dW_s, \quad 0 \leq t \leq T. \tag{3.13}$$

In the sequel, we shall show that $\left((Y_t^n, Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ is Cauchy sequence in the space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right) \times \mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$.

In fact, for each $n \geq 1$ and $m \geq 1$, let $\Delta Y_t^{n,m} = Y_t^n - Y_t^m$ and $\Delta Z_t^{n,m} = Z_t^n - Z_t^m$. Then for each $0 \leq t \leq T$

$$\Delta Y_t^{n,m} = \int_t^T \Delta f^{n,m}\left(s, \Delta Y_s^{n,m}, V_s\right) ds + \int_t^T \Delta g^{n,m}\left(s, \Delta Y_s^{n,m}, V_s\right) dB_s - \int_t^T \Delta Z_s^{n,m} dW_s, \tag{3.14}$$

where

$$\begin{cases} \Delta f^{n,m}\left(s, \Delta Y_s^{n,m}, V_s\right) = f_n\left(s, \Delta Y_s^{n,m} + Y_s^m, V_s\right) - f_m\left(s, Y_s^m, V_s\right), \\ \Delta g^{n,m}\left(s, \Delta Y_s^{n,m}, V_s\right) = g\left(s, \Delta Y_s^{n,m} + Y_s^m, V_s\right) - g\left(s, Y_s^m, V_s\right). \end{cases}$$

It follows from (3.11) that for each $\Delta Y_t^{n,m} \in \mathbb{R}^k$, $dP \times dt - a.e.$,

$$\langle \Delta Y_t^{n,m}, \Delta f^{n,m}(t, \Delta Y_t^{n,m}, V_t)\rangle \leq k\left(|\Delta Y_t^{n,m}|^2\right) + |\Delta Y_t^{n,m}| \left|f_n\left(t, Y_t^m, V_t\right) - f_m\left(t, Y_t^m, V_t\right)\right|.$$

Then the assumption $(H3.6)$ is satisfied for the generator $\Delta f^{n,m}\left(t, \Delta Y_t^{n,m}, V_t\right)$ of BDSDE (3.14) with $\psi(u) = k(u)$, $\lambda = 0$, $\sigma_t = |f_n\left(t, Y_t^m, V_t\right) - f_m\left(t, Y_t^m, V_t\right)|$.

It follows from $(H3.3)\,(ii)$ that $dP \times dt - a.e.$,

$$\left|\Delta g^{n,m}\left(t, \Delta Y_t^{n,m}, V_t\right)\right|^2 \leq c\left|\Delta Y_t^{n,m}\right|^2.$$

Then the assumption $(H3.7)$ is satisfied for the generator $\Delta g^{n,m}\left(t, \Delta Y_t^{n,m}, V_t\right)$ of BDSDE

(3.14) with $\lambda = c$, $\gamma = 0$ and $\eta_t = 0$.

Thus, it follow from Proposition 3.1 (i) with $\delta = 1$ that there exists a constant $K > 0$ depending only on δ, λ and γ such that, for each $0 \leq t \leq T$

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|\Delta Y_s^{n,m}|^2\right) + \mathbb{E}\left(\int_t^T |\Delta Z_s^{n,m}|^2\, ds\right) \leq 2\theta \int_t^T k\left(\mathbb{E}\sup_{s\leq u\leq T}|\Delta Y_u^{n,m}|^2\right) ds \qquad (3.15)$$
$$+ \theta\mathbb{E}\int_t^T \left|f_n\left(s, Y_s^m, V_s\right) - f_m\left(s, Y_s^m, V_s\right)\right|^2 ds,$$

where $\theta = K\exp\left(K\left(T-t\right)\right)$.

On the other hand, it follows from (3.5) that, for each $n, m \geq 1$, $s \in [0,T]$ and each $\Delta Y^{n,m} \in \mathbb{R}^k$, $dP \times dt - a.e.$,

$$\left|f_n\left(t, Y_t^m, V_t\right) - f_m\left(t, Y_t^m, V_t\right)\right| \leq \int_{\{v:|v|\leq 1\}} \left|f\left(t, Y_t^m - \frac{v}{n}, V_t\right) - f\left(t, Y_t^m - \frac{v}{m}, V_t\right)\right| \phi\left(v\right) dv,$$

and also from (3.8), we get

$$\left|f\left(t, Y_t^m - \frac{v}{n}, V_t\right) - f\left(t, Y_t^m - \frac{v}{m}, V_t\right)\right| \leq 2\left(\varphi\left(u+1\right) + c\beta + \beta\right)$$
$$< \infty.$$

Using the continuity of f in y, we have

$$\lim_{n,m\to\infty}\left|f\left(t, Y_t^m - \frac{v}{n}, V_t\right) - f\left(t, Y_t^m - \frac{v}{m}, V_t\right)\right| = 0,$$

applying Lebesgue's dominated convergence theorem, we get

$$\lim_{n,m\to\infty}\left|f_n\left(t, Y_t^m, V_t\right) - f_m\left(t, Y_t^m, V_t\right)\right| = 0.$$

On the other hand, we obtain $dP \times dt - a.e.$,

$$\begin{aligned}\left|f_n(t,Y_t^m,V_t)-f_m(t,Y_t^m,V_t)\right| &\le \int_{\{v:|v|\le 1\}} 2\big(\varphi\big(u+1\big)+c\beta+\beta\big)\phi\left(v\right)dv,\\ &\le 2\big(\varphi\big(u+1\big)+c\beta+\beta\big)<\infty,\end{aligned}$$

applies again Lebesgue's dominated convergence theorem,yields that

$$\lim_{n,m\to\infty}\mathbb{E}\int_t^T \left|f_n\left(s,Y_s^m,V_s\right)-f_m\left(s,Y_s^m,V_s\right)\right|^2 ds=0. \tag{3.16}$$

Now, taking the lim sup in (3.15) and by Fatou's lemma, monotonicity and continuity of $k\left(\cdot\right)$ and (3.16) , we get

$$\begin{aligned}&\lim_{n,m\to\infty}\sup\left(\mathbb{E}\left(\sup_{0\le s\le T}|\Delta Y_s^{n,m}|^2\right)+\mathbb{E}\left(\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right)\right)\\ &\le 2\theta\int_t^T k\left(\lim_{n,m\to\infty}\sup\mathbb{E}\left(\sup_{s\le u\le T}|\Delta Y_u^{n,m}|^2\right)\right)ds.\end{aligned}$$

Thus, in view of $\int_{0^+}k^{-1}\left(u\right)du=\infty$, Bihari's inequality yields that, for each $0\le t\le T$

$$\lim_{n,m\to\infty}\mathbb{E}\left(\sup_{0\le s\le T}|\Delta Y_s^{n,m}|^2\right)+\mathbb{E}\left(\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right)=0,$$

which means that $\left((Y_t^n,Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ is Cauchy sequence in the space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$.

Let $(Y_t,Z_t)_{t\in[0,T]}$ be the limit process of the sequence $\left((Y_t^n,Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ in the process space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$.

On one hand, using (3.5), (3.9) and (3.12), we have

$$\begin{aligned}\left|f_n(s,Y_s^n,V_s)\right| &\le c\beta+\beta+\varphi\left(|Y^n|+1\right),\\ &\le c\beta+\beta+\varphi\left(u+1\right)<\infty,\end{aligned}$$

by definition of f_n and applying $(H3.1)$, we have that f_n converge simply to f. Thus by

Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_0^T \left|f_n\left(s,Y_s^n,V_s\right)-f\left(s,Y_s,V_s\right)\right|ds=0.$$

Other hand, from the continuity properties of the stochastic integral, it follows that

$$\begin{cases}\sup_{0\leq t\leq T}\left|\int_t^T g\left(s,Y_s^n,V_s\right)d\overleftarrow{B}_s-\int_t^T g\left(s,Y_s,V_s\right)d\overleftarrow{B}_s\right|\to 0,\\ \sup_{0\leq t\leq T}\left|\int_t^T Z_s^n dW_s-\int_t^T Z_s dW_s\right|\to 0,\quad \text{as } n\to\infty \text{ in probability.}\end{cases}$$

from wich it follow that Y^n converge uniformly in t to Y i.e., $\lim_{n\to\infty}\left(\sup_{0\leq t\leq T}\left|Y_t^n-Y_t\right|\right)=0$. Finally, we pass to the limit $n\to\infty$ in (3.13), we deduce that $(Y_t,Z_t)_{t\in[0,T]}$ solve BDSDE (3.7). ■

Lemma 3.2 *Let f and g satisfies the hypothesis **(H3.1)–(H3.5)**, $V\in\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$ and $\xi\in\mathbb{L}^2\left(\mathcal{F}_T,\mathbb{R}^k\right)$, if there exists a positive constant β such that*

$$dP-a.s.,\quad \left|\xi\right|\leq\beta\quad dP\times dt-a.e.,\left|g\left(t,\omega,0,0\right)\right|\leq\beta\quad and\quad \left|f\left(t,\omega,0,0\right)\right|\leq\beta. \tag{3.17}$$

Then there exists a unique solution to the BDSDE (3.7).

Proof. In this lemma, we will eliminate the bounded condition with respect to the processes $(V_t)_{t\in[0,T]}$ in Lemma 3.1. For each $n\geq 1$ and $Z\in\mathbb{R}^{k\times d}$, denote $q_n\left(Z\right)=\frac{Z\times n}{\sup(|Z|,n)}$, then $|q_n(Z)|=\left|\frac{Z\times n}{\sup(|Z|,n)}\right|\leq\inf\left(|Z|,n\right)$. It follows from Lemma 3.1, that for each $n\geq 1$, there exists a solution $(Y_t^n,Z_t^n)_{t\in[0,T]}$ to the following BDSDE

$$Y_t^n=\xi+\int_t^T f\left(s,Y_s^n,q_n\left(V_s\right)\right)ds+\int_t^T g\left(s,Y_s^n,q_n\left(V_s\right)\right)d\overleftarrow{B}_s-\int_t^T Z_s^n dW_s,\quad 0\leq t\leq T. \tag{3.18}$$

In the sequel, we shall show that $\left((Y_t^n,Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in the space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$.

In fact, for each $n\geq 1$ and $m\geq 1$, let $\Delta Y_t^{n,m}=Y_t^n-Y_t^m$ and $\Delta Z_t^{n,m}=Z_t^n-Z_t^m$.

Then for each $0 \leq t \leq T$

$$\Delta Y_t^{n,m} = \int_t^T \Delta f^{n,m}\left(s, \Delta Y_s^{n,m}, V_s\right) ds + \int_t^T \Delta g^{n,m}\left(s, \Delta Y_s^{n,m}, V_s\right) d\overleftarrow{B}_s - \int_t^T \Delta Z_s^{n,m} dW_s, \tag{3.19}$$

where

$$\begin{cases} \Delta f^{n,m}\left(s, \Delta Y_s^{n,m}, V_s\right) = f\left(s, \Delta Y_s^{n,m} + Y_s^m, q_n\left(V_s\right)\right) - f\left(s, Y_s^m, q_m\left(V_s\right)\right), \\ \Delta g^{n,m}\left(s, \Delta Y_s^{n,m}, V_s\right) = g\left(s, \Delta Y_s^{n,m} + Y_s^m, q_n\left(V_s\right)\right) - g\left(s, Y_s^m, q_m\left(V_s\right)\right). \end{cases}$$

$(H3.6)$ and $(H3.7)$ is satisfied for the generator $\Delta f^{n,m}\left(t, \Delta Y_t^{n,m}, V_t\right)$ with $\psi(u) = k(u)$, $\lambda = 0$, $\sigma_t = \left|f\left(t, Y^m, q_n\left(V_t\right)\right) - f\left(t, Y^m, q_m\left(V_t\right)\right)\right|$ respectively $\Delta g^{n,m}\left(t, \Delta Y_t^{n,m}, V_t\right)$ with $\gamma = \alpha$ and $\eta_t = 0$ of BDSDE (3.19).

Indeed by $(H3.2)$, we get

$$\left\langle \Delta Y_t^{n,m}, \Delta f\left({}^{n,m}t, \Delta Y_t^{n,m}, V_t\right)\right\rangle \leq k\left(\left|\Delta Y_t^{n,m}\right|^2\right) + \left|\Delta Y_t^{n,m}\right| \left|f\left(t, Y^m, q_n\left(V_t\right)\right) - f\left(t, Y^m, q_m\left(V_t\right)\right)\right|.$$

and by $(H3.3)\,(ii)$, we have

$$\left|\Delta g^{n,m}\left(t, \Delta Y_t^{n,m}, V_t\right)\right|^2 \leq 2c^2 \left|\Delta Y_t^{n,m}\right|^2 + 2\alpha^2 \left|q_n(V_t) - q_m(V_t)\right|^2.$$

Thus, it follow from Proposition 3.1 (i) with $\delta = 1$ that there exists a constant $K > 0$ depending only on δ, λ and γ such that, for each $0 \leq t \leq T$

$$\begin{aligned} &\mathbb{E}\left(\sup_{0 \leq s \leq T} \left|\Delta Y_s^{n,m}\right|^2\right) + \mathbb{E}\left(\int_t^T \left|\Delta Z_s^{n,m}\right|^2 ds\right) \\ &\leq \left(2K \int_t^T k\left(\mathbb{E}\left|\Delta Y_s^{n,m}\right|^2\right) ds + K\mathbb{E} \int_t^T \left|f\left(s, Y_s^m, q_n\left(V_s\right)\right) - f\left(s, Y_s^m, q_m\left(V_s\right)\right)\right|^2 ds \right. \\ &\left. + 2K\alpha^2 \mathbb{E} \int_t^T \left|q_n(V_s) - q_m(V_s)\right|^2 ds\right) \exp\left(K\left(T - t\right)\right), \end{aligned}$$

using $(H3.3)\,(i)$ and $\theta = K\exp\left(K\left(T-t\right)\right)$, we get

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2\right)+\mathbb{E}\left(\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)$$
$$\leq 2\theta\int_t^T k\left(\mathbb{E}\left|\Delta Y_s^{n,m}\right|^2\right)ds+\theta\left(c+2\alpha^2\right)\mathbb{E}\int_t^T\left|q_n\left(V_s\right)-q_m\left(V_s\right)\right|^2ds. \tag{3.20}$$

since $k\left(x\right)\leq A\left(1+x\right)$, we obtain

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2+\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)$$
$$\leq 2\theta AT+2\theta A\int_t^T\mathbb{E}\left(\sup_{s\leq u\leq T}\left|\Delta Y_u^{n,m}\right|^2\right)ds+\theta\left(c+2\alpha^2\right)\mathbb{E}\int_t^T\left|q_n\left(V_s\right)-q_m\left(V_s\right)\right|^2ds.$$

Applying Gronwall's Lemma and $\left(a-b\right)^2\leq a^2+b^2$, yields that for each $t\in[0,T]$ and each $n,m\geq 1$

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2+\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)$$
$$\leq\left(2\theta AT+\theta\left(c+2\alpha^2\right)\mathbb{E}\int_t^T\left(\left|q_n\left(V_s\right)\right|^2+\left|q_m\left(V_s\right)\right|^2\right)ds\right)\exp\left(2\theta AT\right),$$
$$\leq\left(2\theta AT+2\theta\left(c+2\alpha^2\right)\mathbb{E}\int_0^T\left|V_s\right|^2ds\right)\exp\left(2\theta AT\right).$$

By taking the $\limsup$ in (3.20), we have

$$\lim\sup_{n,m\to\infty}\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2+\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)$$
$$\leq\lim\sup_{n,m\to\infty}\left(2\theta\int_t^T k\left(\mathbb{E}\left|\Delta Y_s^{n,m}\right|^2\right)ds+\theta\left(c+2\alpha^2\right)\mathbb{E}\int_t^T\left|q_n(V_s)-q_m(V_s)\right|^2ds\right),$$

by Fatou's lemma, monotonicity and continuity of $k\left(\cdot\right)$, we have

$$\lim\sup_{n,m\to\infty}\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2+\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)$$
$$\leq 2\theta\int_t^T k\left(\lim\sup_{n,m\to\infty}\mathbb{E}\left|\Delta Y_s^{n,m}\right|^2\right)ds+\theta\left(c+2\alpha^2\right)\mathbb{E}\int_t^T\lim\sup_{n,m\to\infty}\left|q_n\left(V_s\right)-q_m\left(V_s\right)\right|^2ds.$$

since

$$\mathbb{E}\int_t^T \lim_{n,m\to\infty}\sup |q_n(V_s) - q_m(V_s)|^2 \, ds = 0.$$

Thus, in view of $\int_{0+} k^{-1}(u)\,du = \infty$, Bihari's inequality yields that, for each $0 \le t \le T$

$$\lim_{n,m\to\infty}\sup \mathbb{E}\left(\sup_{0\le s\le T} |\Delta Y_s^{n,m}|^2 + \int_t^T |\Delta Z_s^{n,m}|^2 \, ds\right) = 0.$$

We know that $\left((Y_t^n, Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ is Cauchy sequence in the process space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right) \times \mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$.

Let $(Y_t, Z_t)_{t\in[0,T]}$ be the limit process of the sequence $\left((Y_t^n, Z_t^n)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ in the process space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right) \times \mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$.

Applying $(H3.1)$, $(H3.3)\,(i)$, $(H3.4)$, (3.17) and Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_0^T \left|f(s, Y_s^n, q_n(V_s)) - f(s, Y_s, V_s)\right| ds = 0,$$

from wich it follow that Y^n converge uniformly in t to Y i.e., $\lim_{n\to\infty}\left(\sup_{0\le t\le T}|Y_t^n - Y_t|\right) = 0$. Finally, we pass to the limit $n\to\infty$ in (3.18), we deduce that $(Y_t, Z_t)_{t\in[0,T]}$ solve BDSDE (3.7). ∎

Lemma 3.3 *Let f and g satisfies the hypothesis **(H3.1)–(H3.5)** and $\xi \in \mathbb{L}^2\left(\mathcal{F}_T, \mathbb{R}^k\right)$, if there exists a positive constant β such that*

$$dP - a.s., \left|\xi\right| \le \beta \quad dP\times dt - a.e., \left|g(t,\omega,0,0)\right| \le \beta \quad and \quad \left|f(t,\omega,0,0)\right| \le \beta. \qquad (3.21)$$

Then there exists a unique solution to the BDSDE $\left(E^{\xi,f,g}\right)$.

Proof. By Lemma 3.2, we can construct the iterative sequence. Let us set as usual $(Y_t^0, Z_t^0) = (0,0)$ and define recursively, for each $n \ge 1$

$$Y_t^n = \xi + \int_t^T f\left(s, Y_s^n, Z_s^{n-1}\right) ds + \int_t^T g\left(s, Y_s^n, Z_s^{n-1}\right) d\overleftarrow{B}_s - \int_t^T Z_s^n dW_s, \qquad t\in[0,T]. \qquad (3.22)$$

It follows from $(H3.2)$ and $(H3.3)\,(i)$ that $dP\times dt-a.e.$,

$$\begin{aligned}\langle Y_s^n, f\left(s,Y_s^n,Z_s^{n-1}\right)\rangle &= \langle Y_s^n, f\left(s,Y_s^n,Z_s^{n-1}\right)-f\left(s,0,Z_s^{n-1}\right)+f\left(s,0,Z_s^{n-1}\right)\rangle,\\ &\leq k\left(\left|Y_s^n\right|^2\right)+\left|Y_s^n\right|\left(c\left|Z_s^{n-1}\right|+\left|f\left(s,0,0\right)\right|\right),\end{aligned}$$

then the assumption $(H3.6)$ is satisfied for the generator $f\left(s,Y_s^n,Z_s^{n-1}\right)$ of BDSDE (3.22) with $\psi\left(u\right)=k\left(u\right),\ \lambda=0,\ \sigma_t=c\left|Z_t^{n-1}\right|+\left|f\left(t,0,0\right)\right|.$

It follows from $(H3.3)\,(ii)$ that $dP\times dt-a.e.$,

$$\left|g\left(t,Y_t^n,Z_t^{n-1}\right)\right|^2\leq 4c^2\left|Y_t^n\right|^2+4\alpha^2\left|Z_t^{n-1}\right|^2+2\left|g\left(t,0,0\right)\right|^2,$$

then the assumption $(H3.7)$ is satisfied for the generator $g\left(t,Y_t^n,Z_t^{n-1}\right)$ of BDSDE (3.22) with $\gamma=4\alpha^2$, $\lambda=4c^2$ and $\eta_t=2\left|g(t,0,0)\right|^2.$

Thus, it follow from Proposition 3.1 (i) that there exists a constant $K>0$ depending only on δ,λ and γ such that, for each $0\leq t\leq T$

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^n\right|^2\right)+\mathbb{E}\left(\int_t^T\left|Z_s^n\right|^2ds\right)\\ &\leq\left(K\mathbb{E}\left|\xi\right|^2+2K\int_t^T k\left(\mathbb{E}\left(\sup_{r\in[s,T]}\left|Y_r^n\right|^2\right)\right)ds+\frac{K}{\delta}\mathbb{E}\int_t^T\left(c\left|Z_s^{n-1}\right|+\left|f(s,0,0)\right|\right)^2ds\right.\\ &\left.+2K\mathbb{E}\int_t^T\left|g(s,0,0)\right|^2ds\right)\exp\left(K(T-t)\right).\end{aligned}$$

By $\theta=K\exp\left(K\left(T-t\right)\right),$ we note $H(t)=\theta\left(\mathbb{E}\left|\xi\right|^2+\frac{2}{\delta}\mathbb{E}\int_t^T\left|f\left(s,0,0\right)\right|^2ds+2\mathbb{E}\int_t^T\left|g\left(s,0,0\right)\right|^2ds\right).$
Using (3.21), we have $H(t)\leq\theta\beta^2\left(1+\frac{2T}{\delta}+2T\right)=\theta h.$ Therefore

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^n\right|^2\right)+\mathbb{E}\left(\int_t^T\left|Z_s^n\right|^2ds\right)\leq\theta h+2\theta\int_t^T k\left(\mathbb{E}\left(\sup_{r\in[s,T]}\left|Y_r^n\right|^2\right)\right)ds+\frac{2\theta c^2}{\delta}\mathbb{E}\int_t^T\left|Z_s^{n-1}\right|^2ds.$$

Since $k(x) \leq A(1+x)$, we get

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2\right)+\mathbb{E}\left(\int_t^T|Z_s^n|^2\,ds\right)$$
$$\leq \theta h+2A\theta T+2A\theta\int_t^T\mathbb{E}\left(\sup_{r\in[s,T]}|Y_r^n|^2\right)ds+\frac{2\theta c^2}{\delta}\mathbb{E}\int_t^T\left|Z_s^{n-1}\right|^2ds.$$

Let us set $\vartheta_1=\max\left\{T-\frac{\ln 2}{K},T-\frac{\ln 2}{4KA},0\right\}$. Then for each $t\in[\vartheta_1,T]$,we have $\exp(K(T-t))\leq 2$, thus $\theta\leq 2K$ and

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2\right)+\mathbb{E}\left(\int_t^T|Z_s^n|^2\,ds\right)$$
$$\leq 2Kh+4KAT+4AK\int_t^T\mathbb{E}\left(\sup_{r\in[s,T]}|Y_r^n|^2\right)ds+\frac{4Kc^2}{\delta}\mathbb{E}\int_t^T\left|Z_s^{n-1}\right|^2ds.$$

we take $\delta=16Kc^2$, obtain

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2+\int_t^T|Z_s^n|^2\,ds\right)\leq 2Kh+4KAT+4AK\int_t^T\mathbb{E}\left(\sup_{r\in[s,T]}|Y_r^n|^2\right)ds+\frac{1}{4}\mathbb{E}\int_t^T\left|Z_s^{n-1}\right|^2ds.$$

Applying Gronwall's lemma yields that for each $t\in[\vartheta_1,T]$

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2+\int_t^T|Z_s^n|^2\,ds\right)\leq\left(2Kh+4KAT+\frac{1}{4}\mathbb{E}\int_t^T\left|Z_s^{n-1}\right|^2ds\right)\exp\left(4AK\left(T-t\right)\right).$$

For each $t\in[\vartheta_1,T]$, we have $\exp\left(4AK\left(T-t\right)\right)<2$, then we deduce for each $n\geq 1$

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2+\int_t^T|Z_s^n|^2\,ds\right)&\leq 4Kh+8KAT+\frac{1}{2}\mathbb{E}\int_t^T\left|Z_s^{n-1}\right|^2ds,\\&\leq 4Kh+8KAT+\frac{1}{2}\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{n-1}\right|^2+\int_t^T\left|Z_s^{n-1}\right|^2ds\right),\\&\leq 4Kh+8KAT+\frac{1}{2}\left(4Kh+8KAT\right)\\&+\frac{1}{2^2}\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{n-2}\right|^2+\int_t^T\left|Z_s^{n-2}\right|^2ds\right),\end{aligned}$$

consequently with $(Y_t^0, Z_t^0) = (0,0)$, we get

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq s\leq T}|Y_s^n|^2 + \int_t^T |Z_s^n|^2\,ds\right)\\
&\leq 4Kh + 8KAT + \frac{1}{2}(4Kh+8KAT) + \cdots + \frac{1}{2^{n-1}}(4Kh+8KAT),\\
&\leq 4Kh + 8KAT + \frac{1}{2}(4Kh+8KAT)\left(\frac{1-\left(\frac{1}{2}\right)^{n-1}}{1-\frac{1}{2}}\right),\\
&\leq 8Kh + 16KAT - (4Kh+8KAT)\left(\frac{1}{2}\right)^n,\\
&\leq 8Kh + 16KAT. \qquad (3.23)
\end{aligned}$$

In the sequel, in each $n \geq 1$ and $m \geq 1$, let $\Delta Y_t^{n,m} = Y_t^n - Y_t^m$ and $\Delta Z_t^{n,m} = Z_t^n - Z_t^m$. Then $\forall t \in [0,T]$

$$\begin{aligned}
\Delta Y_t^{n,m} &= \int_t^T \Delta f^{n,m}\left(s, \Delta Y_s^{n,m}, \Delta Z_s^{n-1,m-1}\right) ds\\
&\quad + \int_t^T \Delta g^{n,m}\left(s, \Delta Y_s^{n,m}, \Delta Z_s^{n-1,m-1}\right) dB_s - \int_t^T \Delta Z_s^{n,m} dW_s, \qquad (3.24)
\end{aligned}$$

where

$$\begin{cases}
\Delta f^{n,m}(s, \Delta Y_s^{n,m}, \Delta Z_s^{n-1,m-1}) = f(s, \Delta Y_s^{n,m} + Y_s^m, \Delta Z_s^{n-1,m-1} + Z_s^{m-1}) - f(s, Y_s^m, Z_s^{m-1}),\\
\Delta g^{n,m}(s, \Delta Y_s^{n,m}, \Delta Z_s^{n-1,m-1}) = g(s, \Delta Y_s^{n,m} + Y_s^m, \Delta Z_s^{n-1,m-1} + Z_s^{m-1}) - g(s, Y_s^m, Z_s^{m-1}).
\end{cases}$$

It follows from $(H3.2)$ and $(H3.3)$ that $dP \times dt - a.e.$,

$$\begin{aligned}
&\langle \Delta Y_t^{n,m}, \Delta f^{n,m}\left(t, \Delta Y_t^{n,m}, \Delta Z_t^{n-1,m-1}\right)\rangle\\
&= \langle \Delta Y_t^{n,m}, f\left(t, \Delta Y_t^{n,m} + Y_t^m, Z_t^{n-1}\right) - f\left(t, Y_t^m, Z_t^{m-1}\right)\rangle,\\
&\leq k\left(|\Delta Y_t^{n,m}|^2\right) + |\Delta Y_t^{n,m}|\left|f\left(t, Y_t^m, Z_t^{n-1}\right) - f\left(t, Y_t^m, Z_t^{m-1}\right)\right|.
\end{aligned}$$

Then the assumption $(H3.6)$ is satisfied for the generator $\Delta f^{n,m}\left(t, \Delta Y_t^{n,m}, \Delta Z_t^{n-1,m-1}\right)$ of BDSDE (3.24) with $\psi(u) = k(u)$, $\lambda = 0$, $\sigma_t = \left|f\left(t, Y_t^m, Z_t^{n-1}\right) - f\left(t, Y_t^m, Z_t^{m-1}\right)\right|$.

It follows from $(H3.3)\,(ii)$ that $dP\times dt-a.e.$,

$$\left|\Delta g^{n,m}\left(t,\Delta Y_t^{n,m},\Delta Z_t^{n-1,m-1}\right)\right|^2\leq 2c^2\left|\Delta Y_t^{n,m}\right|^2+\left|\Delta Z_t^{n-1,m-1}\right|.$$

Then the assumption $(H3.7)$ is satisfied for the generator $\Delta g^{n,m}\left(t,\Delta Y_t^{n,m},\Delta Z_t^{n-1,m-1}\right)$ of BDSDE (3.24) with, $\lambda=2c^2$, $\gamma=2\alpha^2$ and $\eta_t=0$.

Thus, it follow from Proposition 3.1 (i) that there exists a constant $K>0$ depending only on δ,λ and γ such that, for each $0\leq t\leq T$

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2\right)+\mathbb{E}\left(\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)\\&\leq\left(2K\int_t^T k\left(\mathbb{E}\left|\Delta Y_s^{n,m}\right|^2\right)ds+\frac{K}{\delta}\mathbb{E}\int_t^T\left|f(s,Y_s^m,Z_s^{n-1})-f(s,Y_s^m,Z_s^{m-1})\right|^2ds\right)\exp\left(K\left(T-t\right)\right).\end{aligned}$$

Let us set $\vartheta_1=\max\left\{T-\frac{\ln 2}{K},0\right\}$. Then for each $t\in[\vartheta_1,T]$, we have $\exp\left(K(T-t)\right)\leq 2$ and

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2\right)+\mathbb{E}\left(\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)\\&\leq 4K\int_t^T k\left(\mathbb{E}\left|\Delta Y_s^{n,m}\right|^2\right)ds+\frac{2Kc^2}{\delta}\mathbb{E}\int_t^T\left|\Delta Z_s^{n-1,m-1}\right|^2ds,\end{aligned}$$

take $\delta=8Kc^2$, we have

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2\right)+\mathbb{E}\left(\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)\\&\leq 4K\int_t^T k\left(\mathbb{E}\left|\Delta Y_s^{n,m}\right|^2\right)ds+\frac{1}{4}\mathbb{E}\int_t^T\left|\Delta_s Z^{n-1,m-1}\right|^2ds.\end{aligned}\tag{3.25}$$

Taking the lim sup in (3.25), using Fatou's lemma, (3.23), monotonicity and continuity of $k(\cdot)$, we have

$$\begin{aligned}&\limsup_{n,m\to\infty}\left(\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2\right)+\mathbb{E}\left(\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)\right)\\&\leq 4K\int_t^T k\left(\limsup_{n,m\to\infty}\mathbb{E}\left|\Delta Y_s^{n,m}\right|^2\right)ds+\limsup_{n,m\to\infty}\frac{1}{4}\mathbb{E}\int_t^T\left|\Delta Z_s^{n-1,m-1}\right|^2ds.\end{aligned}$$

Thus, in view of $\int_{0^+} k^{-1}(u)\,du = \infty$, Bihari's inequality yields that, for each $\vartheta_1 \leq t \leq T$

$$\lim \sup_{n,m\to\infty} \left(\mathbb{E}\left(\sup_{0\leq s\leq T} |\Delta Y_s^{n,m}|^2 \right) + \mathbb{E}\left(\int_t^T |\Delta Z_s^{n,m}|^2 \, ds \right) \right) = 0,$$

we know that $\left((Y_t^n, Z_t^n)_{t\in[\vartheta_1,T]} \right)_{n\in\mathbb{N}^*}$ is Cauchy sequence in the process space $\mathcal{S}^2\left(\vartheta_1, T, \mathbb{R}^k\right) \times \mathcal{M}^2\left(\vartheta_1, T, \mathbb{R}^{k\times d}\right)$.

Let $(Y_t, Z_t)_{t\in[\vartheta_1,T]}$ be the limit process of the sequence $\left((Y_t^n, Z_t^n)_{t\in[\vartheta_1,T]} \right)_{n\in\mathbb{N}^*}$ in the process space $\mathcal{S}^2\left(\vartheta_1, T, \mathbb{R}^k\right) \times \mathcal{M}^2\left(\vartheta_1, T, \mathbb{R}^{k\times d}\right)$. On the other hand, since Z_t^n converge in $\mathcal{M}^2\left(\vartheta_1, T, \mathbb{R}^{k\times d}\right)$ to Z_t, then there exists a subsequence wich will denote Z_t^n such that $\forall n$, $Z_t^n \to Z_t$, $dt \otimes dP - a.s.$ and $\sup_n |Z_t^n|$ is $dt \otimes dP$ integrable. Therefore by $(H3.3)\,(i)$ and $(H3.4)$, we have

$$\left| f\left(s, Y_s^n, Z_s^{n-1}\right) \right| \leq c\left|Z_s^{n-1}\right| + \left| f(s,0,0) \right| + \varphi\left(\left|Y_s^n\right|\right) < \infty,$$

applying $(H3.1)$ and $(H3.3)\,(i)$, we have

$$\begin{aligned} \lim_{n\to\infty} \left| f\left(s, Y_s^n, Z_s^{n-1}\right) - f(s, Y_s, Z_s) \right| &= \lim_{n\to\infty} \left| f\left(s, Y_s, Z_s^{n-1}\right) - f(s, Y_s, Z) \right| \\ &\leq c \lim_{n\to\infty} \left| Z_s^{n-1} - Z \right| = 0, \end{aligned}$$

thus, $f(s, Y_s^n, Z_s^{n-1})$ converge simply to $f(s, Y_s, Z_s)$. Then by Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_t^T \left| f\left(s, Y_s^n, Z_s^{n-1}\right) - f(s, Y_s, Z_s) \right| ds = 0.$$

From wich it follow that Y^n converge uniformly in $t \in [\vartheta_1, T]$ to Y i.e., $\lim_{n\to\infty} \left(\sup_{\vartheta_1\leq t\leq T} |Y_t^n - Y_t| \right) = 0$. Now, we pass to the limit $n \to \infty$ in (3.22), we follows that $(Y_t, Z_t)_{t\in[\vartheta_1,T]}$ solve BDSDE $\left(E^{\xi,f,g}\right)$. Note that $T - \vartheta_1 \geq 0$ and depends only on c and A, we can repeat the above operation in finite steps to obtain a solution to the BDSDE $\left(E^{\xi,f,g}\right)$ on $[\vartheta_2, \vartheta_1]$, $[\vartheta_3, \vartheta_2]$, ..., and then on $[0, T]$. ■

Proof. of Theorem 3.1. Firstly we approximate $f(t, Y_t, Z_t)$ and ξ by a sequence whose elements satisfy the bound assumption in Lemma 3.3.

For each $n \geq 1$, define $q_n(x) = \frac{x \times n}{\sup(|x|, n)}$ for each $x \in \mathbb{R}^k$, and let

$$\xi_n = q_n(\xi) \quad \text{and} \quad f_n(t, Y_t, Z_t) = f(t, Y_t, Z_t) - f(t, 0, 0) + q_n(f(t, 0, 0)), \tag{3.26}$$

clearly, the f_n satisfies (3.21), we have

$$\lim_{n \to \infty} \mathbb{E}\left|\xi_n - \xi\right|^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}\left(\int_0^T \left|q_n(f(s, 0, 0)) - f(s, 0, 0)\right|^2 ds\right) = 0. \tag{3.27}$$

For each $n \geq 1$, let $(Y_t^n, Z_t^n)_{t \in [0,T]}$ denote the unique solution to the following BDSDE

$$Y_t^n = \xi_n + \int_t^T f_n(s, Y_s^n, Z_s^n)\, ds + \int_t^T g(s, Y_s^n, Z_s^n)\, d\overleftarrow{B}_s - \int_t^T Z_s^n dW_s, \quad 0 \leq t \leq T. \tag{3.28}$$

In the sequel, in each $n \geq 1$ and $m \geq 1$, let $\Delta Y_t^{n,m} = Y_t^n - Y_t^m$, $\Delta Z_t^{n,m} = Z_t^n - Z_t^m$ and $\Delta\xi^{n,m} = \xi_n - \xi_m$. Then $\forall t \in [0, T]$

$$\Delta Y_t^{n,m} = \Delta\xi^{n,m} + \int_t^T \left\{\Delta f^{n,m}(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m})\, ds + \Delta g^{n,m}(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m})\, dB_s - \Delta Z_s^{n,m} dW_s\right\}, \tag{3.29}$$

where

$$\begin{cases} \Delta f^{n,m}(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m}) = f_n(s, \Delta Y_s^{n,m} + Y_s^m, \Delta Z_s^{n,m} + Z_s^m) - f_m(s, Y_s^m, Z_s^m), \\ \Delta g^{n,m}(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m}) = g(s, \Delta Y_s^{n,m} + Y_s^m, \Delta Z_s^{n,m} + Z_s^m) - g(s, Y_s^m, Z_s^m). \end{cases}$$

By add and subtract, we get

$$\begin{aligned} &\langle \Delta Y_s^{n,m}, \Delta f^{n,m}(s, \Delta Y_s^{n,m}, \Delta Z_s^{n,m})\rangle \\ &= \langle \Delta Y_s^{n,m}, f_m(s, \Delta Y_s^{n,m} + Y_s^m, \Delta Z_s^{n,m} + Z_s^m) - f_m(s, Y_s^m, Z_s^m)\rangle \\ &+ \langle \Delta Y_s^{n,m}, f_n(s, \Delta Y_s^{n,m} + Y_s^m, \Delta Z_s^{n,m} + Z_s^m) - f_m(s, \Delta Y_s^{n,m} + Y_s^m, \Delta Z_s^{n,m} + Z_s^m)\rangle. \end{aligned}$$

It follows from $(H3.2)$ and $(H3.3)\,(i)$ and (3.26) that $dP\times dt-a.e.$,

$$
\begin{aligned}
&\langle \Delta Y_s^{n,m}, \Delta f^{n,m}\left(s,\Delta Y_s^{n,m},\Delta Z_s^{n,m}\right)\rangle \\
&=\langle \Delta Y_s^{n,m}, f\left(s,\Delta Y_s^{n,m}+Y_s^m,\Delta Z_s^{n,m}+Z_s^m\right)-f\left(s,Y_s^m,Z_s^n\right)-f\left(s,Y_s^m,Z_s^m\right)+f\left(s,Y_s^m,Z_s^n\right)\rangle \\
&+\langle \Delta Y_s^{n,m}, q_n\left(f\left(s,0,0\right)\right)-q_m\left(f\left(s,0,0\right)\right)\rangle, \\
&\le k\left(|\Delta Y_s^{n,m}|^2\right)+c\,|\Delta Y_s^{n,m}|\,|\Delta Z_s^{n,m}|+|\Delta Y_s^{n,m}|\,|q_n\left(f\left(s,0,0\right)\right)-q_m\left(f\left(s,0,0\right)\right)|\,.
\end{aligned}
$$

Then the assumption $(H3.6)$ is satisfied for the generator $\Delta f^{n,m}\left(s,\Delta Y_s^{n,m},\Delta Z_s^{n,m}\right)$ of BDSDE (3.29) with $\psi\left(u\right)=k\left(u\right),\ \lambda=c,\ \sigma_t=|q_n\left(f\left(t,0,0\right)\right)-q_m\left(f\left(t,0,0\right)\right)|\,.$

It follows from $(H3.3)\,(ii)$ that $dP\times dt-a.e.$,

$$
\begin{aligned}
\left|\Delta g^{n,m}\left(s,\Delta Y_s^{n,m},\Delta Z_s^{n,m}\right)\right|^2 &= \left|g\left(s,\Delta Y_s^{n,m}+Y_s^m,\Delta Z_s^{n,m}+Z_s^m\right)-g\left(s,Y_s^m,Z_s^m\right)\right|^2, \\
&\le 2c^2\left|\Delta Y_s^{n,m}\right|^2+2\alpha^2\left|\Delta Z_s^{n,m}\right|^2.
\end{aligned}
$$

Then the assumption $(H3.7)$ is satisfied for the generator $\Delta g^{n,m}\left(s,\Delta Y_s^{n,m},\Delta Z_s^{n,m}\right)$ of BDSDE (3.29) with, $\lambda=2c^2$, $\gamma=2\alpha^2$ and $\eta_t=0$.

Thus, it follow from Proposition 3.1 (i) with $\delta=1$ that there exists a constant $K>0$ depending only on δ,λ and γ such that, for each $0\le t\le T$

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le T}|\Delta Y_s^{n,m}|^2\right)+\mathbb{E}\left(\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right)\le\theta\mathbb{E}\,|\Delta\xi^{n,m}|^2+2\theta\int_t^T k\left(\mathbb{E}\left(\sup_{0\le r\le s}|\Delta Y_r^{n,m}|^2\right)\right)ds \\
&+\theta\mathbb{E}\int_t^T|q_n\left(f\left(s,0,0\right)\right)-q_m\left(f\left(s,0,0\right)\right)|^2\,ds. \qquad (3.30)
\end{aligned}
$$

where $\theta=K\exp\left(K\left(T-t\right)\right).$ Since $k\left(x\right)\le A\left(1+x\right)$, we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le T}|\Delta Y_s^{n,m}|^2\right)+\mathbb{E}\left(\int_t^T|\Delta Z_s^{n,m}|^2\,ds\right) \\
&\le\theta\mathbb{E}\,|\Delta\xi^{n,m}|^2+2AT\theta+2A\theta\int_t^T\mathbb{E}\left(\sup_{0\le r\le s}|\Delta Y_r^{n,m}|^2\right)ds \\
&+\theta\mathbb{E}\int_t^T|q_n\left(f\left(s,0,0\right)\right)-q_m\left(f\left(s,0,0\right)\right)|^2\,ds.
\end{aligned}
$$

Using (3.27), we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2\right)+\mathbb{E}\left(\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)\\
&\leq 2\theta\mathbb{E}\left|\xi\right|^2+2AT\theta+2A\theta\int_t^T\mathbb{E}\left(\sup_{0\leq r\leq s}\left|\Delta Y_r^{n,m}\right|^2\right)ds+2\theta\mathbb{E}\int_t^T\left|f\left(s,0,0\right)\right|^2ds.
\end{aligned}$$

Applying Gronwall's lemma yields that for each $t\in[0,T]$ and each $n,m\geq 1$

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq s\leq T}\left|\Delta Y_s^{n,m}\right|^2\right)+\mathbb{E}\left(\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)\\
&\leq\left(2\theta AT+2\theta\mathbb{E}\left|\xi\right|^2+2\theta\mathbb{E}\int_t^T\left|f\left(s,0,0\right)\right|^2ds\right)\exp\left(2\theta AT\right),\\
&<\infty.
\end{aligned}$$

Taking the lim sup in (3.30) and by previous inequality, Fatou's lemma, monotonicity and continuity of $k\left(\cdot\right)$, we have

$$\begin{aligned}
&\limsup_{n,m\to\infty}\mathbb{E}\left(\sup_{t\leq r\leq T}\left|\Delta Y_r^{n,m}\right|^2+\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)\\
&\leq\theta\mathbb{E}\left(\limsup_{n,m\to\infty}\left|\xi_n-\xi_m\right|^2\right)+2\theta\int_t^T k\left(\limsup_{n,m\to\infty}\mathbb{E}\left(\sup_{s\leq r\leq T}\left|\Delta Y_r^{n,m}\right|^2\right)\right)ds\\
&+\theta\mathbb{E}\int_t^T\limsup_{n,m\to\infty}\left|q_n\left(f\left(s,0,0\right)\right)-q_m\left(f\left(s,0,0\right)\right)\right|^2ds,\\
&=2\theta\int_t^T k\left(\limsup_{n,m\to\infty}\mathbb{E}\left(\sup_{s\leq r\leq T}\left|\Delta Y_r^{n,m}\right|^2\right)\right)ds.
\end{aligned}$$

Thus, in view of $\int_{0^+}k^{-1}\left(u\right)du=\infty$, Bihari's inequality yields that for each $0\leq t\leq T$

$$\limsup_{n,m\to\infty}\mathbb{E}\left(\sup_{t\leq r\leq T}\left|\Delta Y_r^{n,m}\right|^2+\int_t^T\left|\Delta Z_s^{n,m}\right|^2ds\right)=0.$$

We know that $\left(\left(Y_t^n,Z_t^n\right)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ is Cauchy sequence in the process space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$.

Let $\left(Y_t,Z_t\right)_{t\in[0,T]}$ be the limit process of the sequence $\left(\left(Y_t^n,Z_t^n\right)_{t\in[0,T]}\right)_{n\in\mathbb{N}^*}$ in the process space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$. Using $(H3.3)\,(i)$ and $(H3.4)$, we have

$$\left|f_n\left(t,Y_t^n,Z_t^n\right)\right|=c\left|Z_t^n\right|+\left|f\left(t,0,0\right)\right|+\varphi\left(\left|Y_s^n\right|\right)<\infty,$$

applying $(H3.1)$, and (3.26), we have $f_n(s,Y_s^n,Z_s^n)$ converge simply to $f(s,Y_s,Z_s)$. Then by Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_t^T\left|f_n\left(s,Y_s^n,Z_s^n\right)-f\left(s,Y_s,Z_s\right)\right|ds=0.$$

From wich it follow that Y^n converge uniformly in t to Y. Now, we pass to the limit $n\to\infty$ in (3.28), we deduce that $(Y_t,Z_t)_{t\in[0,T]}$ solve BDSDE $(E^{\xi,f,g})$.

Thus we prove the existence part and finally complete the proof of Theorem 3.1. ■

3.1.3 Application to SPDEs.

In this section we connect BDSDEs with weak monotonicity and general growth generators with the correspondent SPDEs and give the sobolev solution of the SPDEs.

Notation and definition

C_b^k set of function of class C^k, whose partial derivatives of order less then or egal to k are bounded. Given $x\in\mathbb{R}^d$, $b\in C_b^2\left(\mathbb{R}^d,\mathbb{R}^d\right)$ and $\sigma\in C_b^3\left(\mathbb{R}^d,\mathbb{R}^{d\times d}\right)$, denote by $(X_s^{t,x};\ t\leq s\leq T)$ the unique strong solution of the SDEs following

$$dX_s^{t,x}=b\left(X_s^{t,x}\right)ds+\sigma\left(X_s^{t,x}\right)dW_s,\qquad X_t^{t,x}=x. \tag{3.31}$$

It's well know that $\mathbb{E}\left(\sup_{t\leq s\leq T}|X_s^{t,x}|^p\right)<\infty$ for any $p>1$, we recall that the stochastic flow associated to the diffusion processes $(X_s^{t,x};\ t\leq s\leq T)$ is $\left(X_s^{t,x};\ x\in\mathbb{R}^d,\ t\leq s\leq T\right)$ and the inverse flow is denote by $\hat{X}_s^{t,x}$. $x\to\hat{X}_s^{t,x}$ is differentiable and we denote by $J\left(\hat{X}_s^{t,x}\right)$ the determinant of the Jacobian matrix of $\hat{X}_s^{t,x}$, which is positive and satisfies $J\left(\hat{X}_t^{t,x}\right)=1$.
For $\phi\in C_c^\infty\left(\mathbb{R}^d\right)$ we define the process $\phi_t:\Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}$ by $\phi_t\left(s,x\right)=\phi\left(\hat{X}_s^{t,x}\right)J\left(\hat{X}_s^{t,x}\right)$.
Let $\pi:\mathbb{R}^d\to\mathbb{R}_+$ be an integrable continues positive function and $\mathbb{L}^2\left(\mathbb{R}^d,\pi\left(x\right)dx\right)$ be the

weight $\mathbb{L}^2$ space with weight $\pi(x)$ endowed with the following norm

$$||u||_\pi^2 = \int_{\mathbb{R}^d} |u(x)|^2 \pi(x)\, dx.$$

Let us take the weight $\pi(x) = \exp(F(x))$, where $F : \mathbb{R}^d \to \mathbb{R}$ is a continues, moreover we assume that there exist some $R > 0$ such that $F \in C_b^2$ for $|x| > R$, we need the following result of generalized equivalence of norm.

Lemma 3.4 *There exist two positive constant K_1, k_1 wich depend on T, π, such that for any $t \le s \le T$ and $\Phi \in L^1\left(\Omega \times \mathbb{R}^d, \mathbb{P}\otimes\pi(x)\, dx\right)$*

$$k_1 \left(\int_{\mathbb{R}^d} |\Phi(x)| \pi(x)\, dx \right) \le \mathbb{E}\left(\int_{\mathbb{R}^d} \left|\Phi\left(X_s^{t,x}\right)\right| \pi(x)\, dx \right) \le K_1 \left(\int_{\mathbb{R}^d} |\Phi(x)| \pi(x)\, dx \right).$$

Moreover for any $\Psi \in L^1\left(\Omega \times [0,\ T] \times \mathbb{R}^d \times \ \mathbb{P}\otimes dt \otimes \pi(x)\, dx\right)$

$$\begin{aligned} k_1 \left(\int_{\mathbb{R}^d} \int_t^T |\Psi(s,x)|\, ds\ \pi(x)\, dx \right) &\le \mathbb{E}\left(\int_{\mathbb{R}^d} \int_t^T \left|\Psi\left(s, X_s^{t,x}\right)\right| ds\ \pi(x)\, dx \right), \\ &\le K_1 \left(\int_{\mathbb{R}^d} \int_t^T |\Psi(s,x)|\, ds\ \pi(x)\, dx \right). \end{aligned}$$

Proof. Using the change of variable $y = X_s^{t,x}$, we get

$$\begin{aligned} \mathbb{E}\left(\int_{\mathbb{R}^d} \left|\Phi\left(X_s^{t,x}\right)\right| \pi(x)\, dx \right) &= \int_{\mathbb{R}^d} |\Phi(y)|\, \mathbb{E}\left(\pi\left(\hat{X}_s^{t,y}\right) J\left(\hat{X}_s^{t,y}\right) \right) dy, \\ &= \int_{\mathbb{R}^d} \mathbb{E}\left(\frac{J\left(\hat{X}_s^{t,y}\right) \pi\left(\hat{X}_s^{t,y}\right)}{\pi(y)} \right) \pi(y)\, dy. \end{aligned}$$

By Lemma 5.1 in Bally-Matoussi [8], $k_1 \le \mathbb{E}\left(\frac{J\left(\hat{X}_s^{t,y}\right)\pi\left(\hat{X}_s^{t,y}\right)}{\pi(y)} \right) \le K_1$ for any $y \in \mathbb{R}^k$, $s \in [t, T]$, the first claim follows. The second claim can be proved similarly. ■

Now begin to study the following SPDEs

$$\left(\mathcal{P}^{(f,g)}\right) \qquad \begin{cases} u(s,x) = h(x) + \displaystyle\int_s^T \left(\mathcal{L}u(r,x) + f(r,x,u(r,x), \sigma^* \nabla u(r,x)) \right) dr \\ \qquad + \displaystyle\int_s^T g(r,x,u(r,x), \sigma^* \nabla u(r,x))\, d\overleftarrow{B}_r, \quad t \le s \le T, \end{cases}$$

where

$$\mathcal{L} := \frac{1}{2}\sum_{i,j}(a_{ij})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}, \qquad \text{with } (a_{ij}) := \sigma\sigma^*.$$

Let $\mathcal{H}$ be the set of random fields $\{u(t,x), 0 \le t \le T,\ x \in \mathbb{R}^d\}$ such that for every (t,x), $u(t,x)$ is $\mathcal{F}^B_{t,T}$-measurable and

$$\|u\|^2_{\mathcal{H}} = E\Big(\int_{\mathbb{R}^d}\int_0^T (|u(r,x)|^2 + |(\sigma^*\nabla u)(r,x)|^2) dr\ \pi(x)\, dx\Big) < \infty.$$

The couple $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a Banach space.

Definition 3.1 *We say that u is a Sobolev solution to SPDE $(\mathcal{P}^{(f,g)})$, if $u \in \mathcal{H}$ and for any $\varphi \in \mathcal{C}^{1,\infty}_c([0,T]\times\mathbb{R}^d)$*

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_s^T f(r,x,u(r,x),\sigma^*\nabla u(r,x))\varphi(r,x)\,drdx + \int_{\mathbb{R}^d}\int_s^T g(r,x,u(r,x),\sigma^*\nabla u(r,x))\varphi(r,x)\,d\overleftarrow{B}_r dx \\
&= \int_{\mathbb{R}^d}\int_s^T u(r,x)\frac{\partial\varphi(r,x)}{\partial r}(r,x)\,drdx + \int_{\mathbb{R}^d} u(r,x)\varphi(r,x)\,dx - \int_{\mathbb{R}^d} h(x)\varphi(T,x)\,dx \\
&- \frac{1}{2}\int_{\mathbb{R}^d}\int_s^T \sigma^* u(r,x)\sigma^*\varphi(r,x)\,drdx - \int_{\mathbb{R}^d}\int_s^T u div\big((b-A)\varphi\big)(r,x)\,drdx, \qquad (3.32)
\end{aligned}$$

where A is a d-vector whose coordinates are defined by $A_j := \frac{1}{2}\sum_{i=1}^d \frac{\partial a_{ij}}{\partial x_i}$.

In this section well study the Sobolev solution of $(\mathcal{P}^{(f,g)})$ with weak monotonicity and general growth. For $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d} \to \mathbb{R}^k$, $g : [0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d} \to \mathbb{R}^{d\times l}$, $h : \mathbb{R}^d \to \mathbb{R}^d$. The main idea is to connect $(\mathcal{P}^{(f,g)})$ with the following BDSDE for each $s \in [t,T]$

$$\begin{aligned}
Y^{t,x}_s &= h(X^{t,x}_T) + \int_s^T f(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)\,dr \\
&+ \int_s^T g(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)\,d\overleftarrow{B}_r - \int_s^T Z^{t,x}_r dW_r, \qquad (3.33)
\end{aligned}$$

where $(X^{t,x}_s; 0 \le s \le T)$ is the solution of SDEs (3.31).

Our object consists to establish the existence and uniqueness of solutions u to SPDEs $(\mathcal{P}^{(f,g)})$ such that $u(s, X^{t,x}_s) = Y^{t,x}_s$ and $\sigma^*\nabla u(s, X^{t,x}_s) = Z^{t,x}_s$.

We consider the following assumptions **(A3)**:

(A3.1) For (t,x) fixed $dP\times dt$-a.e., $x\in\mathbb{R}^d$, $z\in\mathbb{R}^{k\times d}$ $y\rightarrow f(w,t,x,y,z)$ is continuous and $\int_{\mathbb{R}^d}\int_0^T |f(t,x,0,0)|^2\, dt\pi(x)\, dx<\infty$.

(A3.2) f satisfies the weak monotonicity condition in y, i.e., there exist a nondecreasing and concave function $k(\cdot):\mathbb{R}^+\rightarrow\mathbb{R}^+$ with $k(u)>0$ for $u>0$, $k(0)=0$ and $\int_{0^+}k^{-1}(u)du=+\infty$ such that $dP\times dt$-a.e.,$\forall y_1,y_2\in\mathbb{R}^k$, $z\in\mathbb{R}^{k\times d}$, $x\in\mathbb{R}^d$

$$\langle y_1-y_2, f(t,\omega,x,y_1,z)-f(t,\omega,x,y_2,z)\rangle\leq k\left(\left|y_1-y_2\right|^2\right).$$

(A3.3) i) f is lipschitz in z, uniformly with respect to (ω,t,x,y) i.e., there exists a constant $c\geq 0$ such that $dP\times dt$-a.e.,

$$\left|f(\omega,t,x,y,z)-f(\omega,t,x,y,z')\right|\leq c\left|z-z'\right|.$$

ii) $\int_{\mathbb{R}^d}\int_0^T |g(t,x,0,0)|^2\, dt\pi(x)\, dx<\infty$ and for (t,x) fixed there exists a constant $c>0$ and a constant $0<\alpha\leq\frac{1}{4}$ such that $dP\times dt$-a.e.,

$$\left|g(\omega,t,x,y,z)-g\left(\omega,t,x,y',z'\right)\right|\leq c\left|y-y'\right|+\alpha\left|z-z'\right|.$$

(A3.4) f have a general growth with respect to y, i.e., $dP\times dt$-a.e., $\forall(x,y)\in\mathbb{R}^d\times\mathbb{R}^k$

$$\left|f(t,\omega,x,y,0)\right|\leq\left|f(t,\omega,x,0,0)\right|+\varphi\left(|y|\right),$$

where $\varphi:\mathbb{R}^+\rightarrow\mathbb{R}^+$ is increasing continuous function.

(A3.5) h belongs to $\mathbb{L}^2\left(\mathbb{R}^d,\pi(x)\,dx;\mathbb{R}^k\right)$.

Now by Lemma 3.4, Fubini's theorem and using $(A3.1)$, $(A3.3)(ii)$ and $(A3.5)$, we have for a.e. $x\in\mathbb{R}^d$

$$\mathbb{E}\left(\int_s^T\left|f\left(r,X_r^{t,x},0,0\right)\right|^2 dr+\int_s^T\left|g\left(r,X_r^{t,x},0,0\right)\right|^2 dr+\left|h\left(X_T^{t,x}\right)\right|^2\right)<\infty. \tag{3.34}$$

Hence, it follows from Theorem 3.1, that BDSDEs (3.33) admit a unique solution $(Y_s^{t,x}, Z_s^{t,x})$ such that $Y_s^{t,x}, Z_s^{t,x}$ are $\mathcal{F}_{t,s}^W \vee \mathcal{F}_{s,T}^B$ measurable for any $s \in [0,T]$.

Moreover, by Proposition 3.1 (i) it's easy to check for each $\delta > 0$ that there exists a constant $K > 0$ depending only on δ, λ and γ such that, for each $0 \leq t \leq T$

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{t,x}\right|^2\right) + \mathbb{E}\left(\int_t^T \left|Z_s^{t,x}\right|^2 ds\right) \\
&\leq \left(\mathbb{E}|h\left(X_T^{t,x}\right)|^2 + 2\int_t^T k\left(\mathbb{E}|Y_s^{t,x}|^2\right) ds + \frac{1}{\delta}\mathbb{E}\int_t^T \left|f\left(r, X_r^{t,x}, 0, 0\right)\right|^2 ds \right.\\
&\left. + 2\mathbb{E}\int_t^T g\left(r, X_r^{t,x}, 0, 0\right) ds\right) K\exp\left(K\left(T-T\right)\right),
\end{aligned}$$

using (3.34) and since $k(x) \leq A(1+x)$, we have

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{t,x}\right|^2 + \int_t^T \left|Z_s^{t,x}\right|^2 ds\right) \leq c + 2\theta AT + 2\theta A\int_t^T \mathbb{E}\left(|Y_s^{t,x}|^2\right) ds,$$

where $\theta = K\exp\left(K\left(T-T\right)\right)$. Finally, applying Gronwall's lemma, we obtain

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{t,x}\right|^2 + \int_t^T \left|Z_s^{t,x}\right|^2 ds\right) \leq (c + 2\theta AT)\exp\left(2\theta AT\right) < \infty. \tag{3.35}$$

Now, we are state the main result of this section.

Theorem 3.2 *Under hypothesis (A3), the SPDEs $\left(\mathcal{P}^{(f,g)}\right)$ admits a unique Sobolev solution u. Moreover $u(t,x) = Y_t^{t,x}$, where $(Y_s^{t,x}, Z_s^{t,x})_{t\leq s\leq T}$ is the unique solution of the BDSDEs (3.33) and*

$$u\left(s, X_s^{t,x}\right) = Y_s^{t,x} \quad \text{and} \quad \left(\sigma^*\nabla u\right)\left(s, X_s^{t,x}\right) = Z_s^{t,x}, \quad \text{for a.e. } (s,\omega,x) \text{ in } [t,T]\times\Omega\times\mathbb{R}^d. \tag{3.36}$$

We first consider the following SPDEs:

$$\left(\mathcal{P}^{(f,g,u^n)}\right) \quad \left\{\begin{aligned}
u^n(t,x) &= h(x) + \int_t^T \left(\mathcal{L}u^n(r,x) + f\left(r,x,u^n(r,x), \sigma^*\nabla u^n(r,x)\right)\right)dr \\
&+ \int_t^T g\left(r,x,u^n(r,x), \sigma^*\nabla u^n(r,x)\right) d\overleftarrow{B}_r, \qquad t\leq s\leq T.
\end{aligned}\right.$$

We need the following results.

Proposition 3.2 *Under the assumptions* $(A3)$. *Let* $(X^{t,x})$ *be the unique solution of SDEs* (3.31) *and for a fixed* $n \in \mathbb{N}^*$, *let* $(Y^{n,t,x}, Z^{n,t,x})$ *be the unique solution of the BDSDEs*

$$\begin{aligned} Y_s^{n,t,x} &= h\left(X_T^{t,x}\right) + \int_s^T f\left(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}\right) dr \\ &+ \int_s^T g\left(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}\right) d\overleftarrow{B}_r - \int_t^T Z_r^{n,t,x} dW_r. \end{aligned} \tag{3.37}$$

Then for any $s \in [t,T]$

$$Y_r^{n,s,X_s^{t,x}} = Y_r^{n,t,x}, \qquad Z_r^{n,s,X_s^{t,x}} = Z_r^{n,t,x}, \quad \textit{for a.e. } r \in [s,T]\,,\ x \in \mathbb{R}^d.$$

Proof. The proof is similar to the proof of Proposition 3.4 in Q. Zhang and H. Zhao [32]. ■

Using Proposition 3.1, by the same computation as in (3.35), we have that the sequence $(Y_s^{t,x,n}, Z_s^{t,x,n})$ are bounded in $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right) \times \mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$, i.e.,

$$\sup_n \mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{t,x,n}\right|^2 + \int_t^T \left|Z_s^{t,x,n}\right|^2 ds\right) < \infty. \tag{3.38}$$

Also by Proposition 3.1 applying with $k(\cdot) = \psi(\cdot)$, $\sigma_t = 0$ $\eta_t = 0$, $\lambda = 2c^2$ and $\gamma = 2\alpha^2$, we can proof by the same computation as in Theorem 3.1, that $(Y_s^{t,x,n}, Z_s^{t,x,n})_{s\in[0,T]}$ is a Cauchy sequence in the process space $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right) \times \mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$, i.e., there exists a $(Y_s^{t,x}, Z_s^{t,x})_{s\in[0,T]} \in \mathcal{S}^2\left(0,T,\mathbb{R}^k\right) \times \mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$ such that

$$\mathbb{E}\left(\sup_{0\leq s\leq T}\left|Y_s^{t,x,n} - Y_s^{t,x}\right|^2 + \int_t^T \left|Z_s^{t,x,n} - Z_s^{t,x}\right|^2 ds\right) \to 0, \quad \text{as } n \to \infty. \tag{3.39}$$

Under the assumptions $(A3)$ if we define $u^n(t,x) = Y_t^{n,t,x}$ and $\sigma^*\nabla u^n(t,x) = Z_t^{n,t,x}$. Then by a direct application of Proposition 3.2, and Fubini's Theorem, we have

$$u^n\left(s, X_s^{t,x}\right) = Y_s^{n,t,x}, \qquad \sigma^*\nabla u^n\left(s, X_s^{t,x}\right) = Z_s^{n,t,x}, \quad for\ a.e.\ s \in [t,T]\,, x \in \mathbb{R}^d. \tag{3.40}$$

Theorem 3.3 *Under hypothesis (A3), if we define* $u^n(s,x) = Y_s^{n,t,x}$. *Then the SPDEs* $\left(\mathcal{P}^{(f,g,u^n)}\right)$ *admits a unique Sobolev solution* u^n, *where* $(Y_s^{n,t,x}, Z_s^{n,t,x})_{s\in[t,T]}$ *is the unique solution of the BDSDEs* (3.37) *and*

$$u^n\left(s, X_s^{t,x}\right) = Y_s^{n,t,x} \quad \text{and} \quad \sigma^*\nabla u^n\left(s, X_s^{t,x}\right) = Z_s^{n,t,x}, \quad \text{for } (s,\omega,x) \text{ in } [t,T]\times\Omega\times\mathbb{R}^d. \tag{3.41}$$

Proof. Existence. For each $(s,x) \in [t,T]\otimes\mathbb{R}^d$, define $u^n(s,x) = Y_s^{n,t,x}$ and $\sigma^*\nabla u^n(s,x) = Z_s^{n,t,x}$, where $(Y_s^{n,t,x}, Z_s^{n,t,x}) \in \mathcal{S}^2\left(0,T,\mathbb{R}^k\right) \times \mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$ is the solution of Eq (3.37). Then by (3.40)

$$u^n\left(s, X_s^{t,x}\right) = Y_s^{n,t,x}, \qquad \sigma^*\nabla u^n\left(s, X_s^{t,x}\right) = Z_s^{n,t,x}, \qquad \text{for a.e. } s \in [t,T],\ x \in \mathbb{R}^d.$$

Set

$$\begin{cases} F^n(s,x) = f\left(s,x,u^n(s,x), \sigma^*\nabla u^n(s,x)\right), \\ G^n(s,x) = g\left(s,x,u^n(t,x), \sigma^*\nabla u^n(s,x)\right). \end{cases}$$

Then $(Y_s^{n,t,x}, Z_s^{n,t,x}) \in \mathcal{S}^2\left(0,T,\mathbb{R}^k\right) \times \mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$ solve

$$Y_s^{n,t,x} = h\left(X_T^{t,x}\right) + \int_s^T F^n\left(r, X_r^{t,x}\right)dr + \int_s^T G^n\left(r, X_r^{t,x}\right)d\overleftarrow{B}_r - \int_t^T Z_r^{n,t,x}dW_r.$$

Moreover, by Lemma 3.4 and (3.38), we have

$$\begin{aligned} &\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T \left(|u^n(s,x)|^2 + |\sigma^*\nabla u^n(s,x)|^2\right)ds\pi(x)\,dx\right) \\ &\leq \frac{1}{k_1}\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T \left(\left|Y_s^{n,t,x}\right|^2 + \left|Z_s^{n,t,x}\right|^2\right)ds\pi(x)\,dx\right), \\ &< \infty. \end{aligned}$$

From $(A3.3)\,(i)$ and $(A3.4)$, we have

$$\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T |F^n(s,x)|^2\,ds\;\pi(x)\,dx\right)$$
$$\leq 2\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T \left(c\,|\sigma^*\nabla u^n(s,x)|^2 + |f(s,x,0,0)|^2 + \varphi\left(|u^n(s,x)|\right)^2\right) ds\pi(x)\,dx\right) < \infty.$$

And from $(A3.3)\,(ii)$, we have

$$\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T |G^n(s,x)|^2\,ds\pi(x)\,dx\right) < \infty.$$

Using a some ideas as in the proof of Theorem 2.1 in Bally and Matoussi [8] similar to the argument as in section 4 in [33], we know that $u^n(t,x)$ is the Sobolev solution of the following SPDE:

$$\begin{cases} u^n(t,x) = h(x) + \int_s^T \left(\mathcal{L}u^n(r,x) + F^n(r,x)\right)dr \\ \qquad\qquad + \int_s^T G^n(r,x)\,d\overleftarrow{B}_r, \qquad t \leq s \leq T. \end{cases} \tag{3.42}$$

Noting that by the definition of $F^n(r,x)$ and $G^n(r,x)$, from (3.41), we have that u^n is the Sobolev solution of Eq $\left(\mathcal{P}^{(f,g,u^n)}\right)$.

Uniqueness

Let u^n be a solution of Eq $\left(\mathcal{P}^{(f,g,u^n)}\right)$. Define the same notation in the existence part for F^n and G^n, since u^n is a solution, so $E\left(\int_{\mathbb{R}^d}\int_t^T \left(|u^n(s,x)|^2 + |\sigma^*\nabla u^n(s,x)|^2\right) ds\pi(x)\,dx\right) < \infty$. From a similar computation as in existence part, we have

$$\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T \left(|F^n(s,x)|^2 + |G^n(s,x)|^2\right) ds\;\pi(x)\,dx\right) < \infty.$$

Then, for (3.41) it follows from Proposition 2.3 in Bally and Matoussi [8] that, for and

$\phi \in \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, a.e. $s \in [t,T]$, a.s.

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_s^T u^n(r,x)\, d\phi_t(r,x)dx + \int_{\mathbb{R}^d} u^n(r,x)\,\phi_t(r,x)\,dx \\
&- \int_{\mathbb{R}^d} h(x)\,\phi_t(T,x)\,dx - \int_s^T\int_{\mathbb{R}^d} u^n(r,x)\,\mathcal{L}^*\phi_t(r,x)\,drdx \\
&= \int_{\mathbb{R}^d}\int_s^T F^n(r,x)\,\phi_t(r,x)\,drdx + \int_{\mathbb{R}^d}\int_s^T G^n(r,x)\,\phi_t(r,x)\,d\overleftarrow{B}_r dx.
\end{aligned}$$

Now using $\phi_t(r,x) = \phi\left(\hat{X}_r^{t,x}\right) J\left(\hat{X}_r^{t,x}\right)$ and by a change of variable, we get

$$\begin{aligned}
\int_{\mathbb{R}^d} u^n(r,x)\,\phi_t(r,x)dx &= \int_{\mathbb{R}^d} u^n\left(r, X_r^{t,x}\right)\phi(x)\,dx, \\
\int_{\mathbb{R}^d} h(x)\,\phi_t(T,x)\,dx &= \int_{\mathbb{R}^d} h\left(X_r^{t,x}\right)\phi(x)\,dx, \\
\int_{\mathbb{R}^d}\int_s^T F^n(r,x)\,\phi_t(r,x)\,drdx &= \int_{\mathbb{R}^d}\int_s^T F^n\left(s, X_r^{t,x}\right)\phi(x)\,drdx, \\
\int_{\mathbb{R}^d}\int_s^T G^n(r,x)\,\phi_t(r,x)\,d\overleftarrow{B}_r dx &= \int_{\mathbb{R}^d}\int_s^T G^n\left(s, X_r^{t,x}\right)\phi(x)\,d\overleftarrow{B}_r dx,
\end{aligned}$$

by a change of variable $y = X_r^{t,x}$ and integration by part formula, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_s^T u^n(r,x)\,d\phi_t(r,x)\,dx \\
&= \int_{\mathbb{R}^d}\int_s^T (\sigma^*\nabla u^n)\left(r, X_r^{t,x}\right)\phi(x)\,dW_r dx + \int_{\mathbb{R}^d}\int_s^T u^n(r,x)\,\mathcal{L}^*\phi_t(r,x)\,drdx.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{\mathbb{R}^d} u^n\left(r, X_r^{t,x}\right)\phi(x)dx &= \int_{\mathbb{R}^d} h\left(X_r^{t,x}\right)\phi(x)\,dx + \int_{\mathbb{R}^d}\int_s^T F^n\left(s, X_r^{t,x}\right)\phi(x)\,drdx \\
&+ \int_{\mathbb{R}^d}\int_s^T G^n\left(s, X_r^{t,x}\right)\phi(x)\,d\overleftarrow{B}_r dx - \int_{\mathbb{R}^d}\int_s^T (\sigma^*\nabla u^n)\left(r, X_r^{t,x}\right)\phi(x)\,dW_r dx.
\end{aligned}$$

From the arbitrariness of ϕ we know that $\{u^n(r, X_r^{t,x}), (\sigma^*\nabla u^n)(r, X_r^{t,x}),\ t \le r \le T\}$ is a solution of the following BDSDE

$$Y_s^{n,t,x} = h(X_T^{t,x}) + \int_s^T F^n\left(r, X_r^{t,x}\right)dr + \int_s^T G^n\left(r, X_r^{t,x}\right)d\overleftarrow{B}_r - \int_t^T Z_r^{n,t,x} dW_r,\ t \le s \le T.$$

Then from the definitions of F^n and G^n it follows that $\{u^n(r, X_r^{t,x}), (\sigma^* \nabla u^n)(r, X_r^{t,x}),\ t \leq r \leq T\}$ solve BDSDE (3.37).

If there is another solution $\tilde{u}^n$ to Eq. $(\mathcal{P}^{(f,g,u^n)})$, then by the same procedure, we can find another solution $\left(\tilde{Y}_s^{t,x,n}, \tilde{Z}_s^{t,x,n}\right)$ solve the BDSDE (3.37), where

$$\tilde{u}^n\left(s, X_s^{t,x}\right) = \tilde{Y}_s^{n,t,x}, \qquad \sigma^* \nabla \tilde{u}^n\left(s, X_s^{t,x}\right) = \tilde{Z}_s^{n,t,x}, \quad for\ a.e.s \in [t, T],\ x \in \mathbb{R}^d.$$

By Theorem 3.1, the solution of Eq. (3.37) is unique, therefore

$$\tilde{Y}_s^{n,t,x} = Y_s^{n,t,x}, \qquad for\ a.e.s \in [t, T],\ x \in \mathbb{R}^d.$$

Now, applying Lemma 3.4 again, we have

$$\begin{aligned}
&\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T |\tilde{u}^n(s,x) - u^n(s,x)|^2 ds\pi(x)\,dx\right) \\
&\leq \frac{1}{k_1}\left(\int_{\mathbb{R}^d}\int_t^T \left|\tilde{Y}_s^{n,t,x} - Y_s^{n,t,x}\right|^2 ds\pi(x)\,dx\right) = 0.
\end{aligned}$$

So $\tilde{u}^n(s,x) = u^n(s,x)$, for a.e. $s \in [0,T]$, $x \in \mathbb{R}^d$ a.s..Uniqueness is proved. ■

Proposition 3.3 *Under assumptions* (A), *let* $\left(Y_t^{t,x}, Z_t^{t,x}\right)$ *be the solution of Eq.* (3.33). *If we define* $u(s,x) = Y_s^{t,x}$, *then* $\sigma^* \nabla u(s,x)$ *exists for a.e.* $s \in [t,T]$, $x \in \mathbb{R}^d$ *a.s., and*

$$u(s, X_s^{t,x}) = Y_s^{t,x}, \qquad \sigma^* \nabla u(s, X_s^{t,x}) = Z_s^{t,x}, \quad for\ a.e.\ s \in [t,T],\ x \in \mathbb{R}^d. \tag{3.43}$$

Proof. See Proposition 4.2 in Q. Zhang, and H. Zhao [32]. ■

In the rest part of this section, we study Eq $(\mathcal{P}^{(f,g)})$. Then by Theorem 3.3, Proposition 3.3, Lemma 3.4 and estimation (3.39), we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_t^T \left(|u^n(s,x) - u(s,x)|^2 + |\sigma^*\nabla u^n(s,x) - \sigma^*\nabla u(s,x)|^2\right) ds\pi(x)\,dx \\
&\leq \frac{1}{k_1}\mathbb{E}\left(\int_{\mathbb{R}^d}\int_t^T \left(\left|u^n\left(s, X_s^{t,x}\right) - u\left(s, X_s^{t,x}\right)\right|^2 + \left|\sigma^*\nabla u^n\left(s, X_s^{t,x}\right) - \sigma^*\nabla u\left(s, X_s^{t,x}\right)\right|^2\right) ds\pi(x)\,dx\right), \\
&\to 0, \qquad \text{as } n \to \infty.
\end{aligned} \tag{3.44}$$

With (3.44) we prove the Theorem 3.2 in this section.

Proof. of Theorem 3.2: Existence, by Lemma 3.4 and (3.33), we see that

$$\sigma^{*}\nabla u\left(t,x\right)=Z_{t}^{t,x},\qquad for\ a.e.\ t\in\left[0,T\right],\ x\in\mathbb{R}^{d}.$$

Also, by Lemma 3.4 and (3.35), we have

$$\begin{aligned}&\mathbb{E}\left(\int_{\mathbb{R}^{d}}\int_{t}^{T}\left(\left|u\left(s,x\right)\right|^{2}+\left|\sigma^{*}\nabla u\left(s,x\right)\right|^{2}\right)ds\pi\left(x\right)dx\right)\\&\leq\frac{1}{k_{1}}\mathbb{E}\left(\int_{\mathbb{R}^{d}}\int_{t}^{T}\left(\left|Y_{s}^{t,x}\right|^{2}+\left|Z_{s}^{t,x}\right|^{2}\right)ds\pi\left(x\right)dx\right),\\&<\infty.\end{aligned}$$

Now we will prove that u satisfies the definition 3.1. Let $\varphi\in\mathcal{C}_{c}^{1,\infty}([0,T]\times\mathbb{R}^{d})$, since for any n, u^{n} is a Sobolev solution to the problem $(P^{(f,g,u^{n})})$, we then have

$$\begin{aligned}&\int_{\mathbb{R}^{d}}\int_{s}^{T}u^{n}\left(r,x\right)\frac{\partial\varphi\left(r,x\right)}{\partial r}\left(r,x\right)drdx+\int_{\mathbb{R}^{d}}u^{n}\left(r,x\right)\varphi\left(r,x\right)dx-\int_{\mathbb{R}^{d}}h\left(x\right)\varphi\left(T,x\right)dx\\&-\frac{1}{2}\int_{\mathbb{R}^{d}}\int_{s}^{T}\sigma^{*}u^{n}\left(r,x\right)\sigma^{*}\varphi\left(r,x\right)drdx-\int_{\mathbb{R}^{d}}\int_{s}^{T}u^{n}div\big(\left(b-A\right)\varphi\big)\left(r,x\right)drdx\\&=\int_{\mathbb{R}^{d}}\int_{s}^{T}f(r,x,u^{n}\left(r,x\right),\sigma^{*}\nabla u^{n}\left(r,x\right))\varphi\left(r,x\right)drdx\\&+\int_{\mathbb{R}^{d}}\int_{s}^{T}g(r,x,u^{n}\left(r,x\right),\sigma^{*}\nabla u^{n}\left(r,x\right))\varphi\left(r,x\right)d\overleftarrow{B}_{r}dx,\end{aligned}\tag{3.45}$$

By proving that along a subsequence (3.45) converges to (3.32) in $\mathbb{L}^{2}\left(\Omega\right)$, we have that $u\left(t,x\right)$ satisfies (3.32). We only need to show that along a subsequence as $n\rightarrow\infty$

$$\begin{cases}\int_{\mathbb{R}^{d}}\int_{s}^{T}\big(f\left(r,x,u^{n}\left(r,x\right),\sigma^{*}\nabla u^{n}\left(r,x\right)\right)-f\left(r,x,u\left(r,x\right),\sigma^{*}\nabla u\left(r,x\right)\right)\big)\varphi\left(r,x\right)drdx\rightarrow0,\\\int_{\mathbb{R}^{d}}\int_{s}^{T}\big(g\left(r,x,u^{n}\left(r,x\right),\sigma^{*}\nabla u^{n}\left(r,x\right)\right)-g\left(r,x,u\left(r,x\right),\sigma^{*}\nabla u\left(r,x\right)\right)\big)\varphi\left(r,x\right)d\overleftarrow{B}_{r}dx\rightarrow0.\end{cases}$$

Firstly. Since $\varphi\in C_{c}^{\infty}$ then φ is belong in $\mathbb{L}^{2}\left(\mathbb{R}^{d}\times\left[s,T\right],dt\otimes dx\right)$ and by Cauchy-Schwartz inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\int_s^T\big(f(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))-f(r,x,u(r,x),\sigma^*\nabla u(r,x))\big)\varphi(r,x)\,drdx\right|^2\\&\leq\int_{\mathbb{R}^d}\int_s^T\left|f(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))-f(r,x,u(r,x),\sigma^*\nabla u(r,x))\right|^2\pi(x)\,drdx\\&\times\int_{\mathbb{R}^d}\int_s^T\frac{|\varphi(r,x)|^2}{\pi(x)}drdx,\\&\leq C\int_{\mathbb{R}^d}\int_s^T\left|f(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))-f(r,x,u(r,x),\sigma^*\nabla u(r,x))\right|^2\pi(x)\,drdx.\end{aligned}$$

Also we have by Lemma 3.4, and by definition of $u^n(r,X_r^{s,x})$, $\sigma^*\nabla u^n(r,X_r^{s,x})$ that,

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_s^T\left|f(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))-f(r,x,u(r,x),\sigma^*\nabla u(r,x))\right|^2dr\pi(x)\,dx,\\&\leq\frac{1}{k}\mathbb{E}\int_{\mathbb{R}^d}\int_s^T\left|f(r,x,Y_r^{n,s,x},Z_r^{n,s,x})-f(r,x,Y_r^{s,x},Z_r^{s,x})\right|^2dr\pi(x)\,dx,\end{aligned}$$

using $(A3.3)\,(i)$ and $(a+b)^2\leq 2a^2+2b^2$, we have

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}^d}\int_s^T\left|f(r,x,Y_r^{n,s,x},Z_r^{n,s,x})-f(r,x,Y_r^{s,x},Z_r^{s,x})\right|^2dr\pi(x)\,dx\\&\leq 2c\mathbb{E}\int_{\mathbb{R}^d}\int_s^T\left|Z_r^{n,s,x}-Z_r^{s,x}\right|^2dr\pi(x)\,dx\\&+2\mathbb{E}\int_{\mathbb{R}^d}\int_s^T\left|f(r,x,Y_r^{n,s,x},Z_r^{s,x})-f(r,x,Y_r^{s,x},Z_r^{s,x})\right|^2dr\pi(x)\,dx.\end{aligned}$$

We only need to prove that

$$\mathbb{E}\int_{\mathbb{R}^d}\int_s^T\left|f(r,x,Y_r^{n,s,x},Z_r^{s,x})-f(r,x,Y_r^{s,x},Z_r^{s,x})\right|^2dr\pi(x)\,dx\to 0,\quad as\ n\to\infty.$$

Applying assumption $(A3.1)$, we have

$$\lim_{n\to\infty}\left|f(r,x,Y_r^{n,s,x},Z_r^{s,x})-f(r,x,Y_r^{s,x},Z_r^{s,x})\right|^2=0.$$

Since $\mathbb{E}\int_{\mathbb{R}^d}\int_t^T|Z_s^{t,x,n}|^2\,ds\pi(x)\,dx<\infty$, then there exists a subsequence which we still denote $Z^{t,x,n}\to Z^{s,x}$ such that $\mathbb{E}\int_{\mathbb{R}^d}\int_t^T|Z_s^{t,x}|^2\,ds\pi(x)\,dx<\infty$, using (3.38), $(A3.3)\,(i)$ and $(A3.4)$, we have

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}^d}\int_s^T\left|f\left(r,x,Y_r^{n,s,x},Z_r^{s,x}\right)\right|^2dr\pi\left(x\right)dx\\&\leq\mathbb{E}\int_{\mathbb{R}^d}\int_s^T\left(c\left|Z_r^{s,x}\right|^2+\left|f\left(r,x,0,0\right)\right|^2+\varphi\left(\sup_{t\leq r\leq T}\left|Y_r^{n,s,x}\right|\right)^2\right)dr\pi\left(x\right)dx,\\&<\infty.\end{aligned}$$

According to the Lebesgue's dominated convergence Theorem, it follows that

$$\mathbb{E}\int_{\mathbb{R}^d}\int_s^T\left|f\left(r,x,Y_r^{n,s,x},Z_r^{s,x}\right)-f\left(r,x,Y_r^{s,x},Z_r^{s,x}\right)\right|^2dr\pi\left(x\right)dx\rightarrow0,\quad as\ n\rightarrow\infty,$$

which implies that

$$\begin{aligned}&\lim_{n\rightarrow\infty}\int_{\mathbb{R}^d}\int_s^T f\left(r,x,u^n\left(r,x\right),\sigma^*\nabla u^n\left(r,x\right)\right)\varphi(r,x)drdx\\&=\int_{\mathbb{R}^d}\int_s^T f\left(r,x,u\left(r,x\right),\sigma^*\nabla u\left(r,x\right)\right)\varphi\left(r,x\right)drdx.\end{aligned}$$

Secondly It remains to prove that $\int_{\mathbb{R}^d}\int_s^T g\left(r,x,u^n\left(r,x\right),\sigma^*\nabla u^n\left(r,x\right)\right)\varphi\left(r,x\right)d\overleftarrow{B}_rdx$, tends to $\int_{\mathbb{R}^d}\int_s^T g\left(r,x,u\left(r,x\right),\sigma^*\nabla u\left(r,x\right)\right)\varphi\left(r,x\right)d\overleftarrow{B}_rdx$,as n tends to ∞.

Arguing as in the proof of Theorem 3.1, we get the following limit in probability as $n\rightarrow\infty$, $\int_0^T g\left(r,X_r^{t,x},u^n\left(r,X_r^{t,x}\right),\sigma^*\nabla u^n\left(r,X_r^{t,x}\right)\right)d\overleftarrow{B}_r\rightarrow\int_0^T g\left(r,X_r^{t,x},u\left(s,X_r^{t,x}\right),\sigma^*\nabla u\left(r,X_r^{t,x}\right)\right)d\overleftarrow{B}_r$.

By Lemma 3.4, (3.35) and (3.38), we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\left|\int_s^T\left(g\left(r,x,u^n\left(r,x\right),\sigma^*\nabla u^n\left(r,x\right)\right)-g\left(r,x,u\left(r,x\right),\sigma^*\nabla u\left(r,x\right)\right)\right)\varphi\left(r,x\right)\pi\left(x\right)d\overleftarrow{B}_r\right|\pi^{-1}\left(x\right)dx\\&<\infty,\end{aligned}$$

i.e. $\int_s^T\left(g(r,x,u^n\left(r,x\right),\sigma^*\nabla u^n\left(r,x\right))-g(r,x,u\left(r,x\right),\sigma^*\nabla u\left(r,x\right))\right)\varphi\left(r,x\right)\pi\left(x\right)d\overleftarrow{B}_r$ belongs to $\mathbb{L}^1\left(\mathbb{R}^d,\pi^{-1}\left(x\right)dx\right)$.

Hence, using Lemma 3.4 we get, for every $s \in [0,T]$

$$\int_{\mathbb{R}^d} \left| \int_s^T \left(g\left(r,x,u^n\left(r,x\right),\sigma^*\nabla u^n\left(r,x\right)\right) - g\left(r,x,u\left(r,x\right),\sigma^*\nabla u\left(r,x\right)\right) \right) \varphi\left(r,x\right)\pi\left(x\right) d\overleftarrow{B}_r \right| \pi^{-1}\left(x\right) dx$$
$$\leq \frac{1}{k_1} \int_{\mathbb{R}^d} \mathbb{E} \left| \int_s^T \left(g\left(r,X_r^{t,x},u^n\left(r,X_r^{t,x}\right),\sigma^*\nabla u^n\left(r,X_r^{t,x}\right)\right) - g\left(r,X_r^{t,x},u\left(r,X_r^{t,x}\right),\sigma^*\nabla u\left(r,X_r^{t,x}\right)\right)\right) \right.$$
$$\left. \times \varphi\left(r,X_r^{t,x}\right) \right| \times \pi\left(X_r^{t,x}\right) d\overleftarrow{B}_r \pi^{-1}\left(x\right) dx,$$
$$= \frac{1}{k_1} \int_{\mathbb{R}^d} \mathbb{E} \int_s^T \left(g\left(r,X_r^{t,x},Y_r^{n,t,x},Z_r^{n,t,x}\right) - g\left(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\right)\right) \varphi\left(r,X_r^{t,x}\right) \pi\left(X_r^{t,x}\right) d\overleftarrow{B}_r \pi^{-1}\left(x\right) dx.$$

Since

$$\begin{cases} \sup_n \mathbb{E} \int_s^T \left(g\left(r,X_r^{t,x},Y_r^{n,t,x},Z_r^{n,t,x}\right) - g\left(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\right)\right) \varphi(r,X_r^{t,x}) \pi\left(X_r^{t,x}\right) d\overleftarrow{B}_r < \infty, \\ \textit{and} \\ \int_s^T \left(g\left(r,X_r^{t,x},Y_r^{n,t,x},Z_r^{n,t,x}\right) - g\left(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\right)\right) \varphi\left(r,X_r^{t,x}\right) \pi\left(X_r^{t,x}\right) d\overleftarrow{B}_r \\ \text{converges to 0 in probability,} \end{cases}$$

it follows according to the Lebesgue's dominated convergence theorem that

$$\lim_n \mathbb{E} \int_s^T \left(g\left(r,X_r^{t,x},Y_r^{n,t,x},Z_r^{n,t,x}\right) - g\left(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\right)\right) \varphi\left(r,X_r^{t,x}\right) \pi\left(X_r^{t,x}\right) d\overleftarrow{B}_r = 0.$$

Therefore $u(t,x)$ satisfies (3.32), i.e. it is a Sobolev solution of $(\mathcal{P}^{(f,g)})$. Theorem 3.2. is proved. ■

Part Two:

Reflected Backward Doubly Stochastic Differential Equation-The General Case.

This part is devoted to the study of existence and uniqueness results for Reflected BDSDE's (RBDSDEs in short). In Chapter 4, we present the existence and uniqueness result of RBDSDE under classical Lipshitz conditions see [5]. In Chapter 5, we present our contribution in this part see [20] which is the existence of a minimal and a maximal solution to the Reflected BDSDE with jumps (RBDSDEJ in short) under a linear growth condition and left continuity in y on the generator, the case where the generator has a linear growth and is continuous in (Y, Z, U) is also study, we state a new version of a comparison principle which allows us to compare the solutions to RBDSDEs. In chapter 6 we deal with reflected anticipated backward doubly stochastic differential equations (RABDSDEs) driven by teughels martingales associated with Lévy process see [17], we obtain the existence and uniqueness of solutions to these equations by means of the fixed-point theorem where the coefficients of these BDSDEs depend on the future and present value of the solution (Y, Z), we also show the comparison theorem for a special class of reflected ABDSDEs under some slight stronger conditions, the novelty of our result lies in the fact that we allow the time interval to be infinite, furthermore we get a existence and uniqueness result of the solution to the previous equation when, $S = -\infty$ i.e., $K \equiv 0$.

Chapter 4

Reflected Backward Doubly SDEs.

In this chapter, we study the case where the solution is forced to stay above a given stochastic process, called the obstacle. We obtain the real valued reflected backward doubly stochastic differential equation:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s, Z_s)\, d\overleftarrow{B}_s + \int_t^T dK_s - \int_t^T Z_s dW_s,\ 0 \le t \le T. \quad (4.1)$$

We establish the existence and uniqueness of solutions for equation (4.1) under uniformly Lipschitz condition on the coefficients. We give here a method which allows us to overcome this difficulty in the Lipschitz case. The idea consists to start from the penalized basic RBDSDE where f and g do not depend on $(y; z)$. We transform it to a RBDSDE with $f = g = 0$, for which we prove the existence and uniqueness of a solution by penalization method. The section theorem is then only used in the simple context where $f = g = 0$ to prove that the solution of the RBDSDE (with $f = g = 0$) stays above the obstacle for each time. The (general) case, where the coefficients f, g depend on $(y; z)$, is treated by a Picard type approximation.

Assumption and definition

We consider the following conditions:

(H4.1) $f:[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\rightarrow\mathbb{R}$; $g:[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\rightarrow\mathbb{R}$ be jointly measurable such that for any $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, $f(\cdot,\omega,y,z)\in\mathcal{M}^2(0,T,\mathbb{R})$ and $g(\cdot,\omega,y,z)\in\mathcal{M}^2(0,T,\mathbb{R})$.

(H4.2) There exist constant $C\geq 0$ and a constant $0<\alpha<1$ such that for every $(\omega,t)\in\Omega\times[0,T]$ and $(y,y')\in\mathbb{R}^2$, $(z,z')\in(\mathbb{R}^d)^2$

$$\begin{cases} \left|f(t,\omega,y,z)-f(t,\omega,y',z')\right|^2\leq C\left[\left|y-y'\right|^2+\left|z-z'\right|^2\right], \\ \left|g(t,\omega,y,z)-g(t,\omega,y',z')\right|^2\leq C\left|y-y'\right|^2+\alpha\left|z-z'\right|^2. \end{cases}$$

(H4.3) The terminal value ξ be a given random variable in $\mathbb{L}^2$.

(H4.4) $(S_t)_{t\geq 0}$, is a continuous progressively measurable real valued process satisfying $\mathbb{E}\left(\sup_{0\leq t\leq T}\left(S_t^+\right)^2\right)<+\infty$ and $S_T\leq\xi$, $\mathbb{P}$-almost surely.

Definition 4.1 *A solution of a equation* (4.1) *is a* $\left(\mathbb{R},\mathbb{R}^d,\mathbb{R}_+\right)$*-valued* $\mathcal{F}_t$*-progressively measurable process* $(Y,Z,K)_{t\in[0,T]}$ *wich satisfies*

$$\begin{cases} i)\, Y\in\mathcal{S}^2(0,T,\mathbb{R}),\;\; Z\in\mathcal{M}^2\left(0,T,\mathbb{R}^d\right),\;\; K\in\mathcal{A}^2, \\ ii)\;\; Y_t=\xi+\int_t^T f(s,Y_s,Z_s)ds+\int_t^T g(s,Y_s,Z_s)d\overleftarrow{B}_s+\int_t^T dK_s-\int_t^T Z_s dW_s,\;\; 0\leq t\leq T, \\ ii)\;\; S_t\leq Y_t,\quad 0\leq t\leq T\quad \text{and}\quad \int_0^T\left(Y_t-S_t\right)dK_t=0. \end{cases}$$

Comparison Theorem

Lemma 4.1 *Let* θ^1 *and* θ^2*be two square integrable and* $\mathcal{G}_T$*-measurable random variables and* θ^1, $\theta^2:[0,T]\times\Omega\times\mathbb{R}\rightarrow\mathbb{R}$ *be two measurable functions. For* $j\in\{1,2\}$, *let* (Y^j,Z^j)*be a solution of the following BSDE:*

$$\begin{cases} Y_t^j=\xi+\int_t^T\theta^j(s,Y_s^j)ds+-\int_t^T Z_s^j dW_s, \\ \mathbb{E}\left(\sup_{t\leq T}\left|Y_t^j\right|^2+\int_t^T\left|Z_s^j\right|^2ds\right)<\infty. \end{cases}$$

Assume that,

i) For every $\mathcal{G}_t$*-adapted process* $\{Y_t,\ 0 < t < T\}$ *satisfying* $\mathbb{E}\left(\sup_{t\leq T}\left|Y_t^j\right|^2\right) < \infty$, $\theta^j(s, Y_s)$ *is* $\mathcal{G}_t$*-adapted and satisfies* $\mathbb{E}\left(\int_t^T \left|\theta^j(s, Y_s)\right|^2 ds\right) < \infty$.

ii) θ^1 *uniformly Lipschitz in the variable* y, *uniformly with respect* (t, ω).

iii) $\theta^1 \leq \theta^2$ *a.s.*

iv) $\theta^1(t, Y_t^2) \leq \theta^2(t, Y_t^2)$ $dP \times dt$ *a.e.*

Then,

$$Y_t^1 \leq Y_t^2,\ 0 \leq t \leq T \ a.s.$$

Proof. Applying Itô's formula to $\left|(Y_t^1 - Y_t^2)^+\right|^2$ and using the fact that $\eta^1 \leq \eta^2$, we obtain

$$\begin{aligned} &\left|(Y_t^1 - Y_t^2)^+\right|^2 + \int_t^T 1_{\{Y_s^1 > Y_s^2\}} \left|Z_s^1 - Z_s^2\right|^2 ds \\ &\leq 2\int_t^T \left(Y_s^1 - Y_s^2\right)^+ \left(\theta^1\left(s, Y_s^1\right) - \theta^2\left(s, Y_s^2\right)\right) ds - 2\int_t^T \left(Y_s^1 - Y_s^2\right)^+ \left(Z_s^1 - Z_s^2\right) dW_s. \end{aligned}$$

Using the fact that h^1 is Lipschitz and Gronwall's lemma, we get $(Y_t^1 - Y_t^2)^+ = 0$, for all $0 \leq t \leq T$ a.s. Which implies that $Y_t^1 \leq Y_t^2$, for all $0 \leq t \leq T$, a.s. ■

4.1 Existence and uniqueness result to a RBDSDE with Lipschitz condition.

Theorem 4.1 *Assume that* $(H4.1) - (H4.4)$ *holds. Then Eq* (4.1) *admits a unique solution* $(Y, Z, K) \in \mathcal{S}^2 \times \mathcal{M}^2 \times L^2$.

We first consider the following simple RBDSDE, with f, g independent from (Y, Z).

$$\begin{cases} Y_t = \xi + \int_t^T f(s)\, ds + K_T - K_t + \int_t^T g(s)\, dB_s - \int_t^T Z_s dW_s, \\ Y_t \geq S_t, \quad \forall t \leq T, \quad a.s., \\ \int_0^T (Y_s - S_s)\, dK_s = 0. \end{cases} \tag{4.2}$$

Proposition 4.1 *There exists a unique process* (Y, Z, K) *which solves equation* (4.2).

Proof. By [24], for $n \in \mathbb{N}$, let $(Y_t^n, Z_t^n)_{0\leq t\leq T}$ denotes the unique pair of processes, with values in $\mathbb{R}\times\mathbb{R}^d$ satisfying: $(Y^n, Z^n) \in S^2 \times M^2$ and

$$Y_t^n := \xi + \int_t^T f(s)\,ds + n\int_t^T (S_s - Y_s^n)^+\,ds + \int_t^T g(s)\,dB_s - \int_t^T Z_s^n dW_s.$$

We define,

$$\begin{cases} \overline{\xi} := \xi + \int_0^T f(s)\,ds + \int_0^T g(s)\,dB_s, \\ \overline{S}_t := S_t + \int_0^t f(s)\,ds + \int_0^t g(s)\,dB_s, \\ \overline{Y}_t^n := Y_t^n + \int_0^t f(s)\,ds + \int_0^t g(s)\,dB_s. \end{cases}$$

We have,

$$\overline{Y}_t^n = \overline{\xi} + n\int_t^T \left(\overline{S_s} - \overline{Y}_s^n\right)^+ ds - \int_t^T Z_s^n dW_s. \tag{4.3}$$

Let $\Lambda_t = E^{\mathcal{G}_t}\left[\overline{\xi} \vee sup_{s\leq T}\overline{S}_s\right]$. Then there exists a $\mathcal{G}_t$-predictable process $\gamma \in \mathbb{L}^2([0,T]\times\Omega,\mathbb{R}^d)$ such that

$$\Lambda_t = \Lambda_T - \int_t^T \gamma_s dW_s. \tag{4.4}$$

Since $(\overline{S}_s - \Lambda_s)^+ = 0$, we have

$$\Lambda_t = \Lambda_T + n\int_t^T (\overline{S}_s - \Lambda_s)^+ ds - \int_t^T \gamma_s dW_s. \tag{4.5}$$

By **Lemma 4.1**, we have for all $n \in \mathbb{N}$

$$\overline{Y}_t^0 = E^{\mathcal{G}_t}\left[\overline{\xi}\right] \leq \overline{Y}_t^n \leq \overline{Y}_t^{n+1} \leq \Lambda_t = E^{\mathcal{G}_t}\left[\overline{\xi} \vee sup_{s\leq T}\overline{S}_s\right].$$

Set $\overline{Y}_t := \sup_n \overline{Y}_t^n$ and $Y_t := \sup_n Y_t^n$. Since $\Lambda_s \geq \overline{S}_s$, we then have for every n,

$$(\overline{S_s} - \overline{Y}_s^n)^+(\Lambda_s - \overline{Y}_s^n) = (\overline{S_s} - \overline{Y}_s^n)^+(\Lambda_s - \overline{S}_s) + (\overline{S_s} - \overline{Y}_s^n)^+(\overline{S}_s - \overline{Y}_s^n) \geq 0.$$

Therefore, using Itô's formula, we obtain

$$
\begin{aligned}
|\Lambda_t - \overline{Y}^n_t|^2 + \int_t^T |\gamma_s - Z^n_s|^2 ds &= |\Lambda_T - \overline{\xi}|^2 - 2n\int_t^T (\overline{S_s} - \overline{Y}^n_s)^+(\Lambda_s - \overline{Y}^n_s)ds \\
&\quad - 2\int_t^T (\Lambda_s - \overline{Y}^n_s)(\gamma_s - Z^n_s)dW_s \\
&\leq |\Lambda_T - \overline{\xi}|^2 - 2\int_t^T (\Lambda_s - \overline{Y}^n_s)(\gamma_s - Z^n_s)dW_s.
\end{aligned}
$$

Passing to expectation, we get

$$
E\int_0^T |\gamma_s - Z^n_s|^2 ds \leq E|\sup_{s\leq T}(\overline{S}_s - \overline{\xi})^+|^2.
$$

Coming back to equation (4.3) and using equation (4.4) , we obtain

$$
\begin{aligned}
n\int_0^T (S_s - Y^n_s)^+ ds \quad = n\int_0^T (\overline{S_s} - \overline{Y}^n_s)^+ ds \quad &= (\overline{Y}^n_0 - \overline{\xi}) + \int_0^T Z^n_s dW_s, \\
&\leq (\Lambda_0 - \overline{\xi}) + \int_0^T Z^n_s dW_s, \\
&\leq (\Lambda_T - \overline{\xi}) + \int_0^T (Z^n_s - \gamma_s)dW_s.
\end{aligned}
$$

Which yield that

$$
(n\int_0^T (S_s - Y^n_s)^+ ds)^2 \leq 2(\Lambda_T - \overline{\xi})^2 + 2\int_0^T (Z^n_s - \gamma_s)^2 ds.
$$

Passing to expectation

$$
\begin{aligned}
E(n\int_0^T (S_s - Y^n_s)^+ \, ds)^2 &= E(n\int_0^T \left(\overline{S_s} - \overline{Y}^n_s\right)^+ ds)^2, \\
&\leq 2E(\Lambda_T - \overline{\xi})^2 + 2E\int_0^T (Z^n_s - \gamma_s)^2 ds, \\
&\leq 4E|\sup_{s\leq T}(\overline{S}_s - \overline{\xi})^+|^2.
\end{aligned}
$$

Hence, there exist a nondecreasing and right continuous process K satisfying $E(K_T^2) < \infty$

such that for a subsequence of n (which still denoted n) we have for all $\varphi \in \mathbb{L}^2(\Omega; \mathcal{C}([0,T]))$,

$$\lim_n E\int_0^T \varphi_s n(S_s - Y_s^n)^+ ds = E\int_0^T \varphi_s dK_s.$$

Let $N \in \mathbb{N}^*$ and $n, m \geq N$. We have

$$\begin{aligned}(Y_t^n - Y_t^m)^2 \leq 2\int_t^T \left(S_s - Y_s^N\right) n\left(S_s - Y_s^n\right)^+ ds + 2\int_t^T \left(S_s - Y_s^N\right) m\left(S_s - Y_s^m\right)^+ ds \\ - 2\int_t^T \left(Z_s^n - Z_s^m\right)\left(Y_s^n - Y_s^m\right) dW_s - \int_t^T \left|Z_s^n - Z_s^m\right|^2 ds.\end{aligned}$$

By **B-D-G** inequality's, there exists a constant C such that

$$\limsup_{n,m} \left(E\left(\sup_{t\leq T}\left(Y_t^n - Y_t^m\right)^2\right) + E\int_0^T \left|Z_s^n - Z_s^m\right|^2 ds \right) \leq 2CE\int_0^T \left(S_s - Y_s^N\right) dK_s.$$

Letting N tends to ∞, by using a Lebesgue's theorem we obtain

$$\limsup_{n,m} \left(E\left(\sup_{t\leq T}\left(Y_t^n - Y_t^m\right)^2\right) + E\int_0^T \left|Z_s^n - Z_s^m\right|^2 ds \right) \leq 2CE\int_0^T \left(S_s - Y_s\right) dK_s.$$

Let

$$\widetilde{Y}_t^n := \overline{S}_T + n\int_t^T \left(\overline{S}_s - \widetilde{Y}_s^n\right) ds - \int_t^T \widetilde{Z}_s^n dW_s.$$

Since $\overline{S}_T \leq \overline{\xi}$, the comparison theorem Lemma 4.1, shows that, for every n we have, $\forall t \in [0,T]$, $\overline{Y}_t^n \geq \widetilde{Y}_t^n$ $a.s.$ Let σ be a $\mathcal{G}_t-$stopping time, and $\tau = \sigma \wedge T$. We have

$$\widetilde{Y}_\tau^n = E^{\mathcal{G}_\tau}\left[\overline{S}_T e^{-n(T-\tau)} + n\int_\tau^T \overline{S}_s e^{-n(s-\tau)} ds\right].$$

It is not difficult to see that $\widetilde{Y}^n$ converges to $\overline{S}_\tau$ $a.s.$ Therefore $\overline{Y}_\tau \geq \overline{S}_\tau$ $a.s.$, and hence $Y_\tau \geq S_\tau \quad a.s.$

Using section theorem, we get, $a.s.$ for every $t \in [0,T]$, $Y_t \geq S_t$, which implies that

$$\limsup_{n,m} \left(E\left(\sup_{t\leq T} (Y_t^n - Y_t^m)^2\right) + E\int_0^T |Z_s^n - Z_s^m|^2\, ds \right) = 0,$$
$$\texttt{and} \quad E\int_0^T (S_s - Y_s)\, dK_s = 0.$$

We deduce that (Y,K) is continuous and there exists Z in $\mathbb{L}^2$ such that Z^n converges strongly in $\mathbb{L}^2$ to Z. Finally, it is not difficult to check that (Y,Z,K) satisfies equation (4.2). ∎

Proof. Existence for general case (Theorem 4.1). We define a sequence $(Y_t^n, Z_t^n, K_t^n)_{0\leq t\leq T}$ as follows. Let $Y_t^0 = S_t$, $Z_t^0 = 0$ and for $t \in [0,T]$ and $n \in \mathbb{N}^*$,

$$\begin{cases} Y_t^{n+1} = \xi + \int_t^T f(s, Y_s^n, Z_s^n)\, ds + \int_t^T g(s, Y_s^n, Z_s^n)\, dB_s + \int_t^T dK_s^{n+1} - \int_t^T Z_s^{n+1} dW_s, \\ Y_t^{n+1} \geq S_t \quad a.s., \\ \int_0^T (Y_s^{n+1} - S_s)\, dK_s^{n+1} = 0. \end{cases}$$

Such sequence $(Y^n, Z^n, K^n)_n$ exists by the previous step.

Put $\overline{Y}^{n+1} = Y^{n+1} - Y^n$. By Itô's formula, we have,

$$\begin{aligned} \left|\overline{Y}_t^{n+1}\right|^2 + \int_t^T \left|\overline{Z}_s^{n+1}\right|^2 ds &= 2\int_t^T \overline{Y}_s^{n+1} \left(f(s, Y_s^n, Z_s^n) - f\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right) ds \\ + \int_t^T \overline{Y}_s^{n+1} \left(dK_s^{n+1} - dK_s^n\right) &+ 2\int_t^T \overline{Y}_s^{n+1} \left(g(s, Y_s^n, Z_s^n) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right) dB_s \\ -2\int_t^T \overline{Y}_s^{n+1} \overline{Z}_s^{n+1} dW_s &+ \int_t^T \left|g(s, Y_s^n, Z_s^n) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right|^2 ds. \end{aligned}$$

Therefore, Itô's formula applied to $|y|^2 e^{\beta t}$ shows that:

$$\begin{aligned} &\left|\overline{Y}_t^{n+1}\right|^2 e^{\beta t} - \beta\int_t^T \left|\overline{Y}_s^{n+1}\right|^2 e^{\beta s} ds + \int_t^T e^{\beta s} \left|\overline{Z}_s^{n+1}\right|^2 ds \\ &= 2\int_t^T e^{\beta s} \overline{Y}_s^{n+1} \left(f(s, Y_s^n, Z_s^n) - f\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right) ds + \int_t^T e^{\beta s} \overline{Y}_s^{n+1} \left(dK_s^{n+1} - dK_s^n\right) \\ &+ 2\int_t^T e^{\beta s} \overline{Y}_s^{n+1} \left(g(s, Y_s^n, Z_s^n) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right) dB_s - 2\int_t^T e^{\beta s} \overline{Y}_s^{n+1} \overline{Z}_s^{n+1} dW_s \\ &+ \int_t^T e^{\beta s} \left|g(s, Y_s^n, Z_s^n) - g\left(s, Y_s^{n-1}, Z_s^{n-1}\right)\right|^2 ds. \end{aligned}$$

Using the fact that $\int_t^T e^{\beta s}\overline{Y}_s^{n+1}\left(dK_s^{n+1}-dK_s^n\right)\leq 0$ and taking expectation, we get for every $\delta>0$:

$$\begin{aligned}
&E\left(\left|\overline{Y}_t^{n+1}\right|^2\right)e^{\beta t}-\beta E\left(\int_t^T\left|\overline{Y}_s^{n+1}\right|^2 e^{\beta s}\right)ds+E\int_t^T e^{\beta s}\left|\overline{Z}_s^{n+1}\right|^2 ds\\
&\leq 2L\delta E\int_t^T\left|\overline{Y}_s^{n+1}\right|^2 e^{\beta s}ds+\frac{2L}{\delta}E\int_t^T\left(\left|\overline{Y}_s^n\right|^2+\left|\overline{Z}_s^n\right|^2\right)e^{\beta s}ds\\
&+LE\int_t^T e^{\beta s}\left|\overline{Y}_s^n\right|^2 ds+\alpha E\int_t^T\left|\overline{Z}_s^n\right|^2 e^{\beta s}ds.
\end{aligned}$$

This implies that,

$$\begin{aligned}
&E\left(\left|\overline{Y}_t^{n+1}\right|^2\right)e^{\beta t}-(\beta+2L\delta)E\left(\int_t^T\left|\overline{Y}_s^{n+1}\right|^2 e^{\beta s}\right)ds+E\int_t^T\left|\overline{Z}_s^{n+1}\right|^2 ds\\
&\leq\left(L+\frac{2L}{\delta}\right)E\int_t^T\left|\overline{Y}_s^n\right|^2 e^{\beta s}ds+\left(\alpha+\frac{2L}{\delta}\right)E\int_t^T\left|\overline{Z}_s^n\right|^2 e^{\beta s}ds.
\end{aligned}$$

Choose $\delta=\frac{4L}{(1-\alpha)}$, $\overline{C}=\frac{2}{1+\alpha}\left(L+\frac{1-\alpha}{2}\right)$, and $\beta=-2L\delta-\overline{C}$, we have

$$\begin{aligned}
&E\int_t^T\left(\overline{C}\left|\overline{Y}_s^{n+1}\right|^2+\left|\overline{Z}_s^{n+1}\right|^2\right)e^{\beta s}ds\\
&\leq\left(\frac{1+\alpha}{2}\right)^n E\int_t^T\left(\overline{C}\left|\overline{Y}_s^1\right|^2+\left|\overline{Z}_s^1\right|^2\right)e^{\beta s}ds.
\end{aligned}$$

Since $\frac{1+\alpha}{2}<1$, there exists (Y,Z) in $\mathcal{M}^2\times\mathcal{M}^2$ such that (Y^n,Z^n) converges to (Y,Z) in $\mathcal{M}^2\times\mathcal{M}^2$. It is not difficult to deduce that Y^n converges to Y in $\mathcal{S}^2$.

It remains to prove that (Y,Z,K) is a solution to RBDSDE. By Proposition 4.1, there exists $(\overline{Y},\overline{Z},K)$ which satisfies,

$$\overline{Y}_t=\xi+\int_t^T f\left(s,Y_s,Z_s\right)ds+K_T-K_t+\int_t^T g\left(s,Y_s,Z_s\right)dB_s-\int_t^T\overline{Z}_s dW_s, \tag{4.6}$$

$(\overline{Y},\overline{Z},K_T)\in\mathcal{S}^2\times\mathcal{M}^2\times L^2$, $\overline{Y}_t\geq S_t$, (K_t) is continuous nondecreasing, $K_0=0$ and $\int_0^T\left(\overline{Y}_t-S_t\right)dK_t=0$.

We shall prove that $(Y,Z)=(\overline{Y},\overline{Z})$. By Itô's formula, we have

$$
\begin{aligned}
&(Y_t^{n+1}-\overline{Y}_t)^2-\int_t^T|Z_s^{n+1}-\overline{Z}_s|^2ds\\
&=2\int_t^T(Y_s^{n+1}-\overline{Y}_s)(f(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)ds+2\int_t^T(Y_s^{n+1}-\overline{Y}_s)(dK_s^{n+1}-dK_s)\\
&+\int_t^T(Y_s^{n+1}-\overline{Y}_s)(g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s))dB_s+2\int_t^T(Y_s^{n+1}-\overline{Y}_s)(Z_s^{n+1}-\overline{Z}_s)dW_s\\
&+\int_t^T|g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2ds.
\end{aligned}
$$

Taking expectation and using the fact that $\int_t^T(Y_s^{n+1}-\overline{Y}_s)(dK_s^{n+1}-dK_s)\leq 0$, we get

$$
\begin{aligned}
&E(Y_t^{n+1}-\overline{Y}_t)^2+E\int_t^T|Z_s^{n+1}-\overline{Z}_s|^2ds\\
&\leq 2E\int_t^T(Y_s^{n+1}-\overline{Y}_s)(f(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)ds+E\int_t^T|g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2ds\\
&\leq C\left(E\int_t^T|Y_s^{n+1}-\overline{Y}_s|^2ds+E\int_0^T|Y_s^n-Y_s|^2ds+E\int_0^T|Z_s^n-Z_s|^2ds\right).
\end{aligned}
$$

Using Growall's lemma and letting n tends to ∞ we obtain $\overline{Y}_t=Y_t$ and $\overline{Z}_t=Z_t$, $dP\times dt$ $a.e.$

Uniqueness. It follows from the comparison theorem which will be established below. ■

4.2 RBDSDEs with continuous coefficient

In this section we prove the existence of a solution to RBDSDE where the coefficient is only continuous.

We consider the following assumptions

(H4.5) i) for $a.e$ (t,ω), the map $(y,z)\mapsto f(t,y,z)$ is continuous.

ii) There exist constants $\kappa>0$, $L>0$ and $\alpha\in]0,1[$, such that for every $(t,\omega)\in\Omega\times[0,T]$ and $(y,z)\in\mathbb{R}\times\mathbb{R}^d$,

$$
\begin{cases}
|f(t,y,z)|\leq\kappa(1+|y|+|z|)\\
|g(t,y,z)-g(t,y',z')|^2\leq L|y-y'|^2+\alpha|z-z'|^2.
\end{cases}
$$

Theorem 4.2 *Under assumptions* ***(H4.1)***, ***(H4.3)***, ***(H4.4)*** ***and*** ***(H4.5)***, *the RBDSDE* (4.1) *has an adapted solution* (Y, Z, K) *which is a minimal one, in the sense that, if* $\left(\hat{Y}, \hat{Z}\right)$ *is any other solution we have* $Y \leq \hat{Y} \qquad P - a.s.$

Before giving the proof of **Theorem 4.2**, we recall the following classical lemma.

Lemma 4.2 *Let* $f : [0; T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ *be a measurable function such that:*

(a) *For almost every* $(t, \omega) \in [0; T] \times \Omega$, $x \to f(t, x)$ *is continuous,*

(b) *There exists a constant* $K > 0$*such that for every* $(t, x) \in [0; T] \times \mathbb{R}^d$ $|f(t, x)| \leq K(1 + |x|)$ *a.s.*

Then, the sequence of functions

$$f_n(t, x) = \min_{y \in Q^p} (f(y) - n|x - y|),$$

is well defined for each $n \geq K$ *and satisfies:*

(1) *for every* $(t, x) \in [0; T] \times \mathbb{R}^d$, $|f_n(t, x)| \leq K(1 + |x|)$ *a.s..*

(2) *for every* $(t, x) \in [0; T] \times \mathbb{R}^d$, $x \to f(t, x)$ *is continuous is increasing.*

(3) *for every* $n \geq K$ $(t, x, y) \in [0; T] \times (\mathbb{R}^d)^2$, $|f_n(t, x) - f_n(t, y)| \leq n|x - y|$.

(4) *If* $x_n \to x$ *as* $n \to +\infty$ *then for every* $t \in [0; T]$, $f_n(t, x_n) \to f(t, x)$ *as* $n \to +\infty$.

Since ξ satisfies (**H4.3**), we get from **Theorem 4.1**, that for every $n \in \mathbb{N}^*$, there exists a unique solution $\{(Y_t^n, Z_t^n, K_t^n),\ 0 \leq t \leq T\}$ for the following RBDSDE

$$\begin{cases} Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n)ds + K_T^n - K_t^n + \int_t^T g(s, Y_s^n, Z_s^n)dB_s - \int_t^T Z_s^n dW_s,\ 0 \leq t \leq T, \\ Y_t^n \geq S_t,\ \forall t \leq T,\ a.s., \\ \int_0^T (Y_s^n - S_s)\, dK_s^n = 0. \end{cases} \tag{4.7}$$

We consider the function defined by

$$f^1(t, u, v) := \kappa(1 + |u| + |v|).$$

Since, $|f^1(t,u,v)-f^1(t,u',v')|\leq\kappa(|u-u'|+|v-v'|)$, then similar argument as before shows that there exists a unique solution $((U_s,V_s,K_s),0\leq s\leq T)$ to the following RBDSDE:

$$\begin{cases} U_t=\xi+\int_t^T f^1(s,U_s,V_s)ds+K_T-K_t+\int_t^T g(s,U_s,V_s)dB_s-\int_t^T V_s dW_s,\\ U_t\geq S_t,\ \forall t\leq T,\ a.s.,\\ \int_0^T (U_s-S_s)\,dK_s=0.\end{cases}\tag{4.8}$$

We need also the following comparison theorem.

Theorem 4.3 *Let (ξ,f,g,S) and $\left(\acute{\xi},\acute{f},g,\acute{S}\right)$ be two RBDSDEs. Each one satisfying all the previous assumptions* ***(H4.1), (H4.2), (H4.3)*** *and* ***(H4.4).*** *Assume moreover that:*
i) $\xi\leq\acute{\xi}$ a.s.
ii) $f\,(t;y;z)\leq\acute{f}\,(t;y;z)\ dP\times dt$ a.e $\forall\,(y.z)\in\mathbb{R}\times\mathbb{R}^d$.
iii)$S_t\leq\acute{S}_t\ 0\leq t\leq T$ a.s.
Let (Y,Z,K) be a solution of RBDSDE (ξ,f,g,S) and $\left(\acute{Y},\acute{Z},\acute{K}\right)$ be a solution of RBDSDE $\left(\acute{\xi},\acute{f},g,\acute{S}\right)$
Then,

$$Y_t\leq\acute{Y}_t,\ 0\leq t\leq T\quad a.s.$$

Proof. Applying Itô's formula to $\left|(Y_t-Y_t')^+\right|^2$, and passing to expectation, we have

$$\begin{aligned} &E\left|(Y_t-Y_t')^+\right|^2+E\int_t^T 1_{\{Y_s>Y_s'\}}\left|Z_s-Z_s^{'}\right|^2ds\\ &=2E\int_t^T\left(Y_s-Y_s^{'}\right)^+\left(f\,(s,Y_s,Z_s)-f'\left(s,Y_s^{'},Z_s^{'}\right)\right)ds\\ &+2E\int_t^T\left(Y_s-Y_s^{'}\right)^+(dK_s-dK_s')\\ &+E\int_t^T\left|g\,(s,Y_s,Z_s)-g\left(s,Y_s^{'},Z_s^{'}\right)\right|^2 1_{\{Y_s>Y_s'\}}ds.\end{aligned}$$

Since on the set $\{Y_s>Y_s'\}$, we have $Y_t>S_t'\geq S_t$, then

$$\int_t^T\left(Y_s-Y_s^{'}\right)^+(dK_s-dK_s')=-\int_t^T\left(Y_s-Y_s^{'}\right)^+dK_s'\leq 0.$$

Since f is Lipschitz, we have on the set $\{Y_s > Y'_s\}$,

$$\begin{aligned}
&E\left|(Y_t - Y'_t)^+\right|^2 + E\int_t^T 1_{\{Y_s>Y'_s\}}\left|Z_s - Z'_s\right|^2 ds \\
&\leq \left(3L + \frac{1}{\varepsilon}L^2\right) E\int_t^T \left|Y_s - Y'_s\right|^2 1_{\{Y_s>Y'_s\}}ds \\
&+ (\varepsilon+\alpha)\, E\int_t^T \left|Z_s - Z'_s\right|^2 1_{\{Y_s>Y'_s\}}ds.
\end{aligned}$$

We now choose $\varepsilon = \frac{1-\alpha}{2}$, and $\overline{C} = 3L + \frac{1}{\varepsilon}L^2$, to deduce that

$$E\left|(Y_t - Y'_t)^+\right|^2 \leq \overline{C}E\int_t^T \left|(Y_s - Y'_s)^+\right|^2 ds.$$

The result follows now by using Gronwall's lemma. ■

Lemma 4.3 *Let (Y^n, Z^n) be the process defined by equation (4.7). Then,*

i) For every $n \in \mathbb{N}$, $Y_t^0 \leq Y_t^n \leq Y_t^{n+1} \leq U_t$, $\forall t \leq T$, a.s.

ii) There exists $Z \in \mathcal{M}^2$ such that Z^n converges to $Z \in \mathcal{M}^2$.

Proof. Assertion *i)* follows from Theorem 4.3. We shall prove *ii)*.

Itô's formula yields

$$\begin{aligned}
E|Y_0^n|^2 + E\int_0^T |Z_s^n|^2 ds &= E|\xi|^2 + 2E\int_0^T Y_s^n f_n(s, Y_s^n, Z_s^n)ds + 2E\int_0^T S_s dK_s^n \\
&+ E\int_0^T \left|g(s, Y_s^n, Z_s^n)\right|^2 ds.
\end{aligned}$$

But, assumption **(H4.5)** and the inequality $2ab \leq \dfrac{a^2}{r} + rb^2$ for $r > 0$, show that:

$$\begin{aligned}
2Y_s^n f_n(s, Y_s^n, Z_s^n) &\leq \frac{1}{r}|Y_s^n|^2 + r\left|f_n(s, Y_s^n, Z_s^n)\right|^2 \\
&\leq \frac{1}{r}|Y_s^n|^2 + r\left(\kappa(1 + |Y_s^n| + |Z_s^n|)\right)^2,
\end{aligned}$$

and

$$\left|g(s, Y_s^n, Z_s^n)\right|^2 \leq (1+\varepsilon)L|Y_s^n|^2 + (1+\varepsilon)\alpha\left|Z_s^n\right|^2 + (1+\frac{1}{\varepsilon})\left|g(s,0,0)\right|^2.$$

Hence

$$\begin{aligned}
E\int_0^T |Z_s^n|^2 ds &\leq C + \left(r\kappa^2 + (1+\varepsilon)\alpha\right) E\int_0^T |Z_s^n|^2 ds + 2E\int_0^T S_s dK_s^n, \\
&\leq C + \left(r\kappa^2 + (1+\varepsilon)\alpha\right) E\int_0^T |Z_s^n|^2 ds + \beta E(K_T^n)^2.
\end{aligned}$$

On the other hand, we have from (4.7)

$$K_T^n = Y_0^n - \xi - \int_0^T f_n(s, Y_s^n, Z_s^n) ds - \int_0^T g(s, Y_s^n, Z_s^n) dB_s + \int_0^T Z_s^n dW_s, \qquad (4.9)$$

then

$$E(K_T^n)^2 \leq C\left(1 + E\int_0^T |Z_s^n|^2 ds\right),$$

which yield that

$$E\int_0^T |Z_s^n|^2 ds \leq C + \left(r\kappa^2 + (1+\varepsilon)\alpha + \beta C\right) E\int_0^T |Z_s^n|^2 ds.$$

Choosing $r = \varepsilon = \beta = \frac{1-\alpha}{2(\kappa^2+\alpha+C)}$, we obtain

$$E\int_0^T |Z_s^n|^2 ds \leq C.$$

For $n, p \geq K$, Itô's formula gives:

$$\begin{aligned}
E(Y_0^n - Y_0^p)^2 + E\int_0^T |Z_s^n - Z_s^p|^2 ds &= 2E\int_0^T (Y_s^n - Y_s^p)(f_n(s, Y_s^n, Z_s^n) - f_p(s, Y_s^p, Z_s^p)) ds \\
&\quad + 2E\int_0^T (Y_s^n - Y_s^p) dK_s^n + 2E\int_0^T (Y_s^p - Y_s^n) dK_s^p \\
&\quad + E\int_0^T \left|g(s, Y_s^n, Z_s^n) - g(s, Y_s^p, Z_s^p)\right|^2 ds.
\end{aligned}$$

But

$$E\int_0^T (Y_s^n - Y_s^p) dK_s^n = E\int_0^T (S_s - Y_s^p) dK_s^n \leq 0.$$

Similarly, we have $E\int_0^T (Y_s^p - Y_s^n)dK_s^p \leq 0$.

Therefore,

$$E\int_0^T |Z_s^n - Z_s^p|^2 ds \leq 2E\int_0^T (Y_s^n - Y_s^p)(f_n(s,Y_s^n,Z_s^n) - f_p(s,Y_s^p,Z_s^p))ds$$
$$+ E\int_0^T |g(s,Y_s^n,Z_s^n) - g(s,Y_s^p,Z_s^p)|^2\, ds.$$

By Hôlder's inequality and the fact that g is Lipschitz, we get

$$E\int_0^T |Z_s^n - Z_s^p|^2 ds \leq E\int_0^T (Y_s^n - Y_s^p)^2 ds$$
$$+ CE\int_0^T |Y_s^n - Y_s^p|^2 ds + \alpha E\int_0^T |Z_s^n - Z_s^p|^2 ds.$$

Since $\sup_n E\int_0^T |f_n(s,Y_s^n,Z_s^n)|^2 \leq C$, we obtain,

$$E\int_0^T |Z_s^n - Z_s^p|^2 ds \leq C\left(E\int_0^T (Y_s^n - Y_s^p)^2 ds\right).$$

Hence

$$E\int_0^T |Z_s^n - Z_s^p|^2 ds \longrightarrow 0;\ as\ \ n,p \to \infty.$$

Thus $(Z^n)_{n\geq 1}$ is a cauchy sequence in $\mathcal{M}^2(\mathbb{R}^d)$, which end the proof of this Lemma. ■

Proof. of Theorem 4.2. Put $Y_t := sup_n Y_t^n$. The arguments used in the proof of the previous Lemma allow us to show that $(Y^n, Z^n) \to (Y,Z)$ in $\mathcal{M}^2 \times \mathcal{M}^2$. Then, along a subsequence which we still denote (Y^n, Z^n), we get $(Y^n, Z^n) \to (Y,Z)$, $dt \otimes dP\ a.e.$ Then, using Lemma 4.2, we get $f_n(t,Y_t^n,Z_t^n) \to f(t,Y_t,Z_t)\ dP \times dt\ a.e.$

On the other hand, since $Z^n \longrightarrow Z$ in $\mathcal{M}^2(\mathbb{R}^d)$, then there exists $\Lambda \in \mathcal{M}^2(\mathbb{R})$ and a subsequence which we still denote Z^n such that $\forall n, |Z^n| \leq \Lambda$, $Z^n \longrightarrow Z$, $dt \otimes dP\ a.e.$

Moreover from (**H4.5)**, and Lemma 4.3, we have

$$|f_n(t,Y_t^n,Z_t^n)| \leq \kappa(1 + \sup_n |Y_t^n| + \Lambda_t) \in \mathbf{L}^2([0,T],\ dt), \quad P - a.s.$$

It follows from the dominated convergence theorem that
$E\int_0^T \left|f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)\right|^2 ds \underset{n\to\infty}{\longrightarrow} 0$. By **(H4.2),** we have

$$\mathbb{E}\int_0^T |g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2 ds \le C\mathbb{E}\int_0^T |Y_s^n-Y_s|^2 ds + \alpha\mathbb{E}\int_0^T |Z_s^n-Z_s|^2 ds \underset{n\to\infty}{\longrightarrow} 0,$$

Let

$$\overline{Y}_t = \xi + \int_t^T f\left(s,Y_s,Z_s\right) ds + K_T - K_t + \int_t^T g\left(s,Y_s,Z_s\right) dB_s - \int_t^T \overline{Z}_s dW_s, \tag{4.10}$$

$\overline{Z}\in\mathcal{M}^2$, $\overline{Y}\in\mathcal{S}^2$, $K_T\in\mathbb{L}^2$, $\overline{Y}_t\ge S_t$, (K_t) is continuous and nondecreasing, $K_0=0$ and $\int_0^T\left(\overline{Y}_t-S_t\right)dK_t=0$. By Itô's formula we have

$$\begin{aligned}(Y_t^n-\overline{Y}_t)^2 &= 2\int_t^T (Y_s^n-\overline{Y}_s)(f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)ds + 2\int_t^T (Y_s^n-\overline{Y}_s)(dK_s^n-dK_s)\\ &+\int_t^T (Y_s^n-\overline{Y}_s)(g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s))dB_s + 2\int_t^T (Y_s^n-\overline{Y}_s)(Z_s^n-\overline{Z}_s)dW_s\\ &+\int_t^T |g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2 ds - \int_t^T |Z_s^n-\overline{Z}_s|^2 ds.\end{aligned}$$

Passing to expectation and using the fact that $\int_t^T (Y_s^n-\overline{Y}_s)(dK_s^n-dK_s)\le 0$, we get

$$\begin{aligned}E(Y_t^n-\overline{Y}_t)^2 + E\int_t^T |Z_s^n-\overline{Z}_s|^2 ds &\le 2E\int_t^T (Y_s^n-\overline{Y}_s)(f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)ds\\ &+E\int_t^T |g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2 ds.\end{aligned}$$

Letting n goes to ∞, we have $\overline{Y}_t=Y_t$ and $\overline{Z}_t=Z_t$ $dP\times dt$ $a.e.$

Let (Y^*,Z^*,K^*) be a solution of (4.1). Then, by **Theorem 4.3**, we have for every $n\in\mathbb{N}^*$, $Y^n\le Y^*$. Therefore, $\overline{Y}$ is a minimal solution of (4.1). ■

Remark 4.1 *Using the same arguments and the following approximating sequence*

$$f_n\left(t,x\right)=\sup_{y\in Q^p}\left(f(y)-n\left|x-y\right|\right),$$

one can prove that the RBDSDE (4.1) *as a maximal solution.*

Chapter 5

Reflected Discontinuous Backward Doubly Stochastic Differential Equation With Poisson Jumps.

In this Chapter we prove the existence of a solution to a following Backward Doubly Stochastic Differential Equations with Poisson Jumps (RBDSDEPs) and with one continuous barrier

$$Y_t = \xi + \int_t^T f(s,\Lambda_s)ds + \int_t^T g(s,\Lambda_s)d\overleftarrow{B}_s + \int_t^T dK_s - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\,\tilde{\mu}(ds,de)\,,\ 0 \le t \le T, \tag{5.1}$$

where $\Lambda_s = (Y_s, Z_s, U_s)$ and the generator is continuous and also we study the RBDSDEPs with a linear growth condition and left continuity in y on the generator.

5.1 Preliminaries.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. For $T > 0$, We suppose that $(\mathcal{F}_t)_{t\geq 0}$ is generated by the following three mutually independent processes:

(i) Let $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively, for any $d \in \mathbb{N}^*$.

(ii) Let random Poisson measure μ on $E \times \mathbb{R}_+$ with compensator $\nu(dt, de) = \lambda(de)\, dt$, where the space $E = \mathbb{R} - \{0\}$ is equipped with its Borel field $\mathcal{E}$ such that $\{\tilde{\mu}([0,t] \times A) = (\mu - \nu)([0,t] \times A)\}$ is a martingale for any $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. λ is a σ finite measure on $\mathcal{E}$ and satisfies $\int_E \left(1 \wedge |e|^2\right) \lambda(de) < \infty$.

Let $\mathcal{F}_t^W := \sigma(W_s; 0 \leq s \leq t)$, $\mathcal{F}_t^\mu := \sigma(\mu_s; 0 \leq s \leq t)$ and $\mathcal{F}_{t,T}^B := \sigma(B_s - B_t; t \leq s \leq T)$, completed with P-null sets. We put, $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B \vee \mathcal{F}_t^\mu$. It should be noted that $(\mathcal{F}_t)_{t\geq 0}$ is not an increasing family of sub $\sigma-$fields, and hence it is not a filtration.

- Notice the set $\mathcal{B}^2(\mathbb{R}) = \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbb{R})$.

- Notice also the space $\mathcal{D}^2(\mathbb{R}) = \mathcal{S}^2(0,T,\mathbb{R}) \times \mathcal{M}^2\left(0,T,\mathbb{R}^d\right) \times \mathcal{A}^2 \times \mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R})$ endowd with the norm

$$||(Y,Z,K,U)||_{\mathcal{D}^2(\mathbb{R})} = ||Y||_{\mathcal{S}^2(0,T,\mathbb{R})} + ||Z||_{\mathcal{M}^2\left(0,T,\mathbb{R}^d\right)} + ||K||_{\mathcal{A}^2} + ||U||_{\mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R})},$$

  is a Banach space.

- We may ofter write $|\cdot|$ instead of $||U_t||^2_{L^2\left(E,\mathcal{E},\lambda,\mathbb{R}^d\right)}$ for a sake simplicity.

- For $d \in \mathbb{N}^*$, $|\cdot|$ stands for the Euclidian norm in $\mathbb{R}^d \times [0,T]$.

The result depends on the following extension of the well-krown Itô's formula. Its proof follows the same way as lemma 1.3 of [24]

Lemma 5.1 *Let* $\alpha \in \mathcal{S}^2\left(0,T,\mathbb{R}^k\right)$, $(\beta,\gamma) \in \left(\mathcal{M}^2\left(\mathbb{R}^k\right)\right)^2$, $\eta \in \mathcal{M}^2\left(\mathbb{R}^{k\times d}\right)$ *and* $\sigma \in \mathcal{L}^2\left(0,T,\tilde{\mu},\mathbb{R}^k\right)$ *such that:*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dB_s + \int_0^t \eta_s dW_s + \int_0^t dK_s + \int_0^t \int_E \sigma_s(e)\, \tilde{\mu}(ds, de),$$

then (i)

$$\begin{aligned}|\alpha_t|^2 &= |\alpha_0|^2 + 2\int_0^t \langle \alpha_s, \beta_s\rangle ds + 2\int_0^t \langle \alpha_s, \gamma_s\rangle dB_s + 2\int_0^t \langle \alpha_s, \eta_s\rangle dW_s + 2\int_0^t \langle \alpha_s, dK_s\rangle \\ &+ 2\int_0^t \int_E \langle \alpha_{s-}, \sigma(e)\,\tilde{\mu}(ds,de)\rangle - \int_0^t |\gamma_s|^2 ds + \int_0^t |\eta_s|^2 ds + \int_0^t \int_E |\sigma_s(e)|^2 \lambda(de)\, ds \\ &+ \sum_{0\leq s\leq t} (\Delta\alpha_s)^2,\end{aligned}$$

(ii)

$$\begin{aligned}&\mathbb{E}\,|\alpha_t|^2 + \mathbb{E}\int_t^T |\eta_s|^2 ds + \mathbb{E}\int_t^T \int_E |\sigma_s(e)|^2 \lambda(de)\, ds \\ &\leq \mathbb{E}\,|\alpha_T|^2 + 2\mathbb{E}\int_t^T \langle \alpha_s, \beta_s\rangle ds + 2\mathbb{E}\int_t^T \langle \alpha_s, dK_s\rangle + \mathbb{E}\int_t^T |\gamma_s|^2 ds.\end{aligned}$$

5.1.1 Reflected BDSDE with Jumps.

In this subsection, we assume that f and g satisfy the following assumptions **(H5)** on the data (ξ, f, g, S):

(H5.1) $f : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^2(E,\mathcal{E},\lambda,\mathbb{R}) \to \mathbb{R}$; $g : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^2(E,\mathcal{E},\lambda,\mathbb{R}) \to \mathbb{R}$ be jointly measurable such that for any $(y,z,u) \in \mathbb{R}\times\mathbb{R}^d\times L^2(E,\mathcal{E},\lambda,\mathbb{R})$, $f(\cdot,\omega,y,z,u) \in \mathcal{M}^2(0,T,\mathbb{R})$ and $g(\cdot,\omega,y,z,u) \in \mathcal{M}^2(0,T,\mathbb{R})$.

(H5.2) There exist constant $C \geq 0$ and a constant $0 < \alpha < 1$ such that for every $(\omega,t) \in \Omega\times[0,T]$ and $(y,y') \in \mathbb{R}^2$, $(z,z') \in (\mathbb{R}^d)^2$, $(u,u') \in (L^2(E,\mathcal{E},\lambda,\mathbb{R}))^2$

$$\begin{cases} |f(t,\omega,y,z,u) - f(t,\omega,y',z',u')|^2 \leq C\left[|y-y'|^2 + |z-z'|^2 + |u-u'|^2\right], \\ |g(t,\omega,y,z,u) - g(t,\omega,y',z',u')|^2 \leq C\,|y-y'|^2 + \alpha\left\{|z-z'|^2 + |u-u'|^2\right\}. \end{cases}$$

(H5.3) The terminal value ξ be a given random variable in $\mathbb{L}^2$.

(H5.4) $(S_t)_{t\geq 0}$, is a continuous progressively measurable real valued process satisfying

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left(S_t^+\right)^2\right) < +\infty, \quad \text{where} \quad S_t^+ := \max(S_t, 0).$$

(H5.5) $S_T \leq \xi$, $\mathbb{P}$-almost surely.

Definition 5.1 *A solution of a reflected BDSDEPs is a quadruple of processes* (Y, Z, K, U) *wich satisfies*

$$\begin{cases} i) Y \in \mathcal{S}^2(0,T,\mathbb{R}),\ Z \in \mathcal{M}^2\left(0,T,\mathbb{R}^d\right),\ K \in \mathcal{A}^2,\ U \in \mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R}), \\ ii)\ Y_t = \xi + \int_t^T f(s,Y_s,Z_s,U_s)ds + \int_t^T g(s,Y_s,Z_s,U_s)d\overleftarrow{B}_s \\ +\int_t^T dK_s - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\,\tilde{\mu}(ds,de),\ 0 \leq t \leq T, \\ iii)\ S_t \leq Y_t, \quad 0 \leq t \leq T \quad and \quad \int_0^T (Y_t - S_t)\, dK_t = 0. \end{cases}$$

Theorem 5.1 *Assume that* $(H5.1) - (H5.5)$ *holds. Then Eq* (5.1) *admits a unique solution* $(Y, Z, K, U) \in \mathcal{D}^2(\mathbb{R})$.

Proof. Main method is Snell envelope and the fixed point theorem, see [10]. ■

5.2 Comparison theorem.

Given two parameters (ξ^1, f^1, g, T) and (ξ^2, f^2, g, T), we considere the reflected BDSDEPs, $i = 1, 2$

$$\begin{aligned} Y_t^i &= \xi^i + \int_t^T f^i(s,Y_s^i,Z_s^i,U_s^i)ds + \int_t^T g(s,Y_s^i,Z_s^i,U_s^i)d\overleftarrow{B}_s \\ &\quad + \int_t^T dK_s^i - \int_t^T Z_s^i dW_s - \int_t^T \int_E U_s^i(e)\,\tilde{\mu}(ds,de),\ 0 \leq t \leq T. \end{aligned} \tag{5.2}$$

Theorem 5.2 *Assume that the reflected BDSDEP associated with dates* (ξ^1, f^1, g, T), $\left(resp\ \left(\xi^2, f^2, g, T\right)\right)$ *has a solution* $(Y_t^1, Z_t^1, K_t^1, U_t^1)_{t\in[0,T)]}$, $($ *resp* $(Y_t^2, Z_t^2, K_t^2, U_t^2)_{t\in[0,T]})$. *Each one satisfying the assumption* ***(H5)****, assume moreover that:*

$$\begin{cases} \xi^1 \leq \xi^2, \\ \forall t \leq T,\ S_t^1 \leq S_t^2, \\ f^1(t,Y_t,Z_t,U_t) \leq f^2(t,Y_t,Z_t,U_t). \end{cases}$$

Then we have $\mathbb{P} - a.s.$,

$$Y_t^1 \leq Y_t^2.$$

Proof. Let us show that $\left(Y_t^1 - Y_t^2\right)^+ = 0$, using the equation (5.2), we get

$$\bar{Y}_t = \bar{\xi} + \int_t^T \left(f^1(s, Y_s^1, Z_s^1, U_s^1) - f^2(s, Y_s^2, Z_s^2, U_s^2)\right) ds + \int_t^T \left(dK_s^1 - dK_s^2\right)$$
$$+ \int_t^T \left(g(s, Y_s^1, Z_s^1, U_s^1) - g(s, Y_s^2, Z_s^2, U_s^2)\right) d\overleftarrow{B}_s - \int_t^T \bar{Z}_s dW_s - \int_t^T \int_E \bar{U}_s(e)\, \tilde{\mu}(ds, de).$$

Where $\bar{Y}_t = Y_t^1 - Y_t^2$ and $\bar{Z}_t = Z_t^1 - Z_t^2$. Since $\int_t^T \left(\bar{Y}_s\right)^+ \left(g(s, Y_s^1, Z_s^1, U_s^1) - g(s, Y_s^2, Z_s^2, U_s^2)\right) d\overleftarrow{B}_s$ and $\int_t^T \left(\bar{Y}_s\right)^+ \bar{Z}_s dW_s$ are a uniformly integrable martingale. Then taking expectation, we get by applying Lemma 5.1

$$\mathbb{E}\left|\left(\bar{Y}_t\right)^+\right|^2 + \mathbb{E}\int_t^T 1_{\{\bar{Y}_s>0\}} \left|\left|\bar{Z}_s\right|\right|^2 ds + \mathbb{E}\int_t^T \int_E 1_{\{\bar{Y}_s>0\}} \left|\bar{U}_s(e)\right|^2 \lambda(de)\, ds$$
$$\leq \mathbb{E}\left|\left(\bar{\xi}\right)^+\right|^2 + 2\mathbb{E}\int_t^T \left(\bar{Y}_s\right)^+ \left(f^1(s, Y_s^1, Z_s^1, U_s^1) - f^2(s, Y_s^2, Z_s^2, U_s^2)\right) ds$$
$$+ 2\mathbb{E}\int_t^T \left(\bar{Y}_s\right)^+ \left(dK_s^1 - dK_s^2\right) + \mathbb{E}\int_t^T 1_{\{\bar{Y}_s>0\}} \left|\left|g(s, Y_s^1, Z_s^1, U_s^1) - g(s, Y_s^2, Z_s^2, U_s^2)\right|\right|^2 ds.$$

Since on the set $\{Y_s^1 > Y_s^2\}$, we have $Y_s^1 > S_s^2 \geq S_s^1$, then

$$\begin{cases} (\xi^1 - \xi^2)^+ = 0, \\ \int_t^T \left(\bar{Y}_s\right)^+ (dK_s^1 - dK_s^2) = -\int_t^T (Y_s^1 - Y_s^2)^+ dK_s^2 \leq 0, \end{cases}$$

we get

$$\mathbb{E}\left\{\left|\left(\bar{Y}_t\right)^+\right|^2 + \int_t^T 1_{\{\bar{Y}_s>0\}} \left|\left|\bar{Z}_s\right|\right|^2 ds + \int_t^T \int_E 1_{\{\bar{Y}_s>0\}} \left|\bar{U}_s(e)\right|^2 \lambda(de)\, ds\right\}$$
$$\leq 2\mathbb{E}\int_t^T \left(\bar{Y}_s\right)^+ \left(f^1(s, Y_s^1, Z_s^1, U_s^1) - f^2(s, Y_s^2, Z_s^2, U_s^2)\right) ds$$
$$+ \mathbb{E}\int_t^T 1_{\{\bar{Y}_s>0\}} \left|\left|g(s, Y_s^1, Z_s^1, U_s^1) - g(s, Y_s^2, Z_s^2, U_s^2)\right|\right|^2 ds,$$

we obtain, by hypothesis $(H5.2)$, and Young's inequality the following inequality

$$\begin{aligned}
&2\mathbb{E}\int_t^T (\bar{Y}_s)^+ \left(f^1(s,Y_s^1,Z_s^1,U_s^1) - f^2(s,Y_s^2,Z_s^2,U_s^2)\right) ds \\
&\leq \left(2C + 2\epsilon C^2\right)\mathbb{E}\int_t^T \left|\bar{Y}_s^+\right|^2 ds + \epsilon^{-1}\mathbb{E}\int_t^T \left(1_{\{\bar{Y}_s>0\}}\left|\bar{Z}_s\right|^2 + \int_E 1_{\{\bar{Y}_s>0\}}\left|\bar{U}_s\right|^2 \lambda(de)\right) ds,
\end{aligned}$$

also we applying the assumption $(H5.2)$ for g, we get

$$\left|\left|g(s,Y_s^1,Z_s^1,U_s^1) - g(s,Y_s^2,Z_s^2,U_s^2)\right|\right|^2 \leq C\left|\bar{Y}_s\right|^2 ds + \alpha\left\{\left|\bar{Z}_s\right|^2 + \left|\left|\bar{U}_s\right|\right|^2_{L^2(E,\mathcal{E},\lambda,\mathbb{R})}\right\}.$$

Then, we have the following inequality

$$\begin{aligned}
&\mathbb{E}\left\{\left|(\bar{Y}_t)^+\right|^2 + \int_t^T 1_{\{\bar{Y}_s>0\}}\left|\left|\bar{Z}_t\right|\right|^2 ds + \int_t^T\int_E 1_{\{\bar{Y}_s>0\}}\left|\bar{U}_s(e)\right|^2 \lambda(de)\, ds\right\} \\
&\leq (2C + 2\epsilon C)\,\mathbb{E}\int_t^T \left|\bar{Y}_s^+\right|^2 ds + \epsilon^{-1}\mathbb{E}\int_t^T \left(1_{\{\bar{Y}_s>0\}}\left|\bar{Z}_s\right|^2 + \int_E 1_{\{\bar{Y}_s>0\}}\left|\bar{U}_s(e)\right|^2 \lambda(de)\right) ds \\
&+ C\mathbb{E}\int_t^T 1_{\{\bar{Y}_s>0\}}\left|\bar{Y}_s\right|^2 ds + \alpha\mathbb{E}\left\{\int_t^T 1_{\{\bar{Y}_s>0\}}\left|\bar{Z}_s\right|^2 ds + \int_E 1_{\{\bar{Y}_s>0\}}\left|\bar{U}_s(e)\right|^2 \lambda(de)\, ds\right\}, \\
&= (2C + 2\epsilon C + C)\,\mathbb{E}\int_t^T \left|\bar{Y}_s^+\right|^2 ds \\
&+ \left(\epsilon^{-1} + \alpha\right)\mathbb{E}\left\{\int_t^T 1_{\{\bar{Y}_s>0\}}\left|\bar{Z}_s\right|^2 ds + \int_t^T\int_E 1_{\{\bar{Y}_s>0\}}\left|\bar{U}_s(e)\right|^2 \lambda(de)\, ds\right\},
\end{aligned}$$

chossing ϵ and α such that $0 \leq \epsilon^{-1} + \alpha < 1$, we have

$$\mathbb{E}\left|(\bar{Y}_t)^+\right|^2 \leq (2C + 2\epsilon C + C)\,\mathbb{E}\int_t^T \left|\bar{Y}_s^+\right|^2 ds,$$

using Gronwall's lemma implies that

$$\mathbb{E}\left[\left|(\bar{Y}_t)^+\right|^2\right] = 0,$$

finally, we have, $Y_t^1 \leq Y_t^2$. ■

5.3 Reflected BDSDEPs with continuous coefficient.

In this section we are interested in weakening the conditions on f. We assume that f and g satisfy the following assumptions:

(H5.6) There exists $C > 0$ $s.t.$ for all $(t, \omega, y, z, u) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbb{R})$, $(t, \omega, y', z', u') \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbb{R})$

$$\begin{cases} |f(t,\omega,y,z,u)| \leq C(1+|y|+|z|+|u|), \\ \left|g(t,\omega,y,z,u) - g(t,\omega,y',z',u')\right|^2 \leq C\left|y-y'\right|^2 + \alpha\left\{\left|z-z'\right|^2 + \left|u-u'\right|^2\right\}. \end{cases}$$

(H5.7) For fixed ω and t, $f(t, \omega, \cdot, \cdot, \cdot)$ is continuous.

The theree next Lemmas will be useful in the sequel.

Lemma 5.2 *Let $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbb{R}) \to \mathbb{R}$ be a mesurable function such that:*

1. *For a.s. every $(t, \omega) \in [0,T] \times \Omega$, $f(t, \omega, y, z, u)$ is a continuous.*
2. *There exists a constant $C \geq 0$ such that for every $(t, \omega, y, z, u) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbb{R})$, $|f(t, \omega, y, z, u)| \leq C(1+|y|+|z|+|u|)$.*

Then exists the sequence of fonction f_n

$$f_n(t,\omega,y,z,u) = \inf_{(y',z',u') \in \mathcal{B}^2(\mathbb{R})} \left[f\left(t,\omega,y',z',u'\right) + n\left(\left|y-y'\right| + |z-z'| + \left|u-u'\right|\right)\right],$$

is well defined for each $n \geq C$, and it satisfies, $d\mathbb{P} \times dt - a.s.$

(i) Linear growth: $\forall n \geq 1$, $(y, z, u) \in \mathcal{B}^2(\mathbb{R})$, $|f_n(t, \omega, y, z, u)| \leq C(1+|y|+|z|+|u|)$.

(ii) Monotonicity in n : $\forall$ y, z, u, $f_n(t, \omega, y, z, u)$ is increases in n.

(iii) Convergence: $\forall(t, \omega, y, z, u) \in [0,T] \times \Omega \times \mathcal{B}^2(\mathbb{R})$, if $(t, \omega, y_n, z_n, u_n) \to (t, \omega, y, z, u)$, then $f_n(t, \omega, y_n, z_n, u_n) \to f(t, \omega, y, z, u)$.

(iv) Lipschitz condition: $\forall n \geq 1$, $(t, \omega) \in [0,T] \times \Omega$, $\forall(y, z, u) \in \mathcal{B}^2(\mathbb{R})$ and $\left(y', z', u'\right) \in$

$\mathcal{B}^2(\mathbb{R})$, *we have*

$$\left|f_n(t,\omega,y,z,u)-f_n\left(t,\omega,y',z',u'\right)\right|\leq n\left(\left|y-y'\right|+|z-z'|+\left|u-u'\right|\right).$$

Now given $\xi\in\mathbb{L}^2$, $n\in N$, we consider (Y^n,Z^n,K^n,U^n) and (resp (V,N,K,M)) be solutions of the following reflected BDSDEPs:

$$\begin{cases} Y_t^n=\xi+\int_t^T f_n(s,Y_s^n,Z_s^n,U_s^n)ds+\int_t^T g(s,Y_s^n,Z_s^n,U_s^n)d\overleftarrow{B}_s \\ +\int_t^T dK_s^n-\int_t^T Z_s^n dW_s-\int_t^T\int_E U_s^n(e)\,\tilde{\mu}(ds,de)\,,\ 0\leq t\leq T, \\ S_t\leq Y_t^n,\ 0\leq t\leq T,\quad\text{and}\quad \int_0^T(Y_t^n-S_t)\,dK_t^n=0. \end{cases}\tag{5.3}$$

$$\begin{cases} V_t=\xi+\int_t^T F(s,V_s,N_s,M_s)ds+\int_t^T g(s,V_s,N_s,M_s)d\overleftarrow{B}_s \\ +\int_t^T dK_s-\int_t^T N_s dW_s-\int_t^T\int_E M_s(e)\,\tilde{\mu}(ds,de)\,,\ 0\leq t\leq T, \\ S_t\leq V_t,\ 0\leq t\leq T,\quad\text{and}\quad \int_0^T(V_t-S_t)\,dK_t=0, \end{cases}\tag{5.4}$$

where $F(s,\omega,V,N,M)=C(1+|V|+|N|+|M|)$.

Lemma 5.3 *(i) a.s. for all, t and $\forall n\leq m$, $Y_t^n\leq Y_t^m\leq V_t$.*
(ii) Assume that $(H5.1)$, $(H5.3)-(H5.7)$ is in force. Then there exists a constant $A>0$ depending only on C, α, ξ and T such that:

$$||U^n||_{\mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R})}\leq A,\qquad\qquad ||Z^n||_{\mathcal{M}^2\left(0,T,\mathbb{R}^d\right)}\leq A.$$

Proof. The prove of the (i) follow from comparison theorem. It remains to prove (ii), by Lemma 5.1, we have

$$\begin{aligned}&\mathbb{E}\left|Y_t^n\right|^2+\mathbb{E}\int_t^T\left|Z_s^n\right|^2ds+\mathbb{E}\int_t^T\int_E\left|U_s^n(e)\right|^2\lambda(de)\,ds \\ &\leq\mathbb{E}\left|\xi\right|^2+2\mathbb{E}\int_t^T Y_s^n f_n(s,Y_s^n,Z_s^n,U_s^n)ds+2\mathbb{E}\int_t^T Y_s^n dK_s^n+\mathbb{E}\int_t^T||g(s,Y_s^n,Z_s^n,U_s^n)||^2ds.\end{aligned}\tag{5.5}$$

By (i) in lemma 5.2, we have

$$
\begin{aligned}
2\mathbb{E}\int_t^T Y_s^n f_n(s,Y_s^n,Z_s^n,U_s^n)ds &\leq 2C\mathbb{E}\int_t^T Y_s^n\left(1+|Y_s^n|+|Z_s^n|+|U_s^n|\right)ds \\
&\leq TC^2+\mathbb{E}\left(\int_t^T\left(|Y_s^n|^2+2C\,|Y_s^n|^2\,ds+\frac{C^2}{\gamma_1}\,|Y_s^n|^2\right)ds\right) \\
&+\mathbb{E}\left(\int_t^T\left(\gamma_1\,|Z_s^n|^2+\frac{C^2}{\gamma_2}\,|Y_s^n|^2+\gamma_2\int_E|U_s^n(e)|^2\,\lambda(de)\right)ds\right), \\
&\leq TC^2+\left(1+2C+\frac{C^2}{\gamma_1}+\frac{C^2}{\gamma_2}\right)\mathbb{E}\int_t^T|Y_s^n|^2\,ds \\
&+\mathbb{E}\left(\int_t^T\left(\gamma_1\,|Z_s^n|^2+\gamma_2\int_E|U_s^n(e)|^2\,\lambda(de)\right)ds\right),
\end{aligned}
$$

also by the hypothesis associated with g, we get

$$
\begin{aligned}
||g(s,Y_s^n,Z_s^n,U_s^n)||^2 &\leq (1+\epsilon)\,||g(s,Y_s^n,Z_s^n,U_s^n)-g(s,0,0,0)||^2+\frac{1+\epsilon}{\epsilon}\,||g(s,0,0,0)||^2, \\
&\leq (1+\epsilon)\,C\,|Y_s^n|^2+(1+\epsilon)\,\alpha\left\{|Z_s^n|^2+||U_s^n||^2_{L^2(E,\mathcal{E},\lambda,\mathbb{R})}\right\}+\frac{1+\epsilon}{\epsilon}\,||g(s,0,0,0)||^2.
\end{aligned}
$$

Chossing $\gamma_1=\gamma_2=\frac{\epsilon^2}{2}$. Then, we obtain the following inequality

$$
\begin{aligned}
&\mathbb{E}\left(|Y_t^n|^2+\textstyle\int_t^T|Z_s^n|^2\,ds+\int_t^T\int_E|U_s^n(e)|^2\,\lambda(de)\,ds\right) \\
&\leq \mathbb{E}\,|\xi|^2+TC^2+\left(1+2C+\tfrac{4C^2}{\epsilon^2}+(1+\epsilon)\,C\right)\mathbb{E}\textstyle\int_0^T|Y_s^n|^2\,ds+2\int_0^T Y_s^n dK_s^n \\
&+\left(\tfrac{\epsilon^2}{2}+(1+\epsilon)\,\alpha\right)\left\{\mathbb{E}\textstyle\int_t^T|Z_s^n|^2\,ds+\mathbb{E}\int_0^T\int_E|U_s^n(e)|^2\,\lambda(de)\,ds\right\}+\tfrac{1+\epsilon}{\epsilon}\mathbb{E}\textstyle\int_0^T||g(s,0,0,0)||^2\,ds.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\mathbb{E}\int_t^T\left(|Z_s^n|^2+\int_E|U_s^n(e)|^2\,\lambda(de)\right)ds \\
&\leq\left(\frac{\epsilon^2}{2}+(1+\epsilon)\,\alpha\right)\mathbb{E}\left\{\int_t^T|Z_s^n|^2\,ds+\int_t^T\int_E|U_s^n(e)|^2\,\lambda(de)\,ds\right\}+\Lambda+\theta\mathbb{E}\,|K_T^n-K_t^n|^2,
\end{aligned}
$$

where

$$
\Lambda=\left\{\begin{array}{c}
\mathbb{E}\,|\xi|^2+TC^2+\frac{1+\epsilon}{\epsilon}\mathbb{E}\int_t^T||g(s,0,0,0)||^2\,ds+\frac{1}{\theta}\mathbb{E}\left(\sup_{0\leq s\leq T}(S_s)^2\right) \\
+T\left(1+2C+\frac{4C^2}{\epsilon^2}+(1+\epsilon)\,C\right)\mathbb{E}\left(\sup_t|Y_t^n|^2\right).
\end{array}\right.
$$

Now chossing ϵ and α such that $0 \leq \frac{\epsilon^2}{2} + (1+\epsilon)\,\alpha < 1$, we obtain

$$\mathbb{E}\int_t^T |Z_s^n|^2\,ds + \mathbb{E}\int_t^T\int_E |U_s^n(e)|^2\,\lambda(de)\,ds \leq \Lambda + \theta\mathbb{E}\,|K_T^n - K_t^n|^2\,. \tag{5.6}$$

On the other hand, we have from Eq.(5.3)

$$\begin{aligned} K_T^n - K_t^n &= Y_t^n - \xi - \int_t^T f_n(s, Y_s^n, Z_s^n, U_s^n)ds - \int_t^T g(s, Y_s^n, Z_s^n, U_s^n)d\overleftarrow{B}_s \\ &+ \int_t^T Z_s^n dW_s + \int_t^T\int_E U_s^n(e)\,\tilde{\mu}(ds, de)\,. \end{aligned}$$

Using the Hölder's inequality and assupmtion $(H5.6)$, we have

$$\mathbb{E}\,|K_T^n - K_t^n|^2 \leq C_1 + C_2\left(\mathbb{E}\int_t^T |Z_s^n|^2\,ds + \mathbb{E}\int_t^T\int_E |U_t^n(e)|^2\,\lambda(de)\,ds\right),$$

From inequality (5.6), we get

$$\mathbb{E}\int_0^T\left(|Z_s^n|^2 + \int_E |U_s^n(e)|^2\,\lambda(de)\right)ds \leq \Lambda + \theta C_1 + \theta C_2\mathbb{E}\int_t^T\left(|Z_s^n|^2 + \int_E |U_t^n(e)|^2\,\lambda(de)\right)ds,$$

Finally chossing θ such that $0 \leq \theta C_2 \leq 1$, we obtain

$$\mathbb{E}\int_t^T |Z_s^n|^2\,ds + \mathbb{E}\int_t^T\int_E |U_s^n(e)|^2\,\lambda(de)\,ds \leq \Lambda + \theta C_1 < \infty.$$

The prove of Lemma 5.3 is complet. ■

Lemma 5.4 *Assume that* $(H5.1)$, $(H5.3) - (H5.7)$ *is in force. Then the sequence* (Z^n, U^n) *converges a.s. in* $\mathcal{M}^2\left(0, T, \mathbb{R}^d\right) \times \mathcal{L}^2\left(0, T, \tilde{\mu}, \mathbb{R}\right)$.

Proof. Let $n_0 \geq K$. From Eq.(5.3), we deduce that there exists a process $Y \in \mathcal{S}^2(0, T, \mathbb{R})$ such that $Y^n \to Y$ a.s., as $n \to \infty$. Applying Lemma 5.1 to $|Y_t^n - Y_t^m|^2$, for n, $m \geq n_0$

$$\begin{aligned} &\mathbb{E}\left(|Y_t^n - Y_t^m|^2 + \int_t^T |Z_s^n - Z_s^m|^2\,ds + \int_t^T\int_E |U_s^n(e) - U_s^m(e)|^2\,\lambda(de)\,ds\right) \\ &\leq 2\mathbb{E}\int_t^T (Y_s^n - Y_s^m)\left(f_n(s, Y_s^n, Z_s^n, U_s^n) - f_m(s, Y_s^m, Z_s^m, U_s^m)\right)ds \\ &+2\mathbb{E}\int_t^T (Y_s^n - Y_s^m)\left(dK_s^n - dK_s^m\right) + \mathbb{E}\int_t^T ||g(s, Y_s^n, Z_s^n, U_s^n) - g(s, Y_s^m, Z_s^m, U_s^m)||^2\,ds. \end{aligned}$$

Since $\int_t^T (Y_s^n - Y_s^m)(dK_s^n - dK_s^m) \leq 0$, we deduce that

$$\begin{aligned}
&\mathbb{E}\int_t^T |Z_t^n - Z_t^m|^2 ds + \mathbb{E}\int_t^T \int_E |U_s^n(e) - U_s^m(e)|^2 \lambda(de)\, ds \\
&\leq 2\mathbb{E}\int_t^T (Y_s^n - Y_s^m)(f_n(s, Y_s^n, Z_s^n, U_s^n) - f_m(s, Y_s^m, Z_s^m, U_s^m))\, ds \\
&+ \mathbb{E}\int_t^T ||g(s, Y_s^n, Z_s^n, U_s^n) - g(s, Y_s^m, Z_s^m, U_s^m)||^2 ds.
\end{aligned}$$

Using Hölder's inequality and assumption $(H5.6)$ for g, we deduce that

$$\begin{aligned}
&(1-\alpha)\,\mathbb{E}\left\{\int_t^T |Z_t^n - Z_t^m|^2 ds + \int_t^T \int_E |U_s^n(e) - U_s^m(e)|^2 \lambda(de)\, ds\right\} \\
&\leq 2\mathbb{E}\left(\int_t^T |f_n(s, Y_s^n, Z_s^n, U_s^n) - f_m(s, Y_s^m, Z_s^m, U_s^m)|^2 ds\right)^{\frac{1}{2}} \mathbb{E}\left(\int_t^T |Y_s^n - Y_s^m|^2 ds\right)^{\frac{1}{2}} \\
&+ C\mathbb{E}\int_t^T |Y_s^n - Y_s^m|^2 ds.
\end{aligned}$$

Applying assumption $(H5.6)$ for f and the boundedness of the sequence (Y^n, Z^n, U^n), we deduce that

$$(1-\alpha)\left\{\mathbb{E}\int_t^T |Z_t^n - Z_t^m|^2 ds + \mathbb{E}\int_t^T \int_E |U_s^n(e) - U_s^m(e)|^2 \lambda(de)\, ds\right\} \leq C^{te}\mathbb{E}\int_t^T |Y_s^n - Y_s^m|^2 ds,$$

where the constant $C^{te} > 0$ depend only C, α and T.

Which yields that $(Z^n)_{n\geq 0}$ respectively $(U^n)_{n\geq 0}$ is a cauchy sequence in $\mathcal{M}^2\left(0, T, \mathbb{R}^d\right)$, respectively in $\mathcal{L}^2(0, T, \tilde{\mu}, \mathbb{R})$. Then there exists $(Z, U) \in \mathcal{M}^2\left(0, T, \mathbb{R}^d\right) \times \mathcal{L}^2(0, T, \tilde{\mu}, \mathbb{R})$ such that

$$\mathbb{E}\int_0^T |Z_s^n - Z_s|^2 ds + \mathbb{E}\int_0^T \int_E |U_s^n(e) - U_s(e)|^2 \lambda(de)\, ds \to 0, \quad as\ n \to \infty.$$

■

Theorem 5.3 *Assume that* $(H5.1)$, $(H5.3)-(H5.7)$ *holds. Then Eq* (5.1) *admits a solution* $(Y, Z, K, U) \in \mathcal{D}^2(\mathbb{R})$*. Moreover there is a minimal solution* (Y^*, Z^*, U^*, K^*) *of RBDSDEP* (5.1) *in the sense that for any other solution* (Y, Z, U, K) *of Eq.* (5.1)*, we have* $Y^* \leq Y$.

Proof. From Eq.(5.3), it's readily seen that (Y^n) converges in $\mathcal{S}^2(0,T,\mathbb{R})$, $dt \otimes d\mathbb{P} - a.s.$ to $Y \in \mathcal{S}^2(0,T,\mathbb{R})$. Otherwise thanks to Lemma 5.4 there exists two subsequences still noted as the whole sequence $(Z^n)_{n\geq 0}$ respectively $(U^n)_{n\geq 0}$ such that

$$\mathbb{E}\int_0^T |Z_s^n - Z_s|^2 ds \to 0 \text{ as } n \to \infty, \quad \text{and} \quad \mathbb{E}\int_0^T \int_E |U_s^n(e) - U_s(e)|^2 \lambda(de)\, ds \to 0, \text{ as } n \to \infty.$$

Applying Lemma 5.2, we have $f_n(t, Y^n, Z^n, U^n) \to f(t, Y, Z, U)$ and the linear growth of f_n implies

$$|f_n(t, Y_t^n, Z_t^n, U_t^n)| \leq C\left(1 + \sup_n \left(|Y_t^n| + |Z_t^n| + |U_t^n|\right)\right) \in \mathbb{L}^1([0,T]; dt).$$

Thus by Lebesgue's dominated convergence theorem, we deduce that for almost all ω and uniformly in t, we have

$$\mathbb{E}\int_t^T f_n(s, Y_s^n, Z_s^n, U_s^n)\, ds \to \mathbb{E}\int_t^T f(s, Y_s, Z_s, U_s)\, ds.$$

We have by $(H5.6)$ the following estimation

$$\begin{aligned}
&\mathbb{E}\int_t^T ||g(s, Y_s^n, Z_s^n, U_s^n) - g(s, Y_s, Z_s, U_s)||^2 ds \\
&\leq C\mathbb{E}\int_t^T |Y_s^n - Y_s|^2 ds + \alpha\mathbb{E}\int_t^T |Z_s^n - Z_s|^2 ds + \alpha\mathbb{E}\int_t^T \int_E |U_s^n(e) - U_s(e)|^2 \lambda(de)\, ds \to 0,
\end{aligned}$$

as $n \to \infty$, using Burkhôlder-Davis-Gundy inequality, we have

$$\begin{cases}
\mathbb{E}\sup_{0\leq t\leq T}\left|\int_t^T Z_s^n dW_s - \int_t^T Z_s dW_s\right|^2 \to 0, \\
\mathbb{E}\sup_{0\leq t\leq T}\left|\int_t^T \int_E U_s^n(e)\, \tilde{\mu}(ds, de) - \int_t^T \int_E U_s(e)\, \tilde{\mu}(ds, de)\right|^2 \to 0, \\
\mathbb{E}\sup_{0\leq t\leq T}\left|\int_t^T g(s, Y_s^n, Z_s^n, U_s^n) d\overleftarrow{B}_s - \int_t^T g(s, Y_s, Z_s, U_s) d\overleftarrow{B}_s\right|^2 \to 0, \text{ in probability as, } n \to \infty.
\end{cases}$$

Let the following reflected BDSDEPs with data (ξ, f, g, S)

$$\begin{cases} \hat{Y} \in \mathcal{S}^2(0,T,\mathbb{R}), \quad \hat{Z} \in \mathcal{M}^2\left(0,T,\mathbb{R}^d\right), \quad K \in \mathcal{A}^2, \quad \hat{U} \in \mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R}), \\ \hat{Y}_t = \xi + \int_t^T f(s,Y_s,Z_s,U_s)ds + \int_t^T g(s,Y_s,Z_s,U_s)d\overleftarrow{B}_s + \int_t^T dK_s \\ -\int_t^T \hat{Z}_s dW_s - \int_t^T \int_E \hat{U}_s(e)\,\tilde{\mu}(ds,de), \\ S_t \leq \hat{Y}_t, \quad 0 \leq t \leq T \quad \text{and} \quad \int_0^T \left(\hat{Y}_t - S_t\right) dK_t = 0. \end{cases}$$

Hence along a subsequence, we derive that

$$\mathbb{E}\left|Y_t^n - \hat{Y}_t\right|^2 \leq 2\mathbb{E}\int_t^T \left(Y_s^n - \hat{Y}_s\right)\left(f_n(s,Y_s^n,Z_s^n,U_s^n) - f(s,Y_s,Z_s,U_s)\right)ds$$
$$+2\mathbb{E}\int_t^T \left(Y_s^n - \hat{Y}_s\right)\left(dK_s^n - dK_s\right) + \mathbb{E}\int_t^T ||g(s,Y_s^n,Z_s^n,U_s^n) - g(s,Y_s,Z_s,U_s)||^2 ds$$
$$-\mathbb{E}\int_t^T \int_E \left|U_s^n(e) - \hat{U}_s(e)\right|^2 \lambda(de)\,ds - \mathbb{E}\int_t^T \left|Z_s^n - \hat{Z}_s\right|^2 ds.$$

Using the fact that $\mathbb{E}\int_t^T \left(Y_s^n - \hat{Y}_s\right)\left(dK_s^n - dK_s\right) \leq 0$, we get

$$\mathbb{E}\left|Y_t^n - \hat{Y}_t\right|^2 + \mathbb{E}\int_t^T \int_E \left|U_s^n(e) - \hat{U}_s(e)\right|^2 \lambda(de)\,ds + \mathbb{E}\int_t^T \left|Z_s^n - \hat{Z}_s\right|^2 ds$$
$$\leq 2\mathbb{E}\int_t^T \left(Y_s^n - \hat{Y}_s\right)\left(f_n(s,Y_s^n,Z_s^n,U_s^n) - f(s,Y_s,Z_s,U_s)\right)ds$$
$$+\mathbb{E}\int_t^T ||g(s,Y_s^n,Z_s^n,U_s^n) - g(s,Y_s,Z_s,U_s)||^2 ds,$$

letting $n \to \infty$, we have $Y_t = \hat{Y}_t$, $U_t = \hat{U}_t$ and $Z_t = \hat{Z}_t$ $d\mathbb{P} \times dt - a.e.$

Let (Y^*, Z^*, U^*, K^*) be a solution of (5.1). Then by Theorem 5.2, we have for any $n \in \mathbb{N}^*$, $Y^n \leq Y^*$. Therefore, Y is a minimal solution of (5.1). ■

5.4 Reflected BDSDEPs with discontinuous coefficient.

In this section we are interested in weakening the conditions on f. We assume that f satisfy the following assumptions:

(H5.8) There exists a positive process $f_t \in \mathcal{M}^2(0,T,\mathbb{R})$ such that

$$\forall (t,y,z,u) \in [0,T] \times \mathcal{B}^2(\mathbb{R}), \quad |f(t,y,z,u)| \leq f_t(\omega) + C(|y| + |z| + |u|).$$

(H5.9) $f(t,\cdot,z,u) : \mathbb{R} \to \mathbb{R}$ is a left continuous and $f(t,y,\cdot,\cdot)$ is a continuous.

(H5.10) There exists a continuous fonction $\pi : [0,T] \times \mathcal{B}^2(\mathbb{R})$ satisfying for $y \geq y'$, $(z,z') \in \mathbb{R}^{2d}$, $(u,u') \in (L^2(E,\mathcal{E},\lambda,\mathbb{R}))^2$

$$\begin{cases} |\pi(t,y,z,u)| \leq C(|y| + |z| + |u|), \\ f(t,\omega,y,z,u) - f(t,\omega,y',z',u') \geq \pi(t, y-y', z-z', u-u'). \end{cases}$$

(H5.11) g satisfies $(H5.2)$.

Existence result.

The two next Lemmas will be useful in the sequel.

Lemma 5.5 *Assume that π satisfies $(H5.10)$, g satisfies $(H5.11)$ and h belongs in $\mathcal{M}^2(0,T,\mathbb{R})$. For a continuous function of finite variation A belong in $\mathcal{A}^2$ we consider the processes $(\bar{Y},\bar{Z},\bar{U}) \in \mathcal{S}^2(0,T,\mathbb{R}) \times \mathcal{M}^2(0,T,\mathbb{R}^d) \times \mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R})$ such that:*

$$\begin{cases} (i)\ \bar{Y}_t = \xi + \int_t^T \left(\pi\left(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s\right) + h(s)\right) ds + \int_t^T g(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s) d\overleftarrow{B}_s \\ + \int_t^T dA_s - \int_t^T \bar{Z}_s dW_s - \int_t^T \int_E \bar{U}_s(e)\,\tilde{\mu}(ds,de),\ 0 \leq t \leq T, \\ (ii)\ \int_0^T \bar{Y}_t^- dA_s \geq 0. \end{cases} \tag{5.7}$$

Then, we have

(1) *The* ***RBDSDEPs*** (5.7) *admits a minimal solution* $\left(\tilde{Y}_t,\tilde{Z}_t,A_t,\tilde{U}_t\right) \in \mathcal{D}^2(\mathbb{R})$.

(2) *if $h(t) \geq 0$ and $\xi \geq 0$, we have $\bar{Y}_t \geq 0$, $d\mathbb{P} \times dt - a.s.$*

Proof. (1) Obtained from a previous part.

(2) Applying lemma 5.1 to $\left|\bar{Y}_t^-\right|^2$, we have

$$\begin{aligned}
&\mathbb{E}\left(\left|\bar{Y}_t^-\right|^2 + \int_t^T 1_{\{\bar{Y}_s<0\}} \left|\left|\bar{Z}_s\right|\right|^2 ds + \int_t^T \int_E 1_{\{\bar{Y}_s<0\}} \left|\bar{U}_s(e)\right|^2 \lambda(de)\, ds\right) \\
&\leq \mathbb{E}\left(\left|\xi^-\right|^2 - 2\int_t^T \bar{Y}_s^-\left(\pi\left(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s\right) + h(s)\right) ds - 2\int_t^T \bar{Y}_s^- dA_s\right) \\
&+ \mathbb{E}\int_t^T 1_{\{\bar{Y}_s<0\}} \left|\left|g(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s)\right|\right|^2 ds.
\end{aligned}$$

Since $h(t) \geq 0$, $\xi \geq 0$ and using the fact that $\int_0^T \bar{Y}_t^- dA_s \geq 0$, we obtain

$$\begin{aligned}
&\mathbb{E}\left|\bar{Y}_t^-\right|^2 + \mathbb{E}\int_t^T 1_{\{\bar{Y}_s<0\}} \left|\left|\bar{Z}_s\right|\right|^2 ds + \mathbb{E}\int_t^T \int_E 1_{\{\bar{Y}_s<0\}} \left|\bar{U}_s(e)\right|^2 \lambda(de)\, ds \\
&\leq -2\mathbb{E}\int_t^T \bar{Y}_s^- \pi\left(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s\right) ds + \mathbb{E}\int_t^T 1_{\{\bar{Y}_s<0\}} \left|\left|g(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s)\right|\right|^2 ds.
\end{aligned}$$

According to assumptions $(H5.11)$, we get

$$\begin{aligned}
&\mathbb{E}\left|\bar{Y}_t^-\right|^2 + \mathbb{E}\int_t^T 1_{\{\bar{Y}_s<0\}} \left|\left|\bar{Z}_s\right|\right|^2 ds + \mathbb{E}\int_t^T \int_E 1_{\{\bar{Y}_s<0\}} \left|\bar{U}_s(e)\right|^2 \lambda(de)\, ds \\
&\leq -2\mathbb{E}\int_t^T \bar{Y}_s^- \pi\left(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s\right) ds + C\mathbb{E}\int_t^T 1_{\{\bar{Y}_s<0\}} \left|\bar{Y}_s\right|^2 ds \\
&+ \alpha\mathbb{E}\int_t^T 1_{\{\bar{Y}_s<0\}} \left|\left|\bar{Z}_s\right|\right|^2 ds + \alpha\mathbb{E}\int_t^T \int_E 1_{\{\bar{Y}_s<0\}} \left|\bar{U}_s(e)\right|^2 \lambda(de)\, ds,
\end{aligned}$$

applying assumption $(H5.10)$ and using Young's inequality, we have

$$\begin{aligned}
-2\mathbb{E}\int_t^T \bar{Y}_s^- \pi\left(s,\bar{Y}_s,\bar{Z}_s,\bar{U}_s\right) ds &\leq 2C\mathbb{E}\int_t^T \left|\bar{Y}_s^-\right|^2 ds + \frac{1}{2\epsilon}\mathbb{E}\int_t^T \left|\bar{Y}_s^-\right|^2 ds + 2\epsilon C^2\mathbb{E}\int_t^T \left|\left|\bar{Z}_s\right|\right|^2 ds \\
&+ \frac{1}{2\epsilon}\mathbb{E}\int_t^T \left|\bar{Y}_s^-\right|^2 ds + 2\epsilon C^2\mathbb{E}\int_t^T \int_E \left|\bar{U}_s(e)\right|^2 \lambda(de)\, ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbb{E}\left|\bar{Y}_t^-\right|^2 + \mathbb{E}\int_t^T 1_{\{\bar{Y}_s<0\}} \left|\left|\bar{Z}_s\right|\right|^2 ds + \mathbb{E}\int_t^T \int_E 1_{\{\bar{Y}_s<0\}} \left|\bar{U}_s(e)\right|^2 \lambda(de)\, ds \\
&\leq \left(3C + \epsilon^{-1}\right)\mathbb{E}\int_t^T \left|\bar{Y}_s^-\right|^2 ds + \left(\alpha + 2\epsilon C^2\right)\mathbb{E}\int_t^T 1_{\{\bar{Y}_s<0\}}\left(\left|\left|\bar{Z}_s\right|\right|^2 + \int_E \left|\bar{U}_s(e)\right|^2 \lambda(de)\right) ds
\end{aligned}$$

Therefore, choosing ϵ, α and C such that $0<\alpha+2\epsilon C^2<1$ and using Gronwall's inequality, we have

$$\mathbb{E}\left|\bar{Y}_t^-\right|^2=0,$$

$\mathbf{P}-a.s.$ for all $t\in[0,T]$. Finally implies that $\bar{Y}_t\geq 0$, $\mathbf{P}-a.s.$ for all $t\in[0,T]$. ■

Now by Theorem 5.3 above, we consider the processes $\left(\tilde{Y}_t^0,\tilde{Z}_t^0,\tilde{K}_t^0,\tilde{U}_t^0\right)$, $(Y_t^0,Z_t^0,K_t^0,U_t^0)$ and sequence of processes $\left(\tilde{Y}_t^n,\tilde{Z}_t^n,\tilde{K}_t^n,\tilde{U}_t^n\right)_{n\geq 0}$ respectively minimal solution of the following RBDSDEPs for all $t\in[0,T]$

$$\begin{cases}(i)\ \tilde{Y}_t^0=\xi+\int_t^T\left[-C\left(\left|\tilde{Y}_s^0\right|+\left|\tilde{Z}_s^0\right|+\left|\tilde{U}_s^0\right|\right)-f_s\right]ds+\int_t^T g(s,\tilde{Y}_s^0,\tilde{Z}_s^0,\tilde{U}_s^0)d\overleftarrow{B}_s\\ +\int_t^T d\tilde{K}_s^0-\int_t^T\tilde{Z}_s^0dW_s-\int_t^T\int_E\tilde{U}_s^0(e)\,\tilde{\mu}(ds,de),\ 0\leq t\leq T,\\ (ii)\ \tilde{Y}_t^0\geq S_t,\\ (iii)\ \int_0^T\left(\tilde{Y}_s^0-S_s\right)d\tilde{K}_s^0=0.\end{cases}\tag{5.8}$$

$$\begin{cases}(i)\ Y_t^0=\xi+\int_t^T\left[C\left(|Y_s^0|+|Z_s^0|+|U_s^0|\right)+f_s\right]ds+\int_t^T g(s,Y_s^0,Z_s^0,U_s^0)d\overleftarrow{B}_s\\ +\int_t^T dK_s^0-\int_t^T Z_s^0dW_s-\int_t^T\int_E U_s^0(e)\,\tilde{\mu}(ds,de),\ 0\leq t\leq T,\\ (ii)\ Y_t^0\geq S_t,\\ (iii)\ \int_0^T\left(Y_s^0-S_s\right)dK_s^0=0,\end{cases}\tag{5.9}$$

and

$$\begin{cases}(i)\ \tilde{Y}_t^n=\xi+\int_t^T\left[f(s,\tilde{Y}_s^{n-1},\tilde{Z}_s^{n-1},\tilde{U}_s^{n-1})ds+\pi\left(s,\tilde{Y}_s^n-\tilde{Y}_s^{n-1},\tilde{Z}_s^n-\tilde{Z}_s^{n-1},\tilde{U}_s^n-\tilde{U}_s^{n-1}\right)\right]ds\\ +\int_t^T g(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n)d\overleftarrow{B}_s+\int_t^T d\tilde{K}_s^n-\int_t^T\tilde{Z}_s^ndW_s-\int_t^T\int_E\tilde{U}_s^n(e)\,\tilde{\mu}(ds,de),\ 0\leq t\leq T,\\ (ii)\ \tilde{Y}_t^n\geq S_t,\\ (iii)\ \int_0^T\left(\tilde{Y}_s^n-S_s\right)d\tilde{K}_s^n=0.\end{cases}\tag{5.10}$$

Lemma 5.6 *Under the assumptions $(H5.3)-(H5.5)$ and $(H5.8)-(H5.11)$, we have for any $n\geq 1$ and $t\in[0,T]$*

$$\tilde{Y}_t^0\leq\tilde{Y}_t^n\leq\tilde{Y}_t^{n+1}\leq Y_t^0.$$

Proof. For any $n \geq 0$, we set

$$\begin{cases} \delta\rho_t^{n+1} = \rho_t^{n+1} - \rho_t^n, \\ \text{and} \\ \Delta\psi^{n+1}(s, \delta\tilde{Y}_s^{n+1}, \delta\tilde{Z}_s^{n+1}, \delta\tilde{U}_s^{n+1}) \\ = \psi(s, \delta\tilde{Y}_s^{n+1} + \tilde{Y}_s^n, \delta\tilde{Z}_s^{n+1} + \tilde{Z}_s^n, \delta\tilde{U}_s^{n+1} + \tilde{U}_s^n) - \psi(s, \tilde{Y}_s^n, \tilde{Z}_s^n, \tilde{U}_s^n). \end{cases}$$

Using Eq.(5.10), we have

$$\begin{aligned} \delta\tilde{Y}_t^{n+1} = & \int_t^T \left[\pi\left(s, \delta\tilde{Y}_s^{n+1}, \delta\tilde{Z}_s^{n+1}, \delta\tilde{U}_s^{n+1}\right) + \theta_s^{n+1}\right] ds + \int_t^T \Delta g^{n+1}(s, \delta\tilde{Y}_s^{n+1}, \delta\tilde{Z}_s^{n+1}, \delta\tilde{U}_s^{n+1}) d\overleftarrow{B}_s \\ & + \int_t^T d\left(\delta\tilde{K}_s^{n+1}\right) - \int_t^T \delta\tilde{Z}_s^{n+1} dW_s - \int_t^T \int_E \delta\tilde{U}_s^{n+1}(e)\, \tilde{\mu}(ds, de), \end{aligned}$$

where

$$\begin{cases} \theta_s^{n+1} = \Delta f^{n+1}(s, \delta\tilde{Y}_s^n, \delta\tilde{Z}_s^n, \delta\tilde{U}_s^n) - \pi\left(s, \delta\tilde{Y}_s^n, \delta\tilde{Z}_s^n, \delta\tilde{U}_s^n\right) > 0, \\ \text{and} \\ \theta_s^0 = f(s, \tilde{Y}_s^0, \tilde{Z}_s^0, \tilde{U}_s^0) + C\left(\left|\tilde{Y}_s^0\right| + \left|\tilde{Z}_s^0\right| + \left|\tilde{U}_s^0\right|\right) + f_s > 0, \quad \forall n \geq 0. \end{cases}$$

According to the assumptions on f and g, we can show that θ_s^0 and Δg^{n+1}, $\forall n \geq 0$ satisfy all assumption of lemma 5.5. Moreover, since $\tilde{K}_t^n$ is a continuous and increasing process, for all $n \geq 0$, $\delta\tilde{K}_s^{n+1}$ is a continuous process of finite variation and using the same argument as in first part. We can show that

$$\begin{aligned} \int_0^T \left(\tilde{Y}_t^{n+1} - \tilde{Y}_t^n\right)^- d\left(\delta\tilde{K}_t^{n+1}\right) &= \int_0^T \left(\tilde{Y}_t^{n+1} - \tilde{Y}_t^n\right)^- d\left(\tilde{K}_t^{n+1} - \tilde{K}_t^n\right) \\ &= \int_0^T \left(\tilde{Y}_t^{n+1} - \tilde{Y}_t^n\right)^- d\tilde{K}_t^{n+1} - \int_0^T \left(\tilde{Y}_t^{n+1} - \tilde{Y}_t^n\right)^- d\tilde{K}_t^n \geq 0. \end{aligned}$$

Applying lemma 5.5 we deduce that $\delta\tilde{Y}_t^{n+1} \geq 0$, i.e. $\tilde{Y}_t^{n+1} \geq \tilde{Y}_t^n$ $\forall t \in [0, T]$, we have $\tilde{Y}_t^{n+1} \geq \tilde{Y}_t^n \geq \tilde{Y}_t^0$.

Now we shaw prove that $\tilde{Y}_t^{n+1} \leq Y_t^0$, by definition, we obtain

$$\begin{aligned}
&Y_t^0 - \tilde{Y}_t^{n+1} \\
&= \int_t^T \left\{ \left[C\left(\left|Y_s^0\right| + \left|Z_s^0\right| + \left|U_s^0\right| \right) + f_s \right] - f(s, \tilde{Y}_s^n, \tilde{Z}_s^n, \tilde{U}_s^n) - \pi\left(s, \delta\tilde{Y}_s^{n+1}, \delta\tilde{Z}_s^{n+1}, \delta\tilde{U}_s^{n+1} \right) \right\} ds, \\
&+ \int_t^T \left(g(s, Y_s^0, Z_s^0, U_s^0) - g(s, \tilde{Y}_s^{n+1}, \tilde{Z}_s^{n+1}, \tilde{U}_s^{n+1}) \right) d\overleftarrow{B}_s + \int_t^T \left(dK_s^0 - d\tilde{K}_s^{n+1} \right) \\
&- \int_t^T \left(Z_s^0 - \tilde{Z}_s^{n+1} \right) dW_s - \int_t^T \int_E \left(U_s^0(e) - \tilde{U}_s^{n+1}(e) \right) \tilde{\mu}(ds, de) \\
&= \int_t^T \left(-C\left(\left|Y_s^0 - \tilde{Y}_s^{n+1}\right| + \left|Z_s^0 - \tilde{Z}_s^{n+1}\right| + \left|U_s^0 - \tilde{U}_s^{n+1}\right| \right) + \Lambda_s^{n+1} \right) ds \\
&+ \int_t^T \left(g(s, Y_s^0, Z_s^0, U_s^0) - g(s, \tilde{Y}_s^{n+1}, \tilde{Z}_s^{n+1}, \tilde{U}_s^{n+1}) \right) d\overleftarrow{B}_s \\
&+ \int_t^T \left(dK_s^0 - d\tilde{K}_s^{n+1} \right) + \int_t^T \left(Z_s^0 - \tilde{Z}_s^{n+1} \right) dW_s - \int_t^T \int_E \left(U_s^0(e) - \tilde{U}_s^{n+1}(e) \right) \tilde{\mu}(ds, de),
\end{aligned}$$

where

$\Lambda_s^{n+1} = C\left(\left|Y_s^0 - \tilde{Y}_s^{n+1}\right| + \left|Z_s^0 - \tilde{Z}_s^{n+1}\right| + \left|U_s^0 - \tilde{U}_s^{n+1}\right| + |Y_s^0| + |Z_s^0| + |U_s^0| \right) + f_s - f(s, \tilde{Y}_s^n, \tilde{Z}_s^n, \tilde{U}_s^n) - \pi\left(s, \delta\tilde{Y}_s^{n+1}, \delta\tilde{Z}_s^{n+1}, \delta\tilde{U}_s^{n+1} \right) > 0$. Also using lemma 5.5 we deduce that $Y_t^0 - \tilde{Y}_t^{n+1} \geq 0$, i.e. $Y_t^0 \geq \tilde{Y}_t^{n+1}$, for all $t \in [0, T]$. Thus, we have for all $n \geq 0$

$$Y_t^0 \geq \tilde{Y}_t^{n+1} \geq \tilde{Y}_t^n \geq \tilde{Y}_t^0, \; d\mathbb{P} \times dt - a.s., \quad \forall t \in [0, T].$$

■

Lemma 5.7 *(see saisho [26]) Let $(k^n)_{n\in\mathbb{N}}$ be a sequence of continuous and bounded variation functions from $[0, T]$ to $\mathbb{R}$, such that*

i) $\sup_n Var(k^n) \leq C < +\infty$.

ii) $\lim_{n\to+\infty} k^n = k$ *uniformly on* $[0, T]$.

iii) Let $(f^n)_{n\in\mathbb{N}}$ be a sequence of càdlàg functions from $[0, T]$ to $\mathbb{R}$, such that $\lim_{n\to+\infty} f^n = f$ *uniformly on* $[0, T]$.

Then for any $t \in [0, T]$, we have:

$$\lim_{n\to+\infty} \int_0^t f^n(s)\, dk_s^n = \int_0^t f(s)\, dk_s$$

Theorem 5.4 *Under assumption* $(H5.1)$, $(H5.3)-(H5.5)$ *and* $(H5.8)-(H5.11)$, *the RBDSDEPs* (5.1) *has a solution* $(Y_t, Z_t, K_t, U_t)_{0\leq t\leq T} \in \mathcal{D}^2(\mathbb{R})$.

Proof. Since $\left|\tilde{Y}_t^n\right| \leq \max\left(\tilde{Y}_t^0, Y_t^0\right) \leq \left|\tilde{Y}_t^0\right| + |Y_t^0|$ for all $t \in [0,T]$, we have

$$\sup_n \mathbb{E}\left(\sup_{0\leq t\leq T}\left|\tilde{Y}_t^n\right|^2\right) \leq \mathbb{E}\left(\sup_{0\leq t\leq T}\left|\tilde{Y}_t^0\right|^2\right) + \mathbb{E}\left(\sup_{0\leq t\leq T}\left|Y_t^0\right|^2\right) < \infty.$$

Therefore, we deduce from the Lebesgue's dominated convergence theorem that $\left(\tilde{Y}_t^n\right)_{n\geq 0}$ converges in $\mathcal{S}^2(0,T,\mathbb{R})$ to a limit Y.

On the other hand from (5.10), we deduce that

$$\begin{aligned}
\tilde{Y}_0^{n+1} &= \tilde{Y}_T^{n+1} + \int_0^T \left[f(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n)ds + \pi\left(s,\delta\tilde{Y}_t^{n+1},\delta\tilde{Z}_t^{n+1},\delta\tilde{U}_t^{n+1}\right)\right]ds \\
&+ \int_0^T g(s,\tilde{Y}_s^{n+1},\tilde{Z}_s^{n+1},\tilde{U}_s^{n+1})d\overleftarrow{B}_s + \int_0^T d\tilde{K}_s^{n+1} - \int_0^T \tilde{Z}_s^{n+1}dW_s - \int_0^T\int_E \tilde{U}_s^{n+1}(e)\,\tilde{\mu}(ds,de),
\end{aligned}$$

applying Lemma 5.1, we obtain

$$\begin{aligned}
&\mathbb{E}\left|\tilde{Y}_0^{n+1}\right|^2 + \mathbb{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds + \mathbb{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)\,ds \\
&\leq \mathbb{E}|\xi|^2 + 2\mathbb{E}\int_0^T \tilde{Y}_s^{n+1}\left(f(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n) + \pi\left(s,\delta\tilde{Y}_s^{n+1},\delta\tilde{Z}_s^{n+1},\delta\tilde{U}_s^{n+1}\right)\right)ds \\
&+ 2\int_0^T \tilde{Y}_s^{n+1}d\tilde{K}_s^{n+1} + \int_0^T\left\|g(s,\tilde{Y}_s^{n+1},\tilde{Z}_s^{n+1},\tilde{U}_s^{n+1})\right\|^2 ds.
\end{aligned}$$

From $(H5.8)$ and $(H5.10)$, we get

$$\begin{aligned}
&\tilde{Y}_t^{n+1}\left(f(t,\tilde{Y}_t^n,\tilde{Z}_t^n,\tilde{U}_t^n) + \pi\left(t,\delta\tilde{Y}_t^{n+1},\delta\tilde{Z}_t^{n+1},\delta\tilde{U}_t^{n+1}\right)\right) \\
&\leq \left|\tilde{Y}_t^{n+1}\right|\left\{f_t(\omega) + 2C\left(\left|\tilde{Y}_t^n\right| + \left|\tilde{Z}_t^n\right| + \left|\tilde{U}_t^n\right|\right) + C\left(\left|\tilde{Y}_t^{n+1}\right| + \left|\tilde{Z}_t^{n+1}\right| + \left|\tilde{U}_t^{n+1}\right|\right)\right\}, \\
&\leq \frac{\left|\tilde{Y}_t^{n+1}\right|^2}{2} + \frac{f_t(\omega)}{2} + C^2\left|\tilde{Y}_t^{n+1}\right|^2 + \left|\tilde{Y}_t^n\right|^2 + \frac{2C^2}{\epsilon_1}\left|\tilde{Y}_t^{n+1}\right|^2 + \frac{\epsilon_1}{2}\left|\tilde{Z}_t^n\right|^2 + \frac{2C^2}{\epsilon_2}\left|\tilde{Y}_t^{n+1}\right|^2 + \frac{\epsilon_2}{2}\left|\tilde{U}_t^n\right|^2 \\
&+ C\left|\tilde{Y}_t^{n+1}\right|^2 + \frac{C^2}{2\epsilon_3}\left|\tilde{Y}_t^{n+1}\right|^2 + \frac{\epsilon_3}{2}\left|\tilde{Z}_t^{n+1}\right|^2 + \frac{C^2}{2\epsilon_4}\left|\tilde{Y}_t^{n+1}\right|^2 + \frac{\epsilon_4}{2}\left|\tilde{U}_t^{n+1}\right|^2, \\
&= \pi_t^n,
\end{aligned}$$

where

$$\pi_t^n = \left(\frac{1}{2} + C^2 + \frac{2C^2}{\epsilon_1} + \frac{2C^2}{\epsilon_2} + \frac{C^2}{2\epsilon_3} + \frac{C^2}{2\epsilon_4} + C\right)\left|\tilde{Y}_t^{n+1}\right|^2$$
$$+ \frac{\epsilon_3}{2}\left|\tilde{Z}_t^{n+1}\right|^2 + \frac{\epsilon_4}{2}\left|\tilde{U}_t^{n+1}\right|^2 + \left|\tilde{Y}_t^n\right|^2 + \frac{\epsilon_1}{2}\left|\tilde{Z}_t^n\right|^2 + \frac{\epsilon_2}{2}\left|\tilde{U}_t^n\right|^2 + \frac{f_t(\omega)}{2}.$$

Also applying $(H5.11)$, we obtain the following inequality

$$\left|\left|g(s,\tilde{Y}_s^{n+1},\tilde{Z}_s^{n+1},\tilde{U}_s^{n+1})\right|\right|^2 \le 2\left|\left|g(s,\tilde{Y}_s^{n+1},\tilde{Z}_s^{n+1},\tilde{U}_s^{n+1}) - g(s,0,0,0)\right|\right|^2 + 2\left|\left|g(s,0,0,0)\right|\right|^2,$$
$$\le 2C\left|\tilde{Y}_s^{n+1}\right|^2 + 2\alpha\left\{\left|\tilde{Z}_s^{n+1}\right|^2 + \left|\tilde{U}_s^{n+1}\right|^2\right\} + 2\left|\left|g(s,0,0,0)\right|\right|^2.$$

Using Young's inequality, we get

$$2\mathbb{E}\int_0^T \tilde{Y}_s^{n+1} d\tilde{K}_s^{n+1} \le 2\mathbb{E}\int_0^T S_s d\tilde{K}_s^{n+1} \le \frac{1}{\theta}\mathbb{E}\left(\sup_{0\le t\le T}|S_t|^2\right) + \theta\mathbb{E}\left|\tilde{K}_T^{n+1}\right|^2.$$

Therefore, there exists a constant C independent of n such that for any ϵ_i, where $i = 1:4$, we derive

$$\mathbb{E}\int_0^T \left|\tilde{Z}_s^{n+1}\right|^2 ds + \mathbb{E}\int_0^T\int_E \left|\tilde{U}_s^{n+1}(e)\right|^2 \lambda(de)\,ds$$
$$\le C + (\epsilon_3 + 2\alpha)\,\mathbb{E}\int_0^T \left|\tilde{Z}_s^{n+1}\right|^2 ds + (\epsilon_4 + 2\alpha)\,\mathbb{E}\int_0^T\int_E \left|\tilde{U}_s^{n+1}(e)\right|^2 \lambda(de)\,ds \tag{5.11}$$
$$+ \epsilon_1\mathbb{E}\int_0^T \left|\tilde{Z}_s^n\right|^2 ds + \epsilon_2\mathbb{E}\int_t^T\int_E \left|\tilde{U}_s^n(e)\right|^2 \lambda(de)\,ds + \theta\mathbb{E}\left|\tilde{K}_T^{n+1}\right|^2.$$

Moreover, since

$$\tilde{K}_T^{n+1} = \tilde{Y}_0^{n+1} - \xi - \int_0^T \left[f(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n)ds + \pi\left(s,\delta\tilde{Y}_s^{n+1},\delta\tilde{Z}_s^{n+1},\delta\tilde{U}_s^{n+1}\right)\right] ds$$
$$- \int_0^T g(s,\tilde{Y}_s^{n+1},\tilde{Z}_s^{n+1},\tilde{U}_s^{n+1})d\overleftarrow{B}_s + \int_0^T \tilde{Z}_s^{n+1} dW_s + \int_0^T\int_E \tilde{U}_s^{n+1}(e)\,\tilde{\mu}(ds,de),$$

Using Hölder's inequality and assumption $(H5.8)$, $(H5.10)$, we have that

$$\mathbb{E}\left|\tilde{K}_T^{n+1}\right|^2 \leq C_1+C_2\left(\mathbb{E}\int_0^T\left(\left|\tilde{Z}_s^n\right|^2+\left|\tilde{Z}_s^{n+1}\right|^2\right)ds+\mathbb{E}\int_0^T\int_E\left(\left|\tilde{U}_s^n(e)\right|^2+\left|\tilde{U}_s^{n+1}(e)\right|^2\right)\lambda(de)\,ds\right),$$

we come back to inequality (5.11), we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds+\mathbb{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)\,ds\\
&\leq (C+\theta C_1)+(\epsilon_1+\theta C_2)\,\mathbb{E}\int_0^T\left|\tilde{Z}_s^n\right|^2 ds+(\epsilon_2+\theta C_2)\,\mathbb{E}\int_0^T\int_E\left|\tilde{U}_s^n(e)\right|^2\lambda(de)\,ds\\
&+(\epsilon_3+2\alpha+\theta C_2)\,\mathbb{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds+(\epsilon_4+2\alpha+\theta C_2)\,\mathbb{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)\,ds,
\end{aligned}$$

we taking $\epsilon_1=\epsilon_2=\epsilon_0$ and $\epsilon_3=\epsilon_4=\bar{\epsilon}$, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds+\mathbb{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)\,ds\\
&\leq (C+\theta C_1)+(\epsilon_0+\theta C_2)\left\{\mathbb{E}\int_0^T\left|\tilde{Z}_s^n\right|^2 ds+\mathbb{E}\int_0^T\int_E\left|\tilde{U}_s^n(e)\right|^2\lambda(de)\,ds\right\}\\
&+(\bar{\epsilon}+\theta C_2+2\alpha)\,\mathbb{E}\int_0^T\left(\left|\tilde{Z}_s^{n+1}\right|^2+\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)\right)ds,
\end{aligned}$$

we chossing $\bar{\epsilon}$, θ and α such that $0\leq(\bar{\epsilon}+\theta C_2+2\alpha)<1$, we get

$$\begin{aligned}
&\mathbb{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds+\mathbb{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)\,ds\\
&\leq (C+\theta C_1)+(\epsilon_0+\theta C_2)\left\{\mathbb{E}\int_0^T\left|\tilde{Z}_s^n\right|^2 ds+\mathbb{E}\int_0^T\int_E\left|\tilde{U}_s^n(e)\right|^2\lambda(de)\,ds\right\}\\
&\leq (C+\theta C_1)\sum_{i=0}^{i=n-1}(\epsilon_0+\theta C_2)^i+(\epsilon_0+\theta C_2)^n\left\{\mathbb{E}\int_0^T\left|\tilde{Z}_s^0\right|^2 ds+\mathbb{E}\int_0^T\int_E\left|\tilde{U}_s^0(e)\right|^2\lambda(de)\,ds\right\}.
\end{aligned}$$

Now chossing ϵ_0, θ and C_2 such that $\epsilon_0+\theta C_2<1$ and notting $\mathbb{E}\int_0^T\left(\left|\tilde{Z}_s^0\right|^2+\int_E\left|\tilde{U}_s^0\right|^2\lambda(de)\right)ds<\infty$. Obtain

$$\sup_{n\in\mathbb{N}}\mathbb{E}\int_0^T\left|\tilde{Z}_s^{n+1}\right|^2 ds<\infty \quad\text{and}\quad \sup_{n\in\mathbb{N}}\mathbb{E}\int_0^T\int_E\left|\tilde{U}_s^{n+1}(e)\right|^2\lambda(de)\,ds<\infty,$$

consequently, we deduce that

$$\mathbb{E}\left|\tilde{K}_T^{n+1}\right|^2 < \infty.$$

Now we shall prove that $\left(\tilde{Z}^n, \tilde{K}^n, \tilde{U}^n\right)$ is a Cauchy sequence in $\mathcal{M}^2\left(0,T,\mathbb{R}^d\right) \times \mathcal{A}^2 \times \mathcal{L}^2\left(0,T,\tilde{\mu},\mathbb{R}\right)$, set $\Gamma_s^n = f(s,\tilde{Y}_s^{n-1},\tilde{Z}_s^{n-1},\tilde{U}_s^{n-1}) + \pi\left(s,\delta\tilde{Y}_s^n,\delta\tilde{Z}_s^n,\delta\tilde{U}_s^n\right)$, we have

$$\begin{aligned}
\tilde{Y}_t^n - \tilde{Y}_t^m &= \int_t^T \left(\Gamma_s^n - \Gamma_s^m\right) ds + \int_t^T \left(g(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n) - g(s,\tilde{Y}_s^m,\tilde{Z}_s^m,\tilde{U}_s^m)\right) d\overleftarrow{B}_s \\
&+ \int_t^T \left(d\tilde{K}_s^n - d\tilde{K}_s^m\right) - \int_t^T \left(\tilde{Z}_s^n - \tilde{Z}_s^m\right) dW_s - \int_t^T \int_E \left(\tilde{U}_s^n(e) - \tilde{U}_s^m(e)\right) \tilde{\mu}(ds,de),
\end{aligned}$$

applying Lemma 5.1 to $\left|\delta\tilde{Y}_s^{n,m}\right|^2 = \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2$, we have

$$\begin{aligned}
&\mathbb{E}\left|\tilde{Y}_t^n - \tilde{Y}_t^m\right|^2 + \mathbb{E}\int_t^T \left|\tilde{Z}_s^n - \tilde{Z}_s^m\right|^2 ds + \mathbb{E}\int_t^T \int_E \left|\tilde{U}_s^n - \tilde{U}_s^m\right|^2 \lambda(de)\, ds \\
&\leq 2\mathbb{E}\int_t^T \left(\tilde{Y}_s^n - \tilde{Y}_s^m\right)\left(\Gamma_s^n - \Gamma_s^m\right) ds + 2\mathbb{E}\int_t^T \left(\tilde{Y}_s^{n+1} - \tilde{Y}_s^n\right)\left(d\tilde{K}_s^n - d\tilde{K}_s^m\right) \\
&+ \mathbb{E}\int_t^T \left\|\left(g(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n) - g(s,\tilde{Y}_s^m,\tilde{Z}_s^m,\tilde{U}_s^m)\right)\right\|^2 ds,
\end{aligned}$$

since $\int_t^T \left(\tilde{Y}_s^n - \tilde{Y}_s^m\right)\left(d\tilde{K}_s^n - d\tilde{K}_s^m\right) \leq 0$, we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T \left|\tilde{Z}_s^n - \tilde{Z}_s^m\right|^2 ds + \mathbb{E}\int_t^T \int_E \left|\tilde{U}_s^n(e) - \tilde{U}_s^m(e)\right|^2 \lambda(de)\, ds \\
&\leq 2\mathbb{E}\int_t^T \left(\tilde{Y}_s^n - \tilde{Y}_s^m\right)\left(\Gamma_s^n - \Gamma_s^m\right) ds + \mathbb{E}\int_t^T \left\|\left(g(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n) - g(s,\tilde{Y}_s^m,\tilde{Z}_s^m,\tilde{U}_s^m)\right)\right\|^2 ds.
\end{aligned}$$

Applying Hölder's inequality and assumption $(H5.11)$, we deduce that

$$\begin{aligned}
&(1-\alpha)\left\{\mathbb{E}\int_t^T \left|\tilde{Z}_s^n - \tilde{Z}_s^m\right|^2 ds + \mathbb{E}\int_t^T \int_E \left|\tilde{U}_s^n(e) - \tilde{U}_s^m(e)\right|^2 \lambda(de)\right\} \\
&\leq 2\mathbb{E}\left(\int_t^T \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2 ds\right)^{\frac{1}{2}} \mathbb{E}\left(\int_t^T \left|\Gamma_s^n - \Gamma_s^m\right|^2 ds\right)^{\frac{1}{2}} \\
&+ C\mathbb{E}\int_t^T \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2 ds.
\end{aligned}$$

The boundedness of the sequence $\left(\tilde{Y}^n, \tilde{Z}^n, \tilde{K}^n, \tilde{U}^n\right)$, we deduce that

$$\Lambda = \sup_{n\in\mathbb{N}} \left(\mathbb{E} \int_0^T |\Gamma_s^n|^2 \, ds \right) < \infty.$$

This yields that

$$\begin{aligned}
&(1-\alpha)\, \mathbb{E} \int_t^T \left|\tilde{Z}_s^n - \tilde{Z}_s^m\right|^2 ds + \mathbb{E} \int_t^T \int_E \left|\tilde{U}_s^n(e) - \tilde{U}_s^m(e)\right|^2 \lambda(de)\, ds \\
&\leq 4\Lambda \mathbb{E} \left(\int_t^T \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2 ds \right)^{\frac{1}{2}} + C\mathbb{E} \int_t^T \left|\tilde{Y}_s^n - \tilde{Y}_s^m\right|^2 ds.
\end{aligned}$$

Which yields that $\left(\tilde{Z}^n\right)_{n\geq 0}$ respectively $\left(\tilde{U}^n\right)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{M}^2\left(0, T, \mathbb{R}^d\right)$ respectively in $\mathcal{L}^2(0, T, \tilde{\mu}, \mathbb{R})$. Then there exists $(Z, U) \in \mathcal{M}^2\left(0, T, \mathbb{R}^d\right) \times \mathcal{L}^2(0, T, \tilde{\mu}, \mathbb{R})$ such that,

$$\mathbb{E} \int_t^T \left|\tilde{Z}_s^n - Z_s\right|^2 ds + \mathbb{E} \int_t^T \int_E \left|\tilde{U}_s^n(e) - U_s(e)\right|^2 \lambda(de) \to 0, \quad \text{as } n \to \infty. \tag{5.12}$$

On the other hand, applying Burkholder-Davis-Gundy inequality and (5.12), we obtain

$$\begin{cases}
\mathbb{E} \sup_{0\leq t\leq T} \left| \int_t^T \tilde{Z}_s^n dW_s - \int_t^T Z_s dW_s \right|^2 \leq \mathbb{E} \int_t^T \left|\tilde{Z}_s^n - Z_s\right|^2 ds \to 0, \\
\mathbb{E} \sup_{0\leq t\leq T} \left| \int_t^T \int_E \tilde{U}_s^n(e)\, \tilde{\mu}(ds, de) - \int_t^T \int_E U_s(e)\, \tilde{\mu}(ds, de) \right|^2 \\
\leq \mathbb{E} \int_t^T \int_E \left|\tilde{U}_s^n(e) - U_s(e)\right|^2 \lambda(de)\, ds \to 0, \\
\mathbb{E} \sup_{0\leq t\leq T} \left| \int_t^T g(s, \tilde{Y}_s^n, \tilde{Z}_s^n, \tilde{U}_s^n) d\overleftarrow{B}_s - \int_t^T g(s, Y_s, Z_s, U_s) d\overleftarrow{B}_s \right|^2 \\
\leq C\mathbb{E} \int_t^T \left|\tilde{Y}_s^n - Y_s\right|^2 ds + \alpha \mathbb{E} \int_t^T \left|\tilde{Z}_s^n - Z_s\right|^2 ds + \alpha \mathbb{E} \int_t^T \int_E \left|\tilde{U}_s^n(e) - U_s(e)\right|^2 \lambda(de)\, ds \to 0,
\end{cases}$$

as $n \to \infty$. Therefore, from the properieties of (f, π), we have

$$\Gamma_s^n = f(s, \tilde{Y}_s^{n-1}, \tilde{Z}_s^{n-1}, \tilde{U}_s^{n-1}) + \pi\left(s, \delta\tilde{Y}_s^n, \delta\tilde{Z}_s^n, \delta\tilde{U}_s^n\right) \to f(s, Y_s, Z_s, U_s),$$

$P - a.s.$, for all $t \in [0, T]$ as $n \to \infty$. Then follows by dominated convergence theorem that

$$\mathbb{E} \int_0^T |\Gamma_s^n - f(s, Y_s, Z_s, U_s)|^2 \, ds \to 0.$$

Since $\left(\tilde{Y}_s^n, \tilde{Z}_s^n, \tilde{U}_s^n, \Gamma_s^n\right)$ converges in $\mathcal{S}^2(0,T,\mathbb{R})\times\mathcal{M}^2\left(0,T,\mathbb{R}^d\right)\times\mathcal{L}^2(0,T,\tilde{\mu},\mathbb{R})\times\mathcal{M}^2(0,T,\mathbb{R})$ and

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\tilde{K}_t^n-\tilde{K}_t^m\right|^2\right) &\leq \mathbb{E}\left(\left|\tilde{Y}_0^n-\tilde{Y}_0^m\right|^2+\sup_{0\leq t\leq T}\left(\left|\tilde{Y}_t^n-\tilde{Y}_t^m\right|^2+\left|\int_0^t\left(\tilde{Z}_s^n-\tilde{Z}_s^m\right)dW_s\right|^2\right)\right)\\ &+\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t\left(g(s,\tilde{Y}_s^n,\tilde{Z}_s^n,\tilde{U}_s^n)-g(s,\tilde{Y}_s^m,\tilde{Z}_s^m,\tilde{U}_s^m)\right)d\overleftarrow{B}\right|^2\\ &+\mathbb{E}\int_0^T\left|\Gamma_s^n-\Gamma_s^m\right|^2ds+\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t\int_E\left(\tilde{U}_s^n(e)-\tilde{U}_s^m(e)\right)\tilde{\mu}(ds,de)\right|^2,\end{aligned}$$

for any n, $m\geq 0$, we deduce that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\tilde{K}_t^n-\tilde{K}_t^m\right|^2\right)\rightarrow 0,$$

as n, $m\rightarrow\infty$. Consequently, there exists a $\mathcal{F}_t-$measurable process K wich value in $\mathbb{R}$ such that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\tilde{K}_t^n-K_t\right|^2\right)\rightarrow 0,\quad as\ n\rightarrow\infty. \tag{5.13}$$

Finally, we have

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left(\left|\tilde{Y}_t^n-Y_t\right|^2+\left|\tilde{K}_t^n-K_t\right|^2\right)+\int_t^T\left(\left|\tilde{Z}_s^n-Z_s\right|^2+\int_E\left|\tilde{U}_s^n(e)-U_s(e)\right|^2\lambda(de)\right)ds\right)\rightarrow 0,$$

as $n\rightarrow\infty$. Obviously, $K_0=0$ and $\{K_t; 0\leq t\leq T\}$ is a increasing and continuous process. From (5.10), we have for all $n\geq 0$, $\tilde{Y}_t^n\geq S_t$, $\forall t\in[0,T]$, then $Y_t\geq S_t$, $\forall t\in[0,T]$.

On the other hand, from the result of Saisho, we have

$$\int_0^T\left(\tilde{Y}_s^n-S_s\right)d\tilde{K}_s^n\rightarrow\int_0^T(Y_s-S_s)\,dK_s,\ \mathbf{P}-a.s.,\quad as\ n\rightarrow\infty.$$

Using the identity $\int_0^T\left(\tilde{Y}_s^n-S_s\right)d\tilde{K}_s^n=0$ for all $n\geq 0$, we obtain $\int_0^T(Y_s-S_s)\,dK_s=0$. Letting $n\rightarrow+\infty$ in Eq. (5.10), we prove that $(Y_t,Z_t,K_t,U_t)_{t\in[0,T]}$ is solution to (5.1).

Let (Y_*,Z_*,U_*,K_*) be a solution of (5.1). Then by Theorem 5.2, we have for any $n\in\mathbb{N}^*$, $Y^n\leq Y_*$. Therefore, Y is a minimal solution of (5.1). ■

Chapter 6

Reflected solutions of Anticipated Backward Doubly SDEs driven by Teugels Martingales.

In this chapter, we deal with reflected anticipated backward doubly stochastic differential equations (RABDSDEs) driven by Teugels martingales associated with Lévy process. We obtain the existence and uniqueness of solutions to these equations by means of the fixed-point theorem where the coefficients of these BDSDEs depend on the future and present value of the solution (Y, Z). We also show the comparison theorem for a special class of reflected ABDSDEs under some slight stronger conditions. The novelty of our result lies in the fact that we allow the time interval to be infinite.

Xiaoming Xu in [30] extended of the result introduced by Peng and Yang [24] to the following anticipated BDSDE (ABDSDE in short)

$$\begin{cases} Y_t = \xi + \int_t^T f(s,\Lambda_s,\Lambda_s^{\phi,\psi})ds + \int_t^T g(s,\Lambda_s,\Lambda_s^{\phi,\psi})d\overleftarrow{B}_s - \int_t^T Z_s dW_s, & t \in [0,T], \\ (Y_t, Z_t) = (\eta_t, \vartheta_t), & t \in [T, T+\rho], \end{cases} \tag{6.1}$$

where $\Lambda_s = (Y_s, Z_s)$, $\Lambda_s^{\phi,\psi} = \left(Y_{s+\phi(s)}, Z_{s+\psi(s)}\right)$, and $\phi : [0,T] \to \mathbb{R}_+^*$, and $\psi : [0,T] \to \mathbb{R}_+^*$ are continuous functions satisfying:

(A) There exists a constant $\rho \geq 0$ such that for all $t \in [0,T]$,

$$t + \phi(t) \leq T + \rho, \qquad t + \psi(t) \leq T + \rho.$$

(B) There exists a constant $M \geq 0$ such that for each $t \in [0,T]$ and for all nonnegative integrable functions $h(\cdot)$,

$$\begin{cases} \int_t^T h(s+\phi(s))ds \leq M \int_t^{T+\rho} h(s)ds, \\ \text{and} \\ \int_t^T h(s+\psi(s))ds \leq M \int_t^{T+\rho} h(s)ds. \end{cases}$$

In the paper of Nualart et al [22], a martingale representation theorem associated to Lévy processes was proved, then it is natural to extend BSDEs driven by Brownian motion to BSDEs driven by a Lévy process [23]. In the work of Ren et al [12] and [25], the authors proved the existence and uniqueness of solutions of BDSDEs driven by Teugels martingales associated with a Lévy process without barrier, under Lipschitz conditions on the generator f. These results were important from a pure mathematical point of view as well as from an application point of view in the world of finance.

In this chapter, motivated by the above results and by the result introduced by Xiaoming Xu [30], we establish the existence and uniqueness of the solution to the reflected ABDSDE

(RABDSDEs) driven by teugels martingales associated with a Lévy process,

$$\begin{cases} Y_t = \xi + \int_t^T f(s,\Lambda_s,\Lambda_s^{\phi,\psi})ds + \int_t^T g(s,\Lambda_s,\Lambda_s^{\phi,\psi})d\overleftarrow{B}_s + \int_t^T dK_s - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)}dH_s^{(i)}, & t\in[0,T], \\ (Y_t,Z_t) = (\eta_t,\vartheta_t), & t\in[T,T+\rho], \end{cases} \tag{6.2}$$

and $Y_t \geq S_t$ a.s. for any $t\in[0,T+\rho]$ where $\Lambda_s = (Y_s,Z_s)$, $\Lambda_s^{\phi,\psi} = \left(Y_{s+\phi(s)},Z_{s+\psi(s)}\right)$, is derived by mean of the fixed-point theorem. Furthermore we get a existence and uniqueness result of the solution to the previous equation when, $S=-\infty$ i.e., $K\equiv 0$.

Let $X_t = \{X_t, t\geq 0\}$ be the lévy process defined on a complete probability space $(\Omega,\mathcal{F},P,B_t,L_t;0\leq t\leq T)$. It is well known that X_t has a characteristic function of the form

$$\mathbb{E}^{i\theta X_t} = \exp\left[ia\theta t - \frac{1}{2}\sigma^2\theta^2 t + t\int_{\mathbb{R}}\left(e^{i\theta x} - 1 - i\theta x 1_{\{|x|<1\}}\right)v\left(dx\right)\right],$$

where $a\in\mathbb{R}$, $\sigma^2>0$, and the lévy measure v is a measure defined in $\mathbb{R}^*$ and satisfies:

$$\int_{\mathbb{R}}\left(1\wedge x^2\right)v\left(dx\right)<\infty,$$

$\exists\epsilon>0$, $\int_{(-\epsilon,\epsilon)^c} e^{\lambda|x|}v\left(dx\right)<\infty$, for same $\lambda>0$. This implies that the random variables X_t have moments of all orders, i.e.

$$\int_{\mathbb{R}}|x|^i\,v\left(dx\right)<\infty,\ \forall i\geq 2,$$

and that the characteristic function $\mathbb{E}^{i\theta X_t}$ is analytic in a neighborhood of 0. Moreover, it will ensure the existence of the predictable representation (see [22]), wich we will use in our proofs. We refer to [3] for a detailed account of lévy processes.

Following [22, 23], we define, for every $i=1,2,...$, the so-called power-jump processes $\left\{X_t^{(i)},\ t\geq 0\right\}$ and their compensated version $\left\{Y_t^{(i)} = X_t^{(i)} - \mathbb{E}\left[X_t^{(i)}\right],\ t\geq 0\right\}$, also called the Teugels martingales, as follows:

$$X_t^{(1)} = X_t, \qquad L_t^{(i)} = \sum_{0\leq s\leq t} (\Delta L_s)^i, \text{ for } i \geq 2,$$
$$Y_t^{(i)} = X_t^{(i)} - \mathbb{E}\left[X_t^{(i)}\right] = X_t^{(i)} - t\mathbb{E}\left[X_1^{(i)}\right], \text{ for all } i \geq 1.$$

An orthonormalized procedure can be applied to the martingales $Y_t^{(i)}$ in order to obtain a set of pairwise strongly orthonormal martingales $\left\{H^{(i)}\right\}_{i\geq 1}$ in the sense that each $H^{(i)}$ is a linear combination of the $Y^{(i)}$, $j = 1, ..., i$:

$$H^{(i)} = c_{i,i}Y_t^{(i)} + c_{i,i-1}Y_t^{(i-1)} + ... + c_{i,1}Y_t^{(1)},$$

$\left[H^{(i)}, H^{(j)}\right], i \neq j$ and $\left\{\left[H^{(i)}, H^{(i)}\right]_t - t,\ t \geq 0\right\}$ are uniformly integrable martingale with initial value 0, i.e.,

$$\langle H^{(i)}, H^{(j)}\rangle_t = t\delta_{i,j}.$$

It was shown in [23] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, ...$with respect to the measure $\mu(dx) = x^2 v(dx) + \sigma^2\delta_0(dx)$. The resulting processes $H^{(i)} = \left\{H^{(i)}, t \geq 0\right\}$ are called the orthonormalized ith-power-jump processes.

The following Itô formula, which is a useful tool in our work. Its proof follows the same way as lemma 1.3 of [24]

Lemma 6.1 *Let $\alpha \in \mathcal{S}^2_{\mathcal{H}}([0,T];\mathbb{R})$, β, γ and $\sigma \in \mathcal{M}^2_{\mathcal{H}}([0,T];\mathbb{R})$ such that*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dB_s + \int_0^t dK_s + \sum_{i=1}^{\infty}\int_0^t \sigma_s^{(i)} dH_s^{(i)},$$

then

$$|\alpha_t|^2 = |\alpha_0|^2 + 2\int_0^t \alpha_s\beta_s ds + 2\int_0^t \alpha_s\gamma_s dB_s + 2\int_0^t \alpha_s dK_s + 2\sum_{i=1}^{\infty}\int_0^t \alpha_s\sigma_s^{(i)} dH_s^{(i)}$$
$$- \int_0^t |\gamma_s|^2 ds + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t \sigma_s^{(i)}\sigma_s^{(j)} d\left[H^{(i)}, H^{(j)}\right]_s,$$

note that $\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij}t$, *we have*

$$\mathbb{E}\left|\alpha_t\right|^2 = \mathbb{E}\left(\left|\alpha_0\right|^2 + 2\int_0^t \alpha_s\beta_s ds + 2\int_0^t \alpha_s dK_s - \int_0^t \left|\gamma_s\right|^2 ds + \sum_{i=1}^{\infty}\int_0^t \left(\sigma_s^{(i)}\right)^2 ds\right).$$

Remark 6.1 *In the case where* $S = -\infty$ *(i.e., ABDSDEs without lower barrier), the process* K *has no effect i.e.,* $K \equiv 0$.

Definition 6.1 *A solution of equation* (6.2) *is a triple* (Y, Z, K) *which belongs to the space* $\mathcal{B}^2_{\mathcal{H}}([0,\, T+\rho], \mathbb{R}) \times \mathcal{A}^2$ *and satisfies* (6.2) *such that:*

$$\begin{cases} S_t \leq Y_t,\ \ 0 \leq t \leq T+\rho, \\ \int_0^T \left(Y_{s-} - S_{s-}\right) dK_s = 0. \end{cases}$$

Remark 6.2 *In the setup of Problem* (6.2) *the process* $S\left(\cdot\right)$ *play the role of reflecting barrier.*

Remark 6.3 *The state process* $Y\left(\cdot\right)$ *is forced to stay above the lower barrier* $S\left(\cdot\right)$, *thanks to the action of the increasing reflection process* $K\left(\cdot\right)$.

In the following Proposition, we are going to discuss the equation (6.2) has a unique solution with f, g do not depend on the value or the future value of (Y, Z), i.e., P-a.s.,$f\left(t,\omega,y,z,\pi,\zeta\right) = f\left(t,\omega\right)$ and $g\left(t,\omega,y,z,\pi,\zeta\right) = g\left(t,\omega\right)$, for any $\left(t,y,z,\pi,\zeta\right)$, which will play a key role in the two subsection 6.1.1 and 6.1.2.

Proposition 6.1 *[see [25]] Assume* $\xi_T \in \mathbb{L}^2\left(\mathcal{H}_T\right)$, *there exists a unique triple of processes* $\left(Y_t, Z_t, K_t\right) \in \mathcal{B}^2_{\mathcal{H}}\left(\left[0, T+\rho\right], \mathbb{R}\right) \times \mathcal{A}^2$ *solve the following reflected BDSDEs,*

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B}_s + K_T - K_t - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)}, \qquad t \in \left[0, T\right], \\ Y_t > S_t, \qquad t \in \left[0, T\right], \qquad \int_0^T \left(Y_t - S_t\right) dK_t = 0. \end{cases}$$

6.1 Main results and proofs

Assumptions

We assume that f and g satisfy the following assumptions **(H6)**:

(H6.1) (i) There exist a constant $c > 0$ such that for any $(r, \acute{r}) \in [0, T+\rho]^2$, $(t, \omega, y, z, \pi, \zeta)$, $\left(t, \omega, y, z, \acute{\pi}, \acute{\zeta}\right) \in [0, T] \times \Omega \times \mathbb{R} \times l^2 \times \mathcal{S}^2_{\mathcal{H}}([0, T+\rho]; \mathbb{R}) \times \mathcal{M}^2_{\mathcal{H}}([0, T+\rho]; l^2)$,

$$\begin{aligned}&\left|f(t, \omega, y, z, \pi(r), \zeta(\acute{r})) - f(t, \omega, \acute{y}, \acute{z}, \acute{\pi}(r), \acute{\zeta}(\acute{r}))\right|^2 \\ &\leq c\left(|y-\acute{y}|^2 + ||z-\acute{z}||^2_{l^2} + \mathbb{E}^{\mathcal{F}_t}\left[|\pi(r) - \acute{\pi}(r)|^2 + \left|\left|\zeta(\acute{r}) - \acute{\zeta}(\acute{r})\right|\right|^2_{l^2}\right]\right).\end{aligned}$$

(ii) There exists a constant $c > 0$, $0 < \alpha_1 < \frac{1}{2}$ and $0 < \alpha_2 < \frac{1}{M}$ satisfying $0 < \alpha_1 + \alpha_2 M < \frac{1}{2}$, such that

$$\begin{aligned}&\left|g(t, \omega, y, z, \pi(r), \zeta(r)) - g(t, \omega, \acute{y}, \acute{z}, \acute{\pi}(r), \acute{\zeta}(\acute{r}))\right|^2 \\ &\leq c\left(|y-\acute{y}|^2 + \mathbb{E}^{\mathcal{F}_t}|\pi(r) - \acute{\pi}(r)|^2\right) + \alpha_1 ||z-\acute{z}||^2_{l^2} + \alpha_2 \mathbb{E}^{\mathcal{F}_t}\left|\left|\zeta(\acute{r}) - \acute{\zeta}(\acute{r})\right|\right|^2_{l^2}.\end{aligned}$$

(H6.2) For any $(t, \omega, y, z, \pi, \zeta)$,

$$\mathbb{E}\int_0^T \left(|f(s, \omega, 0, 0, 0, 0)| + |g(s, \omega, y, z, \pi, \zeta)|\right) ds < \infty.$$

(H6.3) The terminal value ξ_T be a given random variable in $\mathbb{L}^2(\mathcal{H}_T)$.

We consider also the following assumptions **(B6)**:

(B6.1) $(S_t)_{t\geq 0}$, is a continuous progressively measurable real valued process satisfying

$$\mathbb{E}\left(\sup_{0\leq t\leq T+\rho}\left(S_t^+\right)^2\right) < +\infty, \quad \text{where} \quad S_t^+ := \max(S_t, 0).$$

(B6.2) For any $t \in [T, T+\rho]$, $S_t \leq \eta_t$, $\mathbb{P}$-almost surely.

(B6.3) $(\eta_t, \vartheta_t) \in \mathcal{S}^2_{\mathcal{H}}([T, T+\rho]; \mathbb{R}) \times \mathcal{M}^2_{\mathcal{H}}([T, T+\rho]; l^2)$.

(B6.4) $(K_t)_{t\in[0,T]}$ is a continuous, increasing process with $K_0 = 0$ and $\mathbb{E}(K_T)^2 < +\infty$.

6.1.1 Existence and uniqueness of solution for the Reflected ABDSDE.

In this subsection we study the anticipated BDSDEs with reflection under Lipschitz continuous generator.

Theorem 6.1 *Let f, g satisfies the hypothesis* (**H6**), (**B6**) *and* ***(A)***, ***(B)*** *are hold. Then the reflected ABDSDEs* (6.2) *has a unique solution* $(Y_t, Z_t, K_t)_{t\in[0,T+\rho]}$.

Proof. Let $\mathcal{D}$ the space of couple process $(U_., V_.) \in \mathcal{S}^2_{\mathcal{H}}\left([0, T+\rho]\,;\mathbb{R}^d\right) \times \mathcal{M}^2_{\mathcal{H}}\left([0, T+\rho]\,;l^2\right)$ such that $U_t \geq S_t$ for $t \in [0,T]$ and $(U_t, V_t) = (\eta_t, \vartheta_t)$ for $t \in [T, T+\rho]$ endowed with the norm

$$||(Y,Z)||_\beta = \left(\mathbb{E}\left[\int_0^{T+\rho} e^{\beta s}\left(|Y_{s-}|^2 ds + \sum_{i=1}^{i=\infty}\left|Z_s^{(i)}\right|^2\right) ds\right]\right)^{\frac{1}{2}}.$$

Given $(U_., V_.) \in \mathcal{D}$, we consider the following ABDSDEs with reflection

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s,\theta_s,\theta_s^{\phi,\psi})ds + \int_t^T g(s,\theta_s,\theta_s^{\phi,\psi})d\overleftarrow{B}_s \\ + \int_t^T dK_s - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)}, & t\in[0,T], \\ (Y_t, Z_t) = (\eta_t, \vartheta_t), & t \in [T, T+\rho], \\ Y_t > S_t, \qquad t\in[0,T], \qquad \int_0^T (Y_t - S_t)\, dK_t = 0, \end{cases} \tag{6.3}$$

where $\theta_s = (U_{s-}, V_s)$, $\theta_s^{\phi,\psi} = \left(U_{s+\phi(s)-}, V_{s+\psi(s)}\right)$, which has a unique solution $(Y_., Z_., K_.) \in \mathcal{S}^2_{\mathcal{H}}([0,T+\rho],\mathbb{R}) \times \mathcal{M}^2_{\mathcal{H}}\left([0,T+\rho]\,;l^2\right) \times \mathcal{A}^2$ according to Proposition 6.1. Construct the mapping Φ is well defined from $\mathcal{D}$ into itself by $(Y_t, Z_t) = \Phi\left(U_{t-}, V_t\right)$, then $(Y_., Z_.)$ is the unique solution of system (6.3).
Let $\left(\tilde{U}_{t-}, \tilde{V}_t\right)$ be another element of $\mathcal{D}$ and define $\left(\tilde{Y}_t, \tilde{Z}_t\right) = \Phi\left(\tilde{U}_{t-}, \tilde{V}_t\right)$, then the couple $(\Delta Y_t, \Delta Z_t)$ solve the ABDSDEs with reflection

$$\begin{cases} \Delta Y_t = \int_t^T \Delta f(s)ds + \int_t^T \Delta g(s) d\overleftarrow{B}_s + \int_t^T d\left(\Delta K_s\right) - \sum_{i=1}^{i=\infty}\int_t^T \Delta Z_s^{(i)} dH_s^{(i)}, \ t\in[0,T], \\ (\Delta Y_t, \Delta Z_t) = (0,0), \qquad t\in[T,T+\rho]. \end{cases}$$

where for a function $h \in \{f, g\}$, $\Delta h(s) = h(s,\theta_s,\theta_s^{\phi,\psi}) - h(s,\tilde{\theta}_s,\tilde{\theta}_s^{\phi,\psi})$, $\tilde{\theta}_s = \left(\tilde{U}_{s-}, \tilde{V}_s\right)$, $\tilde{\theta}_s^{\phi,\psi} = \left(\tilde{U}_{s+\phi(s)-}, \tilde{V}_{s+\psi(s)}\right)$ and $\Delta\Psi_s = \Psi_s - \tilde{\Psi}_s$.

For $\beta \in \mathbb{R}_+^*$, applying Itô's formula for $e^{\beta t}|\Delta Y_t|^2$, we get

$$
\begin{aligned}
e^{\beta t}|\Delta Y_t|^2 + \beta \int_t^T e^{\beta s}|\Delta Y_{s-}|^2 ds &= 2\int_t^T e^{\beta s}\Delta Y_{s-}\Delta f(s)ds + 2\int_t^T e^{\beta s}\Delta Y_{s-}\Delta g(s)d\overleftarrow{B}_s \\
&+ 2\int_t^T e^{\beta s}\Delta Y_{s-}d(\Delta K_s) - 2\int_t^T \sum_{i=1}^{i=\infty} e^{\beta s}\Delta Y_{s-}\Delta Z_s^{(i)}dH_s^{(i)} \\
&+ \int_t^T e^{\beta s}|\Delta g(s)|^2 ds - \int_t^T e^{\beta s}\Delta Y_{s-}\sum_{i,j=1}^{\infty}\Delta Z_s^{(i)}\Delta Z_s^{(j)} d\left[H_s^{(i)}, H_s^{(j)}\right].
\end{aligned}
$$

Noting that $\int_t^T e^{\beta s}\Delta Y_{s-}d(\Delta K_s) \leq 0$, using that $\int_0^t e^{\beta s}\Delta Y_{s-}\Delta g(s)d\overleftarrow{B}_s$, $\int_0^t \sum_{i=1}^{i=\infty} e^{\beta s}\Delta Y_{s-}\Delta Z_s^{(i)}dH_s^{(i)}$ $\forall\ i \geq 1$ and $\int_0^t \sum_{i,j=1}^{\infty} e^{\beta s}\Delta Z_s^{(i)}\Delta Z_s^{(j)} d\left(\left[H_s^{(i)}, H_s^{(j)}\right] - \langle H_s^{(i)}, H_s^{(j)}\rangle\right)$ for $i \neq j$ are uniformly integrable martingales and taking the mathematical expectation on bath sides, we obtain

$$
\begin{aligned}
&\mathbb{E}e^{\beta t}|\Delta Y_t|^2 + \beta\mathbb{E}\int_t^T e^{\beta s}|\Delta Y_{s-}|^2 ds + \mathbb{E}\int_t^T \sum_{i=1}^{i=\infty} e^{\beta s}\left|\Delta Z_s^{(i)}\right|^2 ds \\
&\leq 2\mathbb{E}\int_t^T e^{\beta s}\Delta Y_{s-}\Delta f(s)ds + \mathbb{E}\int_t^T e^{\beta s}|\Delta g(s)|^2 ds,
\end{aligned}
$$

Hence for inequality $2ab \leq \epsilon_1 a^2 + \frac{b^2}{\epsilon_1}$ and hypothesis $(H6)$,

$$
2\mathbb{E}\int_t^T e^{\beta s}\Delta Y_s \Delta f(s)ds \leq \mathbb{E}\int_t^{T+\rho}\left[\epsilon_1 e^{\beta s}|\Delta Y_{s-}|^2 + \left(\frac{c+cM}{\epsilon_1}\right)e^{\beta s}\left(|\Delta U_{s-}|^2 + ||\Delta V_s||_{l^2}^2\right)\right]ds,
$$

and also

$$
\mathbb{E}\int_t^T e^{\beta s}|\Delta g(s)|^2 ds \leq \mathbb{E}\int_t^{T+\rho}\left[(c+cM)e^{\beta s}|\Delta U_{s-}|^2 + (\alpha_1+\alpha_2 M)e^{\beta s}||\Delta V_s||_{l^2}^2\right]ds.
$$

Then, we have

$$
\begin{aligned}
&\mathbb{E}e^{\beta t}|\Delta Y_t|^2 + (\beta-\epsilon_1)\mathbb{E}\int_t^T e^{\beta s}|\Delta Y_{s-}|^2 ds + \mathbb{E}\int_t^{T+\rho} e^{\beta s}||\Delta Z_s||_{l^2}^2 ds \\
&\leq \mathbb{E}\left(\int_t^{T+\rho}\left[\left(\frac{c+cM}{\epsilon_1} + c + cM\right)e^{\beta s}|\Delta U_{s-}|^2 + \left((\alpha_1+\alpha_2 M) + \left(\frac{c+cM}{\epsilon_1}\right)\right)e^{\beta s}||\Delta V_s||_{l^2}^2\right]ds\right),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&(\beta-\epsilon_1)\,\mathbb{E}\int_t^{T+\rho} e^{\beta s}\left|\Delta Y_{s-}\right|^2 ds+\mathbb{E}\int_t^{T+\rho} e^{\beta s}\left\|\Delta Z_s\right\|_{l^2}^2 ds\\
&\leq \mathbb{E}\left(\int_t^{T+\rho}\left[\left(\frac{c+cM}{\epsilon_1}+c+cM\right)e^{\beta s}\left|\Delta U_{s-}\right|^2+\left((\alpha_1+\alpha_2 M)+\left(\frac{c+cM}{\epsilon_1}\right)\right)e^{\beta s}\left\|\Delta V_s\right\|_{l^2}^2\right]ds\right),\\
&\leq\left((\alpha_1+\alpha_2 M)+\left(\frac{c+cM}{\epsilon_1}\right)\right)\left(\epsilon_2\mathbb{E}\int_t^{T+\rho} e^{\beta s}\left|\Delta U_{s-}\right|^2 ds+\mathbb{E}\int_t^{T+\rho} e^{\beta s}\left\|\Delta V_s\right\|_{l^2}^2 ds\right),
\end{aligned}
$$

where $\epsilon_2=\frac{\frac{c+cM}{\epsilon_1}+c+cM}{(\alpha_1+\alpha_2M)+\left(\frac{c+cM}{\epsilon_1}\right)}$. Hence if we choose $\epsilon_1,\ \alpha_1,\ \alpha_2$ such that $\hat{c}=(\alpha_1+\alpha_2M)+\left(\frac{c+cM}{\epsilon_1}\right)<1$ and choose $\beta=\epsilon_1+\epsilon_2$, then we deduce

$$
\mathbb{E}\int_t^{T+\rho}\epsilon_2 e^{\beta s}\left|\Delta Y_{s-}\right|^2 ds+\mathbb{E}\int_t^{T+\rho} e^{\beta s}\left\|\Delta Z_s\right\|_{l^2}^2 ds\leq\hat{c}\mathbb{E}\int_t^{T+\rho} e^{\beta s}\left(\epsilon_2\left|\Delta U_{s-}\right|^2+\left\|\Delta V_s\right\|_{l^2}^2\right)ds.
$$

Thus, the mapping Φ is a strict contraction on $\mathcal{D}$ and it has a unique fixed point $(Y_\cdot, Z_\cdot)\in\mathcal{D}$, according to Proposition 6.1, we know $Y_\cdot\in\mathcal{S}^2_{\mathcal{H}}\left([0,T+\rho]\,;\mathbb{R}^d\right)$.

Consequently, $(Y_\cdot, Z_\cdot)\in\mathcal{S}^2_{\mathcal{H}}\left([0,T+\rho]\,;\mathbb{R}^d\right)\times\mathcal{M}^2_{\mathcal{H}}\left([0,T+\rho]\,;l^2\right)$ is the unique solution of reflected ABDSDE (6.2). The proof is complete. ∎

In the next subsection, we will study Problem (6.1) in the case where $S_t=-\infty$, that is, we will establish the existence and uniqueness of the solution to the backward doubly stochastic differential equation with teughles martingales associated by lévy process (6.1).

6.1.2 Existence and uniqueness of solution for the ABDSDE

In this subsection we study the anticipated BDSDEs without reflection under Lipschitz continuous generator.

Theorem 6.2 *Assume that* ***(A), (B)****,* ***(H6)*** *and* ***(B6.3)*** *are satisfied. Then the equation* (6.1) *has a unique solution* $(Y_t, Z_t)\in\mathcal{S}^2_{\mathcal{H}}\left([0,T+\rho]\,;\mathbb{R}\right)\times\mathcal{M}^2_{\mathcal{H}}\left([0,T+\rho]\,;l^2\right)$.

Firstly we start proving equation (6.1) has a unique solution with f, g do not depend on the value or the future value of (Y, Z). More precisely, given f, g such that

$$E\left(\int_0^T |f(t)|^2 dt\right) < \infty,$$
$$E\left(\int_0^T |g(t)|^2 dt\right) < \infty.$$

Proposition 6.2 *Given $\xi_T \in \mathbb{L}^2(\mathcal{H}_T)$, the following BDSDEs,*

$$Y_t = \xi_T + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B}_s - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} dH_s^{(i)}, \quad t \in [0, T],$$

has a unique solution $(Y_t, Z_t) \in \mathcal{S}^2_{\mathcal{H}}([0, T+\rho]; \mathbb{R}) \times \mathcal{M}^2_{\mathcal{H}}([0, T+\rho]; l^2)$.

Proof. Existence. We consider the following filtration

$$\mathcal{G}_t := \mathcal{F}_t^L \vee \mathcal{F}_{T+\rho}^B,$$

and the $\mathcal{G}_t$ square integrable martingale

$$M_t = \mathbb{E}^{\mathcal{G}_t}\left(\xi_T + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B}_s\right), \quad t \in [0, T].$$

Thank's to the prédictable representation property in Nualart et al [22] yields that there exist $Z \in \mathcal{M}^2_{\mathcal{G}}([0, T]; l^2)$ such that

$$M_t = M_0 + \sum_{i=1}^{i=\infty} \int_0^t Z_s^{(i)} dH_s^{(i)},$$

hence

$$M_T = M_t + \sum_{i=1}^{i=\infty} \int_t^T Z_s^{(i)} dH_s^{(i)}.$$

Let

$$\begin{aligned} Y_t &= M_t - \int_t^T f(s)ds - \int_t^T g(s)d\overleftarrow{B}_s, \\ &= \mathbb{E}^{\mathcal{G}_t}\left(\xi_T + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B}_s\right), \\ &= M_T - \sum_{i=1}^{i=\infty}\int_t^T Z_s^{(i)}dH_s^{(i)} - \int_0^t f(s)ds - \int_0^t g(s)d\overleftarrow{B}_s, \end{aligned}$$

from which, we deduce that

$$Y_t = \xi + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B}_s - \sum_{i=1}^{i=\infty}\int_t^T Z_s^{(i)}dH_s^{(i)},$$

we deduce that the triplet (Y, Z) solves (6.1). Next we show that (Y, Z) are in fact $\mathcal{H}_t$-adapted, it is obvious that

$$Y_t = \mathbb{E}\left(\Gamma|\, \mathcal{H}_t \vee \mathcal{F}_{0,t}^B\right),$$

where

$$\Gamma = \xi_T + \int_0^T f(s)ds + \int_0^T g(s)d\overleftarrow{B}_s,$$

is $\mathcal{F}_{0,T}^W \vee \mathcal{F}_{0,T+\rho}^B$-mesurable. Using the fact that $\mathcal{F}_{0,t}^B$ is independent of $\mathcal{H}_t \vee \sigma(\Gamma)$, we deduce that $Y_t = \mathbb{E}^{\mathcal{G}_t}(\Gamma)$. Moreover, we have

$$\sum_{i=1}^{i=\infty}\int_t^T Z_s^{(i)}dH_s^{(i)} = \xi_T + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B}_s - Y_t,$$

and the right-hand side is $\mathcal{F}_{0,T}^W \vee \mathcal{F}_{0,T+\rho}^B$-mesurable.

Uniqueness. Let (Y, Z) and $\left(\tilde{Y}, \tilde{Z}\right)$ be two solution of (6.1) and define $\theta \in \{Y, Z\}$, $\Delta\theta = \theta - \tilde{\theta}$. Then the triplet $(\Delta Y, \Delta Z)$ solves the equation

$$\Delta Y_t + \sum_{i=1}^{j=\infty}\int_t^T \Delta Z_s^{(i)}dH_s^{(i)} = 0, \qquad t \in [0, T].$$

Itô's formula implies

$$\mathbb{E}\left|\Delta Y_t\right|^2+\mathbb{E}\int_t^T\sum_{i=1}^{i=\infty}e^{\beta s}\left|\Delta Z_s^{(i)}\right|^2ds=0,\qquad t\in[0,T].$$

The proof of Proposition 6.2 is complete. ■

We are now in a position to give the proof of Theorem 6.2.

Proof. It remains to show the existence which will be obtained via a fixed point of the contraction of the function Φ defined as follows

$$\Phi:\ \mathcal{D}\ \rightarrow\ \mathcal{D}$$

where $\mathcal{D}$ the space of couple process $(Y_{\cdot},Z_{\cdot})\in\mathcal{S}_{\mathcal{H}}^2([0,T+\rho];\mathbb{R})\times\mathcal{M}_{\mathcal{H}}^2([0,T+\rho];l^2)$, such that $(Y_t,Z_t)_{T\leq t\leq T+\rho}=(\eta_t,\vartheta_t)$ endowed with the norm

$$||(Y,Z)||_{\beta}=\left(\mathbb{E}\left[\int_0^{T+\rho}e^{\beta s}\left(\left|Y_{s-}\right|^2ds+\sum_{i=1}^{i=\infty}\left|Z_s^{(i)}\right|^2\right)ds\right]\right)^{\frac{1}{2}}.$$

Let Φ be the map from $\mathcal{D}$ into itself which to (Y,Z) associates $\Phi(Y,Z)=\left(\tilde{Y},\tilde{Z}\right)$ where the couple $(Y_t,Z_t)_{0\leq t\leq T}\in\mathcal{D}$ is such that $(Y_t,Z_t)_{T\leq t\leq T+\rho}=(\eta_t,\vartheta_t)$ and satisfies the equation (6.1). Thanks to Proposition 6.2, the mapping Φ is well defined. Let $\left(\tilde{Y},\tilde{Z}\right)$ and $\left(\tilde{Y}',\tilde{Z}'\right)$ be two elements of $\mathcal{D}$ such that

$$(Y,Z)=\Phi\left(\tilde{Y},\tilde{Z}\right),\qquad\left(\acute{Y},\acute{Z}\right)=\Phi\left(\tilde{Y}',\tilde{Z}'\right),$$

where $\left(\tilde{Y},\tilde{Z}\right)$ and $\left(\tilde{Y}',\tilde{Z}'\right)$ is the solution of the ABDSDE (6.1) associated with $\left(\xi,f(s,\theta_s,\theta_s^{\phi,\psi}),g(s,\theta_s,\theta_s^{\phi,\psi})\right)$ and $\left(\xi,f(s,\tilde{\theta}'_s,\tilde{\theta}'^{\phi,\psi}_s),g(s,\tilde{\theta}'_s,\tilde{\theta}'^{\phi,\psi}_s)\right)$ such that $\theta_s=\left(\tilde{Y}_{s-},\tilde{Z}_s\right)$, $\theta_s^{\phi,\psi}=\left(\tilde{Y}_{s+\phi(s)-},\tilde{Z}_{s+\psi(s)}\right)$, $\tilde{\theta}'_s=\left(\tilde{Y}'_{s-},\tilde{Z}'_s\right)$ and $\tilde{\theta}'^{\phi,\psi}_s=\left(\tilde{Y}'_{s+\phi(s)-},\tilde{Z}'_{s+\psi(s)}\right)$. We use the following notation for $h\in\{f,g\}$, $\Delta h(s)=h(s,\theta_s,\theta_s^{\phi,\psi})-h(s,\tilde{\theta}'_s,\tilde{\theta}'^{\phi,\psi}_s)$, $\Delta\tilde{\Psi}_s=\tilde{\Psi}_s-\tilde{\Psi}'_s$ and $\Delta\Psi_s=\Psi_s-\Psi'_s$.

Then, to obtain this result, we use the same calculation used in subsection 6.1.1, but we take

$S = -\infty$ i.e., $K = 0$. For $\beta \in \mathbb{R}_+^*$, we get

$$\mathbb{E}\int_t^{T+\rho} e^{\beta s}\left(\epsilon_2 \left|\Delta Y_{s-}\right|^2 + ||\Delta Z_s||_{l^2}^2\right) ds \leq \hat{c}\mathbb{E}\int_t^{T+\rho} e^{\beta s}\left(\epsilon_2 \left|\Delta \tilde{Y}_{s-}\right|^2 + \left|\left|\Delta \tilde{Z}_s\right|\right|_{l^2}^2\right) ds,$$

where $0 < \hat{c} < 1$ and $\epsilon_2 > 0$. Thus, the mapping Φ is a strict contraction on $\mathcal{D}$ and it has a unique fixed point $(Y_., Z_.) \in \mathcal{D}$.

Consequently, $(Y_., Z_.) \in \mathcal{S}_{\mathcal{H}}^2([0, T+\rho];\mathbb{R}) \times \mathcal{M}_{\mathcal{H}}^2([0, T+\rho];l^2)$ is the unique solution of ABDSDE (6.1). Finally we complete the proof of Theorem 6.2. ■

Remark 6.4 *In* (6.2)*, if* $\int_t^T g(s, Y_s, Z_s, Y_{s+\phi(s)}, Z_{s+\psi(s)})d\overleftarrow{B}_s \equiv 0$*,* $S_. = -\infty$ *and* $K_. = 0$*, then we have*

$$\begin{cases} Y_t = \xi + \int_t^T f(s, \Lambda_s, \Lambda_s^{\phi,\psi})ds - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)}, & t \in [0, T], \\ \\ (Y_t, Z_t) = (\eta_t, \vartheta_t), & t \in [T, T+\rho]. \end{cases}$$

G. Zong [37] study the previous anticipated BSDE driven by teugels martingale and obtained an existence and uniqueness theorem.

6.1.3 Comparison theorem

In general we do not have a comparison result for solutions of BDSDEs driven by Lévy process, reflected or not.

In this subsection our objective is to obtain a comparison result for the following equations for $j = 1, 2$

$$\begin{cases} Y_t^j = \xi_T^j + \int_t^T f^j(s, Y_{s-}^j, Z_s^j, Y_{s+\phi(s)-}^j)ds + \int_t^T g(s, Y_{s-}^j, Y_{s+\phi(s)-}^j)d\overleftarrow{B}_s \\ + \int_t^T dK_s^j - \sum_{i=1}^{\infty}\int_t^T Z_s^{j,(i)} dH_s^{(i)}, & t \in [0, T], \\ Y_t^j = \eta_t^j, & t \in [T, T+\rho], \end{cases}$$

Theorem 6.3 *Assume that* ***(H6)***, ***(B6)*** *and* ***(A)***, ***(B)*** *are satisfied. Assume moreover that:*

- For all $t \in [0,T]$, $y \in \mathbb{R}$, $z \in l^2$, $f^2(t,y,z,\cdot)$ is increasing.
- For any $t \in [T,T+\rho]$, $\xi_t^2 \leq \xi_t^1$, $\mathbb{P}$-almost surely.
- For any $t \in [0,T+\rho]$, $S_t^2 \leq S_t^1$, $\mathbb{P}$-almost surely.
- $f^2\left(t, y_{t-}^1, z_t^1, y_{t+\phi(t)-}^1\right) \leq f^1\left(t, y_{t-}^1, z_t^1, y_{t+\phi(t)-}^1\right)$, $dt \times dP - a.s.$.
- For all $i \in \mathbb{N}$ let $\tilde{Z}^i$ denote the l^2-valued stochastic process such that its i first component are equal to those of Z^2 and its $\mathbb{N} \setminus \{1,2,...,i\}$ last components are equal to those of Z^1. With this notation, we define, for $i \in \mathbb{N}$

$$\pi_t^i := \frac{f^1(t, Y_{t-}^2, \tilde{Z}_t^{i-1}, Y_{t+\phi(t)-}^1) - f^1(t, Y_{t-}^2, \tilde{Z}_t^i, Y_{t+\phi(t)-}^1)}{\left(Z_t^{1,(i)} - Z_t^{2,(i)}\right) 1_{\left\{Z_t^{1,(i)} \neq Z_t^{2,(i)}\right\}}},$$

  satisfying that $\sum_{i=1}^{\infty} \pi_t^i \Delta H_t^{(i)} > -1$, $P - a.s,$.

Then, we have that almost surely for any time t, $Y_t^2 \leq Y_t^1$.

Proof. Set the following reflected BDSDE,

$$\begin{cases} Y_t^3 = \xi_T^2 + \int_t^T f^2(s, Y_{s-}^3, Z_s^3, Y_{s+\phi(s)-}^1) ds + \int_t^T g(s, Y_{s-}^3, Y_{s+\phi(s)-}^1) d\overleftarrow{B}_s \\ + \int_t^T dK_s^3 - \sum_{i=1}^{\infty} \int_t^T Z_s^{3,(i)} dH_s^{(i)}, & t \in [0,T], \\ Y_t^3 = \eta_t^2, & t \in [T, T+\rho]. \end{cases}$$

We set the following notations

$$\bar{f}_t = f^1(t, Y_{t-}^1, Z_t^1, Y_{t+\phi(t)-}^1) - f^2(t, Y_{t-}^1, Z_t^1, Y_{t+\phi(t)-}^1) \geq 0,$$

and

$$\bar{\eta} = \eta^1 - \eta^2, \qquad \bar{\xi}_T = \xi_T^1 - \xi_T^2, \qquad \bar{Y} = Y^1 - Y^3, \qquad \bar{Z} = Z^1 - Z^3, \qquad \bar{K} = K^1 - K^3.$$

Then the triple process $\left(\bar{Y}, \bar{Z}, \bar{K}\right)$ can be regarded as the solution to the following linear reflected BDSDE

$$\begin{cases} \bar{Y}_t = \bar{\xi}_T + \int_t^T \left[\left(\bar{f}_s + a_s \bar{Y}_{s-} + \sum_{i=1}^{\infty} \pi_s^i \bar{Z}_s^{(i)}\right) ds + b_s \bar{Y}_{s-} dB_s + d\bar{K}_s\right] - \sum_{i=1}^{\infty} \int_t^T \bar{Z}_s^{(i)} dH_s^{(i)}, \ t \in [0,T], \\ \bar{Y}_t = \bar{\eta}_t, \qquad t \in [T, T+\rho], \end{cases}$$

where,

$$a_t = \frac{f^2(t, Y^1_{t-}, Z^1_t, Y^1_{t+\phi(t)-}) - f^2(t, Y^3_{t-}, Z^1_t, Y^1_{t+\phi(t)-})}{(Y^1_{t-} - Y^3_{t-})\, 1_{\left\{Y^1_{t-} \neq Y^3_{t-}\right\}}},$$
$$b_t = \frac{g(s, Y^1_{s-}, Y^1_{s+\phi(s)-}) - g(s, Y^3_{s-}, Y^1_{s+\phi(s)-})}{(Y^1_{t-} - Y^3_{t-})\, 1_{\left\{Y^1_{t-} \neq Y^3_{t-}\right\}}}.$$

Let $\Gamma_s,\ s \in [t,T]$, be solution of the linear stochastic differential equation

$$\Gamma_t = 1 + \int_0^t \Gamma_s d\Lambda_s,$$

where $\Lambda_t = \int_0^t a_s ds + \int_0^t b_s dB_s + \sum_{i=1}^{\infty} \int_0^t \pi_s^i dH_s^{(i)}$. Now applying Itô's formula to $\Gamma_t \bar{Y}_t$, we get

$$\begin{aligned} \Gamma_T \bar{Y}_T &= \Gamma_t \bar{Y}_t + \int_t^T \Gamma_{s-} d\bar{Y}_s + \int_t^T \bar{Y}_{s-} d\Gamma_s + \int_t^T d\left[\Gamma, \bar{Y}\right]_s, \\ &= \Gamma_t \bar{Y}_t - \int_t^T \Gamma_{s-} \left(\bar{f}_s + a_s \bar{Y}_{s-} + \sum_{i=1}^{\infty} \pi_s^i \bar{Z}_s^{(i)}\right) ds - \int_t^T \Gamma_{s-} b_s \bar{Y}_{s-} dB_s - \int_t^T \Gamma_{s-} d\bar{K}_s \\ &+ \sum_{i=1}^{\infty} \int_t^T \Gamma_{s-} \bar{Z}_s^{(i)} dH_s^{(i)} + \int_t^T \bar{Y}_{s-} \Gamma_{s-} a_s ds + \int_t^T \Gamma_{s-} b_s \bar{Y}_{s-} dB_s \\ &+ \sum_{i=1}^{\infty} \int_t^T \bar{Y}_{s-} \Gamma_s \pi_s^i dH_s^{(i)} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_t^T \Gamma_{s-} \pi_s^i \bar{Z}_s^{(j)} d\left[H^{(i)}, H^{(j)}\right]_s \\ &= \Gamma_t \bar{Y}_t - \int_t^T \Gamma_{s-} \bar{f}_s ds - \int_t^T \Gamma_{s-} d\bar{K}_s - \sum_{i=1}^{\infty} \int_t^T \Gamma_{s-} \pi_s^i \bar{Z}_s^{(i)} ds + \sum_{i=1}^{\infty} \int_t^T \Gamma_{s-} \bar{Z}_s^{(i)} dH_s^{(i)} \\ &+ \sum_{i=1}^{\infty} \int_t^T \bar{Y}_{s-} \Gamma_s \pi_s^i dH_s^{(i)} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_t^T \Gamma_{s-} \pi_s^i \bar{Z}_s^{(j)} d\left[H^{(i)}, H^{(j)}\right]_s. \end{aligned}$$

Taking conditional expectation w.r.t. $\mathcal{H}_t$, we get

$$\bar{Y}_t = \mathbb{E}\left(\Gamma_T \bar{Y}_T + \int_t^T \Gamma_{s-} \bar{f}_s ds + \int_t^T \Gamma_{s-} d\bar{K}_s \middle| \mathcal{H}_t\right),$$

since $\bar{Y}_T = \bar{\xi}_T \geq 0$, $\Gamma_t \geq 0$, $\bar{f}_s \geq 0$ and $d\bar{K}_s \geq 0$, we have

$$\bar{Y}_t \geq 0,$$

we conclude that $Y_t^1 \geq Y_t^3, a.s.$.

Set

$$\begin{cases} Y_t^4 = \xi_T^2 + \int_t^T f^2(s, Y_{s-}^3, Z_s^4, Y_{s+\phi(s)-}^3)ds + \int_t^T g(s, Y_{s-}^3, Y_{s+\phi(s)-}^3)d\overleftarrow{B}_s \\ + \int_t^T dK_s^4 - \sum_{i=1}^{\infty} \int_t^T Z_s^{4,(i)} dH_s^{(i)}, \qquad t \in [0,T], \\ Y_t^4 = \eta_t^2, \qquad t \in [T, T+K]. \end{cases}$$

since $t \in [0,T]$, $y \in \mathbb{R}$, $z \in l^2$, $f^2(t,y,z,\cdot)$ is increasing and $Y_t^1 \geq Y_t^3$, we know that for almost all t, $Y_t^3 \geq Y_t^4$, $\quad a.s.$.

For $n = 5, 6, ...$, we consider the following ABDSDEs:

$$\begin{cases} Y_t^n = \xi_T^2 + \int_t^T f^2(s, Y_{s-}^{n-1}, Z_s^n, Y_{s+\phi(s)-}^{n-1})ds + \int_t^T g(s, Y_{s-}^{n-1}, Y_{s+\phi(s)-}^{n-1})d\overleftarrow{B}_s \\ + \int_t^T dK_s^n - \sum_{i=1}^{\infty} \int_t^T Z_s^{n,(i)} dH_s^{(i)}, \qquad t \in [0,T], \\ Y_t^n = \eta_t^2, \qquad t \in [T, T+\rho], \end{cases}$$

similarly, for almost all t

$$Y_t^4 \geq Y_t^5 \geq Y_t^6 \geq \cdots \geq Y_t^n \geq \cdots, \quad a.s..$$

From the proof of Theorem 6.1, we know that (Y^n, Z^n, K^n) is a Cauchy sequence in $\mathcal{S}_{\mathcal{H}}^2\left([0,T+\rho];\mathbb{R}^d\right) \times \mathcal{M}_{\mathcal{H}}^2\left([0,T+\rho];l^2\right) \times \mathcal{A}^2$.

Denoting their limits by (Y, Z, K), and taking limits in the above iterative equations, we have that (Y, Z, K) satisfies the following ABDSDE:

$$\begin{cases} Y_t = \xi_T^2 + \int_t^T f^2(s, Y_{s-}, Z_s, Y_{s+\phi(s)-})ds + \int_t^T g(s, Y_{s-}, Y_{s+\phi(s)-})d\overleftarrow{B}_s \\ + \int_t^T dK_s - \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} dH_s^{(i)}, & t \in [0, T], \\ Y_t = \eta_t^2, & t \in [T, T+\rho]. \end{cases}$$

By Theorem 6.1, we know for almost all t, $Y_t = Y_t^2$, a.s..

Since for almost all t, $Y_t^1 \geq Y_t^3 \geq Y_t^4 \geq Y_t$, a.s.. it hold immediately for almost all t

$$Y_t^1 \geq Y_t^2, \ a.s..$$

Then the proof is complete. ■

Remark 6.5 *By the same way used in the proof of Theorem 6.3 we can easily proof the comparison theorem of the ABDSDE without reflection (i.e., $S = -\infty$), for this, it is enough to take $K = 0$.*

General conclusion

In this work, we discussed three new existence results for different categories to backward doubly stochastic differential equations (BDSDE for short). In this Phd thesis, we have the existence result to the BDSDE with weak assumptions and related to quasi linear stochastic partial differential equations (SPDEs). Also we have extended some results for BDSDE driven by a Brownian motions to case of BDSDEs with jumps.

Finally, following this study, several perspectives are considered. It would be interesting to prove the existence result in the following problems:

- Reflected Backward Boubly SDE with jumps in infinite-horizon under weak assumptions.
- Reflected Mean field Backward Doubly Stochastic Differential Equations with jumps.
- BDSDE with a quadratic coefficient.

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