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By

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Title

Qualitative study of certain evolution problems

Under the direction of

Dr. Mohamed BERBICHE, Prof, University of Biskra

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Option: **Analyse**

Par

Messaouda TERCHI

Titre

Etude qualitative de certains problèmes d'évolution

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Dédicace

Je dédie cette thèse

$\mathcal{A}$ mes très chers parents

qui veillent sans cesse sur moi avec leurs prières et leurs recommandations. Que Dieu le tout puissant les protège et leur réserve une longue et meilleure vie.

$\mathcal{A}$ mes très chers frères et sœurs.

$\mathcal{A}$ toute ma famille

$\mathcal{A}$ mes chères amies.

Terchi Messaouda

Abstract

In this thesis, we are interested in the study of the existence and uniqueness of global solutions, as well as, the blow up in finite time of solutions for a certains systems of semi-linear Volterra integro differential equations of parabolic and hyperbolic type. Especially the non-linear part is defined by an integral terms over the past history of the nonlinear forcing containing fractional time-dependent convolution kernels. We study this type of generalized problems to obtain similar results to those obtained in the case of an equation. We will see that under certain conditions on the exponents, the order of the temporal fractional derivatives there is a critical value of the dimension space for which the global with small data solution results as well as the explosion in finite time with initial conditions having positive average are obtained.

The methodology to be followed to demonstrate the global existence and the asymptotic behavior based essentially on the use of the semi-group method combined with a priori estimates in the Lebesgue spaces.

In parallel, in the study of the blow-up in finite time result, we will focus on the concept of weak solutions and its connection with the mild ones and thus via the test functions method's get the desired results.

Keywords

Damped wave equation, Heat system, damped wave system, Local existence, Global existence, Asymptotic behavior, Finite time blow-up.

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Symbols and Abbreviations

Sets:

$\mathbb{R}^N$	the real Euclidean space of dimension $N \geq 1$.

Functions and functions spaces:

$C([0,T],X)$	the space of continuous functions on $[0,T]$ to values in X.
$C_0\left(\mathbb{R}^N\right)$	space of all continuous functions decaying to zero at infinity.
$AC([0,T])$	the space of absolutely continuous functions on $[0,T]$.
$AC^{n+1}[0,T]$	$\{f : [0,T] \to R, \text{and } \partial_t^n f \in AC[0,T]\}$ and ∂_t^n is the usual n times derivative.
$C_c(I,X)$	the space of continuous functions with compact support from I to X.
$C_b(I,X)$	the space of continuous and bounded functions from I to X.
$\mathcal{S}\left(\mathbb{R}^N\right)$	The Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^N$.
$F(resp. F^{-1})$	Fourier transform(resp.Fourier transform reverse).
$S(t)$	the heat semigroup on $\mathbb{R}^N$.
$L^p\left(\mathbb{R}^N\right)$	the space of measurable functions on $\mathbb{R}^N$ such that $\|u\|^p$ is integrable.
$L^p([0,T],X)$	the space of measurable functions u on $[0,T]$ to values in X such that $\|\|u\|\|_X^p$ is integrable $(1 \leq p < \infty)$.
$L^\infty\left(\mathbb{R}^N\right)$	The space of measurable functions u on $\mathbb{R}^N$ such that there exists k such that $\|u(x)\| \leq k$ for almost every $x \in \mathbb{R}^N$.
$W^{m,p}\left(\mathbb{R}^N\right)$	the usual Sobolev space.
$H^m\left(\mathbb{R}^N\right)$	$W^{m,2}\left(\mathbb{R}^N\right) = \{f \in L^2\left(\mathbb{R}^N\right), D^\alpha f \in L^2\left(\mathbb{R}^N\right)$ for all $\alpha \in N^N$ such that $\|\alpha\| \leq m\}$.

Norms:

$$||u||_p := \left(\int_{\mathbb{R}^N} |u|^p\right)^{1/p} \text{ for } u \in L^p\left(\mathbb{R}^N\right).$$

$$||u||_{p,q,T} := \sup_{0\leq t\leq T}\left(\int_0^t ||u||_p^q\right)^{1/q}.$$

$$||u||_\infty := \inf\{k>0, |u(x)|<k \text{ almost every where}\}, \text{ for } u \in L^\infty\left(\mathbb{R}^N\right).$$

$$||u||_{W^{m,p}} := \sum_{\alpha\leq m} ||D^\alpha u||_{L^P} \text{ for } u \in W^{m,p}\left(\mathbb{R}^N\right).$$

$$||u||_{H^m} := \left(\sum_{\alpha\leq m} (||D^\alpha u||_{L^P})^2\right)^{\frac{1}{2}} \text{ for } u \in H^m\left(\mathbb{R}^N\right). \text{ such that}$$

$$D^\alpha := \frac{\partial^\alpha}{\partial^{\alpha_1}x_1...\partial^{\alpha_N}x_N}, \alpha = (\alpha_1, ..., \alpha_N), |\alpha| = \sum_{i=1}^N \alpha_i.$$

Mathematical operators:

$*$	The convolution product.
$\lvert.\rvert$	Absolute value.
Δ	The classical Laplace operator: $\Delta u(x,t) = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}(x,t)$.
J_t^α	the fractional integral operator in Riemann liouville sense.
$a \lesssim b$	*i.e* $a \leq Cb$.

Contents

Chapter 1

General Introduction

The theory of Volterra integro-differential equations is an exciting branch of mathematics. It is a mixture of Ode's and integral equations and is still one of the actively developing branch of the theory of differential equations. We cite a few monographs which are the classical source of fundamental facts and approaches in this field [7], [8],[16],[56],[64],. Over the last 50 years or so the theory of VIDE's has been revealed as a very powerful and important tool in the study of nonlinear phenomena. It has arisen in many applications where the current behavior of a system depends not only on the present state, but also on the entire history of states since some fixed starting time. Many such problems arise in environmental modeling (in models of evolution, population dynamics, pollution) as well as in model equations from engineering and the physical sciences. For example, in viscoelasticity, thermodynamics of phase transition, image processing, control theory, theory of heat conduction with memory, compression of visco-elastic media and in the theory of nuclear reactor dynamics (see e.g [9],[11],[19],[70],[74],[75]). The interplay between applied sciences and mathematics leads to the development of initial value problems for nonlinear partial VIDE's to model physical systems.

In the last few years considerable progress has been achieved in the investigation of VIDE's of parabolic and hyperbolic type. Some important results concerning existence, uniqueness, asymptotic behavior and blow-up in finite time in this theory may be found in the works cited in our references. In particular, there has now been a great deal of research on purely time dependent systems with memory delay, and on reaction-diffusion systems containing terms which involve time memory delays. Some authors have proved results on global convergence in some rather general settings (e.g. Pozio[36] 1983; Yamada[74] 1984). In most of these works the nonlinear term forcing is written as $f\left(u(t,x), \int_0^t g(t,x,s,u(s,x))ds\right)$, $t > 0$, $x \in \Omega$. A particular case widely encountered in population dynamics are Volterra diffusion equations, where the nonlocal "hereditary" term takes the form of a convolution with a kernel $g(t,x,s,u(s,x)) = k(t-s)h(u(s,x))$ with a monotonicity assumption on h, thus allowing the nonlinear term to contribute to the global existence and blow-up of solutions via comparaison principle (e.g see [4], [9], [22], [48], [49]). These results are mostly based on the boundedness and the positivity of the initial data in establishing the finite-time blow-up results.

However, it seems that there are not so many results concerning global existence study and finite-time blow-up theories for PVIDEs with more general kernels, for example, with weakly singular kernels of the form $k(t) = t^{-\alpha}$, $0 < \alpha < 1$, as compared with above cited works (see [2], [28], [38], [45], [46],[66] and the references therein). Their approach is completely different from those described above. Precisely, they do not require any condition on the sign of the initial data and the solution, also they may consider unbounded initial data.

On the other hand the study of the critical exponent problem traces back to 1966, when Fujita considered the initial value problem

$$\begin{cases} u_t = \Delta u + u^p,\ t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

with $p > 1$. He showed in his pioneering work [25] that if $1 < p < p_F$, where $p_F = 1 + \frac{2}{N}$ then the equation (1.1) has no global (in time) positive solutions, but if $p > p_F$, then for

initial values bounded by a sufficiently small Gaussian the solution is global. Later on Hayakawa [29] and Kobayashi et *al.* *[30]* proved that the critical case $p = p_F$ belongs to the blow-up region. The power p_F is called the Fujita critical exponent. In [14] Weissler showed that if the initial data is sufficiently small in $L^{q_{sc}}(\mathbb{R}^N)$, where $q_{sc} = \frac{N(p-1)}{2} > 1$, then the solution is global. Namely, he established that p_{sc} corresponding to q_{sc} is the one of Fujita exponent ($p_F = p_{sc}$). That is the Fujita critical exponent p_F can be predicted from the scaling properties: In other words if u is a solution of nonlinear parabolic equation on $\mathbb{R}^N$ with initial value $u_0(x)$ then, $\forall \lambda > 0, \lambda^a u(\lambda^b t, \lambda x)$ with initial value $\lambda^a u_0(\lambda x)$ is also solution of (1.1). Let us mention that the Fujita critical exponent can be detected from rescaling argument by many others equations using the same procedure such as $u_t - \Delta u = t^k |x|^\sigma u^p$, $u_t - \nabla.(u^\sigma \nabla u) = u^p$, $u_t - \Delta u = u(t,0)^{p-q} u(t,x)$ with $p > q \geq 1$.
Recently, Souplet [48] studied the boundary value problem $u_t = \Delta u + \displaystyle\int_0^t u^p(s)ds$, $t > 0$, and in particular he showed that all positive solutions blow up in finite time for all $p > 1$ ($p_F = \infty$).

More recently, Cazenave et *al.* [66] investigate the global existence/blow-up properties of the following parabolic equation with non-local in time non-linearity

$$\begin{cases} u_t - \Delta u = \displaystyle\int_0^t (t-s)^{-\gamma} |u|^p u(s) ds \quad \text{in } (0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \text{ in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

with $p > 1, 0 < \gamma < 1$, they showed that, if

$$p_\gamma = 1 + \frac{4 - 2\gamma}{(N - 2 + 2\gamma)^+},$$

and $p_* = \max\left\{\frac{1}{\gamma}, p_\gamma\right\} \in (0, +\infty]$, where $u_+ = \max(u, 0)$, then the behavior of solutions can be divided into the following way:

(i) If $\gamma \neq 0, p \leq p_*$, and $u_0 \geq 0, u_0 \neq 0$, then u blows up in finite time.

(ii) If $\gamma \neq 0, p > p_*$, and $u_0 \in L_{q_{sc}}(\mathbb{R}^N)$ $\left(\text{where } q_{sc} = \frac{N(p-1)}{4-2\gamma}\right)$ with $||u_0||_{L_{q_{sc}}}$, sufficiently small, then u exists globally.

Their study reveals surprising the fact that for equation (1.2) the critical exponent is not the one predicted by scaling argument, as we have seen above the well known scaling technique is efficient for detecting the Fujita exponent for several equations of nonlinear parabolic type one. Needless to say that the equation considered by Cazenave et al is a genuine extension of the one considered by Fujita in his pioneering work [25].

The results obtained in [66] are later extended by some authors to the weakly coupled parabolic systems, damped wave equations, weakly coupled damped wave systems, we refer the reader to [6, 21, 37, 46, 47, 50, 71, 72].

From the point of view of diffusion phenomenon, it is expected that for the damped wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \Delta u(t,x) + \partial_t u(t,x) = f(u),\ t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = u_0(x),\ \ u_t(0,x) = u_1(x);\ \ x\in\mathbb{R}^N, \end{cases} \tag{1.3}$$

the same result hold, for $f = |u|^{p-1}u$, existence of classical solutions has been investigated for a long time (see [5, 65]), for $f = |u|^p$ Todorova and Yordanov proved in [20] that the critical exponent of (1.3) is p_F. More precisely, they proved if $p > p_F$ there exists a unique global solution of (1.3) for sufficiently small initial data in the weighted energy space while, if $1 < p < p_F$ every solution with initial data having positive average must blow-up in finite time. Later Zhang in [51] proved that the exponent p_F belongs to the blow-up region.

In the case of damped wave equation with nonlinear memory

$$u_{tt} - \Delta u + u_t = \int_0^t (t-\tau)^{-\gamma} |u|^\alpha d\tau \quad \text{in } (0,T)\times\mathbb{R}^N \tag{1.4}$$

Fino in [2] addressed the global small data solutions and their asymptotic behavior as $t \to +\infty$ to (1.4) when $N = 1, 2, 3$. he was shown that the solution of (1.4) behaves as that of the corresponding diffusive equation (1.2), indeed, he showed that when $t \to +\infty$ the solution decays exponentially outside every ball $B(t^{\frac{1}{2}+\delta}), \delta > 0$. Namely

$$||Du(t,.)||_{L^2\left(\mathbb{R}^N \setminus B(t^{\frac{1}{2}+\delta})\right)} = \mathcal{O}(e^{-t^{2\delta/4}}).$$

Furthermore, he proved that:

1. Let $\alpha > 1, \gamma \in \left(\frac{1}{2}, 1\right)$ for $N = 1, 2$ and $\gamma \in \left(\frac{11}{16}, 1\right)$ for $N = 3$. If $p_N < \alpha$, where

$$p_1 = 1 + \frac{2(3-2\gamma)}{(N-2+2\gamma)^+}, p_2 = 1 + \frac{4(3-2\gamma)}{(N-4+4\gamma)^+}, p_3 = 1 + \frac{N+2(5-4\gamma)}{(N-2+4\gamma)^+}.$$

   Then, the problem (1.4) admits a unique global mild solution with small data. While

2. let $1 < \alpha < \frac{N}{N-2}$ for $N = 3, \alpha \in (1, +\infty)$ for $N = 1, 2, \frac{N-2}{N} < \gamma < 1$ and $(u_0, u_1) \in H^1\left(\mathbb{R}^N\right) \times L^2\left(\mathbb{R}^N\right)$ such that $\int_{\mathbb{R}^N} u_i(x)dx > 0, i = 0, 1$. if $\alpha \leq p_*$, then the mild solution of the problem (1.4) blow-up in finite time.

In [38], Berbiche studied the global existence of solution for (1.4) and he proved that if $p > 2$, $p > p_*$ for $N = 1, 2, 3$, then the mild solution with small initial data with low regularity and not necessarily in $L^1(\mathbb{R}^N)$ exists globally in time. In addition, if $2 < p < 5$, then some of these solutions have the same behavior of the self-similar solutions of the corresponding heat equation with nonlinear memory (1.2).

Before closing this section, we just briefly mention the result obtained by M.Loayza, I.G.Quinteiro which is directly connected to the problem proposed in chapter 3. In [46] the authors discussed the following weakly coupled parabolic system

$$\begin{cases} u_t - \Delta u = \displaystyle\int_0^t (t-s)^{-\gamma_1} |v|^{p-1} v(s) ds, t > 0, x \in \mathbb{R}^N, \\ v_t - \Delta v = \displaystyle\int_0^t (t-s)^{-\gamma_2} |u|^{q-1} u(s) ds, t > 0, x \in \mathbb{R}^N, \\ u(0,x) = u_0(x); u_t(0,x) = u_1(x), x \in \mathbb{R}^N, \\ v(0,x) = v_0(x); v_t(0,x) = v_1(x), x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

with $p,q \geq 1, 0 \leq \gamma_1,\gamma_2 < 1$ and $u_0,v_0 \in C_0\left(\mathbb{R}^N\right)$, they interested to find conditions on parameters p,q,γ_1,γ_2 to determine when solutions of system (1.5) either blow up in finite time or exist globally in smooth bounded domain of $\mathbb{R}^N$ with Dirichlet boundary conditions.They established the following result: Assume that, $pq > 1$

(i) if $\begin{cases} 1 - p\gamma_2 + p\left(1 - q\gamma_1\right) + p\left(q+1\right) < \frac{N}{2}\left(pq-1\right), \\ 1 - q\gamma_1 + q\left(1 - p\gamma_2\right) + q\left(p+1\right) < \frac{N}{2}\left(pq-1\right), \end{cases}$

(ii) if $\begin{cases} 1 - p\gamma_2 + p\left(1 - q\gamma_1\right) < 0, \\ 1 - q\gamma_1 + q\left(1 - p\gamma_2\right) < 0, \end{cases}$

(iii) and if $\left\{ \left(\frac{p}{r_2} - \frac{4}{N}\right) < \frac{1}{q}; \left(\frac{q}{r_1} - \frac{4}{N}\right) < \frac{1}{p}, \right.$

with $r_1 = \dfrac{N(pq-1)}{2[2-\gamma_1+p(2-\gamma_2)]}, r_2 = \dfrac{N(pq-1)}{2[2-\gamma_2+q(2-\gamma_1)]}.$

Then global solution with small initial data exists.

Whereas if

(i) $\begin{cases} \frac{N}{2}\left(pq-1\right) \leq 1 - p\gamma_2 + p\left(1 - q\gamma_1\right) + p\left(q+1\right), \\ \frac{N}{2}\left(pq-1\right) \leq 1 - q\gamma_1 + q\left(1 - p\gamma_2\right) + q\left(p+1\right), \end{cases}$

(ii) $1 - p\gamma_2 + p\left(1 - q\gamma_1\right)$, or $1 - q\gamma_1 + q\left(1 - p\gamma_2\right) \geq 0.$

Then every nontrivial solution blows up in finite time.The condition (i) in both cases are eliminated in bounded domain Ω.

1.1 Structure of Thesis

1. In the next section, we show a summary of our main results which are developed in the chapters of this thesis.

2. In order to provide the reader with a sufficient background, we recall in chapter 2 some definitions and basic results support our subject, besides we will expose the well posedness of the damped wave equation with nonlinear source terms.

3. The third chapter is devoted to show the blow-up result for damped wave system with nonlinear memory through which we extend the study of fino [2], and we give conditions relating the space dimension N with the parameters γ_1, γ_2, p, q for which the solution with initial data have positive average blow-up in finite time. We apply the method of test functions developed by Mitidieri and Pohozaev [12],[13] and Zhang [51] to prove this result.

4. In the chapter 4, we are interested to study the existence and uniqueness of the local solution for Cauchy problem of wave equation with both frictional and displacement dependent damping terms with nonlinear memory in multi-dimensional space $\mathbb{R}^N$, as well as, we give a sufficient conditions on parameters in order to show a blow-up of weak solution result for any dimension space.

5. In the last chapter, we establish a results to more general class of Cauchy problem contains strongly coupled semi-linear heat equations with some kind of nonlinearity in $\mathbb{R}^N$, we see under some conditions on the exponents and on the dimension N, that the existence and uniqueness of time-global solutions for small data and their asymptotic behaviors are obtained. The observation will be applied to the corresponding system of the damped wave equations in low dimensional space.

1.2 Presentation of the Obtained Results

Chapter 4: On The Nonexistence of Global Solution For Wave Equations With Double Damping Terms and Nonlinear Memory

This chapter is devoted to study the following Cauchy problem:

$$\begin{cases} \Box u(t,x) + |u(t,x)|^{m-1}\partial_t u(t,x) + \partial_t u(t,x) = \displaystyle\int_0^t (t-\tau)^{-\gamma}\,|u(\tau,x)|^p d\tau, \text{ in } [0,T]\times\mathbb{R}^N, \\ u(0,x) = u_0(x)\,, u_t(0,x) = u_1(x)\,, \quad \text{in } \mathbb{R}^N. \end{cases} \tag{1.6}$$

Where, $\Box = \partial_t^2 - \Delta$, $p, m > 1; 0 < \gamma < 1, N \geq 1$, $u_i(x), i = 0, 1$ are given initial data.We prove the existence and uniqueness of local solution by using some estimates in sobolev space and we get the following result:

Theorem 1.1 *Let $N \geq 1, s > \frac{N}{2} + 1$ and $m, p \in (0, +\infty) \cap (s - 1, +\infty)$. Then for any $u_0 \in H^s\left(\mathbb{R}^N\right)$ and $u_1 \in H^{s-1}\left(\mathbb{R}^N\right)$, (1.6) admits a unique solution*

$$u \in C\left([0,T]; H^s\left(\mathbb{R}^N\right)\right) \cap C^1\left([0,T]; H^{s-1}\left(\mathbb{R}^N\right)\right),$$

with some positive T, which depends only on $||u_0||_{H^s} + ||u_1||_{H^{s-1}}$.

In the second part of this chapter we deal with the blow-up case and we use the test method function to show the result below

Theorem 1.2 *Let $N \geq 1, 0 < \gamma < 1$ and p, m such that $p > m > 1$. Assume that the initial data (u_0, u_1) satisfy*

$$\int_{\mathbb{R}^N} u_0(x)dx > 0, \int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x)dx > 0, \text{ and } \int_{\mathbb{R}^N} u_1(x)dx > 0.$$

Then if

$$p \leq \left\{ \frac{N+2}{(N-2+2\gamma)_+}, \frac{1}{\gamma} \right\}.$$

The solution of (1.6) does not exist globally in time.

Chapter 5: Global Small Data Solution For a System of Semilinear Heat Equations and The Corresponding System of Damped Wave Equations With Nonlinear Memory

This chapter is contains two sections, the first one is devoted to study the Cauchy problem for a strongly coupled semi-linear heat equations

$$\begin{cases} u_t - \Delta u = \displaystyle\int_0^t (t-s)^{-\gamma_1} |u(s)|^{p_1} |v(s)|^{q_1} ds \text{ in } (0,T)\times\mathbb{R}^N, \\ v_t - \Delta v = \displaystyle\int_0^t (t-s)^{-\gamma_2} |u(s)|^{p_2} |v(s)|^{q_2} ds \text{ in } (0,T)\times\mathbb{R}^N, \\ u(0,x) = u_0, v(0,x) = v_0, \qquad x \in \mathbb{R}^N. \end{cases} \tag{1.7}$$

Where the unknown functions $u := u(t,x), v := v(t,x)$ are real valued, $N \geq 1, p_1, q_1, p_2, q_2 \geq 1, 0 < \gamma_1, \gamma_2 < 1$ and u_0, v_0 are given initial data. We show an important result concerns the existence and uniqueness of local solution for the system (1.7) then, we give conditions relating the space dimension N with the parameters of the system for which the mild solution exists globally in time and satisfy $L^p - L^q$ estimates with the norms of initial data sufficiently small.

Theorem 1.3 *Let N be positive integer. Let the real numbers $p_1, p_2, q_1, q_2 \geq 1$, $0 < \gamma_1, \gamma_2 < 1$ be such that*

$$\begin{aligned} \left[(1-\gamma_1)(p_2-1) - (1-\gamma_2) q_1\right]\left((p_1-1)(p_2-1) - q_1q_2\right) &> 0, \\ \left[(1-\gamma_2)(p_1-1) - (1-\gamma_1) q_2\right]\left((p_1-1)(p_2-1) - q_1q_2\right) &> 0, \\ \left[p_2 - q_1 - 1\right]\left((p_1-1)(p_2-1) - q_1q_2\right) &> 0, \\ \left[p_1 - q_2 - 1\right]\left((p_1-1)(p_2-1) - q_1q_2\right) &> 0. \end{aligned}$$

And $u_0, v_0 \in C_0(\mathbb{R}^N)$. Let $(u,v) \in \left\{C\left((0,T_{\max}), C_0(\mathbb{R}^N)\right)\right\}^2$.

Assume that

$$\begin{cases} \dfrac{N}{2} > 1 - \gamma_1 + \dfrac{(2-\gamma_1)(p_2-1) - (2-\gamma_2) q_1}{(p_1-1)(p_2-1) - q_1q_2}, \\ \dfrac{N}{2} > 1 - \gamma_2 + \dfrac{(2-\gamma_2)(p_1-1) - (2-\gamma_1) q_2}{(p_1-1)(p_2-1) - q_1q_2}, \end{cases} \tag{1.8}$$

$$\begin{cases} \left((p_1-1)(p_2-1) - q_1q_2\right) \times \left[p_2(\gamma_1 p_1 - 1) - \gamma_1 p_1 + q_1(1-\gamma_1 q_2) + 1 - \gamma_2 q_1\right], \\ \left((p_1-1)(p_2-1) - q_1q_2\right) \times \left[p_1(\gamma_2 p_2 - 1) - \gamma_2 p_2 + q_2(1-\gamma_2 q_1) + 1 - \gamma_1 q_2\right], \end{cases} \tag{1.9}$$

and

$$\begin{cases} \dfrac{Np_1}{2}\left[\dfrac{p_1}{r_1}+\dfrac{q_1}{r_2}\right]+\dfrac{Nq_1}{2}\left[\dfrac{p_2}{r_2}+\dfrac{q_2}{r_1}\right]<2\left(p_1+q_1\right)+\dfrac{N}{2}, \\ \dfrac{Np_2}{2}\left[\dfrac{p_2}{r_2}+\dfrac{q_2}{r_1}\right]+\dfrac{Nq_2}{2}\left[\dfrac{p_1}{r_1}+\dfrac{q_1}{r_2}\right]<2\left(p_2+q_2\right)+\dfrac{N}{2}. \end{cases} \tag{1.10}$$

Then there exists a constant $\varepsilon > 0$ *such that if the initial data satisfy* $(u_0, v_0) \in L^{r_1}\left(\mathbb{R}^N\right) \times L^{r_2}\left(\mathbb{R}^N\right)$ *and*

$$||u_0||_\infty + ||v_0||_\infty + ||u_0||_{r_1} + ||v_0||_{r_2} \leq \varepsilon,$$

the problem (1.7) admits global solution $(u, v) \in C\left([0,+\infty); L^{r_1}\left(\mathbb{R}^N\right) \times L^{r_2}\left(\mathbb{R}^N\right) \cap C_0(\mathbb{R}^N)\right)^2$ *satisfies the following decay estimates*

$$||u||_\infty \leq C\left(1+t\right)^{-\alpha_1}, ||v||_\infty \leq C\left(1+t\right)^{-\beta_1}, t \geq 0,$$

where

$$r_1 = \frac{N\left[\left(p_1-1\right)\left(p_2-1\right)-q_1q_2\right]}{2\left[\left(2-\gamma_1\right)\left(p_2-1\right)-\left(2-\gamma_2\right)q_1\right]}, \quad \alpha_1 = \frac{\left(1-\gamma_1\right)\left(p_2-1\right)-\left(1-\gamma_2\right)q_1}{\left(p_1-1\right)\left(p_2-1\right)-q_1q_2},$$

$$r_2 = \frac{N\left[\left(p_1-1\right)\left(p_2-1\right)-q_1q_2\right]}{2\left[\left(2-\gamma_2\right)\left(p_1-1\right)-\left(2-\gamma_1\right)q_2\right]}, \quad \beta_1 = \frac{\left(1-\gamma_2\right)\left(p_1-1\right)-\left(1-\gamma_1\right)q_2}{\left(p_1-1\right)\left(p_2-1\right)-q_1q_2}.$$

In the second part, we consider the following damped wave system in low dimensional space:

$$\begin{cases} u_{tt}-\Delta u+u_t = \displaystyle\int_0^t \left(t-s\right)^{-\gamma_1}|u(s)|^{p_1}|v(s)|^{q_1}ds \text{ in } (0,T)\times\mathbb{R}^N, \\ v_{tt}-\Delta v+v_t = \displaystyle\int_0^t \left(t-s\right)^{-\gamma_2}|u(s)|^{p_2}|v(s)|^{q_2}ds \text{ in } (0,T)\times\mathbb{R}^N, \\ u(0,x)=u_0(x), u_t(0,x)=u_1(x), v(0,x)=v_0(x), v_t(0,x)=v_1(x) \quad x\in\ \mathbb{R}^N. \end{cases} \tag{1.11}$$

Where the unknown functions $u := u(t,x), v := v(t,x)$ are real valued, $N \geq 1, p_1, q_1, p_2, q_2 \geq 1, 0 < \gamma_1, \gamma_2 < 1$ and u_0, v_0 are given initial data. We prove the local existence, uniqueness and global existence theorems similar to the one presented in previous part in different spaces.

Theorem 1.4 *Let* $1 \leq N \leq 3$ *be positive integer. Let the real numbers* $p_1, p_2, q_1, q_2 \geq 1, 0 < \gamma_1, \gamma_2 < 1$ *be such that*

$$\begin{aligned}
\left[(1-\gamma_1)(p_2-1)-(1-\gamma_2)q_1\right]\left((p_1-1)(p_2-1)-q_1q_2\right) &> 0, \\
\left[(1-\gamma_2)(p_1-1)-(1-\gamma_1)q_2\right]\left((p_1-1)(p_2-1)-q_1q_2\right) &> 0, \\
\left[p_2-q_1-1\right]\left((p_1-1)(p_2-1)-q_1q_2\right) &> 0, \\
\left[p_1-q_2-1\right]\left((p_1-1)(p_2-1)-q_1q_2\right) &> 0.
\end{aligned}$$

Assume that the conditions (1.8), (1.9), (1.10) are verified. Then there exists a constant $\varepsilon > 0$ *such that if the initial data satisfy* $(u_i, v_i) \in \left\{W^{1-i,1}\left(\mathbb{R}^N\right) \times W^{1-i,\infty}\left(\mathbb{R}^N\right)\right\}^2, i = 0, 1$ *and*

$$||u_0||_{W^{1,1}\cap W^{1,\infty}} + ||v_0||_{W^{1,1}\cap W^{1,\infty}} + ||u_1||_{L^1\cap L^\infty} + ||v_1||_{L^1\cap L^\infty} \leq \varepsilon,$$

the problem (1.11) admits global solution

$$\begin{aligned}
&(u,v) \in C\left([0,+\infty); L^{r_1}\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right) \cap H^1\left(\mathbb{R}^N\right)\right) \cap C^1\left([0,+\infty); L^2\left(\mathbb{R}^N\right)\right) \\
&\quad \times C\left([0,+\infty); L^{r_2}\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right) \cap H^1\left(\mathbb{R}^N\right)\right) \cap C^1\left([0,+\infty); L^2\left(\mathbb{R}^N\right)\right),
\end{aligned}$$

satisfies the following decay estimates

$$||u||_\infty \leq C(1+t)^{-\alpha_1}, ||v||_\infty \leq C(1+t)^{-\beta_1}, t \geq 0,$$

where

$$\begin{aligned}
r_1 &= \frac{N\left[(p_1-1)(p_2-1)-q_1q_2\right]}{2\left[(2-\gamma_1)(p_2-1)-(2-\gamma_2)q_1\right]}, \quad \alpha_1 = \frac{(1-\gamma_1)(p_2-1)-(1-\gamma_2)q_1}{(p_1-1)(p_2-1)-q_1q_2}, \\
r_2 &= \frac{N\left[(p_1-1)(p_2-1)-q_1q_2\right]}{2\left[(2-\gamma_2)(p_1-1)-(2-\gamma_1)q_2\right]}, \quad \beta_1 = \frac{(1-\gamma_2)(p_1-1)-(1-\gamma_1)q_2}{(p_1-1)(p_2-1)-q_1q_2}.
\end{aligned}$$

Chapter 2

Preliminary Concepts

In this chapter, we recall some definitions and classical theorems of functional analysis that are necessary for the study of semi linear evolution equations. The proofs can be found in [65].

2.1 Definitions and Basic Results

Definition 2.1 *(L^p space). The L^pnorm of function $f:\mathbb{R}^N \to \mathbb{R}$, denoted by $||f||_{L^p}$, is*

$$||f||_{L^p} := \begin{cases} \left(\int_{\mathbb{R}} |f(t)|^p dt\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ ess \sup\limits_{x\in\mathbb{R}^N} |f|, & p = \infty. \end{cases}$$

If $||f||_{L^p} < \infty$, then $f \in L^p\left(\mathbb{R}^N\right)$.

Definition 2.2 *(Sobolev Space) Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, +\infty]$, then the sobolev space*

$$W^{k,p}(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) : D^\alpha u \in L^p(\mathbb{R}^N),\ \forall \alpha \text{ with } |\alpha| \leq k\right\}.$$

This space is equipped with the norm

$$||u||_{W^{k,p}(\mathbb{R}^N)} := \sum_{|\alpha|\leq k} ||D^\alpha u||_{L^p(\mathbb{R}^N)}.$$

Definition 2.3 *($H^{\mathbf{s}}$ norm).The $H^{\mathbf{s}}$ norm of function $f : \mathbb{R}^N \to \mathbb{R}$, denoted by $||f||_{H^s}$, is*

$$||f||_{H^s} = \left[\int_{\mathbb{R}^N} \left(1 + |\xi|^2\right)^s |\hat{f}|^2 d\xi\right]^{1/2},$$

where ξ is the Fourier transform variable of $\hat{f}$. If $||f||_{H^s} < \infty$, then $f \in H^s$.

Definition 2.4 *(X^s Space). The X^s norm of function $u(t,x) : \mathbb{R}^N \times [0,T) \to \mathbb{R}$, denoted by $||u||_{X^s}$*

$$||u||_{X^s} = \sup_{t\in[0,T]} \left(||u(t)||_{H^s} + ||\partial_t u(t)||_{H^{s-1}}\right).$$

where $T \leq \infty$ is positive constant. If $||u||_{X^s} < \infty$, then $u \in X^s$.

Theorem 2.1 *(Sobolev Embedding). For any N-dimensional space, the function $u(t_0) \in H^s$, if $s > \frac{N}{2}$, then*

$$||u(t_0)||_{L^\infty} \lesssim ||u(t_0)||_{H^s}.$$

Theorem 2.2 *(Banach Contraction-Mapping Principal). Let (X,d) be a complete metric space and $G : X \to X$ a map such that there exists $\theta \in [0,1)$ satisfying $d\left(G(x), G(y)\right) \leq \theta d(x,y)$ for all $x,y \in X$. Then, there exists a unique $x_0 \in X$ such that $G(x_0) = x_0$.*

Theorem 2.3 *(Gronwall's inequality). Let f be a nonnegative, continuous functions on $[0,T]$, which satisfies*

$$f(t) \lesssim \int_0^t f(s)\, ds,$$

for all $t \in [0,T]$. Then, $f(t) = 0$ for all $t \in [0,T]$.

Theorem 2.4 *(Leibnitz Integral Rule). For* $-\infty < a(x) < b(x) < \infty$,

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x)).\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \partial_x f(x,t)dt. \quad (2.1)$$

2.1.1 Fractional integrals and derivatives

Definition 2.5 *(Fractional Riemann-Liouville Integral): Let* $f : [a,b) \to \mathbb{R}^N$. *Fractional Riemann-Liouville integral of* f *of order* $\alpha > 0$ *is the integral defined by the following formula:*

$$J_{a|t}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds. \quad (2.2)$$

Where α *is real or complex number.* $\Gamma(\alpha)$ *is Euler gamma function.*

Definition 2.6 *(Riemann-Liouville fractional derivatives): Let* $\alpha \in [m-1, m[$,
$m \in \mathbb{N}^*$. *We say the left derivative of order* α *in the sense of Riemann-Liouville the function defined by :*

$$\begin{aligned} D_{a|t}^{\alpha} f(t) &:= \left(\frac{d}{dt}\right)^m \left[\left(J_{a|t}^{m-\alpha} f\right)(t)\right], \\ &:= \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t (t-s)^{m-\alpha-1} f(s)ds. \end{aligned}$$

We say the right derivative of order α *in the sense of Riemann-Liouville the function defined by :*

$$\begin{aligned} D_{t|b}^{\alpha} f(t) \ &: \ = \left(\frac{d}{dt}\right)^m \left[\left(J_{t|b}^{m-\alpha} f\right)(t)\right], \\ &: \ = \frac{1}{\Gamma(m-\alpha)} \left(-\frac{d}{dt}\right)^m \int_t^b (s-t)^{m-\alpha-1} f(s)ds. \end{aligned}$$

Definition 2.7 *(Caputo fractional derivatives): Let* $\alpha \in [m-1, m[, m \in \mathbb{N}^*$. *We say the left derivative of order* α *in the sense of Caputo the function defined by :*

$$^{C}D^{\alpha}_{a|t}f(t) := \frac{1}{\Gamma\left(m-\alpha\right)}\int_{a}^{t}\left(t-s\right)^{m-\alpha-1}f(s)^{(m)}ds,\ t>a.$$

We say the left derivative of order α in the sense of Caputo the function defined by :

$$^{C}D^{\alpha}_{t|b}f(t) := \frac{\left(-1\right)^{m}}{\Gamma\left(m-\alpha\right)}\int_{t}^{b}\left(t-s\right)^{m-\alpha-1}f(s)^{(m)}ds,\ t<b.$$

Proposition 2.1 *Let α and β are real number, and f is continuous function in the interval $[a,b)$*

(1) $J^{\alpha}_{a|t}\left(J^{\beta}_{a|t}f\right)(t) = J^{\alpha+\beta}_{a|t}f,\ \ (\alpha,\beta>0)\,.$

(2) $\frac{d}{dt}\left(J^{\alpha}_{a|t}f\right)(t) = J^{\alpha-1}_{a|t}f(t), \quad \alpha>1.$

(3) $\lim\limits_{\alpha\to 0}\left(J^{\alpha}_{a|t}f\right)(t) = f(t), \quad \alpha>0.$

Lemma 2.1 *(Formula of integration y parts) for every $f,g\in C([0,T])$ such that $D^{\alpha}_{0|t}f(t)$, $D^{\alpha}_{t|T}f(t)$ exist and are continuous, for all $t\in[0,T]$, $0<\alpha<1$ we have the formula of integration by parts*

$$\int_{0}^{T}\left(D^{\alpha}_{0|t}f\right)(t)g(t)dt = \int_{0}^{T}f(t)\left(D^{\alpha}_{t|T}g\right)(t)dt. \tag{2.3}$$

Note also that, for all $f\in AC^{n+1}[0,T]$ and all integers $n\geq 0$, we have

$$(-1)^{n}\partial_{t}^{n}D^{\alpha}_{t|T}f(t) = D^{n+\alpha}_{t|T}f(t). \tag{2.4}$$

Moreover, for all $1\leq q\leq\infty$, the following formula

$$D^{\alpha}_{0|t}J^{\alpha}_{0|t} := Id_{L^{q}(0,T)}. \tag{2.5}$$

holds almost everywhere on $[0,T]$.

Lemma 2.2 *if* $w_1(t) = (1 - t/T)_+^{\sigma}$, $t \geq 0$, $T > 0$, $\sigma >> 1$, *then*

$$D_{t|T}^{\alpha} w_1(t) = CT^{-\sigma}(T-t)_+^{\sigma-\alpha}, D_{t|T}^{\alpha+1} w_1(t) = CT^{-\sigma}(T-t)_+^{\sigma-\alpha-1}, \tag{2.6}$$

$$D_{t|T}^{\alpha+2} w_1(t) = CT^{-\sigma}(T-t)_+^{\sigma-\alpha-2}, \text{ for all } \alpha \in (0,1), \tag{2.7}$$

$$\left(D_{t|T}^{\alpha} w_1\right)(T) = 0, \left(D_{t|T}^{\alpha} w_1\right)(0) = T^{-\alpha}, \left(D_{t|T}^{\alpha+1} w_1\right)(T) = 0, \left(D_{t|T}^{\alpha+1} w_1\right)(0) = T^{-\alpha-1}. \tag{2.8}$$

For the proof of these results, see [1]

2.2 Semigroup of bounded operators

2.2.1 m-dissipative operators

In this part, we recall some facts concerning semigroup of bounded operators. Let X is Banach space.

Definition 2.8 ***(Unbounded operators in Banach spaces)***. *A linear unbounded operator in X is a pair $(D(\mathcal{A}), \mathcal{A})$, where $D(\mathcal{A})$ is a linear subspace of X and $\mathcal{A}$ is a linear mapping $D(\mathcal{A}) \to X$. We say that $\mathcal{A}$ is bounded if there exists $c > 0$ such that*

$$||\mathcal{A}u|| \leq c,$$

for all $u \in \{x \in D(\mathcal{A}), ||x|| \leq 1\}$. Otherwise, $\mathcal{A}$ is not bounded.

Definition 2.9 ***(dissipative operators).*** *An operator $\mathcal{A}$ in X is dissipative if*

$$||u - \lambda \mathcal{A}u|| \geq ||u||,$$

for all $u \in D(\mathcal{A})$ and all $\mathcal{A} > 0$.

Definition 2.10 ***(m-dissipative operators)*** *An operator $\mathcal{A}$ in X is m-dissipative if*

1. *$\mathcal{A}$ is dissipative.*

2. *for all $\mathcal{A} > 0$ and all $f \in X$, there exists $u \in D(\mathcal{A})$ such that $u - \lambda \mathcal{A} u = f$.*

If X is a Hilbert space, we have the following proposition:

Proposition 2.2 *$\mathcal{A}$ is dissipative in X if and only if $\langle u, \mathcal{A}u \rangle \leq 0$, for all $u \in D(\mathcal{A})$.*

2.2.2 The Laplacian in an open subset of $\mathbb{R}^N$: L^2, C_0 theories

Let Ω be any open subset of $\mathbb{R}^N$, we define the linear operator $\mathcal{A}$ in $L^2(\Omega)$ by

$$\begin{cases} D(\mathcal{A}) = \left\{ u \in H_0^1(\Omega) \cap L^2(\Omega); \Delta u \in L^2(\Omega) \right\}, \\ \mathcal{A}u = \Delta u, \forall u \in D(\mathcal{A}). \end{cases}$$

Proposition 2.3 *$\mathcal{A}$ is m-dissipative with dense domain.*

Let Ω be any open subset of $\mathbb{R}^N$, we define the linear operator $\mathcal{B}$ in $C_0(\Omega)$ by

$$\begin{cases} D(\mathcal{B}) = \left\{ u \in H_0^1(\Omega) \cap C_0(\Omega); \Delta u \in C_0(\Omega) \right\}, \\ \mathcal{B}u = \Delta u, \forall u \in D(\mathcal{B}). \end{cases}$$

Proposition 2.4 *Assume that Ω has Lipschitz continuous boundary. Then $\mathcal{B}$ is m-dissipative, with dense domain.*

2.3 Contraction semigroups

Definition 2.11 *(Contraction semigroups) one-parameter family $(S(t))_{t \geq 0}$ of linear operators is a contraction semigroup in X provided that*

1. *$||S(t)|| \leq 1$ for all $t \geq 0$,*

2. $S(0) = I$,

3. $S(t+s) = S(t)S(s)$ *for all* $s, t \geq 0$,

4. *for all* $x \in X$, *the function* $t \to S(t)x$ *belongs to* $C([0,\infty), X)$.

Definition 2.12 *(Infinitesimal generator) The generator of* $(S(t))_{t>o}$ *is the linear operator* L *defined by*

$$D(L) = \left\{ x \in X; \frac{S(t)x - x}{h} \text{ has a limit in } X \text{ as } h \downarrow 0 \right\},$$

and

$$Lx = \lim_{h \downarrow 0} \frac{S(t)x - x}{h}, \text{ for all } x \in D(L).$$

Proposition 2.5 *Let* $(S(t))_{t>o}$ *be a contraction semigroup in* X *and let* L *be its generator. Then* L *is m-dissipative and* $D(L)$ *is dense in* X.

Theorem 2.5 *(The Hille-Yosida-Phillips Theorem) A linear operator* $\mathcal{A}$ *is the generator of a contraction semigroup in* X *if and only if* $\mathcal{A}$ *is m-dissipative with dense domain.*

2.3.1 Heat semigroup

In this part, we denote by $(S(t))_{t>o}$ the semigroup generated by $\mathcal{B}$ in $L^2(\Omega)$.

Lemma 2.3 *The embedding* $D(\mathcal{B}) \hookrightarrow H_0^1(\Omega)$ *is continuous.*

Homogeneous equations

Proposition 2.6 *Let* $\varphi \in L^2(\Omega)$ *and let* $u(t) = S(t)\varphi$ *for* $t \geq 0$. *Then* u *is unique solution solution of the problem*

$$\begin{cases} u \in C([0,+\infty), L^2(\Omega)) \cap C^1((0,+\infty), L^2(\Omega)), \Delta u \in C([0,+\infty), L^2(\Omega)), \\ u'(t) = \Delta u(t), \forall t > 0, \\ u(0) = \varphi. \end{cases}$$

In addition we have

$$\begin{aligned} u &\in C\left([0,+\infty), H_0^2(\Omega)\right), \\ ||\Delta u(t)||_{L^2} &\leq \frac{1}{t\sqrt{2}}||\varphi||_{L^2}, \forall t>0, \\ ||\Delta u(t)||_{L^2} &\leq \frac{1}{\sqrt{t}\sqrt{2}}||\varphi||_{L^2}, \forall t>0. \end{aligned}$$

Assuming more regularity of φ, the solution u is also more regular.

Lemma 2.4 *For $t>0$, we define $K(t)\in\mathcal{S}\left(\mathbb{R}^N\right)$ by $K(t)x=(4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}$. Let $\psi\in C_c\left(\mathbb{R}^N\right)$ and let $v(t)=K(t)*\psi$. Then $v\in C\left([0,+\infty);C_b\left(\mathbb{R}^N\right)\right)\cap C^\infty\left([0,+\infty);C_b^2\left(\mathbb{R}^N\right)\right)$ and, for all $1\leq p\leq\infty$, we have $v\in C\left([0,+\infty);L^p\left(\mathbb{R}^N\right)\right)\cap C^\infty\left([0,+\infty);L^p\left(\mathbb{R}^N\right)\right)$. In addition:*

(i) *$v_t=\Delta v$ for all $t>0$.*

(ii) *$v(0)=\psi$.*

(iii) *$||v(t)||_{L^p}\leq(4\pi t)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}||\varphi||_{L^q}$, for $1\leq q\leq p\leq\infty$ and for all $t>0$.*

Lemma 2.5 *Let $\varphi\in Y,\varphi\geq0$ a.e. on Ω. Then, for all $t>0$, we have $S(t)\varphi\geq0$ a.e. on Ω. From the lemma 2.4 and 2.5 we have the following proposition:*

Proposition 2.7 *Let $1\leq q\leq p\leq\infty$. Then*

$$||S(t)\varphi||_{L^p}\leq(4\pi t)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}||\varphi||_{L^q},$$

for all $t>0$, and $\varphi\in X$.

Semilinear problems

Proposition 2.8 *: Let $\varphi \in C_0(\Omega), T > 0$, and let $u \in C([0,T], C_0(\Omega))$. Then u solution of*

$$\begin{cases} u \in C([0,T], C_0(\Omega)) \cap C((0,T], H_0^1(\Omega)) \cap C^1((0,T], L^2(\Omega)), \\ \Delta u \in C((0,T], L^2(\Omega)), \\ u_t - \Delta u = F(u), \forall t \in (0,T], \\ u(0) = \varphi, \end{cases}$$

if and only if u satisfies

$$u(t) = S(t)\varphi + \int_0^t S(t-s)F(u(s))ds, \forall t \in [0,T].$$

Remark 2.1 *For the proofs of these results and more details you can see [65].*

2.4 Damped Wave Equation

The original concept of damped wave equation, was appeared in 1965, when James Maxwell formulated a unifying theory of electricity and magnetism, his theory was concluded by the following electromagnetic wave equation

$$\nabla \times \overrightarrow{B} = \mu_0 \overrightarrow{J} + \mu_0 \epsilon_0 \partial_t \overrightarrow{E}. \tag{2.9}$$

Where $\overrightarrow{E}$ is the electric field, $\overrightarrow{B}$ is the magnetic field, $\overrightarrow{J}$ is the current density in the medium, μ_0, ϵ_0 are fundamental constants.

recently, Justin Richman added some conditions to the equation (2.9), he considered it inside ohmic materials and proposed that Ohm's law holds everywhere in the medium, i-e $\overrightarrow{J} = \sigma \overrightarrow{E}$, and the conductivity of material $\sigma = \mu_0 = \epsilon_0 = 1$.

using some mathematical tools, then, Maxwell's equation takes the following form

$$\partial_t^2 \overrightarrow{E} + \partial_t \overrightarrow{E} - \Delta \overrightarrow{E} = 0. \tag{2.10}$$

The new form of Maxwell's equation (2.10) neglects some physical effect, indeed, the assumption that the conductivity is constant is good approximation for most metals, but some effects like heating can cause the conductivity changes over time. However, perturbations to the system can be reintroduced by adding terms that may depend on position, time, and the solution itself, generalized as some function $F(\overrightarrow{E})$, which for our study be the nonlinear source term.

2.4.1 Well-Posedness

Definition 2.13 *(Damped Wave Equation).*

$$\begin{cases} \partial_t^2 u + \partial_t u - \Delta u = F(u), (x,t) \in \mathbb{R}^N \times [0,\infty), \\ u(x,0) = g, x \in \mathbb{R}^N, \\ u_t(x,0) = h, x \in \mathbb{R}^N. \end{cases} \tag{2.11}$$

Definition 2.14 *A partial differential equation is called* ***well-posed*** *if the following are satisfied:*

1. *The solution exists in some function space given initial data which is contained in the function space.*

2. *The solution is unique within this function space for given initial data.*

3. *The solution depends continuously on the initial data.*

The second part of this section is devoted to solve the damped wave equation (2.11) on the Fourier transform side. This will provide us necessary formulas for bounding solutions in X^s space later.

Lemma 2.6 *if*

$$\hat{u}_0 = \begin{cases} \hat{g}e^{-1/2t}\cosh\left(1/2t\sqrt{1-4|\xi|^2}\right) + \frac{\hat{g}+2\hat{h}}{\sqrt{1-4|\xi|^2}}e^{-1/2t}\sinh\left(1/2t\sqrt{1-4|\xi|^2}\right), |\xi| < 1/2, \\ \hat{g}e^{-1/2t} + \left(1/2\hat{g}+\hat{h}\right)te^{-1/2t}, \qquad |\xi| = 1/2, \\ \hat{g}e^{-1/2t}\cos\left(1/2t\sqrt{4|\xi|^2-1}\right) + \frac{\hat{g}+2\hat{h}}{\sqrt{4|\xi|^2-1}}e^{-1/2t}\sin\left(1/2t\sqrt{4|\xi|^2-1}\right), \quad |\xi| > 1/2, \end{cases} \tag{2.12}$$

then u_0 solves the equation (2.11) for $F \equiv 0$ (the homogeneous case).

Proof. Taking the spatial Fourier transform of the homogeneous damped wave equation yields the following:

$$\hat{u}_{tt} + \hat{u}_t - |\xi|^2 u = 0. \tag{2.13}$$

This is a second order ordinary differential equation with respect to time. this equation has the following characteristic equation

$$r^2 + r + |\xi|^2 = 0,$$

in which the solution is

$$r = \frac{-1 \pm \sqrt{1-4|\xi|^2}}{2}.$$

The solution of (2.13) depends on whether the roots of the characteristic equation are real, imaginary, or double roots. Depending on the value of $|\xi|$ these are all possible, so the solution must be computed piecewise.

Case 1, $|\xi| < \frac{1}{2}$: In this case, the roots are both real, and so the solution of (2.13) takes the form

$$\hat{u} = Ae^{-1/2t}\cosh\left(1/2t\sqrt{1-4|\xi|^2}\right) + Be^{-1/2t}\sinh\left(1/2t\sqrt{1-4|\xi|^2}\right). \tag{2.14}$$

Evaluating this at $t = 0$, we get

$$\hat{u}(\xi, 0) = A.$$

Since $u(x,0) = g$, it follow that $\hat{u}(x,0) = \hat{g}$, and so $A = \hat{g}$.

Taking the partial derivative with respect to time of (2.14) and evaluating at $t = 0$ gives

$$\hat{u}_t(\xi, 0) = -1/2A + 1/2B\sqrt{1 - 4|\xi|^2}.$$

From equation (2.11) we have $\hat{u}_t(\xi, 0) = \hat{h}$, so $B = \frac{\hat{g}+\hat{h}}{\sqrt{1-4|\xi|^2}}$

Case 2, $\xi = 1/2$: In this case, there is a double root, so the solution takes the form

$$\hat{u} = Ae^{-1/2t} + Bte^{-1/2t}.$$

Evaluating at $t = 0$ gives $A = \hat{g}$. Evaluating the partial derivative with respect to time at $t = 0$ gives $B = 1/2\hat{g} + \hat{h}$.

Case 3, $\xi < 1/2$: In this case, both roots are imaginary. The solution takes the form

$$\hat{u} = Ae^{-1/2t}\cos\left(1/2t\sqrt{4|\xi|^2 - 1}\right) + Be^{-1/2t}\sin\left(1/2t\sqrt{4|\xi|^2 - 1}\right).$$

Evaluating at $t = 0$ gives $A = \hat{g}$. Evaluating the partial derivative with respect to time at $t = 0$ gives

$$\hat{u}_t(\xi, 0) = -1/2A + 1/2B\sqrt{4|\xi|^2 - 1}.$$

From (2.11) we have $\hat{u}_t(\xi, 0) = \hat{h}$, so $B = \frac{\hat{g}+\hat{h}}{\sqrt{4|\xi|^2-1}}$. Thus, we have found the homogeneous solution to the equation (2.11) on the Fourier transform side. ■

To find a particular solution, we use Duhamel's Principal.

Lemma 2.7 *(Duhamel's Principal) .Suppose* $w = \int_0^t v(x, t-s, s)ds$, *where* v *solves*

$$\partial_t^2 v + \partial_t v - \Delta v = 0, \quad v_t(x, 0, s) = F(x, s). \tag{2.15}$$

Then, w *solves equation (2.11) for* $g \equiv 0$ *and* $h \equiv 0$ *(particular solution).*

Proof. plugging w into (2.11) and using Theorem 2.4, you can get the proof easily. ■

Consequently, we can deduce the following corollary:

Corollary 2.1 *For v describe as in Lemma **2.7***

$$\hat{v}\left(\xi, t-s; s\right) = \begin{cases} \frac{2\hat{F}(\xi,s)}{\sqrt{1-4|\xi|^2}} e^{-1/2(t-s)} \sinh\left(1/2(t-s)\sqrt{1-4|\xi|^2}\right), |\xi| < 1/2, \\ \hat{F}\left(\xi, s\right) t e^{-1/2(t-s)}, \quad |\xi| = 1/2, \\ \frac{2\hat{F}(\xi,s)}{\sqrt{4|\xi|^2-1}} e^{-1/2(t-s)} \sin\left(1/2(t-s)\sqrt{4|\xi|^2-1}\right), \quad |\xi| > 1/2. \end{cases} \tag{2.16}$$

Proof. This follows directly from Lemma 2.6 for $g \equiv 0$ and $h = F(x, s)$. ■

Now, we have an explicit formula for the solution of equation (2.11) in terms of its Fourier transform:

Theorem 2.6 *If $u = u_0 + w$, with u_0 describe as in Lemma 2.6 and w describe as in Lemma **2.7** then u solves equation (2.11).*

Proof. u satisfies the initial condition:

$$\begin{aligned} u\left(x, 0\right) &= u_0(x, 0) + w(x, 0) = g + 0 = g, \\ \partial_t u\left(x, 0\right) &= \partial_t u_0(x, 0) + \partial_t w(x, 0) = h + 0 = h. \end{aligned}$$

Plugging u into equation (2.11) yields

$$\partial_t^2 u + \partial_t u - \Delta u = \partial_t^2\left(u_0 + w\right) + \partial_t\left(u_0 + w\right) - \Delta\left(u_0 + w\right) = 0 + F = F. \tag{2.17}$$

As shown in Lemma 2.6 and (2.7). Since u is the sum of the homogeneous solution u_0 and a particular solution (w), u solves equation (2.11). ■

Theorem 2.7 *[53] for the solution u to equation (2.11)*

$$\begin{aligned} ||u||_{H^s} &\lesssim ||g||_{H^s} + ||h||_{H^{s-1}} + \int_0^t ||F(t')||_{H^{s-1}} dt', t \in [0, +\infty), \\ ||\partial_t u||_{H^{s-1}} &\lesssim ||g||_{H^s} + ||h||_{H^{s-1}} + \int_0^t ||F(t')||_{H^{s-1}} dt', t \in [0, +\infty). \end{aligned}$$

Chapter 3

Blow-up Of Solution For Damped Wave System With Nonlinear Memory

3.1 Introduction

In this chapter, we are going to extend the result of M.Loayza, I.G.Quinteiro presented in chapter 1, and we deal the blow-up case of damped wave system

$$\begin{cases} u_{tt} - \Delta u + u_t = \int_0^t (t-s)^{-\gamma_1} |v(s)|^p ds; t > 0, x \in \mathbb{R}^N, \\ v_{tt} - \Delta v + v_t = \int_0^t (t-s)^{-\gamma_2} |u(s)|^q ds; t > 0, x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), x \in \mathbb{R}^N, \\ v(0,x) = v_0(x), v_t(0,x) = v_1(x); x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

where, $p, q \geq 1$ satisfy $pq > 1$, and $0 < \gamma_1, \gamma_2 < 1$, $u_0, v_0 \in C_0\left(\mathbb{R}^N\right)$. There is a wide literature on the qualitative properties of solutions to the heat equations and the damped wave equations with polynomial nonlinearities, see for example, [10]-[50], and the references therein. These works deal with the questions of global existence, asymptotic behavior, blow-up in finite time and so forth as well as a variety of methods used to study these questions.

In [72], Xu considered the problem (3.1), he proved the global existence and asymptotic behavior as $t \to \infty$ of small data solutions in the case when $N = 1$, also, he showed under some positive data the nonexistence of nonnegative weak solution for $N \geq 1$. The method used in [72] is inspired from the weighted energy method developed by Todorova and Yordanov [20]. As we have seen, Xu restricts himself in the case of compactly supported data and the dimension $N = 1$.

Recently Berbiche [37], studied the problem (3.1), he obtained the small data global solution result in low-dimensional space $1 \leq N \leq 3$ with non compactly supported initial data and obtained the L^∞-decay estimates.

More recently [47], Wu et *al.* studied the problem (3.1) with $\gamma_1 = \gamma_2 = \gamma \in (0, 1/2)$, when $N = 1$, and obtained the critical exponent

$$F(p,q,\gamma) := \max\left\{1-\gamma+\frac{\gamma(p+1)}{pq-1}, 1-\gamma+\frac{\gamma(q+1)}{pq-1}\right\} - \frac{1}{2}.$$

They proved that if $F(p,q,\gamma) < 0$ there exists a unique global small data solution of (3.1) and if $F(p,q,\gamma) \geq 0$ the non-existence of global solution can be derived with the initial data having positive average value.

Before setting the result concerning the nonexistence of global solution of 3.1, let us define the weak solution of (3.1)

Definition 3.1 (*Weak Solution*). *Let* $T > 0, \gamma_1, \gamma_2 \in (0,1)$ *and* $u_0, u_1 \in L^1_{loc}(\mathbb{R}^N)$. *We say that* (u,v) *is a weak solution if* $(u,v) \in L^q\left((0,T_{\max}); L^q_{loc}(\mathbb{R}^N)\right) \times L^p\left((0,T_{\max}); L^p_{loc}(\mathbb{R}^N)\right)$ *and satisfies*

$$\begin{aligned}
&\Gamma(\alpha_1)\int_0^T\int_{\mathbb{R}^N}\varphi J^{\alpha_1}_{0|t}(|v|^p)\,dxdt + \int_{\mathbb{R}^N}u_1(x)\varphi(0,x)dx + \int_{\mathbb{R}^N}u_0(x)\left(\varphi(0,x)-\varphi_t(0,x)\right)dx \\
= &\int_0^T\int_{\mathbb{R}^N}u\varphi_{tt}dxdt - \int_0^T\int_{\mathbb{R}^N}u\varphi_t dxdt - \int_0^T\int_{\mathbb{R}^N}u\Delta\varphi dxdt
\end{aligned}$$

and

$$\begin{aligned}&\Gamma(\alpha_2)\int_0^T\int_{\mathbb{R}^N}\psi J_{0|t}^{\alpha_2}\left(|u|^q\right)dxdt+\int_{\mathbb{R}^N}v_1(x)\psi(0,x)dx+\int_{\mathbb{R}^N}v_0(x)\left(\psi(0,x)-\psi_t(0,x)\right)dx\\ =\;&\int_0^T\int_{\mathbb{R}^N}v\psi_{tt}dxdt-\int_0^T\int_{\mathbb{R}^N}v\psi_t dxdt-\int_0^T\int_{\mathbb{R}^N}v\Delta\psi dxdt \qquad (3.2)\end{aligned}$$

holds for any test functions $(\varphi,\psi)\in\left(C^2\left([0,T]\times\mathbb{R}^N\right)\right)^2$ *and satisfying* $\varphi(T,.)=\varphi_t(T,.)=0$ *and* $\psi(T,.)=\psi_t(T,.)=0,$ *where* $\alpha_1=1-\gamma_1,\alpha_2=1-\gamma_2.$

3.2 Blow-up Theorem and its Proof

Theorem 3.1 *Let* $N\geq 1, p,q>1,$ *and* $0<\gamma_1,\gamma_2<1.$ *Assume that*

$$\frac{N}{2}\leq\min\left\{\left(\frac{(2-\gamma_2)p+(1-\gamma_1)pq+1}{pq-1};\frac{(2-\gamma_1)q+(1-\gamma_2)pq+1}{pq-1}\right)\right\}, \qquad (3.3)$$

or

$$\begin{cases}1-p\gamma_2+p(1-q\gamma_1)\geq 0,\\ or\\ 1-q\gamma_1+q(1-p\gamma_2)\geq 0.\end{cases} \qquad (3.4)$$

If the initial data $(u_i,v_i), i=0,1,$ *satisfy*

$$\int_{\mathbb{R}^N}u_i(x)dx>0 \text{ and} \int_{\mathbb{R}^N}v_i(x)dx>0, i=0,1. \qquad (3.5)$$

Then the solution $(u(t,x),v(t,x))$ *of problem (3.1) does not exist globally.*

Proof. The proof is by contradiction. Suppose that (u,v) is a nontrivial weak solution of (3.1) which exists globally in time. Furthermore, let define the following test functions

$$\varphi(t,x)=D_{t|T}^{\alpha_1}\left(\tilde{\varphi}(t,x)\right):=D_{t|T}^{\alpha_1}\left(\varphi_1^l(x)\,\varphi_2(t)\right)$$
$$\psi(t,x)=D_{t|T}^{\alpha_2}\left(\tilde{\varphi}(t,x)\right):=D_{t|T}^{\alpha_2}\left(\varphi_1^l(x)\,\varphi_2(t)\right)$$

with $\varphi_1(x) = \Phi\left(\frac{|x|}{T^{\frac{1}{2}}}\right)$, $\varphi_2(t) = \left(1 - \frac{t}{T}\right)_+^l$ and $\Phi(r)$ is a smooth non increasing function such that

$$\Phi(r) = \begin{cases} 1 & \text{if } \ 0 \leq r \leq 1, \\ 0 & \text{if } \ r \geq 2, \end{cases}$$

with $0 \leq \Phi \leq 1$. The constant $l > 1$ in the definition of φ and ψ will be chosen later. We have from the definition of the weak solution

$$\begin{aligned} &\Gamma(\alpha_1) \int_0^T \int_{\mathbb{R}^N} J_{0|t}(|v|^p) D_{t|T}^{\alpha_1} \tilde{\varphi} dx dt + \int_{\mathbb{R}^N} u_1(x) D_{t|T}^{\alpha_1} \tilde{\varphi}(0,x) dx \\ &+ \int_{\mathbb{R}^N} u_0(x) \left(D_{t|T}^{\alpha_1} \tilde{\varphi}(0,x) - \partial_t D_{t|T}^{\alpha_1} \tilde{\varphi}(0,x) \right) dx \\ = \ &\int_0^T \int_{\mathbb{R}^N} u \partial_t^2 D_{t|T}^{\alpha_1} \tilde{\varphi} dx dt - \int_0^T \int_{\mathbb{R}^N} u \partial_t D_{t|T}^{\alpha_1} \tilde{\varphi} dx dt - \int_0^T \int_{\mathbb{R}^N} u \Delta D_{t|T}^{\alpha_1} \tilde{\varphi} dx dt, \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} &\Gamma(\alpha_2) \int_0^T \int_{\mathbb{R}^N} J_{0|t}(|u|^q) D_{t|T}^{\alpha_2} \tilde{\varphi} dx dt + \int_{\mathbb{R}^N} v_1(x) D_{t|T}^{\alpha_2} \tilde{\varphi}(0,x) dx \\ &+ \int_{\mathbb{R}^N} v_0(x) \left(D_{t|T}^{\alpha_2} \tilde{\varphi}(0,x) - \partial_t D_{t|T}^{\alpha_2} \tilde{\varphi}(0,x) \right) dx \\ = \ &\int_0^T \int_{\mathbb{R}^N} v \partial_t^2 D_{t|T}^{\alpha_2} \tilde{\varphi} dx dt - \int_0^T \int_{\mathbb{R}^N} v \partial_t D_{t|T}^{\alpha_2} \tilde{\varphi} dx dt - \int_0^T \int_{\mathbb{R}^N} v \Delta D_{t|T}^{\alpha_2} \tilde{\varphi} dx dt. \end{aligned} \quad (3.7)$$

Using the formulas (2.3) and (2.8) in the left-hand sides of (3.6) and (3.7), while in the right-hand sides using (2.4), we conclude that

$$\begin{aligned} &\Gamma(\alpha_1) \int_0^T \int_{\mathbb{R}^N} D_{0|t}^{\alpha_1} J_{0|t}^{\alpha_1}(|v|^p) \tilde{\varphi} dx dt + C T^{-\alpha_1} \int_{\mathbb{R}^N} u_1(x) \varphi_1^l(x) dx \\ &+ C\left(T^{-\alpha_1} + T^{-1-\alpha_1}\right) \int_{\mathbb{R}^N} u_0(x) \varphi_1^l(x) dx \\ &= \int_0^T \int_{\mathbb{R}^N} u \left(D_{t|T}^{2+\alpha_1} \tilde{\varphi} + D_{t|T}^{1+\alpha_1} \tilde{\varphi} \right) dx dt - \int_0^T \int_{\mathbb{R}^N} u \Delta D_{t|T}^{\alpha_1} \tilde{\varphi} dx dt, \end{aligned} \quad (3.8)$$

and

$$\Gamma(\alpha_2)\int_0^T\int_{\mathbb{R}^N} D_{0|t}^{\alpha_2}J_{0|t}^{\alpha_2}\left(|u|^q\right)\tilde{\varphi}dxdt + CT^{-\alpha_2}\int_{\mathbb{R}^N} v_1(x)\varphi_1^l(x)dx$$
$$+C\left(T^{-\alpha_2}+T^{-1-\alpha_2}\right)\int_{\mathbb{R}^N} v_0(x)\varphi_1^l(x)dx$$
$$=\int_0^T\int_{\mathbb{R}^N} v\left(D_{t|T}^{2+\alpha_2}\tilde{\varphi}+D_{t|T}^{1+\alpha_2}\tilde{\varphi}\right)dxdt-\int_0^T\int_{\mathbb{R}^N} v\Delta D_{t|T}^{\alpha_2}\tilde{\varphi}dxdt. \tag{3.9}$$

From (2.5), we may write

$$\Gamma(\alpha_1)\int_0^T\int_{\mathbb{R}^N}|v|^p\tilde{\varphi}dxdt + CT^{-\alpha_1}\int_{\mathbb{R}^N} u_1(x)\varphi_1^l(x)dx + C\left(T^{-\alpha_1}+T^{-(1+\alpha_1)}\right)$$
$$\times\int_{\mathbb{R}^N} u_0(x)\varphi_1^l(x)dx$$
$$=\int_0^T\int_{\mathbb{R}^N} u\left(D_{t|T}^{2+\alpha_1}\tilde{\varphi}+D_{t|T}^{1+\alpha_1}\tilde{\varphi}\right)dxdt-\int_0^T\int_{\mathbb{R}^N} u\Delta\varphi_1^l(x)D_{t|T}^{\alpha_1}\varphi_2 dxdt \tag{3.10}$$

and

$$\Gamma(\alpha_2)\int_0^T\int_{\mathbb{R}^N}|u|^q\tilde{\varphi}dxdt + CT^{-\alpha_2}\int_{\mathbb{R}^N} v_1(x)\varphi_1^l(x)dx + C\left(T^{-\alpha_2}+T^{-(1+\alpha_2)}\right)$$
$$\times\int_{\mathbb{R}^N} v_0(x)\varphi_1^l(x)dx$$
$$=\int_0^T\int_{\mathbb{R}^N} v\left(D_{t|T}^{2+\alpha_2}\tilde{\varphi}+D_{t|T}^{1+\alpha_2}\tilde{\varphi}\right)dxdt-\int_0^T\int_{\mathbb{R}^N} v\Delta\varphi_1^l(x)D_{t|T}^{\alpha_2}\varphi_2 dxdt \tag{3.11}$$

Using the fact that the support of φ_1 is included in $\Omega := \left\{x\in\mathbb{R}^N : |x|\le 2T^{\frac{1}{2}}\right\}$, we get

$$\Gamma(\alpha_1)\int_{Q_T}|v|^p\tilde{\varphi}dxdt + CT^{-\alpha_1}\int_\Omega u_1(x)\varphi_1^l(x)dx + C\left(T^{-\alpha_1}+T^{-(1+\alpha_1)}\right)\int_\Omega u_0(x)\varphi_1^l(x)dx$$
$$=\int_{Q_T} u\left(D_{t|T}^{2+\alpha_1}\tilde{\varphi}+D_{t|T}^{1+\alpha_1}\tilde{\varphi}\right)dxdt-\int_{Q_T} u\Delta\varphi_1^l(x)D_{t|T}^{\alpha_1}\varphi_2 dxdt,$$

and

$$\Gamma(\alpha_2)\int_{Q_T}|u|^q\tilde{\varphi}dxdt + CT^{-\alpha_2}\int_\Omega v_1(x)\varphi_1^l(x)dx + C\left(T^{-\alpha_2}+T^{-(1+\alpha_2)}\right)\int_\Omega v_0(x)\varphi_1^l(x)dx$$
$$\int_{Q_T} v\left(D_{t|T}^{2+\alpha_2}\tilde{\varphi}+D_{t|T}^{1+\alpha_2}\tilde{\varphi}\right)dxdt-\int_{Q_T} v\Delta\varphi_1^l(x)D_{t|T}^{\alpha_2}\varphi_2 dxdt.$$

Where $Q_T := \{(x,t);(x,t)\in\Omega\times[0,T]\}$. In addition, the condition (3.5) implies that

$\int_\Omega v_i\varphi_1^l(x)dx > 0$ and $\int_\Omega u_i\varphi_1^l(x)dx > 0$ for $i = 0, 1$. Indeed since

$$\lim_{T\to+\infty} u_i(x)\varphi_1^l(x) = u_i(x), i = 0, 1,$$

and $u_i(x) \in L^1_{loc}\left(\mathbb{R}^N\right)$, for $i = 0, 1$ then by Lebesgue dominated convergence Theorem, we obtain

$$\begin{aligned}\lim_{T\to+\infty}\int_\Omega u_i(x)\varphi_1^l(x)dx &= \int_{\mathbb{R}^N} u_i(x)dx.\ i = 0, 1.\\ \lim_{T\to+\infty}\int_\Omega v_i(x)\varphi_1^l(x)dx &= \int_{\mathbb{R}^N} v_i(x)dx.\ i = 0, 1.\end{aligned}$$

So $\int_{\mathbb{R}^N} u_idx > 0$ (resp. $\int_{\mathbb{R}^N} v_idx > 0$) implies that $\int_\Omega u_i\varphi_1^l(x)dx > 0$ (resp. $\int_\Omega v_i\varphi_1^l(x)dx > 0$), for T large. From what we have seen, we can write

$$\begin{aligned}\int_{Q_T} |v|^p\tilde{\varphi}dxdt &\le C\int_{Q_T} |u|\left(D^{2+\alpha_1}_{t|T}\tilde{\varphi} + D^{1+\alpha_1}_{t|T}\tilde{\varphi}\right)dxdt\\ &\quad +C\int_{Q_T} |u|\varphi_1^{l-2}\left(|\Delta\varphi_1| + |\nabla\varphi_1|^2\right)D^{\alpha_1}_{t|T}\varphi_2dxdt, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\int_{Q_T} |u|^q\tilde{\varphi}dxdt &\le C\int_{Q_T} |v|\left(D^{2+\alpha_2}_{t|T}\tilde{\varphi} + D^{1+\alpha_2}_{t|T}\tilde{\varphi}\right)dxdt\\ &\quad +C\int_{Q_T} |v|\varphi_1^{l-2}\left(|\Delta\varphi_1| + |\nabla\varphi_1|^2\right)D^{\alpha_2}_{t|T}\varphi_2dxdt, \end{aligned} \tag{3.13}$$

where we have used the formula $\Delta\varphi_1^l = \left(l\varphi_1^{l-1}\Delta\varphi_1 + l\,(l-1)\,|\nabla\varphi|^2\right)$ and $\varphi_1 \le 1$. Using Hö lder's inequality, with parameters p and p' (resp. q and q'), to the right-hand side of the inequalities (3.12) and (3.13), we get

$$\int_{Q_T} |v|^p\tilde{\varphi}dxdt \le \left(\int_{Q_T} |u|^q\tilde{\varphi}dxdt\right)^{\frac{1}{q}}\mathcal{A}, \tag{3.14}$$

$$\int_{Q_T} |u|^q \tilde{\varphi} dxdt \leq \left(\int_{Q_T} |v|^p \tilde{\varphi} dxdt \right)^{\frac{1}{p}} \mathcal{B}, \tag{3.15}$$

where

$$\begin{aligned}
\mathcal{A} \ : \ &= C \left(\int_{Q_T} \varphi_1^l \varphi_2^{-\frac{1}{q-1}} \left(\left(D_{t|T}^{2+\alpha_1} \varphi_2 \right)^{q'} + \left(D_{t|T}^{1+\alpha_1} \varphi_2 \right)^{q'} \right) dxdt \right)^{\frac{1}{q'}} \\
&+ C \left(\int_{Q_T} \varphi_1^{l-2q'} \varphi_2^{-\frac{1}{q-1}} \left(|\Delta \varphi_1| + |\nabla \varphi_1|^2 \right)^{q'} |D_{t|T}^{\alpha_1} \varphi_2|^{q'} dxdt \right)^{\frac{1}{q'}}, \\
\mathcal{B} \ : \ &= C \left(\int_{Q_T} \varphi_1^l \varphi_2^{-\frac{1}{p-1}} \left(\left(D_{t|T}^{2+\alpha_2} \varphi_2 \right)^{p'} + \left(D_{t|T}^{1+\alpha_2} \varphi_2 \right)^{p'} \right) dxdt \right)^{\frac{1}{p'}} \\
&+ C \left(\int_{Q_T} \varphi_1^{l-2p'} \varphi_2^{-\frac{1}{p-1}} \left(|\Delta \varphi_1| + |\nabla \varphi_1|^2 \right)^{p'} |D_{t|T}^{\alpha_2} \varphi_2|^{p'} dxdt \right)^{\frac{1}{p'}},
\end{aligned}$$

with $p' = \frac{p}{p-1}$ (resp $q' = \frac{q}{q-1}$) Now, combining (3.14) and (3.15), we obtain

$$\left(\int_{Q_T} |v|^p \tilde{\varphi} dxdt \right)^{1-\frac{1}{pq}} \leq C \mathcal{B}^{\frac{1}{q}} \mathcal{A}, \tag{3.16}$$

and

$$\left(\int_{Q_T} |u|^q \tilde{\varphi} dxdt \right)^{1-\frac{1}{pq}} \leq C \mathcal{A}^{\frac{1}{p}} \mathcal{B}. \tag{3.17}$$

Next, we consider the scaled variables $t = T\tau$, $x = T^{\frac{1}{2}} y$; in the right-hand sides of (3.16),(3.17), and using (2.6), (2.7) we find

$$\begin{cases} \left(\int_{Q_T} |v|^p \tilde{\varphi} dxdt \right)^{1-\frac{1}{pq}} \leq CT^{\delta_1}, \\ \left(\int_{Q_T} |u|^q \tilde{\varphi} dxdt \right)^{1-\frac{1}{pq}} \leq CT^{\delta_2}, \end{cases} \tag{3.18}$$

with

$$\begin{aligned}
\delta_1 &= \frac{1}{q'} \left(-\left(1+\alpha_1\right) q' + \frac{N}{2} + 1 \right) + \frac{1}{qp'} \left(-\left(1+\alpha_2\right) p' + \frac{N}{2} + 1 \right), \\
\delta_2 &= \frac{1}{p'} \left(-\left(1+\alpha_2\right) p' + \frac{N}{2} + 1 \right) + \frac{1}{pq'} \left(-\left(1+\alpha_1\right) q' + \frac{N}{2} + 1 \right).
\end{aligned}$$

The condition (3.3) leads to either

- The case $\delta_1 < 0$ or $\delta_2 < 0$, then, as $T \to +\infty$ the right-hand side of the first equation (resp. second equation) of (3.18) tends to zero and the left-hand side converges to

$$\left(\int_{\mathbb{R}^N \times \mathbb{R}_+} |v|^p dxdt\right)^{1-\frac{1}{pq}} \left(\text{resp} \left(\int_{\mathbb{R}^N \times \mathbb{R}_+} |u|^q dxdt\right)^{1-\frac{1}{pq}}\right).$$

Consequently, the couple $(u, v) \equiv (0, 0)$.

- The case $\delta_1 = 0$ (resp $\delta_2 = 0$), in this case, using (3.18), we conclude that

$$v \in L^p\left((0,\infty), L^p\left(\mathbb{R}^N\right)\right), \; u \in L^q\left((0,\infty), L^q\left(\mathbb{R}^N\right)\right). \tag{3.19}$$

Now, we need to modify the test function $\varphi_1(x)$ by introducing a new parameter B $(1 << B < T)$ as follows $\varphi_1(x) := \Phi\left(\frac{|x|}{B^{-\frac{1}{2}}T^{\frac{1}{2}}}\right)$. From (3.12) and (3.13), we get

$$\begin{aligned}\int_{\Omega_1} |v|^p \tilde{\varphi} dxdt \;\; &\leq \;\; C\int_{\Omega_1} |u| \left(D_{t|T}^{2+\alpha_1}\tilde{\varphi} + D_{t|T}^{1+\alpha_1}\tilde{\varphi}\right) dxdt \\ &\quad + C\int_{\Sigma} |u|\varphi_1^{l-2}\left(\Delta|\varphi_1| + \nabla|\varphi_1|^2\right) D_{t|T}^{\alpha_1}\varphi_2 dxdt \end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\int_{\Omega_1} |u|^q \tilde{\varphi} dxdt \;\; &\leq \;\; C\int_{\Omega_1} |v| \left(D_{t|T}^{2+\alpha_2}\tilde{\varphi} + D_{t|T}^{1+\alpha_2}\tilde{\varphi}\right) dxdt \\ &\quad + C\int_{\Sigma} |v|\varphi_1^{l-2}\left(\Delta|\varphi_1| + \nabla|\varphi_1|^2\right) D_{t|T}^{\alpha 2}\varphi_2 dxdt \end{aligned} \tag{3.21}$$

where $\Omega_1 := [0,T]\times\left\{x \in \mathbb{R}^N, |x| \leq 2B^{-\frac{1}{2}}T^{\frac{1}{2}}\right\}, \Sigma := [0,T]\times\left\{x \in \mathbb{R}^N, \frac{T^{\frac{1}{2}}}{B^{\frac{1}{2}}} \leq |x| \leq 2\frac{T^{\frac{1}{2}}}{B^{\frac{1}{2}}}\right\}.$

It follows from (3.19), that

$$\lim_{T\to\infty}\int_{\Sigma} |v|^p \tilde{\varphi} dxdt = 0 \;\text{ or }\; \lim_{T\to\infty}\int_{\Sigma} |u|^q \tilde{\varphi} dxdt = 0. \tag{3.22}$$

By using Hölder inequality again, we get

$$\begin{cases} \displaystyle\int_{\Omega_1} |v|^p \tilde{\varphi} dxdt \leq C \left(\int_{\Omega_1} |u|^q dxdt \right)^{\frac{1}{q}} \mathcal{A}_1 + C \left(\int_{\Sigma} |u|^q dxdt \right)^{\frac{1}{q}} \mathcal{C}_1, \\ \displaystyle\int_{\Omega_1} |u|^q \tilde{\varphi} dxdt \leq C \left(\int_{\Omega_1} |v|^p dxdt \right)^{\frac{1}{p}} \mathcal{B}_1 + C \left(\int_{\Sigma} |v|^p dxdt \right)^{\frac{1}{p}} \mathcal{C}_2, \end{cases} \tag{3.23}$$

where

$$\mathcal{A}_1 := \left(\int_{\Omega_1} \varphi_1^l \varphi_2^{-\frac{1}{q-1}} \left(\left(D_{t|T}^{2+\alpha_1} \varphi_2 \right)^{q'} + \left(D_{t|T}^{1+\alpha_1} \varphi_2 \right)^{q'} \right) dxdt \right)^{\frac{1}{q'}}.$$

$$\mathcal{B}_1 := \left(\int_{\Omega_1} \varphi_1^l \varphi_2^{-\frac{1}{p-1}} \left(\left(D_{t|T}^{2+\alpha_2} \varphi_2 \right)^{p'} + \left(D_{t|T}^{1+\alpha_2} \varphi_2 \right)^{p'} \right) dxdt \right)^{\frac{1}{p'}}.$$

$$\mathcal{C}_1 := \left(\int_{\Sigma} \varphi_1^{l-2q'} \varphi_2^{-\frac{1}{q-1}} \left(\Delta|\varphi_1| + \nabla|\varphi_1|^2 \right)^{q'} \left(D_{t|T}^{\alpha_1} \varphi_2 \right)^{q'} dxdt \right)^{\frac{1}{q'}}.$$

$$\mathcal{C}_2 := \left(\int_{\Sigma} \varphi_1^{l-2p'} \varphi_2^{-\frac{1}{p-1}} \left(\Delta|\varphi_1| + \nabla|\varphi_1|^2 \right)^{p'} \left(D_{t|T}^{\alpha_2} \varphi_2 \right)^{p'} dxdt \right)^{\frac{1}{p'}}.$$

If we set $Y := \displaystyle\int_{\Omega_1} |v|^p \tilde{\varphi} dxdt$, $Z := \displaystyle\int_{\Omega_1} |u|^q \tilde{\varphi} dxdt$. It follows from (3.23) that

$$\begin{cases} Y \leq C Z^{\frac{1}{q}} \mathcal{A}_1 + C \left(\displaystyle\int_{\Sigma} |u|^q \tilde{\varphi} dxdt \right)^{\frac{1}{q}} \mathcal{C}_1. \\ Z \leq C Y^{\frac{1}{p}} \mathcal{B}_1 + C \left(\displaystyle\int_{\Sigma} |v|^p \tilde{\varphi} dxdt \right)^{\frac{1}{p}} \mathcal{C}_2. \end{cases} \tag{3.24}$$

On the other hand by integrating (1.4) on Σ, we obtain

$$\begin{cases} \left(\displaystyle\int_{\Sigma} |u|^q \tilde{\varphi} dxdt \right) \leq C \left(\displaystyle\int_{\Sigma} |v|^p \tilde{\varphi} dxdt \right)^{\frac{1}{p}} (\mathcal{B}_2 + \mathcal{C}_2). \\ \left(\displaystyle\int_{\Sigma} |v|^p \tilde{\varphi} dxdt \right) \leq C \left(\displaystyle\int_{\Sigma} |u|^q \tilde{\varphi} dxdt \right)^{\frac{1}{q}} (\mathcal{A}_2 + \mathcal{C}_1). \end{cases} \tag{3.25}$$

Where

$$\mathcal{A}_2 \ : \ = \left(\int_{\Sigma} \varphi_1^l \varphi_2^{-\frac{1}{q-1}} \left(\left(D_{t|T}^{2+\alpha_1} \varphi_2 \right)^{q'} + \left(D_{t|T}^{1+\alpha_1} \varphi_2 \right)^{q'} \right) dxdt \right)^{\frac{1}{q'}}.$$

$$\mathcal{B}_2 \ : \ = \left(\int_{\Sigma} \varphi_1^l \varphi_2^{-\frac{1}{p-1}} \left(\left(D_{t|T}^{2+\alpha_2} \varphi_2 \right)^{p'} + \left(D_{t|T}^{1+\alpha_2} \varphi_2 \right)^{p'} \right) dxdt \right)^{\frac{1}{p'}}.$$

Combining (3.24) and (3.25), we obtain

$$\begin{cases} Y \leq C\mathcal{A}_1\mathcal{B}_1^{\frac{1}{q}}Y^{\frac{1}{pq}} + C\left(\int_\Sigma |v|^p\tilde{\varphi}dxdt\right)^{\frac{1}{pq}}\left[\mathcal{A}_1\mathcal{C}_2^{\frac{1}{q}} + \mathcal{B}_2^{\frac{1}{q}}\mathcal{C}_1 + \mathcal{C}_1\mathcal{C}_2^{\frac{1}{q}}\right], \\ Z \leq C\mathcal{A}_1^{\frac{1}{p}}\mathcal{B}_1Z^{\frac{1}{pq}} + C\left(\int_\Sigma |u|^q\tilde{\varphi}dxdt\right)^{\frac{1}{pq}}\left[\mathcal{A}_2^{\frac{1}{p}}\mathcal{C}_2 + \mathcal{B}_1\mathcal{C}_1^{\frac{1}{p}} + \mathcal{C}_1^{\frac{1}{p}}\mathcal{C}_2\right]. \end{cases}$$

Using ε-Young inequality, we get

$$Y \leq C\mathcal{A}_1^{\frac{pq}{pq-1}}\mathcal{B}_1^{\frac{p}{pq-1}} + C\left(\int_\Sigma |v|^p\tilde{\varphi}dxdt\right)^{\frac{1}{pq}}\left[\mathcal{A}_1\mathcal{C}_2^{\frac{1}{q}} + \mathcal{B}_2^{\frac{1}{q}}\mathcal{C}_1 + \mathcal{C}_1\mathcal{C}_2^{\frac{1}{q}}\right], \tag{3.26}$$

$$Z \leq C\mathcal{A}_1^{\frac{q}{pq-1}}\mathcal{B}^{\frac{pq}{pq-1}} + C\left(\int_\Sigma |u|^q\tilde{\varphi}dxdt\right)^{\frac{1}{pq}}\left[\mathcal{A}_2^{\frac{1}{p}}\mathcal{C}_2 + \mathcal{B}_1\mathcal{C}_1^{\frac{1}{p}} + \mathcal{C}_1^{\frac{1}{p}}\mathcal{C}_2\right]. \tag{3.27}$$

Now, using the scaled variables (y,τ) defined by $\tau = T^{-1}t, y = T^{\frac{-1}{2}}B^{\frac{1}{2}}x$, in the right-hand sides of (3.26) and (3.27), we then have the estimates

$$\begin{cases} Y \leq CT^{\frac{pq}{pq-1}\delta_1}B^{\frac{pq}{pq-1}k_1} + C\left(\int_\Sigma |v|^p\tilde{\varphi}dxdt\right)^{\frac{1}{pq}}T^{\delta_1}\left(B^{k_2} + B^{k_3} + B^{k_4}\right), \\ Z \leq CT^{\frac{pq}{pq-1}\delta_2}B^{\frac{pq}{pq-1}l_1} + C\left(\int_\Sigma |u|^q\tilde{\varphi}dxdt\right)^{\frac{1}{pq}}T^{\delta_2}\left(B^{l_2} + B^{l_3} + B^{l_4}\right). \end{cases}$$

where

$$\begin{aligned} k_1 &:= -\frac{N}{2}\left(\frac{1}{qp'} + \frac{1}{q'}\right), k_2 := \frac{1}{q} - \frac{N}{2}\left(\frac{1}{qp'} + \frac{1}{q'}\right), k_3 := 1 - k_1, k_4 := 1 + k_2, \\ l_1 &:= -\frac{N}{2}\left(\frac{1}{pq'} + \frac{1}{p'}\right), l_2 := \frac{1}{p} - \frac{N}{2}\left(\frac{1}{pq'} + \frac{1}{p'}\right), l_3 := 1 - l_1, l_4 := 1 + l_2. \end{aligned}$$

The last two inequalities with $\delta_1 = 0$ (resp. $\delta_2 = 0$) imply that

$$Y \leq C\mathcal{B}^{\frac{pq}{pq-1}k_1} + C\left(\int_\Sigma |v|^p\tilde{\varphi}dxdt\right)^{\frac{1}{pq}}\left(B^{k_2} + B^{k_3} + B^{k_4}\right), \tag{3.28}$$

$$Z \leq C\mathcal{B}^{\frac{pq}{pq-1}l_1} + C\left(\int_\Sigma |u|^q\tilde{\varphi}dxdt\right)^{\frac{1}{pq}}\left(B^{l_2} + B^{l_3} + B^{l_4}\right). \tag{3.29}$$

We obtain via (3.22) after passing to the limit in (3.28), (resp.(3.29)) when $T \to +\infty$

$$\begin{cases} \int_{\mathbb{R}^N \times \mathbb{R}_+} |v|^p dxdt \leq CB^{\frac{pq}{pq-1}k_1}, \\ \int_{\mathbb{R}^N \times \mathbb{R}_+} |u|^q dxdt \leq CB^{\frac{pq}{pq-1}l_1}. \end{cases} \tag{3.30}$$

Finally, as $k_1 < 0$ and $l_1 < 0$ computing the limit in (3.30) when $B \to \infty$ we infer that $u \equiv 0; v \equiv 0,$ which is contradiction.

- When (3.4) is satisfied, we argue as in the case ($\delta_1 < 0, \delta_2 < 0$) by choosing the following function

$$\varphi_1^l(x) := \Phi^l\left(\frac{|x|}{R}\right)$$

We repeat the same computation as above by using the new variables $t = T^{-1}\tau$ and $x = R^{-1}y$ in both sides of (3.16) and (3.17) we find

$$\left(\int_{Q_T} |v|^p \tilde{\varphi} dxdt\right)^{1-\frac{1}{pq}} \leq C_1(T,R), \left(\int_{Q_T} |u|^q \tilde{\varphi} dxdt\right)^{1-\frac{1}{pq}} \leq C_2(T,R), \tag{3.31}$$

where $T > R > 1$,

$$C_1(T,R) := T^{\lambda_1}R^{\beta_1} + T^{\lambda_2}R^{\beta_2} + T^{\lambda_3}R^{\beta_3} + T^{\lambda_4}R^{\beta_4},$$

$$C_2(T,R) := T^{\mu_1}R^{\sigma_1} + T^{\mu_2}R^{\sigma_2} + T^{\mu_3}R^{\sigma_3} + T^{\mu_4}R^{\sigma_4}.$$

With

$$\lambda_1 := \tfrac{1}{qp'}\left(1 - p'(\alpha_2 + 1)\right) + \tfrac{1}{q'}\left(1 - q'(\alpha_1 + 1)\right),$$

$$\lambda_2 := \tfrac{1}{qp'}\left(1 - p'(\alpha_2 + 1)\right) + \tfrac{1}{q'}\left(1 - q'\alpha_1\right),$$

$$\lambda_3 := \tfrac{1}{q}\left(\tfrac{1}{p'} - \alpha_2\right) + \tfrac{1}{q'}\left(1 - q'(\alpha_1 + 1)\right),$$

$$\lambda_4 := \tfrac{1}{q}\left(\tfrac{1}{p'} - \alpha_2\right) + \tfrac{1}{q'}\left(1 - q'\alpha_1\right).$$

$$\beta_1 := \tfrac{N}{p'q} + \tfrac{N}{q'}, \beta_2 := \beta_1 - 2, \beta_3 := \tfrac{1}{q}\left(\tfrac{N}{p'} - 2\right) + \tfrac{N}{q'}, \beta_4 := \beta_3 - 2.$$

$\mu_1 := \frac{1}{pq'}\left(1 - q'\left(\alpha_1 + 1\right)\right) + \frac{1}{p'}\left(1 - p'\left(\alpha_2 + 1\right)\right),$

$\mu_2 := \frac{1}{pq'}\left(1 - q'\left(\alpha_1 + 1\right)\right) + \frac{1}{p'}\left(1 - p'\alpha_2\right),$

$\mu_3 := \frac{1}{pq'}\left(1 - \alpha_1 q'\right) + \frac{1}{p'}\left(1 - p'\left(\alpha_2 + 1\right)\right),$

$\mu_4 := \frac{1}{pq'}\left(1 - \alpha_1 q'\right) + \frac{1}{p'}\left(1 - p'\alpha_2\right),$

and

$\sigma_1 := \frac{N}{pq'} + \frac{N}{p'}, \sigma_2 := \sigma_1 - 2, \sigma_3 := \frac{1}{p}\left(\frac{N}{q'} - 2\right) + \frac{N}{p'}, \sigma_4 := \sigma_3 - 2.$

The condition (3.4) equivalent to or $\lambda_4 = 0$(resp $\mu_4 = 0$).

Firstly, if $\lambda_4 < 0$(resp $\mu_4 < 0$). Passing to the limit in (3.31) as $T \to \infty$ we infer, as $\lambda_i < 0$(resp $\mu_i < 0$) for $i = 1, ..., 4$ that $\int_{\mathbb{R}_+}\int_{|x|\leq 2R} |v|^p \tilde{\varphi}\left(x,t\right) dxdt = 0$, and

$\int_{\mathbb{R}_+}\int_{|x|\leq 2R} |u|^q \tilde{\varphi}(x,t) dxdt = 0,$

by letting $R \to \infty$, we get $\int_{\mathbb{R}^N \times \mathbb{R}_+} |v|^p \tilde{\varphi}\left(x,t\right) dxdt = 0, \int_{\mathbb{R}^N \times \mathbb{R}_+} |u|^q \tilde{\varphi}(x,t) dxdt = 0$,which implies that $u = v \equiv 0$. This is contradiction.

- When $\lambda_4 = 0$ or $\mu_4 = 0$, we get from (3.31) after passing to the limit when $T \to \infty$,

$$\int_{\mathbb{R}_+}\int_{|x|\leq 2R} |v|^p \tilde{\varphi}\left(x,t\right) dxdt \leq CR^{\beta_4}, \int_{\mathbb{R}_+}\int_{|x|\leq 2R} |u|^q \tilde{\varphi}(x,t) dxdt \leq CR^{\sigma_4}. \quad (3.32)$$

Precisely, if $\beta_4 < 0$ or $\sigma_4 < 0$ in particular when $\frac{N-2}{N} \leq \gamma_1, \gamma_2 < 1$, passing to the limit in (3.32) as $R \to \infty$, we find $u = v \equiv 0$. This is contradiction.

■

Chapter 4

On The Nonexistence of Global Solution For Wave Equations With Double Damping Terms and Nonlinear Memory

4.1 Introduction

In this chapter we study whether or not there exist solutions to the initial value problem:

$$\begin{cases} \Box u(t,x) + |u(t,x)|^{m-1}\partial_t u(t,x) + \partial_t u(t,x) = \displaystyle\int_0^t (t-\tau)^{-\gamma} |u(\tau,x)|^p d\tau, \ t \in [0,T] \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x), \ \ x \in \mathbb{R}^N, \end{cases} \tag{4.1}$$

where $\Box = \partial_t^2 - \Delta$, u is the unknown real-valued function, $N \geq 1$, $m \geq 1$, $p > 1$, $\gamma \in (0,1)$ and $u_0(x)$, $u_1(x)$ are the given initial data.

There are many literatures concerning this type of equations and all researchers turn around the fact that the asymptotic behavior of solutions of semi-linear damped wave equations is similar to the one of the corresponding semi-linear heat equations, see for example, ([20], [21], [39], [51], [55], [69]), and the references therein. In order to motivate our results, it must be recall some facts concern the Cauchy problem for the dissipative nonlinear wave equation.

A natural extension of (1.4) consists in introducing a displacement-dependent damping coefficient, thus leading to the following problem

$$u_{tt} - \Delta u + |u|^{m-1} u_t = \int_0^t (t-s)^{-\gamma} |u(s,.)|^p ds, \tag{4.2}$$

where $0 < \gamma < 1$, p,$m > 1$ and $N \geq 1$, $u_0 \in C_0(\mathbb{R}^N)$. Berbiche and Hakem [39], showed the local existence and blow-up for the problem (4.2). More precisely, they proved that, if $p > m > 1$ and the initial conditions satisfy

$$\int_{\mathbb{R}^N} u_0(x)dx > 0; \int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x)dx > 0; \text{ and } \int_{\mathbb{R}^N} u_1(x)dx > 0, \tag{4.3}$$

and if

$$N \leq \min\left\{ \frac{2(m+(1-\gamma)p}{p-1+(1-\gamma)(m-1)}, \frac{2(1+(2-\gamma)p}{\left(\frac{(p-1)(2-\gamma)}{(p-m)} + \gamma - 1\right)(p-1)} \right\} \text{ or } p \leq \frac{1}{\gamma},$$

the solution of problem (4.2) does not exist globally in time.

It should be emphasized that the natural space where solutions are found is the one of the energy $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. However, such energy inequalities do not seem to fit in this setting for the following reasons: some difficulties appear due to the lack of Lipschitz continuity of function the $|u|^{m-1} v$ $(1 < m < 2)$ with respect to $(u,v) \in \mathbb{R}^N \times \mathbb{R}^N$, and the Sobolev embedding $H^1(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$ is true only when $N = 1$, another difficulty comes from the absence of regularity created by the singular kernel involving in the nonlinear source term.

In this work, we will focus on the interaction between the nonlinear nonlocal source term involving fractional integral kernel and the dissipation term. In particular, we will give conditions relating the space dimension N with parameters p,γ, m, for which the solution of (4.1) with initial data having positive average does not exist globally. We also emphasize here that no effect of displacement-dependent damping on the critical exponent as unlike to what the authors saw in the article from [39].

In order to show that the problem (4.1) is well-posed, we use an idea developed by Lions and Strauss [27], Katayama et al.[61] combined with energy estimates in higher order Sobolev spaces (see [53]). Moreover, our approach to derive Fujita exponent is based on the test-function method, developed by Mitidieri & Pohozaev [12], [13], Pohozaev & Tesei [60], Pohozaev & Veron [35] and Zhang [51].

The rest of the chapter is divided into three sections. In Section 2, we present some lemmas which will be needed later in our proof. Section 3 is devoted to the local existence result and Section 4 contains a nonexistence of global weak solutions result.

The positive constants C will be change from line to line.

4.2 Some preliminary results

The following Lemmas will be used in the proof of Theorems 3.1.

Lemma 4.1 *(See [44], Proposition 2.4, p. 5) If* $s > \frac{N}{2}$*, then*

$$H^s\left(\mathbb{R}^N\right) \subset C\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right),$$

where the inclusion is continuous. In fact

$$\|u\|_{L^\infty} \leq C\,\|u\|_{H^s}$$

The next Lemma is consequence of lemma 2.1 and Proposition 3.7, p. 10 in [44]

Lemma 4.2 *(See [44]) Assume that* $s_1, s_2 \geq s > \frac{N}{2}$, *then for* $u \in H^{s_1}\left(\mathbb{R}^N\right)$, $v \in H^{s_2}\left(\mathbb{R}^N\right)$, *we have the estimates*

$$\|uv\|_{H^s(\mathbb{R}^N)} \leq C\|u\|_{H^{s_1}(\mathbb{R}^N)}\|v\|_{H^{s_2}(\mathbb{R}^N)}$$

where C *is constant independent of* u *and* v.

The last Lemma uses the equivalent norm of $\|u\|_{H^{s-1}}$ (see [1, Theorem 7.48, p. 214]). We omit their proof since it can be found in (see [24])

Lemma 4.3 *For any* $s \in (1,2) \cup \mathbb{N}^*$ *and* $p \in (1,+\infty) \cap (s-1,+\infty)$ *we have for a nonnegative function* $f \in L^\infty\left(\mathbb{R}^N\right) \cap H^{s-1}\left(\mathbb{R}^N\right)$, $f^p \in H^{s-1}\left(\mathbb{R}^N\right)$ *and there exists a positive constant* C *such that*

$$\|f^p\|_{H^{s-1}(\mathbb{R}^N)} \leq C\|f\|^{p-1}_{L^\infty(\mathbb{R}^N)}\|f\|_{H^{s-1}(\mathbb{R}^N)}.$$

Now, proceed with the following linear damped wave equation:

$$\begin{cases} u_{tt} - \Delta u + u_t = F(t,x), \;\; x \in \mathbb{R}^N, \, t>0, \\ u(x,0) = g\left(x\right), \; u_t(x,0) = h\left(x\right), \, x \in \mathbb{R}^N. \end{cases} \tag{4.4}$$

Let us give some results which will be used in the following.

Lemma 4.4 *Let* $s \in \mathbb{R}$. *Let* $(g,h) \in H^s\left(\mathbb{R}^N\right) \times H^{s-1}\left(\mathbb{R}^N\right)$ *and* $F(t,x) \in L^1\left([0,T], H^{s-1}\left(\mathbb{R}^N\right)\right)$. *Then for every* $T>0$, *there is a unique solution* $u \in C\left([0,T], H^s\left(\mathbb{R}^N\right)\right) \cap C^1\left([0,T], H^{s-1}\left(\mathbb{R}^N\right)\right)$ *of cauchy problem of (4.4). Moreover,* u *satisfies*

$$\|u\|_{H^s(\mathbb{R}^N)} + \|u_t\|_{H^{s-1}(\mathbb{R}^N)} \leq C\left(\|g\|_{H^s(\mathbb{R}^N)} + \|h\|_{H^{s-1}(\mathbb{R}^N)} + \int_0^t \|F(\tau,.)\|_{H^{s-1}(\mathbb{R}^N)}\,d\tau\right),$$

for all $0 \leq t \leq T$, *where* C *only depends on* s.

In addition, if $g \in H^{s+1}\left(\mathbb{R}^N\right)$, $h \in H^s\left(\mathbb{R}^N\right)$ *and* $F(t,x) \in L^1\left([0,T], H^s\left(\mathbb{R}^N\right)\right) \cap L^\infty\left([0,T], H^{s-1}\left(\mathbb{R}^N\right)\right)$, $u_{tt} \in L^\infty\left([0,T], H^{s-1}\left(\mathbb{R}^N\right)\right)$ *satisfies*

$$\|u_{tt}\|_{H^{s-1}(\mathbb{R}^N)} \leq C\left(\|h\|_{H^s(\mathbb{R}^N)} + \|g\|_{H^{s+1}(\mathbb{R}^N)} + \|F(t)\|_{H^{s-1}(\mathbb{R}^N)} + \int_0^t \|F(t')\|_{H^s(\mathbb{R}^N)}\, dt'\right),$$

for all $0 \leq t \leq T$, *where* C *only depends on* s.

Proof. The first estimate is already shown in [53]. Arguing as in [53] to show the second estimate. It is known that the first derivative of $\hat{u}$ satisfies in the domain $|\xi| < 1/2$ (see Theorem 2.6),

$$\begin{aligned}
\hat{u}_t(t) &= -\frac{1}{2}\hat{u}(t) + \frac{1}{2}\hat{g}\sqrt{1-4|\xi|^2}e^{-\frac{t}{2}}\sinh\left(\frac{1}{2}t\sqrt{1-4|\xi|^2}\right) \\
&\quad + \frac{1}{2}\left(\hat{g}+2\hat{h}\right)e^{-\frac{t}{2}}\cosh\left(\frac{1}{2}t\sqrt{1-4|\xi|^2}\right) \\
&\quad + \int_0^t \hat{F}(t')e^{-\frac{1}{2}(t-t')}\cosh\left(\frac{1}{2}(t-t')\sqrt{1-4|\xi|^2}\right)dt',
\end{aligned}$$

we derive this expression with respect to t, we find

$$\begin{aligned}
&\hat{u}_{tt}(t) = -\tfrac{1}{2}\hat{u}_t(t) - \tfrac{1}{4}\hat{g}\sqrt{1-4|\xi|^2}e^{-\frac{t}{2}}\sinh\left(\tfrac{1}{2}t\sqrt{1-4|\xi|^2}\right) + \tfrac{1}{4}\hat{g}\left(1-4|\xi|^2\right) \\
&\times e^{-\frac{t}{2}}\cosh\left(\tfrac{1}{2}t\sqrt{1-4|\xi|^2}\right) - \tfrac{1}{4}\left(\hat{g}+2\hat{h}\right)e^{-\frac{t}{2}}\cosh\left(\tfrac{1}{2}t\sqrt{1-4|\xi|^2}\right) \\
&+\tfrac{1}{4}\left(\hat{g}+2\hat{h}\right)e^{-\frac{t}{2}}\sqrt{1-4|\xi|^2}\sinh\left(\tfrac{1}{2}t\sqrt{1-4|\xi|^2}\right) \\
&+\hat{F}(t) + \int_0^t \hat{F}(t')\left[-\tfrac{1}{2}e^{-\frac{1}{2}(t-t')}\cosh\left(\tfrac{1}{2}(t-t')\sqrt{1-4|\xi|^2}\right)\right. \\
&\left.+\tfrac{1}{2}\sqrt{1-4|\xi|^2}e^{-\frac{1}{2}(t-t')}\sinh\left(\tfrac{1}{2}(t-t')\sqrt{1-4|\xi|^2}\right)\right]dt',
\end{aligned}$$

or

$$\begin{aligned}\hat{u}_{tt}(t) &= -\frac{1}{2}\hat{u}_t(t) - \hat{g}\,|\xi|^2 e^{-\frac{t}{2}} \cosh\left(\frac{1}{2}t\sqrt{1-4|\xi|^2}\right) - \frac{1}{2}\hat{h}e^{-\frac{t}{2}}\cosh\left(\frac{1}{2}t\sqrt{1-4|\xi|^2}\right) \\ &+\frac{1}{2}\hat{h}e^{-\frac{t}{2}}\sqrt{1-4|\xi|^2}\sinh\left(\frac{1}{2}t\sqrt{1-4|\xi|^2}\right) + \hat{F}(t) \\ &+\int_0^t \hat{F}(t')\left[-\frac{1}{2}e^{-\frac{1}{2}(t-t')}\cosh\left(\frac{1}{2}(t-t')\sqrt{1-4|\xi|^2}\right)\right. \\ &\left.+\frac{1}{2}\sqrt{1-4|\xi|^2}e^{-\frac{1}{2}(t-t')}\sinh\left(\frac{1}{2}(t-t')\sqrt{1-4|\xi|^2}\right)\right]dt'. \end{aligned} \tag{4.5}$$

Now we consider the case for $|\xi| > 1/2$. From Theorem 2.6, we have

$$\begin{aligned}\hat{u}_t(t) &= -\frac{1}{2}\hat{u}(t) - \frac{1}{2}\hat{g}\sqrt{4|\xi|^2-1}e^{-\frac{t}{2}}\sin\left(\frac{1}{2}t\sqrt{4|\xi|^2-1}\right) \\ &+\frac{1}{2}\left(\hat{g}+2\hat{h}\right)e^{-\frac{t}{2}}\cos\left(\frac{1}{2}t\sqrt{4|\xi|^2-1}\right) \\ &+\int_0^t \hat{F}(t')e^{-\frac{1}{2}(t-t')}\cos\left(\frac{1}{2}(t-t')\sqrt{4|\xi|^2-1}\right)dt'.\end{aligned}$$

By taking the derivative of $\hat{u}_t(t)$, we get

$$\begin{aligned}&\hat{u}_{tt}(t) = -\tfrac{1}{2}\hat{u}_t(t) + \tfrac{1}{4}\hat{g}\sqrt{4|\xi|^2-1}e^{-\frac{t}{2}}\sin\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right) - \tfrac{1}{4}\hat{g}\left(4|\xi|^2-1\right) \\ &\times e^{-\frac{t}{2}}\cos\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right) - \tfrac{1}{4}\left(\hat{g}+2\hat{h}\right)e^{-\frac{t}{2}}\cos\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right) - \tfrac{1}{4}\left(\hat{g}+2\hat{h}\right)e^{-\frac{t}{2}} \\ &\times\sqrt{4|\xi|^2-1}\sin\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right) + \hat{F}(t) + \int_0^t \hat{F}(t')\left[-\tfrac{1}{2}e^{-\frac{1}{2}(t-t')}\cos\left(\tfrac{1}{2}(t-t')\sqrt{4|\xi|^2-1}\right)\right. \\ &\left.-\tfrac{1}{2}\sqrt{4|\xi|^2-1}e^{-\frac{1}{2}(t-t')}\sin\left(\tfrac{1}{2}(t-t')\sqrt{4|\xi|^2-1}\right)\right]dt',\end{aligned}$$

which gives

$$\begin{aligned}&\hat{u}_{tt}(t) = -\tfrac{1}{2}\hat{u}_t(t) - \hat{g}\,|\xi|^2 e^{-\frac{t}{2}}\cos\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right) \\ &-\tfrac{1}{2}\hat{h}e^{-\frac{t}{2}}\cos\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right) - \tfrac{1}{2}\hat{h}e^{-\frac{t}{2}}\sqrt{4|\xi|^2-1}\sin\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right) \\ &+\hat{F}(t) + \int_0^t \hat{F}(t')\left[-\tfrac{1}{2}e^{-\frac{1}{2}(t-t')}\cos\left(\tfrac{1}{2}(t-t')\sqrt{4|\xi|^2-1}\right)\right. \\ &\left.-\tfrac{1}{2}\sqrt{4|\xi|^2-1}e^{-\frac{1}{2}(t-t')}\sin\left(\tfrac{1}{2}(t-t')\sqrt{4|\xi|^2-1}\right)\right]dt'.\end{aligned} \tag{4.6}$$

Now, using the definition of the H^s norm

$$\|u_{tt}\|_{H^{s-1}(\mathbb{R}^n)} = \left[\int_{\mathbb{R}^N} \left(1+|\xi|^2\right)^{s-1} |\hat{u}_{tt}(t)|^2 d\xi\right]^{1/2}$$
$$= \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^{s-1} |\hat{u}_{tt}(t)|^2 d\xi + \int_{|\xi|>1/2} \left(1+|\xi|^2\right)^{s-1} |\hat{u}_{tt}(t)|^2 d\xi\right]^{1/2}$$
$$\leq \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^{s-1} |\hat{u}_{tt}(t)|^2 d\xi\right]^{1/2} + \left[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^{s-1} |\hat{u}_{tt}(t)|^2 d\xi\right]^{1/2} := I_1 + I_2, \tag{4.7}$$

where we have used the Triangle Inequality.

Let's, estimate the first integrand. From (4.5), we have

$$I_1 \leq \|u_t\|_{H^{s-1}(\mathbb{R}^n)} + \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^{s-1} \tfrac{(1+|\xi|^2)}{1+|\xi|^2} |\hat{g}|^2 d\xi\right]^{1/2}$$
$$+C\left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^{s-1} \left|\hat{h}(\xi)\right|^2 d\xi\right]^{1/2} + \|F(t)\|_{H^{s-1}(\mathbb{R}^n)} + \int_0^t \|F(t')\|_{H^{s-1}(\mathbb{R}^n)} dt'$$
$$\leq C\left(\|u_t\|_{H^{s-1}(\mathbb{R}^n)} + \|h\|_{H^{s-1}(\mathbb{R}^n)} + \|g\|_{H^{s}(\mathbb{R}^n)} + \|F(t)\|_{H^{s-1}(\mathbb{R}^n)} + \int_0^t \|F(t')\|_{H^{s-1}(\mathbb{R}^n)} dt'\right).$$

Where we have used that $(1-y^2)\, e^{-\frac{t}{2}} \cosh\left(\frac{1}{2}ty\right) \leq 1$, $e^{-\frac{t}{2}} y \sinh\left(\frac{1}{2}ty\right) \leq 1$ for $y \in [0,1]$, $t > 0$.

Inserting the estimate of $\|u_t\|_{H^{s-1}(\mathbb{R}^n)}$ in the last inequality it yields

$$I_1 \leq C\left(\|h\|_{H^{s-1}(\mathbb{R}^n)} + \|g\|_{H^{s}(\mathbb{R}^n)} + \|F(t)\|_{H^{s-1}(\mathbb{R}^n)} + \int_0^t \|F(t')\|_{H^{s-1}(\mathbb{R}^n)} dt'\right). \tag{4.8}$$

Estimate for the second integrand in (4.7). We have

$$\left[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^{s-1} |\hat{u}_{tt}(t)|^2 d\xi\right]^{1/2} \leq \|u_t\|_{H^{s-1}(\mathbb{R}^N)} +$$
$$\left[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^{s-1} \left(1+|\xi|^2\right)^2 \left|\hat{g}\tfrac{|\xi|^2}{1+|\xi|^2} e^{-\frac{t}{2}} \cos\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right)\right|^2 d\xi\right]^{1/2}$$
$$+\left[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^{s-1} \tfrac{(1+|\xi|^2)}{1+|\xi|^2} \left|\hat{h}\left(1+\sqrt{4|\xi|^2-1}\sin\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right)\right)\right|^2 d\xi\right]^{1/2} + \|F(t)\|_{H^{s-1}(\mathbb{R}^N)}$$
$$+\left[\int_0^t \int_{|\xi|>1/2} \left(1+|\xi|^2\right)^{s-1} \left(1+|\xi|^2\right) \left|\hat{F}(t')\left(1+\tfrac{\sqrt{4|\xi|^2-1}}{\sqrt{1+|\xi|^2}} \sin\left(\tfrac{1}{2}t\sqrt{4|\xi|^2-1}\right)\right)\right|^2 dt' d\xi\right]^{1/2}$$

$$\leq \|u_t\|_{H^{s-1}(\mathbb{R}^N)} + \|h\|_{H^s(\mathbb{R}^N)} + \|g\|_{H^{s+1}(\mathbb{R}^N)} + \|F(t)\|_{H^{s-1}(\mathbb{R}^N)} + \int_0^t \|F(t')\|_{H^s(\mathbb{R}^N)}\, dt'. \tag{4.9}$$

Using the estimate of $\|u_t\|_{H^{s-1}(\mathbb{R}^N)}$ into (4.9) and by Sobolev's embedding theorem, we find that

$$I_2 \leq C\left(\|h\|_{H^s(\mathbb{R}^N)} + \|g\|_{H^{s+1}(\mathbb{R}^N)} + \|F(t)\|_{H^{s-1}(\mathbb{R}^N)} + \int_0^t \|F(t')\|_{H^s(\mathbb{R}^N)}\, dt\right). \tag{4.10}$$

From (4.8) and (4.10), we conclude the proof of Lemma 2.4 . ∎

4.3 Local-existence

In this section, we first present the following Theorem concerning the existence and uniqueness of the local solution to the problem (4.1). Next, we will prove this result.

Theorem 4.1 *Let $N \geq 1$, $s > \frac{N}{2} + 1$ and $m, p \in (1, +\infty) \cap (s-1, +\infty)$. Then for any $u_0 \in H^s\left(\mathbb{R}^N\right)$ and $u_1 \in H^{s-1}\left(\mathbb{R}^N\right)$, (4.1) admits a unique solution*

$$u \in C\left([0,T]; H^s\left(\mathbb{R}^N\right)\right) \cap C^1\left([0,T]; H^{s-1}\left(\mathbb{R}^N\right)\right)$$

with some positive T, which depends only on $\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}$.

Proof. The idea of the proof is based on the fact that we have $m\,|u|^{m-1} u_t = \partial_t\left(|u|^{m-1} u\right)$. By introducing a new unknown v satisfying $u = v_t$, the problem (4.1) is reduced so to the following problem of the known case where the nonlinear term is locally lipschitz

$$\begin{cases} v_{tt}(t,x) - \Delta v(t,x) + v_t(t,x) = -\frac{1}{m}\,|v_t|^{m-1}\, v_t(t,x) + \frac{1}{1-\gamma}\displaystyle\int_0^t (t-\tau)^{1-\gamma}\,|v_t(\tau,x)|^p\, d\tau \\ +\frac{1}{m}\,|u_0|^{m-1}\, u_0(x) + u_0(x) + u_1(x),\ t > 0,\ x \in \mathbb{R}^N, \end{cases} \tag{4.11}$$

where $v(0,x)=0$, $v_t(0,x)=u_0(x)$, $x\in\mathbb{R}^N$. It is not difficult to check that $u\in C\left([0,T];H^s\left(\mathbb{R}^N\right)\right)\cap C^1\left([0,T];H^{s-1}\left(\mathbb{R}^N\right)\right)$ solution to (4.1) if and only if v is solution to (4.11) in the class

$$\begin{cases} v\in C\left([0,T];H^s\left(\mathbb{R}^N\right)\right), \\ v_t\in C^1\left([0,T];H^{s-1}\left(\mathbb{R}^N\right)\right),\ v_{tt}\in C^1\left([0,T];H^{s-1}\left(\mathbb{R}^N\right)\right). \end{cases} \tag{4.12}$$

Let us define

$$X_T:=C\left([0,T];H^s\left(\mathbb{R}^N\right)\right)\cap C^1\left([0,T];H^{s-1}\left(\mathbb{R}^N\right)\right),$$

$$Y_T:=L^\infty\left([0,T];H^s\left(\mathbb{R}^N\right)\right)\cap W^{1,\infty}\left([0,T];H^{s-1}\left(\mathbb{R}^N\right)\right),$$

$$Y_{T,M}:=\left\{u\in Y_T;\ \sup_{0\le t\le T}\left(\|v(t,.)\|_{H^s}+\|v_t(t,.)\|_{H^{s-1}}\right)\le M\right\}.$$

Next, set $X_{T,M}=Y_{T,M}\cap X_T$. Obviously $X_T\subset Y_T$ and $X_{T,M}\subset Y_{T,M}$. Set

$$G(v_t)=-\tfrac{1}{m}|v_t|^{m-1}v_t(t,x)+\tfrac{1}{1-\gamma}\int_0^t(t-\tau)^{1-\gamma}|v_t(\tau,x)|^p\,ds$$
$$+\tfrac{1}{m}|u_0|^{m-1}u_0(x)+u_0(x)+u_1(x).$$

For any $w\in Y_T$, define $\Phi[w]=v$, where $v\in X_T$ is a solution to

$$\begin{cases} v_{tt}-\Delta v+v_t=G(w_t) & \text{in } (0,T)\times\mathbb{R}^N, \\ v(x,0)=0,\ v_t(x,0)=u_0(x) & x\in\mathbb{R}^N. \end{cases} \tag{4.13}$$

Since we have $G(w_t)\in L^1\left([0,T];H^{s-1}\left(\mathbb{R}^N\right)\right)$ for any $w\in Y_T$ by Sobolev's embedding theorem, existence and uniqueness of such $v\in X_T$ is guaranteed by the Cauchy problems for linear damped wave equations (Lemma 4.4). Let $M=4\left(\|u_0\|_{H^s}^m+\|u_0\|_{H^s}+\|u_1\|_{H^{s-1}}\right)$. We first claim that $w\in Y_{T,M}$ implies that $\Phi[w]\in X_{T,M}$ for sufficiently small $T>0$. Set $\Phi[w]=v$. From the Lemma 4.4, we have

$$\|v(t,.)\|_{H^s}+\|v_t(t,.)\|_{H^{s-1}}\le C\left(\|v_0\|_{H^s}+\|v_1\|_{H^{s-1}}+\int_0^t\|G(w_t)(\tau,.)\|_{H^{s-1}}\,d\tau\right). \tag{4.14}$$

From Lemma 4.3, since $s > \frac{N}{2}+1$, we have

$$\int_0^t \|G(w_t)(\tau,.)\|_{H^{s-1}}\, d\tau \le \tfrac{1}{m}\int_0^t \left\| |w_t|^{m-1} w_t(\tau,.)\right\|_{H^{s-1}} d\tau$$
$$+\tfrac{1}{1-\gamma}\int_0^t (t-s)^{1-\gamma}\, \||w_t|^p\|_{H^{s-1}}(s)\, ds + T\left\|\tfrac{1}{m}|u_0|^{m-1}u_0(x)+u_0(x)+u_1(x)\right\|_{H^{s-1}}$$
$$\le CT\|w_t\|_{L^\infty}^{m-1}\sup_{0\le t\le Y}\|w_t\|_{H^{s-1}} + C\int_0^t (t-s)^{1-\gamma}\|w_t\|_{L^\infty}^{p-1}\|w_t\|_{H^{s-1}}(s)\, ds$$
$$+CT\left(\|u_0\|_{H^{s-1}}^m + \|u_0\|_{H^{s-1}} + \|u_1\|_{H^{s-1}}\right).$$

Then by the Lemma 4.1, we find

$$\begin{aligned}\int_0^t \|G(w_t)(\tau,.)\|_{H^{s-1}}\, d\tau &\le CT\sup_{0\le t\le T}\|w_t\|_{H^{s-1}}^m + CT^{2-\gamma}\sup_{0\le t\le T}\|w_t\|_{H^{s-1}}^p \\ &+CT\left(\|u_0\|_{H^s}^m+\|u_0\|_{H^s}+\|u_1\|_{H^{s-1}}\right) \\ &\le C\left(TM^m + T^{2-\gamma}M^p + T\left(\|u_0\|_{H^s}^m+\|u_0\|_{H^s}+\|u_1\|_{H^{s-1}}\right)\right).\end{aligned} \tag{4.15}$$

From (4.14) and (4.15), we get

$$\begin{aligned}\sup_{0\le t\le T}\left(\|v(t,.)\|_{H^s}+\|v_t(t,.)\|_{H^{s-1}}\right) &\le C\left(\|u_0\|_{H^{s-1}} + TM^m + T^{2-\gamma}M^p\right. \\ &\left.+T\left(\|u_0\|_{H^s}^m+\|u_0\|_{H^s}+\|u_1\|_{H^{s-1}}\right)\right) \\ &\le C\left(M+TM^m+T^{2-\gamma}M^p+TM\right).\end{aligned} \tag{4.16}$$

By (4.16) we arrive at

$$\sup_{0\le t\le T}\left(\|v(t,.)\|_{H^s}+\|v_t(t,.)\|_{H^{s-1}}\right) \le C_{T,M}M,$$

where $C_{T,M} = C\left(\frac{1}{4}+TM^{m-1}+CT^{2-\gamma}M^{p-1}+\frac{T}{4}\right)$. Since we can find $T_1 > 0$ such that $C_{T,M} \le 1$ for any $T \in (0, T_1]$, this implies the claim.

We next prove that Φ is a contraction mapping in $X_{T,M}$ for small T by using the Lemma 4.4 and the mean value theorem. Suppose that $w_1, w_2 \in Y_{T,M}$, then we have $\Phi[w_1], \Phi[w_2] \in X_{T,M}$.

Let v_i $(i=1,2)$ be solutions to the following problems

$$\begin{cases} (v_i)_{tt} - \Delta v_i + (v_i)_t = -\frac{1}{m} \left|(w_i)_t\right|^{m-1} (w_i)_t (t,x) + \frac{1}{1-\gamma} \displaystyle\int_0^t (t-\tau)^{1-\gamma} \left|(w_i)_t (\tau,x)\right|^p d\tau \\ +\frac{1}{m} \left|u_0\right|^{m-1} u_0 (x) + u_0 (x) + u_1 (x), \ \text{ in } t>0,\ x \in \mathbb{R}^N, \\ v_i(0,x) = 0,\ (v_i)_t (0,x) = u_0 (x) \ \text{ in } \mathbb{R}^N. \end{cases} \tag{4.17}$$

Set $\tilde{v} = v_1 - v_2$, we have $\tilde{v}$ verifies

$$\begin{cases} \tilde{v}_{tt} - \Delta \tilde{v} + \tilde{v}_t = -\frac{1}{m} \left|(w_1)_t\right|^{m-1} (w_1)_t (t,x) + \frac{1}{m} \left|(w_2)_t\right|^{m-1} (w_2)_t (t,x) \\ +\frac{1}{1-\gamma} \displaystyle\int_0^t (t-\tau)^{1-\gamma} \left(\left|(w_1)_t (\tau,x)\right|^p - \left|(w_2)_t (\tau,x)\right|^p\right) d\tau \quad \text{ in } (0,T) \times \mathbb{R}^N, \\ \tilde{v}(0,x) = \tilde{v}_t(0,x) = 0 \quad \text{in } \mathbb{R}^N. \end{cases} \tag{4.18}$$

for $i=1,2$, respectively.

Since $w_i \in Y_{T,M}$ implies that $\displaystyle\int_0^t (t-s)^{1-\gamma} \left|(w_i)_t\right|^p (s)\, ds$ and $\left|(w_i)_t\right|^{m-1} (w_i)_t$ are functions in $L^\infty\left(0,T;H^{s-1}\right)$ by Sobolev's embedding theorem, we have $v_i \in X_T$ and

$$\tilde{v} \in C\left([0,T];H^s\left(\mathbb{R}^N\right)\right) \cap C^1\left([0,T];H^{s-1}\left(\mathbb{R}^N\right)\right). \tag{4.19}$$

By (4.17), (4.18) and (4.20), the higher order energy inequality

$$\begin{aligned} &\left\|\tilde{v}_t (t,.)\right\|_{H^{s-1}} + \left\|\tilde{v} (t,.)\right\|_{H^s} \\ &\leq C \int_0^t \int_0^s (s-\tau)^{1-\gamma} \left\|\left(\left|(w_1)_t\right|^p - \left|(w_2)_t\right|^p\right)(\tau,.)\right\|_{H^{s-1}} d\tau ds \\ &+C \int_0^t \left\|\left|(w_1)_t\right|^{m-1} (w_1)_t (\tau,.) - \left|(w_2)_t\right|^{m-1} (w_2)_t (\tau,.)\right\|_{H^{s-1}} d\tau. \end{aligned} \tag{4.20}$$

Note that since $|v|^{l-1} v$ with $l>1$ is a C^1 function, the mean value theorem implies

$$\left|\left|v_1\right|^{l-1} v_1 - \left|v_2\right|^{l-1} v_2\right| \leq C \left(\left|v_1\right|^{l-1} + \left|v_2\right|^{l-1}\right) \left|v_1 - v_2\right|. \tag{4.21}$$

Applying the inequality (4.21) with Sobolev's inequality $\|\tilde{v}\|_{L^\infty} \leq C \|\tilde{v}\|_{H^{s-1}}$ to right-hand of (4.20), it follows for $s > \frac{N}{2} + 1$

$$\begin{aligned}
&\|\tilde{v}_t(t,.)\|_{H^{s-1}} + \|\tilde{v}(t,.)\|_{H^s} \\
&+C\int_0^t (t-\tau)^{1-\gamma} \left(\|(w_1)_t\|_{H^{s-1}}^{p-1} + \|(w_2)_t\|_{H^{s-1}}^{p-1}\right) \|(w_1-w_2)_t(\tau,.)\|_{H^{s-1}} \, d\tau \\
&+C\int_0^t \left(\|(w_1)_t\|_{H^{s-1}}^{m-1} + \|(w_2)_t\|_{H^{s-1}}^{m-1}\right) \|(w_1-w_2)_t(\tau,.)\|_{H^{s-1}} \, d\tau \\
&\leq C\left(T^{2-\gamma}M^{p-1} + TM^{m-1}\right) \sup_{0\leq\tau\leq T} \|(w_1-w_2)_t(\tau,.)\|_{H^{s-1}} \text{ for } 0 \leq t \leq T.
\end{aligned} \tag{4.22}$$

Then by (4.22), we get

$$\|\Phi[w_1] - \Phi[w_2](t,.)\|_{X_{T,M}} \leq C\left(T^{2-\gamma}M^{p-1} + TM^{m-1}\right) \times \sup_{0\leq\tau\leq T} \|(w_1-w_2)_t(\tau,.)\|_{H^{s-1}}. \tag{4.23}$$

In the following, we fix $T \in (0, T_1]$ which is small enough to have

$$C\left(T^{2-\gamma}M^{p-1} + TM^{m-1}\right) < \frac{1}{2}.$$

Therefore

$$\|\Phi[w_1] - \Phi[w_2](t,.)\|_{X_{T,M}} \leq \frac{1}{2} \|(w_1-w_2)(\tau,.)\|_{X_{T,M}} \qquad \text{for such } T. \tag{4.24}$$

Finally, define

$$\begin{cases} v^{(0)}(t,x) = v_t(0,x) = u_0(x), \\ v^{(k)} = \Phi\left[v^{(k-1)}\right] \quad k = 1,2,3,.... \end{cases}$$

By (4.24), there exists some $v \in C([0,T];H^s)$ such that $v^{(k)} \to v$ in $C([0,T];H^s)$ as $k \to \infty$. Now, we will show that this v belongs to X_T and is solution to (4.11). Since $v^{(k)} \in X_{T,M}$, $\{v^{(k)}\}$ (resp. $\left\{v_t^{(k)}\right\}$) has a weak-* convergent subsequence in $L^\infty(0,T;H^s)$ (resp. in $L^\infty(0,T;H^{s-1})$). Since $v^{(k)} \to v$ in $C([0,T];H^s)$, the above subsequence of $\{v^{(k)}\}$ (resp. $\left\{v_t^{(k)}\right\}$) converges weakly-* to v (resp. to v_t) in $L^\infty(0,T;H^s)$ (resp. in

$L^{\infty}(0,T;H^{s-1})$), and consequently we see that $v \in L^{\infty}(0,T;H^{s})$ and $v_t \in L^{\infty}(0,T;H^{s-1})$. Then we can see that $v \in Y_{T,M}$, and then we get $\Phi[v] \in X_{T,M}$. Hence we can apply (4.24) to have

$$\sup_{0\leq t\leq T} \left\| \Phi[v] - \Phi\left[v^{(k)}\right](t,.) \right\|_{X_{T,M}} \leq \frac{1}{2} \sup_{0\leq t\leq T} \left\| \left(v - v^{(k)}\right)(t,.) \right\|_{Y_{T,M}}. \tag{4.25}$$

Since the right-hand side of (4.25) tends to 0 as $k \to \infty$, we get $\Phi\left[v^{(k)}\right] \to \Phi[v]$ in $C([0,T];H^{s})$. Since we have showed that $v^{(k)} \to v$ in $C([0,T];H^{s})$, passing to the limit in $v^{(k)} = \Phi\left[v^{(k-1)}\right]$, we obtain $v = \Phi[v] \in X_{T,M}$. This v apparently the desired solution. The uniqueness of weak solutions in $X_{T,M}$ follows immediately from Gronwall's inequality. As $u = v_t$, we have $u \in C([0,T];H^{s-1})$. This completes the proof of Theorem 3.1. ■

Remark 4.1 *If we take* $(u_0,u_1) \in H^{s+1}\left(\mathbb{R}^N\right) \times H^{s}\left(\mathbb{R}^N\right)$, *with* $s > N/2-1$, *then from Lemma 2.4 the solution of problem (4.1)* $(u,u_t) \in C\left([0,T];H^{s}\left(\mathbb{R}^N\right)\right) \times C\left([0,T];H^{s-1}\left(\mathbb{R}^N\right)\right)$.

4.4 Blow-up results

This section is devoted to the blow-up of solutions of the problem (4.1). We start by introducing the definition of the weak solution of (4.1).

Definition 4.1 *Let* $T>0$, $0<\gamma<1$ *and* $u_0 \in L^1_{loc}\left(\mathbb{R}^N\right) \cap L^m_{loc}\left(\mathbb{R}^N\right)$, $u_1 \in L^1_{loc}\left(\mathbb{R}^N\right)$. *We say that* u *is a weak solution if* $u \in L^p\left((0,T), L^p_{loc}\left(\mathbb{R}^N\right)\right) \cap L^m\left((0,T), L^m_{loc}\left(\mathbb{R}^N\right)\right)$ *and satisfies*

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^N} J^{\alpha}_{0|t}\left(|u|^p\right)\varphi dxdt + \int_{\mathbb{R}^N} u_1(x)\varphi(0,x)dx - \int_{\mathbb{R}^N} u_0(x)\varphi_t(0,x)dx \\
&+\int_{\mathbb{R}^N} u_0(x)\varphi(0,x)dx + \tfrac{1}{m}\int_{\mathbb{R}^N} |u_0|^{m-1}u_0(x)\varphi(0,x)dx \\
&= \int_0^T\int_{\mathbb{R}^N} u\varphi_{tt}dxdt - \tfrac{1}{m}\int_0^T\int_{\mathbb{R}^N} |u|^{m-1}u\varphi_t(t,x)dxdt - \int_0^T\int_{\mathbb{R}^N} u\varphi_t(t,x)dxdt \\
&-\int_0^T\int_{\mathbb{R}^N} u\Delta\varphi dxdt,
\end{aligned} \tag{4.26}$$

for all nonnegative test function $\varphi \in C^2\left([0,T]\times\mathbb{R}^N\right)$ *such that* $\varphi(T,.) = \varphi_t(T,.) = 0$, *where* $\alpha = 1-\gamma$.

Theorem 4.2 *Let $N \geq 1$, $0 < \gamma < 1$ and let p, m such that $p > m > 1$. Assume that the initial data (u_0, u_1) satisfy*

$$\int_{\mathbb{R}^N} u_0(x)\, dx > 0, \ \int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x) > 0, \ and \int_{\mathbb{R}^N} u_1(x)\, dx > 0.$$

Then if

$$p \leq \max\left\{ \frac{m(N+2)}{(N-2+2\gamma)_+}, \frac{1}{\gamma} \right\}. \tag{4.27}$$

The solution of problem (4.1) does not exist globally in time.

Proof. The proof is by contradiction. Suppose that u is non trivial weak solution of the problem (4.1) which exists globally in time. Therefore, let us define

$$\varphi(t,x) = D^{\alpha}_{t|T}\tilde{\varphi}(t,x) := \varphi_1^l(x) D^{\alpha}_{t|T}\varphi_2(t).$$

With $\varphi_1^l(x) := \Phi\left(\frac{|x|^2}{T}\right)^l, \varphi_2(t) := \left(1-\frac{t}{T}\right)_+^{\eta},$ where $l, \eta \gg 1$ and $\Phi \in C^\infty(\mathbb{R}_+)$ be cut-off non-increasing function such that

$$\Phi(z) = \begin{cases} 1 & \text{if } 0 \leq z \leq 1, \\ 0 & \text{if } z \geq 2, \end{cases}, \quad \text{with} \quad 0 \leq \Phi \leq 1.$$

We have from the definition of weak solution

$$\begin{aligned}
&\Gamma(\alpha) \int_0^T \int_{\mathbb{R}^N} J^{\alpha}_{0|t}(|u|^p) D^{\alpha}_{t|T}\tilde{\varphi}(t,x)dxdt + \int_{\mathbb{R}^N} u_1(x) D^{\alpha}_{t|T}\tilde{\varphi}(0,x)\,dx - \int_{\mathbb{R}^N} u_0(x)\, \partial_t D^{\alpha}_{t|T}\tilde{\varphi}_t(0,x)\,dx \\
&+\tfrac{1}{m}\int_{\mathbb{R}^N} |u_0|^{m-1} u_0(x) D^{\alpha}_{t|T}\tilde{\varphi}(0,x)\,dx + \int_{\mathbb{R}^N} u_0(x) D^{\alpha}_{t|T}\tilde{\varphi}(0,x)\,dx \\
&= \int_0^T \int_{\mathbb{R}^N} u\partial_t^2 D^{\alpha}_{t|T}\tilde{\varphi}(t,x)\,dxdt - \tfrac{1}{m}\int_0^T \int_{\mathbb{R}^N} |u|^{m-1} u\partial_t D^{\alpha}_{t|T}\tilde{\varphi}(t,x)\,dxdt \\
&- \int_0^T \int_{\mathbb{R}^N} u\partial_t D^{\alpha}_{t|T}\tilde{\varphi}(t,x)\,dxdt - \int_0^T \int_{\mathbb{R}^N} u\Delta D^{\alpha}_{t|T}\tilde{\varphi}(t,x)dxdt.
\end{aligned} \tag{4.28}$$

Using the formulas(2.3),(2.4) and (2.8) in the left-hand side of (4.28), while in the right-hand side using (2.4), we get

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^N}(|u|^p)\varphi_1^l(x)\varphi_2(t)dxdt+\int_{\mathbb{R}^N}u_1(x)\,D_{t|T}^{\alpha}\varphi_1^l(x)\varphi_2(0)dx\\
&+\int_{\mathbb{R}^N}u_0(x)\,D_{t|T}^{\alpha+1}\varphi_1^l(x)\varphi_2(0)dx+\tfrac{1}{m}\textstyle\int_{\mathbb{R}^N}|u_0|^{m-1}u_0(x)\,D_{t|T}^{\alpha}\varphi_1^l(x)\varphi_2(0)dx\\
&+\int_{\mathbb{R}^N}u_0(x)\,D_{t|T}^{\alpha}\varphi_1^l(x)\varphi_2(0)dx, \qquad (4.29)\\
&=\int_0^T\int_{\mathbb{R}^N}uD_{t|T}^{\alpha+2}\varphi_1^l(x)\varphi_2(t)dxdt+\tfrac{1}{m}\int_0^T\int_{\mathbb{R}^N}|u|^{m-1}uD_{t|T}^{\alpha+1}\varphi_1^l(x)\varphi_2(t)dxdt\\
&+\int_0^T\int_{\mathbb{R}^N}uD_{t|T}^{\alpha+1}\varphi_1^l(x)\varphi_2(t)dxdt-\int_0^T\int_{\mathbb{R}^N}u\Delta D_{t|T}^{\alpha}\varphi_1^l(x)\varphi_2(t)dxdt.
\end{aligned}$$

From the fact that $\Delta\varphi_1^l=l\varphi_1^{l-1}\Delta\varphi_1+l(l-1)|\nabla\varphi_1|^2$, and the support of φ_1 is included in

$$\Omega_T:=\left\{x\in\mathbb{R}^N:|x|\le(2T)^{1/2}\right\},$$

we may write

$$\begin{aligned}
&\int_{\Omega_T}(|u|^p)\tilde{\varphi}dxdt+CT^{\gamma-1}\int_{\mathbb{R}^N}u_1(x)\,\varphi_1^l(x)dx+C\left(T^{\gamma-2}+T^{\gamma-1}\right)\int_{\mathbb{R}^N}u_0(x)\,\varphi_1^l(x)dx\\
&+CT^{\gamma-1}\int_{\mathbb{R}^N}|u_0|^{m-1}u_0(x)\,\varphi_1^l(x)dx\le C\int_{Q_T}|u|\varphi_1^l(x)\left|D_{t|T}^{3-\gamma}\varphi_2(t)\right|dxdt\\
&+C\int_{Q_T}|u|^m\,\varphi_1^l(x)\left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|dxdt+C\int_{Q_T}|u|\varphi_1^l(x)\left|D_{t|T}^{2-\gamma}\varphi_2(t)\right|dxdt\\
&-C\int_{Q_T}u\varphi_1^{l-2}(x)\left(|\Delta\varphi_1|+|\nabla\varphi_1|^2\right)\left|D_{t|T}^{1-\gamma}\varphi_2(t)\right|dxdt,
\end{aligned} \qquad (4.30)$$

where $Q_T:=[0,T]\times\mathbb{R}^N$. By the Lebesgue dominated convergence theorem, we can obtain for all $i=0,1$ the following limits

$$\begin{aligned}
\lim_{T\to\infty}\int_{\Omega_T}u_i(x)\varphi_1^l(x)dx &= \int_{\mathbb{R}^N}u_i(x)dx,\\
\lim_{T\to\infty}\int_{\Omega_T}|u_0(x)|^{m-1}u_0(x)\varphi_1^l(x)dx &= \int_{\mathbb{R}^N}|u_0(x)|^{m-1}u_0(x)dx,
\end{aligned}$$

since we have $\lim_{T\to\infty} u_i(x)\varphi_1^l(x) = u_i(x)$, $\lim_{T\to\infty} |u_0(x)|^{m-1}u_0(x)\varphi_1^l(x) = |u_0(x)|^{m-1}u_0(x)$, and $(u_i, u_0) \in L^1_{loc}(\mathbb{R}^N) \times L^m_{loc}(\mathbb{R}^N)$. So
$\int_{\mathbb{R}^N} u_i(x)dx > 0 \ \left(\text{resp.} \int_{\mathbb{R}^N} |u_0(x)|^{m-1}u_0(x)dx > 0\right)$
implies that
$\int_{\mathbb{R}^N} u_i(x)\varphi_1^l(x)dx > 0 \ \left(\text{resp.} \int_{\mathbb{R}^N} |u_0(x)|^{m-1}u_0(x)\varphi_1^l(x)dx > 0\right),$
for T large. Now, By applying the following ε-Young's inequality

$$XY \le \varepsilon X^p + C(\varepsilon)Y^{p'}, p + p' = pp', X \ge 0, Y \ge 0,$$

to the right-hand side of (4.30), we get

$$\begin{aligned}&\int_{Q_T} |u|\varphi_1^l(x)\left|\left(D^{3-\gamma}_{t|T} + D^{2-\gamma}_{t|T}\right)\varphi_2(t)\right|dxdt = \int_{Q_T} |u|\tilde{\varphi}^{\frac{1}{p}}\tilde{\varphi}^{-\frac{1}{p}}\varphi_1^l(x)\left|\left(D^{3-\gamma}_{t|T} + D^{2-\gamma}_{t|T}\right)\varphi_2(t)\right|dxdt \\ &\le \varepsilon\int_{Q_T} |u|^p\tilde{\varphi}dxdt + C(\varepsilon)\int_{Q_T} \varphi_1^l(x)\varphi_2^{-\frac{1}{p-1}}(t)\left|\left(D^{3-\gamma}_{t|T} + D^{2-\gamma}_{t|T}\right)\varphi_2(t)\right|^{\frac{p}{p-1}}dxdt.\end{aligned} \tag{4.31}$$

For $\varepsilon > 0$, also, we have the estimate

$$\begin{aligned}&\int_{Q_T} |u|^m\varphi_1^l(x)\left|D^{2-\gamma}_{t|T}\varphi_2(t)\right|dxdt = \int_{Q_T} |u|^m\tilde{\varphi}^{\frac{m}{p}}\tilde{\varphi}^{-\frac{m}{p}}\varphi_1^l(x)\left|D^{2-\gamma}_{t|T}\varphi_2(t)\right|dxdt \\ &\le \varepsilon\int_{Q_T} |u|^p\tilde{\varphi}dxdt + C(\varepsilon)\int_{Q_T} \varphi_1^l(x)\varphi_2^{-\frac{m}{p-m}}(t)\left|D^{2-\gamma}_{t|T}\varphi_2(t)\right|^{\frac{p}{p-m}}dxdt.\end{aligned} \tag{4.32}$$

The same is true for the third part of the right-hand side of(4.30)

$$\begin{aligned}&\int_{Q_T} u\varphi_1^{l-2}\left(|\Delta\varphi_1| + |\nabla\varphi_1|^2\right)\left|D^{1-\gamma}_{t|T}\varphi_2(t)\right|dxdt \le \varepsilon\int_{Q_T} |u|^p\tilde{\varphi}dxdt \\ &+C(\varepsilon)\int_{Q_T} \varphi_1^{l-2\frac{p}{p-1}}\left(|\Delta\varphi_1|^{\frac{p}{p-1}} + |\nabla\varphi_1|^{2\frac{p}{p-1}}\right)\varphi_2^{-\frac{1}{p-1}}(t)\left|D^{1-\gamma}_{t|T}\varphi_2(t)\right|^{\frac{p}{p-1}}dxdt.\end{aligned} \tag{4.33}$$

Combining (4.31), (4.32),(4.33), with ε small enough, and some positive constant C, we obtain

$$\begin{aligned}&\int_{Q_T} |u|^p\tilde{\varphi}dxdt \le C\int_{Q_T} \varphi_1^l(x)\varphi_2^{-\frac{1}{p-1}}(t)\left|\left(D^{3-\gamma}_{t|T} + D^{2-\gamma}_{t|T}\right)\varphi_2(t)\right|^{\frac{p}{p-1}}dxdt \\ &+C\int_{Q_T} \varphi_1^l(x)\varphi_2^{-\frac{m}{p-m}}(t)\left|D^{2-\gamma}_{t|T}\varphi_2(t)\right|^{\frac{p}{p-m}}dxdt \\ &+C\int_{Q_T} \varphi_1^{l-2\frac{p}{p-1}}(x)\left(|\Delta\varphi_1|^{\frac{p}{p-1}} + |\nabla\varphi_1|^{2\frac{p}{p-1}}\right)\varphi_2^{-\frac{1}{p-1}}(t)\left|D^{1-\gamma}_{t|T}\varphi_2(t)\right|^{\frac{p}{p-1}}dxdt.\end{aligned} \tag{4.34}$$

Using the change of variables

$$y = T^{-\frac{1}{2}}x, s = T^{-1}t.$$

The equation (4.34) leads to

$$\int_{Q_T} |u|^p \tilde{\varphi} dxdt \leq C \left(T^{(\gamma-3)\frac{p}{p-1}+\frac{N}{2}+1} + 2T^{(\gamma-2)\frac{p}{p-1}+\frac{N}{2}+1} + T^{(\gamma-2)\frac{p}{p-m}+\frac{N}{2}+1} \right) \equiv T^\sigma, \quad (4.35)$$

with $\sigma = \max \left\{ (\gamma-2)\frac{p}{p-1} + \frac{N}{2} + 1, (\gamma-2)\frac{p}{p-m} + \frac{N}{2} + 1 \right\} = (\gamma-2)\frac{p}{p-1} + \frac{N}{2} + 1$. At this stage, we have to distinguish two cases: The case $\sigma < 0$: we pass to the limit in (4.35) as $T \to \infty$, we get

$$\lim_{T\to\infty} \int_{Q_T} |u|^p \tilde{\varphi} dxdt = 0.$$

Using the continuity in time and space of u and the fact that $\lim_{T\to\infty} \tilde{\varphi}(t,x) = 1$, then we based on the Lebesgue dominated convergence theorem, we can conclude the following

$$\int_0^\infty \int_{\Omega_T} |u|^p \tilde{\varphi} dxdt = 0 \Rightarrow u \equiv 0,$$

which is contradiction. Now, we move to the second case : case $\sigma = 0$: using inequality (4.35) with $T \to \infty$, we have

$$u \in L^p\left((0,\infty); L^p(\mathbb{R}^N)\right),$$

which implies that

$$\lim_{T\to\infty} \int_0^T \int_{\sum_T} |u|^p \tilde{\varphi}(x,t) dxdt = 0,$$

where

$$\Omega_T := \left\{ x \in \mathbb{R}^N,\ |x|^2 \leq 2T \right\}, \ \ \sum\nolimits_T := \left\{ x \in \mathbb{R}^n,\, T \leq |x|^2 \leq 2T \right\}.$$

On the other hand, using Hölder's inequality instead of Young's one to the term

$$\int_0^T\int_{\mathbb{R}^N}|u|\varphi_1^{l-2}(|\Delta\varphi_1|+|\nabla\varphi_1|^2)\left|D^{\alpha}_{t|T}\varphi_2\left(t\right)\right|dxdt,$$

we find

$$\int_0^T\int_{\mathbb{R}^N}|u|\varphi_1^{l-2}(|\Delta\varphi_1|+|\nabla\varphi_1|^2)\left|D^{\alpha}_{t|T}\varphi_2\left(t\right)\right|dxdt\leq\left(\int_0^T\int_{\sum_{TR^{-1}}}|u|^p\tilde{\varphi}(t,x)dxdt\right)^{1/p}$$
$$\times\left(\int_0^T\int_{\sum_{TR^{-1}}}\varphi_1^{l-2p'}\varphi_2^{-p'}(|\Delta\varphi_1|^{p'}+|\nabla\varphi_1|^{2p'})\left|D^{\alpha}_{t|T}\varphi_2\left(t\right)\right|^{p'}dxdt\right)^{1/p'}.$$

We repeat the same calculation as above by taking in this time $\varphi_1(x):=\Phi\left(\frac{|x|^2}{R^{-1}T}\right)$ where R is fixed number such that $1<R<T$. Using the change of variables $y=R^{\frac{1}{2}}T^{-\frac{1}{2}}x$, $\tau=T^{-1}t$, and the fact $\sigma=0$, we get

$$\int_0^T\int_{\Omega_{TR^{-1}}}|u|^p\tilde{\varphi}dxdt\leq C\left(\varepsilon\right)\int_0^T\int_{\Omega_{TR^{-1}}}\varphi_1^l(x)\varphi_2^{-\frac{1}{p-1}}(t)\left(\left|D^{3-\gamma}_{t|T}\varphi_2(t)\right|^{p'}+\left|D^{2-\gamma}_{t|T}\varphi_2(t)\right|^{p'}\right)dxdt$$
$$+\int_0^T\int_{\Omega_{TR^{-1}}}\varphi_1^l(x)|\varphi_2(t)|^{-\frac{m}{p-m}}\left|D^{2-\gamma}_{t|T}\varphi_2(t)\right|^{\frac{p}{p-m}}dxdt+\left(\int_0^T\int_{\sum_{TR^{-1}}}|u|^p\tilde{\varphi}(x,t)dxdt\right)^{1/p}$$
$$\times\left(\int_0^T\int_{\sum_{TR^{-1}}}\varphi_1^{l-2p'}(x)\varphi_2(t)^{-p'}(|\Delta\varphi_1|^{p'}+|\nabla\varphi_1|^{2p'})\left|D^{1-\gamma}_{t|T}\varphi_2\left(t\right)\right|^{p'}dxdt\right)^{1/p'},$$

which yields

$$\int_0^T\int_{\left\{|x|\leq\sqrt{2}R^{-\frac{1}{2}}T^{\frac{1}{2}}\right\}}|u|^p\tilde{\varphi}dxdt\leq\left(T^{(\gamma-3)\frac{p}{p-1}+\frac{N}{2}+1}+T^{(\gamma-2)\frac{p}{p-1}+\frac{N}{2}+1}+T^{(\gamma-2)\frac{p}{p-m}+\frac{N}{2}+1}\right)R^{-\frac{N}{2}}$$
$$+C\left(T^{(\gamma-2)\frac{p}{p-1}+\frac{N}{2}+1}\right)^{1/p'}R^{1-\frac{N}{2p'}}\times\left(\int_0^T\int_{\sum_{TR^{-1}}}|u|^p\tilde{\varphi}(x,t)dxdt\right)^{1/p'}.$$

Because of $\sigma=0$, we can get

$$\int_0^T\int_{\left\{|x|\leq\sqrt{2}R^{-\frac{1}{2}}T^{\frac{1}{2}}\right\}}|u|^p\tilde{\varphi}dxdt\leq CR^{-\frac{N}{2}}+R^{1-\frac{N}{2p'}}\times\left(\int_0^T\int_{\sum_{TR^{-1}}}|u|^p\tilde{\varphi}(x,t)dxdt\right)^{1/p}. \tag{4.36}$$

Passing to the limit in (4.36) as $T \to \infty$, we get

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p dxdt \leq CR^{-\frac{N}{2}},$$

and then $R \to \infty$ which give a contradiction.

The case $p \leq \frac{1}{\gamma}$ we choose $\varphi_1(x) := \Phi\left(\frac{|x|^2}{R}\right), \varphi_2(t) := \left(1 - \frac{t}{T}\right)_+^\eta$, then by taking the change variables $x = R^{\frac{1}{2}} y$, $t = T\tau$, it follows from (4.34) that

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^N} |u|^p \tilde{\varphi} dxdt \quad &\leq \quad CR^{\frac{n}{2}}\left(T^{(\gamma-3)p'+1} + T^{(\gamma-2)p'+1} + T^{(\gamma-2)\frac{p}{p-m}+1}\right) \\ &\quad +CR^{(\frac{n}{2}-p')}T^{(\gamma-1)p'+1}.\end{aligned}$$

Now, passing to the limit as $T \to \infty$ in the last inequality by taking account that $p < 1/\gamma$, we deduce that

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p \tilde{\varphi} dxdt = 0.$$

Then, by taking $R \to \infty$, we get contradiction.

Precisely, in the case $p = \frac{1}{\gamma}$, we have to use condition $\frac{N}{2} - \frac{p}{p-1} < 0$, which is equivalent to $\gamma > \frac{N-2}{2}$ to obtain the desired convergence. This completes the proof ■

Remark 4.2 *When $m = 1$, we recover the case studied by Fino [2].*

Chapter 5

Global Small Data Solution For a System of Semilinear Heat Equations and The Corresponding System of Damped Wave Equations With Nonlinear Memory

5.1 Introduction

We consider the two Cauchy problems for a systems of strongly coupled semilinear integro-differential equations of parabolic type:

$$\begin{cases} u_t - \Delta u = \displaystyle\int_0^t (t-s)^{-\gamma_1} |u(s)|^{p_1} |v(s)|^{q_1} ds,\ t>0,\ x\in\mathbb{R}^N, \\ v_t - \Delta v = \displaystyle\int_0^t (t-s)^{-\gamma_2} |v(s)|^{p_2} |u(s)|^{q_2} ds,\ t>0,\ x\in\mathbb{R}^N, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ x\in\mathbb{R}^N, \end{cases} \tag{5.1}$$

and hyperbolic type:

$$\begin{cases} u_{tt} - \Delta u + u_t = \displaystyle\int_0^t (t-s)^{-\gamma_1} |u(s)|^{p_1} |v(s)|^{q_1} ds, \ t > 0, x \in \mathbb{R}^N, \\ v_{tt} - \Delta v + v_t = \displaystyle\int_0^t (t-s)^{-\gamma_2} |v(s)|^{p_2} |u(s)|^{q_2} ds, \ t > 0, x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), v(0,x) = v_0(x), \ v_t(0,x) = v_1(x), \ x \in \mathbb{R}^N, \end{cases} \tag{5.2}$$

where the unknown functions $u := u(t,x)$, $v := v(t,x)$ are real-valued, $N \geq 1$, p_1,q_1,p_2,$q_2 \geq 1$, $0 < \gamma_1$,$\gamma_2 < 1$ and $u_0(x)$, $v_0(x)$ are the given initial data.

Motivated by the results cited in the Chapter 3 and the papers [37], [46], [47], [66], [72], we consider the problem (5.1)((5.2) respectively), we will give conditions relating the space dimension N with the system of parameters γ_1, γ_2, p_1, q_1, p_2 and q_2 for which the solution of (5.1) ((5.2) respectively) exists globally in time as well as L^∞ decay estimates.

The best way to do this is to consider appropriately Lebesgue space where we can expect global well-posedness for this model, we observe that if (u,v) is a solution for the system (5.1) with initial data (u_0, v_0), then for all $\lambda > 0$, $(u_\lambda, v_\lambda) = \left(\lambda^{k_1} u(\lambda^2 t, \lambda x), \lambda^{k_2} u(\lambda^2 t, \lambda x)\right)$ where

$$k_1 = \frac{(4-2\gamma_1)(p_2-1) - (4-2\gamma_2) q_1}{((p_1-1)(p_2-1) - q_1 q_2)}, \ k_2 = \frac{(4-2\gamma_2)(p_1-1) - (4-2\gamma_1) q_2}{((p_1-1)(p_2-1) - q_1 q_2)},$$

is also a solution of (5.1). If $(u_0, v_0) \in L^{r_1}\left(\mathbb{R}^N\right) \times L^{r_2}\left(\mathbb{R}^N\right)$, then the norms in $L^{r_1}\left(\mathbb{R}^N\right)$ and $L^{r_2}\left(\mathbb{R}^N\right)$ are preserved if and only if

$$r_1 = \frac{N((p_1-1)(p_2-1) - q_1 q_2)}{2[(2-\gamma_1)(p_2-1) - (2-\gamma_2) q_1]}, \ r_2 = \frac{N((p_1-1)(p_2-1) - q_1 q_2)}{2[(2-\gamma_2)(p_1-1) - (2-\gamma_1) q_2]}. \tag{5.3}$$

So we could expect that if $r_1 > 1$ and $r_2 > 1$ the mild solution of (5.1) with small initial data would exist globally. We will show in this chapter that this result partially is not true. Using the diffusion phenomenon properties, we can obtain similar critical exponent results for the corresponding system of semilinear damped wave equations in low dimension space. The rest is organized as follows. In the next section, we present some preliminary lemmas that we will need in the proof main results of this part. We collect some basic facts and useful tools such as smoothing effect of the heat semigroup, L^p-L^q estimates of the fundamental solutions of the damped wave equation. The local existence and the continuation results are presented in Section 3. Finally the proof of main results are presented in Sections 4 and 5.

In all the chapter, C is positive constant which may have different values at different places.

For any $1 \leq p \leq \infty$; $W^{1,p}\left(\mathbb{R}^N\right)$ denotes the usual Sobolev space

$$W^{1,p}\left(\mathbb{R}^N\right) := \left\{ f : \mathbb{R}^N \to \mathbb{R};\;\; \|f\|_{W^{1,p}(\mathbb{R}^N)} = \|f\|_{L^p(\mathbb{R}^N)} + \|\nabla f\|_{L^p(\mathbb{R}^N)} < +\infty \right\}.$$

For any Banach space B, we denote by $C([0,T]\,;B)$ the space of continuous functions from $[0,T]$ into B equipped with the uniform convergence $\sup\limits_{t\in[0,T]} \|.\|_B$, and
$H^l\left(\mathbb{R}^N\right) := W^{2,l}\left(\mathbb{R}^N\right)$ $(l \in \mathbb{N})$ stands for the usual Sobolev space equipped with the norm $\|f\|^2_{H^l(\mathbb{R}^N)} = \sum\limits_{k=0}^{l} \left\|\partial_x^k\right\|^2_{L^2(\mathbb{R}^N)} < +\infty.$

5.2 Preliminary lemmas

Heat semigroup: Let us recall the definition of the so-called smoothing effect of the heat semigroup on $\mathbb{R}^N$ and some related basic facts. For a complete presentation and more details, we refer the reader to [58].

Lemma 5.1 *[58] Let $1 \leq r \leq s \leq \infty$. There exists a constant $C > 0$ such that*

$$\|S(t)\,u_0\|_{L^s} \leq Ct^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{s}\right)} \|u_0\|_{L^r}\,,\;\; t > 0 \tag{5.4}$$

for all $u_0 \in L^r$. In particular for $u_0 \in L^r\left(\mathbb{R}^N\right) \cap L^s\left(\mathbb{R}^N\right)$, $1 \leq r \leq s \leq \infty$, there exists

$C = C(r, s)$ *such that*

$$\|S(t)\, u_0\|_{L^s} \leq C\,(t+1)^{-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{s}\right)} \left(\|u_0\|_{L^r} + \|u_0\|_{L^s}\right) \tag{5.5}$$

for $t \geq 0$.

We will use also the following interpolation inequality

$$\|u\|_{L^s} \leq \|u\|_{L^{s_1}}^{\theta}\, \|u\|_{L^{s_2}}^{1-\theta}, \tag{5.6}$$

for $u \in L^{s_1}\left(\mathbb{R}^N\right) \cap L^{s_2}\left(\mathbb{R}^N\right)$, where $s \in [s_1, s_2]$, $\theta \in [0,1]$ with $\frac{1}{s} = \frac{\theta}{s_1} + \frac{1-\theta}{s_2}$.

We will need the following lemma which used in the proofs of Theorems 5.1 and 5.2.

Lemma 5.2 *[58] Let* $0 \leq a < 1$, $b \geq 0$. *Then there exists a constant* $C > 0$ *depending only on* a *and* b *such that for all* $t \geq 0$,

$$\int_0^t (t-s)^{-a}\,(1+s)^{-b}\,ds \leq \begin{cases} C(1+t)^{-\min(a,b)} & \text{if } \max(a,b) > 1, \\ C(1+t)^{-\min(a,b)}\ln(2+t) & \text{if } \max(a,b) = 1, \\ C\,(1+t)^{1-a-b}, & \text{if } \max(a,b) < 1. \end{cases} \tag{5.7}$$

$$\int_0^t (t-s+1)^{-a}\,(1+s)^{-b}\,ds \leq C\,(1+t)^{-b}, \; \text{for } t > 0, a > 1, a \geq b, \tag{5.8}$$

and

$$\int_0^t e^{-a(t-s)}\,(t-s)\,(1+s)^{-b}\,ds \leq C\,(1+t)^{-b}, \; a,b > 0. \tag{5.9}$$

Linear damped wave equation :

Now, we recall some preliminary results concerning $L^p - L^q$ estimates of the fundamental solutions $K_0(t)$ and $K_1(t)$ to the linear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, \;\; (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ u(0,x) = v_0(x), \; u_t(0,x) = v_1(x), \; x \in \mathbb{R}^N. \end{cases} \tag{5.10}$$

The solution $u(t)$ of linear equation (5.10) is given through the Fourier transform by $K_0(t)$ and $K_1(t)$ as

$$u(t) := K_0(t)v_0 + K_1(t)\left(\frac{1}{2}v_0 + v_1\right).$$

Similarly, we introduce the evolution operators of the linear wave equation as follows:

$$W_0(t)f := \mathcal{F}^{-1}\left[\cos\left(t\left|\xi\right|\right)\hat{f}\right], W_1(t)g := \mathcal{F}^{-1}\left[\frac{\sin\left(t\left|\xi\right|\right)}{\left|\xi\right|}\hat{g}\right]. \tag{5.11}$$

In the following, we will consider the properties of these operators.

Lemma 5.3 *([5]) If $f \in L^m(\mathbb{R}^N) \cap H^{k+|\nu|-1}(\mathbb{R}^N)$ $(1 \leq m \leq 2)$, then*

$$\|\partial_t^k \nabla_x^\nu K_1(t) * f\|_2 \leq C(1+t)^{-N/4-N/(2m)-|\nu|/2-k}(\|f\|_m + \|f\|_{H^{k+|\nu|-1}(\mathbb{R}^N)}).$$

Lemma 5.4 *Let $1 \leq N \leq 3$, $1 \leq p \leq \infty$, $f \in W^{1,p}(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$. Then there exist some constants $C > 0$ such that*

$$\|W_0(t)f\|_{L^p(\mathbb{R}^N)} \leq C\left(1+|t|\right)\|f\|_{W^{1,p}(\mathbb{R}^N)}, t \neq 0,$$

$$\|W_1(t)g\|_{L^p(\mathbb{R}^N)} \leq C\left|t\right|\|g\|_{L^p(\mathbb{R}^N)}, t \neq 0,$$

where $W_0(t)f$ and $W_1(t)g$ are defined by (5.11).

The proof of Lemma 5.4 is well known (cf. [32, 67]).

The following Lemma will be used later and the proof of this Lemma can be found in

[31, 32, 67, 68, 69]), so we omit it here.

Lemma 5.5 *Let $1 \leq N \leq 3$, $1 \leq q \leq p \leq \infty$ and $g \in L^q(\mathbb{R}^N)$. Then, there exist some constants $C > 0$ such that for all $t > 0$,*

$$\left\| \left(K_0(t) - e^{-\frac{t}{2}} W_0(t) - e^{-\frac{t}{2}} \frac{t}{8} W_1(t) \right) g \right\|_{L^p(\mathbb{R}^N)} \leq C \left(1+t\right)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|g\|_{L^q(\mathbb{R}^N)},$$

$$\left\| K_1(t)g - e^{-\frac{t}{2}} W_1(t) g \right\|_{L^p(\mathbb{R}^N)} \leq C \left(1+t\right)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|g\|_{L^q(\mathbb{R}^N)},$$

where $K_0(t)g$ and $K_1(t)g$ are defined by (5.18)and (5.19).

5.3 Main results

Before presenting the main theorems we introduce the de notion of mild solutions.

Definition 5.1 *For a mild solution of (5.1) we mean a function $(u, v) \in \left\{C([0,T); L^1\left(\mathbb{R}^N\right))\right\}^2 \cap \left\{C(0,T; C_0\left(\mathbb{R}^N\right))\right\}^2$ satisfying the integral system*

$$\begin{cases} u(t) = S(t)u_0 + \displaystyle\int_0^t \int_0^s (s-\tau)^{-\gamma_1} S(t-s) |u\left(s\right)|^{p_1} |v\left(s\right)|^{q_1} d\tau ds, \\ v(t) = S(t)v_0 + \displaystyle\int_0^t \int_0^s (s-\tau)^{-\gamma_2} S(t-s) |v\left(s\right)|^{p_2} |u\left(s\right)|^{q_2} d\tau ds, \end{cases} \tag{5.12}$$

where $\{S(t)\}_{t\geq 0}$ as the family of convolution operators with corresponding Gauss kernels $g(t,x) = (4\pi t)^{-N/2} e^{-|x|^2/(4t)}, t > 0,\ x \in \mathbb{R}^N$, that is $S(t)f = g(t,.) * f$, here $*$ denotes the convolution product.

Our first result concerns the existence and uniqueness of mild solutions for the system (5.1).

Proposition 5.1 (Local existence of the heat system) *Let $N \geq 1$, $p_1, q_1, p_2, q_2 \geq 1$, $\gamma_1, \gamma_2 \in [0,1)$ and $u_0, v_0 \in C_0(\mathbb{R}^N)$. There exists a unique function $(u, v) \in \left\{C\left((0, T_{\max}), C_0(\mathbb{R}^N)\right)\right\}^2$ solution of (5.1) such that either* *(i)* $T_{\max} = \infty$ *(the solution is global) or else*

(ii) $T_{\max} < \infty$ *and* $\lim_{t \to T_{\max}} \left(\|u(t)\|_{\infty} + \|v(t)\|_{\infty} \right) = \infty$ *(the solution blows up in finite time).*

Moreover if $(u_0, v_0) \in L^{r_1}\left(\mathbb{R}^N\right) \times L^{r_2}\left(\mathbb{R}^N\right)$ *with* $r_1 \geq 1$ *and* $r_2 \geq 1$*, then* $(u, v) \in C\left((0, T_{\max}), L^{r_1}(\mathbb{R}^N)\right) \times C\left((0, T_{\max}), L^{r_2}(\mathbb{R}^N)\right)$ *and*

$$\lim_{t \to T_{\max}} \left(\|u(t)\|_{L^{r_1} \cap L^{\infty}} + \|v(t)\|_{L^{r_2} \cap L^{\infty}} \right) = \infty, \tag{5.13}$$

when $T_{\max} < \infty$.

Under the above notations, our global existence result for the Cauchy problem (5.1) can be stated as in the following

Theorem 5.1 (Global existence of the heat system) *Let N be a positive integer. Let the real numbers* $p_1, q_1, p_2, q_2 \geq 1$, $0 < \gamma_1, \gamma_2 < 1$ *be such that*

$$[(1-\gamma_1)(p_2-1) - (1-\gamma_2)q_1]\left((p_1-1)(p_2-1) - q_1 q_2\right) > 0,$$

$$[(1-\gamma_2)(p_1-1) - (1-\gamma_1)q_2]\left((p_1-1)(p_2-1) - q_1 q_2\right) > 0,$$

$$[p_2 - q_1 - 1]\left((p_1-1)(p_2-1) - q_1 q_2\right) > 0,$$

$$[p_1 - q_2 - 1]\left((p_1-1)(p_2-1) - q_1 q_2\right) > 0.$$

and $u_0, v_0 \in C_0\left(\mathbb{R}^N\right)$. *Let* $(u, v) \in \left\{C\left((0, T_{\max}), C_0(\mathbb{R}^N)\right)\right\}^2$.
Assume that

$$\begin{cases} \frac{N}{2} > 1 - \gamma_1 + \frac{(2-\gamma_1)(p_2-1) - (2-\gamma_2)q_1}{(p_1-1)(p_2-1) - q_1 q_2}, \\ \frac{N}{2} > 1 - \gamma_2 + \frac{(2-\gamma_2)(p_1-1) - (2-\gamma_1)q_2}{(p_1-1)(p_2-1) - q_1 q_2}, \end{cases} \tag{5.14}$$

$$\begin{cases} \left((p_1-1)(p_2-1) - q_1 q_2\right) \times [p_2(\gamma_1 p_1 - 1) - \gamma_1 p_1 + q_1(1 - \gamma_1 q_2) + 1 - \gamma_2 q_1] > 0, \\ \left((p_1-1)(p_2-1) - q_1 q_2\right) \times [p_1(\gamma_2 p_2 - 1) - \gamma_2 p_2 + q_2(1 - \gamma_2 q_1) + 1 - \gamma_1 q_2] > 0, \end{cases} \tag{5.15}$$

and

$$\begin{cases} \frac{Np_1}{2}\left[\frac{p_1}{r_1}+\frac{q_1}{r_2}\right]+\frac{Nq_1}{2}\left[\frac{p_2}{r_2}+\frac{q_2}{r_1}\right]<2\left(p_1+q_1\right)+\frac{N}{2}, \\ \frac{Np_2}{2}\left[\frac{p_2}{r_2}+\frac{q_2}{r_1}\right]+\frac{Nq_2}{2}\left[\frac{p_1}{r_1}+\frac{q_1}{r_2}\right]<2\left(p_2+q_2\right)+\frac{N}{2}. \end{cases} \tag{5.16}$$

Then there exists a constant $\varepsilon>0$ *such that if the initial data satisfy* $(u_0,v_0)\in L^{r_1}\left(\mathbb{R}^N\right)\times L^{r_2}\left(\mathbb{R}^N\right)$ *and*

$$\|u_0\|_\infty+\|v_0\|_\infty+\|u_0\|_{r_1}+\|v_0\|_{r_2}\leq\varepsilon$$

the problem (5.1) admits global solution $(u,v)\in C\left([0,\infty);L^{r_1}\left(\mathbb{R}^N\right)\times L^{r_2}\left(\mathbb{R}^N\right)\cap\left\{C_0\left(\mathbb{R}^N\right)\right\}^2\right)$ *satisfies the following decay estimates*

$$\|u\|_\infty\leq C\left(t+1\right)^{-\alpha},\|v\|_\infty\leq C\left(t+1\right)^{-\beta},\ \forall t\geq0,$$

where r_1, r_2 *given by (5.3) and*

$$\alpha=\frac{\left(1-\gamma_1\right)\left(p_2-1\right)-\left(1-\gamma_2\right)q_1}{\left(p_1-1\right)\left(p_2-1\right)-q_1q_2},\ \beta=\frac{\left(1-\gamma_2\right)\left(p_1-1\right)-\left(1-\gamma_1\right)q_2}{\left(p_1-1\right)\left(p_2-1\right)-q_1q_2}.$$

Similar consideration to the system for heat equations can be applied to the Cauchy problem (5.1) for the system of damped wave equations (5.2) in low dimensional space. Let us give the definition of a mild solution for the Cauchy problem (5.2).

Definition 5.2 *Let* $u\in C([0,T);L^1\left(\mathbb{R}^N\right))\cap L^\infty([0,T);L^\infty\left(\mathbb{R}^N\right))$. *Then the function* (u,v) *is said to be a mild solution for the Cauchy problem (5.2) if there holds*

$$\begin{cases} u(t)=K_0(t)u_0+K_1(t)\left(\frac{1}{2}u_0+u_1\right)+\int_0^t\int_0^s(s-\tau)^{-\gamma_1}K_1(t-s)|u\left(s\right)|^{p_1}|v\left(s\right)|^{q_1}d\tau ds, \\ v(t)=K_0(t)v_0+K_1(t)\left(\frac{1}{2}v_0+v_1\right)+\int_0^t\int_0^s(s-\tau)^{-\gamma_2}K_1(t-s)|v\left(s\right)|^{p_2}|u\left(s\right)|^{q_2}d\tau ds, \end{cases} \tag{5.17}$$

for all $(x,t)\in\mathbb{R}^N\times[0,T)$, *where the evolution operators* $K_0(t)$ *and* $K_1(t)$ *solutions of the* *linear damped wave equation are given by*

$$(K_0(t)\phi)(x) := \mathcal{F}^{-1}\left[e^{-\frac{t}{2}}\cos\left(t\sqrt{|\xi|^2-\frac{1}{4}}\right)\mathcal{F}[\phi]\right](x), \tag{5.18}$$

$$(K_1(t)\phi)(x) := \mathcal{F}^{-1}\left[e^{-\frac{t}{2}}\frac{\sin\left(t\sqrt{|\xi|^2-\frac{1}{4}}\right)}{\sqrt{|\xi|^2-\frac{1}{4}}}\mathcal{F}[\phi]\right](x). \tag{5.19}$$

Here we denote the Fourier and Fourier inverse transform by $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively. In particular, when $N = 1, 2, 3$, like in the paper [69], (5.17) can also be written as follows:

$$\begin{cases} u\,(t,.) = K_1(t)(\frac{1}{2}u_0+u_1)+K_0(t)u_0+\displaystyle\int_0^t\Big(K_1(t-s)-e^{-\frac{t-s}{2}}W_1(t-s)\Big) \\ \times\displaystyle\int_0^s(s-\tau)^{-\gamma_1}|u(\tau,.)|^{p_1}|v(\tau,.)|^{q_1}d\tau ds+\int_0^t e^{-\frac{t-s}{2}}W_1(t-s) \\ \times\displaystyle\int_0^s(s-\tau)^{-\gamma_1}|u(\tau,.)|^{p_1}|v(\tau,.)|^{q_1}d\tau ds, \\ v\,(t,.) = K_1(t)(\frac{1}{2}v_0+v_1)+K_0(t)v_0+\displaystyle\int_0^t\Big(K_1(t-s)-e^{-\frac{t-s}{2}}W_1(t-s)\Big) \\ \times\displaystyle\int_0^s(s-\tau)^{-\gamma_2}|v(\tau,.)|^{p_2}|u(\tau,.)|^{q_2}d\tau ds+\int_0^t e^{-\frac{t-s}{2}}W_1(t-s) \\ \displaystyle\int_0^s(s-\tau)^{-\gamma_2}|v(\tau,.)|^{p_2}|u(\tau,.)|^{q_2}d\tau ds. \end{cases} \tag{5.20}$$

Proposition 5.2 (Local existence for the damped wave system) *Let $1 \le N \le 3$, $p_1, q_1, p_2, q_2 \ge 1, \gamma_1, \gamma_2 \in [0,1)$ and (u_0, u_1), $(v_0, v_1) \in W^{1,\infty}\left(\mathbb{R}^N\right) \times L^\infty(\mathbb{R}^N)$. There exists a unique function $(u, v) \in \left\{C\left((0, T_{\max}), L^\infty(\mathbb{R}^N)\right)\right\}^2$ solution of (5.2) such that either*

(i) $T_{\max} = \infty$ (the solution is global) or else

(ii) $T_{\max} < \infty$ and $\lim\limits_{t\to T_{\max}}(\|u(t)\|_\infty + \|v(t)\|_\infty) = \infty$ (the solution blows up in finite time).

Moreover if $(u_0, u_1) \in W^{1,r_1}\left(\mathbb{R}^N\right) \times L^1\left(\mathbb{R}^N\right)$ and $(v_0, v_1) \in W^{1,r_2}(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$, then $(u, v) \in C\left([0, T_{\max}), L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N)\right)$ for any $r_1 \ge 1$ and $r_2 \ge 1$,

and $\lim_{t \to T_{\max}} \left(\|u(t)\|_{L^{r_1} \cap L^\infty} + \|v(t)\|_{L^{r_2} \cap L^\infty} \right) = \infty$, *when* $T_{\max} < \infty$.
Furthermore the solution $u,v \in C\left([0, T_{\max}), H^1(\mathbb{R}^N)\right) \cap C^1\left([0, T_{\max}), L^2(\mathbb{R}^N)\right)$.

Remark 5.1 *The proof of Proposition 5.2 is omitted here since it follows by combining the proofs of Propositions 5.1 and 3.6 in [37] with Lemmas 5.2, 5.3, 5.5 togethers.*

The main purpose of our next theorem is to show that the same result holds also for the Cauchy problem (5.2)

Theorem 5.2 (Global existence) *Let* $1 \le N \le 3$ *be a positive integer. Let the real numbers* $p_1, q_1, p_2, q_2 \ge 1$, $0 < \gamma_1, \gamma_2 < 1$ *be such that*

$$
\begin{aligned}
&\left[(1-\gamma_1)(p_2-1) - (1-\gamma_2)q_1\right]\left((p_1-1)(p_2-1) - q_1q_2\right) > 0,\\
&\left[(1-\gamma_2)(p_1-1) - (1-\gamma_1)q_2\right]\left((p_1-1)(p_2-1) - q_1q_2\right) > 0,\\
&\left[p_2 - q_1 - 1\right]\left((p_1-1)(p_2-1) - q_1q_2\right) > 0,\\
&\left[p_1 - q_2 - 1\right]\left((p_1-1)(p_2-1) - q_1q_2\right) > 0.
\end{aligned}
$$

Assume that

$$
\begin{cases}
\frac{N}{2} > 1 - \gamma_1 + \frac{(2-\gamma_1)(p_2-1)-(2-\gamma_2)q_1}{(p_1-1)(p_2-1)-q_1q_2},\\[2mm]
\frac{N}{2} > 1 - \gamma_2 + \frac{(2-\gamma_2)(p_1-1)-(2-\gamma_1)q_2}{(p_1-1)(p_2-1)-q_1q_2},
\end{cases}
\tag{5.21}
$$

$$
\begin{cases}
\left((p_1-1)(p_2-1) - q_1q_2\right) \times \left[p_2(\gamma_1 p_1 - 1) - \gamma_1 p_1 + q_1(1-\gamma_1 q_2) + 1 - \gamma_2 q_1\right] > 0,\\[2mm]
\left((p_1-1)(p_2-1) - q_1q_2\right) \times \left[p_1(\gamma_2 p_2 - 1) - \gamma_2 p_2 + q_2(1-\gamma_2 q_1) + 1 - \gamma_1 q_2\right] > 0,
\end{cases}
\tag{5.22}
$$

and

$$
\begin{cases}
\frac{Np_1}{2}\left[\frac{p_1}{r_1} + \frac{q_1}{r_2}\right] + \frac{Nq_1}{2}\left[\frac{p_2}{r_2} + \frac{q_2}{r_1}\right] < 2(p_1+q_1) + \frac{N}{2},\\[2mm]
\frac{Np_2}{2}\left[\frac{p_2}{r_2} + \frac{q_2}{r_1}\right] + \frac{Nq_2}{2}\left[\frac{p_1}{r_1} + \frac{q_1}{r_2}\right] < 2(p_2+q_2) + \frac{N}{2}.
\end{cases}
\tag{5.23}
$$

Then there exists a positive constant $\varepsilon > 0$ *such that if the initial data satisfy*

$(u_i, v_i) \in \left\{W^{1-i,1}\left(\mathbb{R}^N\right) \times W^{1-i,\infty}\left(\mathbb{R}^N\right)\right\}^2$, $i = 0, 1$, *and*

$$\|u_0\|_{W^{1,1}\cap W^{1,\infty}} + \|v_0\|_{W^{1,1}\cap W^{1,\infty}} + \|u_1\|_{L^1\cap L^\infty} + \|v_1\|_{L^1\cap L^\infty} \leq \varepsilon,$$

the corresponding problem (5.2) admits global solution

$$\begin{aligned}(u, v) \in\ & C\left([0,\infty); L^{r_1}\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right) \cap H^1\left(\mathbb{R}^N\right)\right) \cap C^1\left([0,\infty); L^2\left(\mathbb{R}^N\right)\right) \\ & \times C\left([0,\infty); L^{r_2}\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right) \cap H^1\left(\mathbb{R}^N\right)\right) \cap C^1\left([0,\infty); L^2\left(\mathbb{R}^N\right)\right),\end{aligned}$$

satisfies the following decay estimates

$$\|u\|_\infty \leq C\,(t+1)^{-\alpha}, \qquad \|v\|_\infty \leq C\,(t+1)^{-\beta}, \ \forall t \geq 0,$$

where r_1, r_2 *given by (5.3) and*

$$\alpha = \frac{(1-\gamma_1)(p_2-1) - (1-\gamma_2)\,q_1}{(p_1-1)(p_2-1) - q_1 q_2}, \ \beta = \frac{(1-\gamma_2)(p_1-1) - (1-\gamma_1)\,q_2}{(p_1-1)(p_2-1) - q_1 q_2}.$$

1) From the definition of r_1 and r_1, we note that the left hand sides of inequalities (5.16) and (5.23) are independent of the dimension N.

2) Notice that the above results remain true for $p_1, p_2 \geq 0$, q_1, $q_2 \geq 1$ with $p_1 + q_1 > 1$ and $p_2 + q_2 > 1$.

3) Theorem 5.1 and 5.2 are sharp in the case (5.16) (respectively, (5.23)), see (4)–(14) and generalizes Theorem 1.1 of [66]. In fact let $p_1 = p_2 = 0$, $q_1 = q_2 = q$ and $\gamma_1 = \gamma_2 = \gamma$. Conditions (5.14) and (5.15) reduce to $q\left(\frac{N}{2} + \gamma - 1\right) \geq \frac{N}{2} + 1$ or $q\gamma \geq 1$ respectively. From these facts, it is possible to conclude that, if (5.15) and (5.23) are valid, then the value of the Fujita critical exponent is p_*.

4) When $p_1 = p_2 = 0$. These results are in agreement with results obtained by [46,

Theorem 2] and [37, Theorem 2.1, 2.2]. In particular the condition (5.23) of Theorem 5.2 improves the one in [37, Formula (10)]

5) The same result can be stated for more general nonlinearities, namely for f_i, $i = 1, 2$

$$\begin{aligned}
|f_1(u,v) - f_1(\bar{u},v)| &\le C|u-\bar{u}|\left(|u|^{p_1-1} + |\bar{u}|^{p_1-1}\right)|v|^{q_1},\\
|f_1(u,v) - f_1(u,\bar{v})| &\le C|v-\bar{v}|\left(|v|^{q_1-1} + |\bar{v}|^{q_1-1}\right)|u|^{p_1},\\
|f_2(u,v) - f_2(\bar{u},v)| &\le C|u-\bar{u}|\left(|u|^{q_2-1} + |\bar{u}|^{q_2-1}\right)|v|^{p_2},\\
|f_2(u,v) - f_2(u,\bar{v})| &\le C|v-\bar{v}|\left(|v|^{p_2-1} + |\bar{v}|^{p_2-1}\right)|u|^{q_2}.
\end{aligned}$$

Proof of Proposition 5.1. The proof relies in the Banach fixed point theorem. Given $M > 0$ such that let

$$K := \left\{(u,v) \in \left\{L^\infty\left([0,T), C_0\left(\mathbb{R}^N\right)\right)\right\}^2; \|u(t)\|_\infty \le M+1, \|v(t)\|_\infty \le M+1\right\}, \quad (5.24)$$

where $T > 0$ will be chosen later. The space K equipped with the metric

$$d((u,v),(\bar{u},\bar{v})) = \sup_{t\in(0,T)} \|u(t) - \bar{u}(t)\|_\infty + \sup_{t\in(0,T)} \|v(t) - \bar{v}(t)\|_\infty,$$

is a complete metric space. Define the mapping $\Phi : K \times K \to \left\{L^\infty\left([0,T), C_0\left(\mathbb{R}^N\right)\right)\right\}^2$ as the following

$$\Phi(u,v) = (\Phi_1(u,v), \Phi_2(u,v)), \ (u,v) \in K$$

where

$$\begin{cases}
\Phi_1(u,v) = S(t)u_0 + \displaystyle\int_0^t\int_0^s S(t-s)(s-\tau)^{-\gamma_1}|u(\tau)|^{p_1}|v(\tau)|^{q_1}d\tau ds,\\
\Phi_2(u,v) = S(t)v_0 + \displaystyle\int_0^t\int_0^s S(t-s)(s-\tau)^{-\gamma_2}|v(\tau)|^{p_2}|u(\tau)|^{q_2}d\tau ds.
\end{cases} \quad (5.25)$$

First,we claim that Φ maps K it into itself, and is contraction when T is chosen appropriately. We have for $(u,v)\in K$ that

$$\begin{aligned}\|\Phi_1(u,v)\|_\infty &\leq C\|u_0\|_\infty+\int_0^t\int_0^s (s-\tau)^{-\gamma_1}\|u\|_\infty^{p_1}\|v\|_\infty^{q_1}\,d\tau ds\\ &\leq C\|u_0\|_\infty+T^{2-\gamma_1}(M+1)^{p_1+q_1}.\end{aligned}$$

Similarly

$$\begin{aligned}\|\Phi_2(u,v)\|_\infty &\leq C\|u_0\|_\infty+\int_0^t\int_0^s (s-\tau)^{-\gamma_2}\|v\|_\infty^{p_2}\|u\|_\infty^{q_2}\,d\tau ds\\ &\leq C\|v_0\|_\infty+T^{2-\gamma_2}(M+1)^{p_2+q_2}.\end{aligned}$$

We will prove that for $T>0$ sufficiently small, Φ is a contraction map from K into itself. We consider (u_1,u_2), $(v_1,v_2)\in K$. For any $t\in[0,T)$, we have

$$\begin{aligned}\|\Phi_1(u_1,u_2)-\Phi_1(v_1,v_2)\|_\infty &\leq \int_0^t\int_0^s \big\|S(t-s)\,(s-\tau)^{-\gamma_1}\left(|u_1|^{p_1}|u_2|^{q_1}-|v_1|^{p_1}|u_2|^{q_1}\right.\\ &\quad \left.+|v_1|^{p_1}|u_2|^{q_1}-|v_1|^{p_1}|v_2|^{q_1}\right)\big\|_\infty\,d\tau ds.\end{aligned}$$

Then, thanks to Lemma 5.1, we get

$$\begin{aligned}\|\Phi_1(u_1,u_2)-\Phi_1(v_1,v_2)\|_\infty &\leq C\int_0^t\int_0^s (s-\tau)^{-\gamma_1}\|u_2\|_\infty^{q_1}\left(\|u_1\|_\infty^{p_1-1}+\|v_1\|_\infty^{p_1-1}\right)\|u_1-v_1\|_\infty\\ &\quad +\|v_1\|_\infty^{p_1}\left(\|u_2\|_\infty^{q_1-1}+\|v_2\|_\infty^{q_1-1}\right)\|u_2-v_2\|_\infty\,d\tau ds.\end{aligned}$$

Therefore

$$\|\Phi_1(u_1,u_2)-\Phi_1(v_1,v_2)\|_\infty\leq 2CT^{2-\gamma_1}(M+1)^{p_1+q_1-1}\left(\sup_{t\in(0,T)}\|u_1-v_1\|+\sup_{t\in(0,T)}\|u_2-v_2\|_\infty\right). \tag{5.26}$$

Similarly, we have

$$\|\Phi_2(u_1,u_2)-\Phi_2(v_1,v_2)\|_\infty \le 2CT^{2-\gamma_2}(M+1)^{p_2+q_2-1}\left(\sup_{t\in(0,T)}\|u_1-v_1\|_\infty+\sup_{t\in(0,T)}\|u_2-v_2\|_\infty\right), \tag{5.27}$$

for some positive constant $C>0$. From (5.24)-(5.27) it follows that if T sufficiently small, then Φ is a strict contraction from K into itself, so Φ has a unique fixed point (u,v) in K, which is a solution of (5.1).

It is easy to prove that for each $T'>0$ system (5.25) has at most one solution which lies to $K(T')$. In fact, let (u_1,v_1), $(u_2,v_2)\in K(T')$ be two solutions of system (5.12); then

$$\begin{aligned} u_1(t,.)-u_2(t,.) &= \int_0^t S(t-s)\int_0^s (s-\tau)^{-\gamma_1}\left(\left(|u_1(\tau)|^{p_1}-|u_2(\tau)|^{p_1}\right)|v_1(\tau)|^{q_1}\right.\\ &\quad \left.+\left(|v_1(\tau)|^{q_1}-|v_2(\tau)|^{q_1}\right)|u_2(\tau)|^{p_1}\right)d\sigma ds \end{aligned} \tag{5.28}$$

$$\begin{aligned} v_1(t,.)-v_2(t,.) &= \int_0^t S(t-s)\int_0^s (s-\tau)^{-\gamma_2}\left(\left(|v_1(\tau)|^{p_2}-|v_2(\tau)|^{p_2}\right)|u_1(\tau)|^{q_2}\right.\\ &\quad \left.+|v_2(\tau)|^{p_2}\left(|u_1(\tau)|^{q_2}-|u_2(\tau)|^{q_2}\right)\right)d\sigma ds \end{aligned} \tag{5.29}$$

By the help of the following inequality

$$\left||u_1|^k-|u_2|^k\right|\le C|u_1-u_2|\left(|u_1|^{k-1}+|u_2|^{k-1}\right),$$

for every u_1, $u_2\in\mathbb{R}$ and all $k\ge 1$ and by the definition of $K(T')$. Thus from (5.28), (5.29), Lemma 5.1, we get

$$\begin{aligned} &\sup_{t\in[0,T']}\|u_1(t,.)-u_2(t,.)\|_\infty+\sup_{t\in[0,T']}\|v_1(t,.)-v_2(t,.)\|_\infty\le\\ &C(T')\int_0^t\left(\sup_{\tau\in[0,s]}\|u_1(\tau,.)-u_2(\tau,.)\|_\infty+\sup_{t\in[0,s]}\|v_1(\tau,.)-v_2(\tau,.)\|_\infty\right)ds, \end{aligned} \tag{5.30}$$

where $C(T')$ constant dependent on T'. From (5.30) and Gronwall's inequality, we find $u_1 = u_2$ and $v_1 = v_2$, i.e. system (5.25) has at most one solution which belongs to $K(T')$. Due to uniqueness, it follows that the solution (u, v) can be extended to a maximal interval $[0, T_{\max})$. Note that if $0 \le t \le t + \tau \le T_{\max}$, we have

$$\begin{aligned} u(t+\tau) &= S(\tau)u(t) + \int_0^\tau S(\tau - s)\int_0^\tau (\tau-\sigma)^{-\gamma_1}|u(t+\sigma)|^{p_1}|v(t+\sigma)|^{q_1}d\sigma ds \\ &\quad + \int_0^\tau S(\tau - s)\int_0^\tau (t+s-\sigma)^{-\gamma_1}|u(\sigma)|^{p_1}|v(\sigma)|^{q_1}d\sigma ds, \\ v(t+\tau) &= S(\tau)v(t) + \int_0^\tau S(\tau - s)\int_0^\tau (\tau-\sigma)^{-\gamma_2}|v(t+\sigma)|^{p_2}|u(t+\sigma)|^{q_2}d\sigma ds \\ &\quad + \int_0^\tau S(\tau - s)\int_0^\tau (t+s-\sigma)^{-\gamma_2}|v(\sigma)|^{p_2}|u(\sigma)|^{q_2}d\sigma ds. \end{aligned} \tag{5.31}$$

By the fixed point argument, it follows from (5.31) that if $\|u(t)\|_{L^\infty((0,T)\times\mathbb{R}^N)} + \|v(t)\|_{L^\infty((0,T)\times\mathbb{R}^N)} < \infty$, then (u, v) can be extended to interval $[0, T')$ with $T' > T$. This shows that if $T_{\max} < \infty$, then $\lim_{t\to T_{\max}} \|u(t)\|_\infty + \|v(t)\|_\infty = \infty$.

To show the remaining part of Theorem 5.1, we use again a fixed point argument. Consider the space $E = L^\infty\left([0,T), L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N) \cap \{C_0(\mathbb{R}^N)\}^2\right)$, and

$K = \left\{\bar{u} = (u, v) \in E;\ \|u(t)\|_\infty, \|v(t)\|_\infty, \|u(t)\|_{r_1},\ \|v(t)\|_{r_2} < M+1, \text{ for all } t \in (0,T)\right\}$,

where $M \ge \max\left\{\|u_0\|_\infty, \|v_0\|_\infty, \|u_0\|_{r_1}, \|v_0\|_{r_2}\right\}$. The space (K, d) with the metric

$$d(\bar{u}, \bar{v}) = \max_{i=1,2}\left\{\sup_{t\in(0,T)}\|u_i(t) - v_i(t)\|_\infty, \sup_{t\in(0,T)}\|u_i(t) - v_i(t)\|_{r_i}\right\},$$

where $\bar{u} = (u_1, u_2)$, $\bar{v} = (v_1, v_2)$ is a complete metric space.

Since $r_1 \ge 1$ and $r_2 \ge 1$, we can choose $\xi, \omega, \xi_1, \omega_1 \ge 1$ by taking $\frac{1}{\xi} = \frac{1-k}{r_1}$, $\frac{1}{\omega} = \frac{1-k}{r_2}$, $\frac{1}{\xi_1} = \frac{1-k_1}{r_1}$ and $\frac{1}{\omega_1} = \frac{1-k_1}{r_2}$ for some constants $0 < k,\ k_1 < 1$ satisfying

$$\begin{aligned} &\tfrac{p_1}{\xi} + \tfrac{q_1}{\omega} \le 1,\ 0 \le \tfrac{N}{2}\left(\tfrac{p_1}{\xi} + \tfrac{q_1}{\omega} - \tfrac{1}{r_1}\right) \le 1,\ \tfrac{1}{\xi} \le \tfrac{1}{r_1},\ \tfrac{1}{\omega} \le \tfrac{1}{r_2}, \\ &\tfrac{p_2}{\omega_1} + \tfrac{q_2}{\xi_1} \le 1,\ 0 \le \tfrac{N}{2}\left(\tfrac{p_2}{\omega_1} + \tfrac{q_2}{\xi_1} - \tfrac{1}{r_2}\right) \le 1,\ \tfrac{1}{\omega_1} \le \tfrac{1}{r_2},\ \tfrac{1}{\xi_1} \le \tfrac{1}{r_1}. \end{aligned} \tag{5.32}$$

Using the smoothing effect of the heat semigroup (5.4), interpolation inequality (5.6) and (5.32), we find

$$\begin{aligned}
\|\Phi_1(u,v)\|_{r_1} &= \|u_0\|_{r_1} + \int_0^t \left\| S(t-s) \int_0^s (s-\tau)^{-\gamma_1} |u(\tau)|^{p_1} |v(\tau)|^{q_1} \right\|_{r_1} d\tau ds \\
&\leq \|u_0\|_{r_1} + \int_0^t \int_0^s (t-s)^{-\frac{N}{2}\left(\frac{p_1}{\xi}+\frac{q_1}{\omega}-\frac{1}{r_1}\right)} (s-\tau)^{-\gamma_1} \|u\|_{\xi}^{p_1} \|v\|_{\omega}^{q_1} d\tau ds \\
&\leq \|u_0\|_{r_1} + \int_0^t \int_0^s (t-s)^{-\frac{N}{2}\left(\frac{p_1}{\xi}+\frac{q_1}{\omega}-\frac{1}{r_1}\right)} (s-\tau)^{-\gamma_1} \|u\|_{r_1}^{\frac{r_1}{\xi}p_1} \|u\|_{\infty}^{(1-\frac{r_1}{\xi})p_1} \\
&\times \|v\|_{r_2}^{\frac{r_2}{\omega}q_1} \|v\|_{\infty}^{(1-\frac{r_2}{\omega})q_1} d\tau ds \\
&\leq \|u_0\|_{r_1} + \int_0^t \int_0^s (t-s)^{-\frac{N}{2}\left(\frac{p_1}{\xi}+\frac{q_1}{\omega}-\frac{1}{r_1}\right)} (s-\tau)^{-\gamma_1} \|u\|_{r_1}^{\frac{r_1}{\xi}p_1} \|u\|_{\infty}^{(1-\frac{r_1}{\xi})p_1} \\
&\times \|v\|_{r_2}^{\frac{r_2}{\omega}q_1} \|v\|_{\infty}^{(1-\frac{r_2}{\omega})q_1} d\tau ds \\
&\leq \|u_0\|_{r_1} + T^{2-\gamma_1-\frac{N}{2}\left(\frac{p_1}{\xi}+\frac{q_1}{\omega}-\frac{1}{r_1}\right)} (M+1)^{p_1+q_1}
\end{aligned}$$

Thus

$$\|\Phi_1(u,v)\|_{r_1} \leq M+1,$$

if T is small enough. Analogously, taking T eventually smaller, we get

$$\|\Phi_2(u,v)\|_{r_2} \leq \|v_0\|_{r_2} + T^{2-\gamma_2-\frac{N}{2}\left(\frac{p_2}{\omega_1}+\frac{q_2}{\xi_1}-\frac{1}{r_2}\right)} (M+1)^{p_2+q_2} \leq M+1.$$

Taking a smaller T if necessary, show that Φ is a contraction in K, indeed

$$|\Phi_1(u_1,u_2) - \Phi_1(v_1,v_2)| \leq \int_0^t \int_0^s \left| S(t-s)(s-\tau)^{-\gamma_1} \left(|u_1|^{p_1}|u_2|^{q_1} - |v_1|^{p_1}|v_2|^{q_1}\right)\right| d\tau ds$$

$$\begin{aligned}
|\Phi_1(u_1,u_2) - \Phi_1(v_1,v_2)| &\leq \int_0^t \int_0^s S(t-s)(s-\tau)^{-\gamma_1} \times ||u_1|^{p_1} [|u_2|^{q_1} - |v_2|^{q_1}]| d\tau ds \\
&+ \int_0^t \int_0^s S(t-s)(s-\tau)^{-\gamma_1} |v_2|^{q_1} ||u_1|^{p_1} - |v_1|^{p_1}| d\tau ds
\end{aligned}$$

Therefore, from (5.4) and the Hölder inequality, we have

$$\begin{aligned}\|\Phi_1(u_1,u_2)-\Phi_1(v_1,v_2)\|_{r_1} \;\le\; & \int_0^t\int_0^s (t-s)^{-\frac{N}{2}\left(\frac{p_1}{\xi}+\frac{q_1}{\omega}-\frac{1}{r_1}\right)}(s-\tau)^{-\gamma_1} \\ & \times\Big[\|u_1\|_\eta^{p_1}\left(\|u_2\|_\omega^{q_1-1}+\|v_2\|_\omega^{q_1-1}\right)\|u_2-v_2\|_\omega \\ & +\|v_2\|_\omega^{q_1}\left(\|u_1\|_\xi^{p_1-1}+\|v_1\|_\xi^{p_1-1}\right)\|u_1-v_1\|_\xi\Big]\,d\tau ds.\end{aligned}$$

Hence

$$\|\Phi_1(\bar{u})-\Phi_1(\bar{v})\|_{r_1}\le 2\,(M+1)^{p_1+q_1-1}\int_0^t\int_0^s (t-s)^{-\frac{N}{2}\left(\frac{p_1}{\xi}+\frac{q_1}{\omega}-\frac{1}{r_1}\right)}(s-\tau)^{-\gamma_1}\,d\tau ds\times d(\bar{u},\bar{v}).$$

That is

$$\|\Phi_1(\bar{u})-\Phi_1(\bar{v})\|_{r_1}\le CT^{2-\gamma_1-\frac{N}{2}\left(\frac{q_1}{\omega}+\frac{p_1}{\eta}-\frac{1}{r_1}\right)}(M+1)^{p_1+q_1-1}\,d(\bar{u},\bar{v}).$$

By analogous computations one can prove that

$$\begin{aligned}\|\Phi_2(u)\|_\infty &\le M+T^{2-\gamma_2}(M+1)^{p_2+q_2},\\ \|\Phi_2(u)-\Phi_2(\bar{u})\|_\infty &\le CT^{2-\gamma_2}(M+1)^{p_2+q_2-1}\,d(u,\bar{u}),\\ \|\Phi_2(u)\|_{r_2} &\le M+T^{2-\gamma_2-\frac{N}{2}\left(\frac{p_2}{\omega_1}+\frac{q_2}{\xi_1}-\frac{1}{r_2}\right)}(M+1)^{p_2+q_2},\\ \|\Phi_2(u)-\Phi_2(\bar{u})\|_{r_2} &\le CT^{2-\gamma_2-\frac{N}{2}\left(\frac{p_2}{\omega_1}+\frac{q_2}{\xi_1}-\frac{1}{r_2}\right)}(M+1)^{p_2+q_2-1}\,d(u,\bar{u}),\end{aligned}$$

if T is suitably small such that $CT^{1-\gamma_2}(M+1)^{p_2+q_2-1}\le 1/2$, we get the claimed result. Therefore the application Φ is a contraction in K and by contraction mapping principle there exists a unique $(u,v)\in K$ satisfying $\Phi[(u,v)]=(u,v)$ and it is the solution to the semilinear problem (5.1). ■

5.4 Proof of global existence theorem of the heat system

Proof of Theorem 5.1.

This section is devoted to prove the first main result.

To do this, let $(u_0, v_0) \in \left\{C_0\left(\mathbb{R}^N\right)\right\}^2 \cap L^{r_1}\left(\mathbb{R}^N\right) \times L^{r_2}\left(\mathbb{R}^N\right)$ where r_1, r_2 are given by (5.3). Let (u, v) be a corresponding solution given by Theorem 5.1. Since $r_1 > 1$ and $r_2 > 1$ from (5.14), we have that

$$(u, v) \in C\left([0, T_{\max}), L^{r_1}\left(\mathbb{R}^N\right) \cap C_0\left(\mathbb{R}^N\right)\right) \times C\left([0, T_{\max}), L^{r_2}\left(\mathbb{R}^N\right) \cap C_0\left(\mathbb{R}^N\right)\right)$$

and (5.13) holds. Let us consider

$$\varphi(t) = \|u(t)\|_{r_1} + (t+1)^{\frac{N}{2}\left(\frac{1}{r_1} - \frac{1}{\eta_1}\right)} \|u(t)\|_{\eta_1} + (t+1)^{\alpha} \|u(t)\|_{\infty}, \tag{5.33}$$

$$\psi(t) = \|v(t)\|_{r_2} + (t+1)^{\frac{N}{2}\left(\frac{1}{r_1} - \frac{1}{w_1}\right)} \|v(t)\|_{w_1} + (t+1)^{\beta} \|v(t)\|_{\infty}, \tag{5.34}$$

be functions defined for $t \in [0, T_{\max})$. We show that there exists ε_0 such that if $\varphi(0) + \psi(0) \leq \varepsilon_0$ and $T \in (0, T_{\max})$, then φ, ψ are bounded on $[0, T]$, where $\alpha = \frac{(1-\gamma_1)(p_2-1)-(1-\gamma_2)q_1}{((p_1-1)(p_2-1)-q_1q_2)}$, $\beta = \frac{(1-\gamma_2)(p_1-1)-(1-\gamma_1)q_2}{((p_1-1)(p_2-1)-q_1q_2)}$ and $\eta_1, w_1 > 0$ are given by

$$\frac{1}{\eta_1} = \frac{1}{r_1} - \frac{2}{N}[\alpha + \mu]; \frac{1}{w_1} = \frac{1}{r_2} - \frac{2}{N}[\beta + \lambda], \tag{5.35}$$

with $\mu, \lambda > 0$ satisfying

$$\mu < \min\left\{\gamma_1 - \alpha, \frac{N}{2r_1} - \alpha\right\}, \qquad \lambda < \min\left\{\gamma_2 - \beta, \frac{N}{2r_2} - \beta\right\}, \tag{5.36}$$

$$\begin{aligned} \frac{N}{2}\left(\frac{p_1}{r_1} + \frac{q_1}{r_2} - 1\right) - (\alpha p_1 + \beta q_1) &< (p_1\mu + q_1\lambda) < 1 - \alpha, \\ \frac{N}{2}\left(\frac{p_2}{r_2} + \frac{q_2}{r_1} - 1\right) - (\beta p_2 + \alpha q_2) &< (p_2\lambda + q_2\mu) < 1 - \beta, \end{aligned} \tag{5.37}$$

If this statement is proved, then $T^* = \infty$ and (u, v) is global. From (5.35)–(5.37), we have that

$$\begin{aligned} &\tfrac{p_1}{\eta_1}+\tfrac{q_1}{w_1}<1,\ 0<\tfrac{N}{2}\left(\tfrac{p_1}{\eta_1}+\tfrac{q_1}{w_1}-\tfrac{1}{r_1}\right)<1,\\ &\tfrac{p_2}{w_1}+\tfrac{q_2}{\eta_1}<1,\ 0<\tfrac{N}{2}\left(\tfrac{p_2}{w_1}+\tfrac{q_2}{\eta_1}-\tfrac{1}{r_1}\right)<1. \end{aligned} \tag{5.38}$$

Moreover, from (5.14), it easy to check that

$$\begin{cases} \frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}=\frac{2}{N}\left[\frac{(1-\gamma_1)((p_1-1)(p_2-1)-q_1q_2)+(2-\gamma_1)(p_2-1)-(2-\gamma_2)q_1}{((p_1-1)(p_2-1)-q_1q_2)}\right]<1,\\ \frac{p_2}{r_2}+\frac{q_2}{r_1}-\frac{2}{N}=\frac{2}{N}\left[\frac{(1-\gamma_2)((p_1-1)(p_2-1)-q_1q_2)+(2-\gamma_2)(p_1-1)-(2-\gamma_1)q_2}{((p_1-1)(p_2-1)-q_1q_2)}\right]<1. \end{cases} \tag{5.39}$$

From the definition of α, β, it yields

$$\begin{cases} q_2\alpha+(p_2-1)\,\beta=(1-\gamma_2)\,,\\ q_1\beta+(p_1-1)\,\alpha=(1-\gamma_1)\,. \end{cases} \tag{5.40}$$

An estimate for (u,v) in $L^{r_1}\times L^{r_2}$. From (5.35) and (5.40), we have that

$$\frac{p_1}{\eta_1}+\frac{q_1}{w_1}<\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}\left(\alpha p_1+q_1\beta\right)<\frac{2}{N}+\frac{1}{r_1}.$$

So, we can select $w\in(r_2,w_1)$ and $\xi\in(r_1,\eta_1)$ such that

$$\max\left\{\frac{1}{r_1},\frac{p_1}{\eta_1}+\frac{q_1}{w_1},\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}\right\}<\frac{p_1}{\xi}+\frac{q_1}{w}<\min\left\{\frac{2}{N}+\frac{1}{r_1},\frac{p_1}{r_1}+\frac{q_1}{r_2},1\right\}. \tag{5.41}$$

To see this, let us take $\frac{1}{\xi}=\frac{(1-k)}{\eta_1}+\frac{k}{r_1}$, $\frac{1}{w}=\frac{(1-k)}{w_1}+\frac{k}{r_2}$, $0<k<1$,

$$\frac{p_1}{\xi}+\frac{q_1}{w}=\frac{p_1}{\eta_1}+\frac{q_1}{w_1}+k\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta_1}+\frac{q_1}{w_1}\right)\right).$$

It's easy to check that for some $0<k<1$ the inequality (5.41) holds.

Observe that by (5.41)

$$0<\frac{N}{2}\left(\frac{p_1}{\xi}+\frac{q_1}{w}-\frac{1}{r_1}\right)<1,\ 0<\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\xi}+\frac{q_1}{w}\right)\right)<1. \tag{5.42}$$

Making use of the interpolation inequality, we get

$$\begin{aligned} \|u\|_{\xi} \le \|u\|_{r_1}^{\theta} \|u\|_{\eta_1}^{1-\theta} \le \|u\|_{r_1}^{\theta} \left[(t+1)^{\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)} \|u\|_{\eta_1}\right]^{1-\theta} (t+1)^{-\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\xi}\right)}, \\ \|v\|_{w} \le \|v\|_{r_2}^{\theta'} \|v\|_{w_1}^{1-\theta'} \le \|v\|_{r_2}^{\theta'} \left[(t+1)^{\frac{N}{2}\left(\frac{1}{r_2}-\frac{1}{w_1}\right)} \|v\|_{w_1}\right]^{1-\theta'} (t+1)^{-\frac{N}{2}\left(\frac{1}{r_2}-\frac{1}{w}\right)}, \end{aligned} \tag{5.43}$$

where $\frac{1}{\xi} = \frac{\theta}{r_1} + \frac{1-\theta}{\eta_1}$, $\frac{1}{w} = \frac{\theta'}{r_2} + \frac{1-\theta'}{w_1}$, $\theta,\, \theta' \in (0,1)$. We easily check that

$$2-\gamma_1 - \frac{N}{2}\left(\frac{p_1}{\xi}+\frac{q_1}{w}-\frac{1}{r_1}\right) - \frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\xi}+\frac{q_1}{w}\right)\right) = 0,$$

we see from (5.12), (5.42) and (5.43) that

$$\begin{aligned} \|u(t)\|_{r_1} &\le \|u_0\|_{r_1} + \int_0^t (t-s)^{-\frac{N}{2}\left(\frac{p_1}{\xi}+\frac{q_1}{w}-\frac{1}{r_1}\right)} \int_0^s (s-\tau)^{-\gamma_1} \|u(\tau)\|_{\xi}^{p_1} \|v(\tau)\|_{w}^{q_1}\, d\tau ds \\ &\le \|u_0\|_{r_1} + \int_0^t (t-s)^{-\frac{N}{2}\left(\frac{p_1}{\xi}+\frac{q_1}{w}-\frac{1}{r_1}\right)} \int_0^s (s-\tau)^{-\gamma_1} \tau^{-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\xi}+\frac{q_1}{w}\right)\right)} d\tau ds \\ &\le \|u_0\|_{r_1} + C\left[\sup_{s\in(0,t)} \varphi(s)\right]^{p_1} \left[\sup_{s\in(0,t)} \psi(s)\right]^{q_1}. \end{aligned}$$

An estimate for (u,v) in $L^{\eta_1}\times L^{w_1}$. From (5.16) as $\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N} < \frac{2}{N}+\frac{1}{\eta_1}$, $\frac{p_1}{r_1}+\frac{q_1}{r_2} < 1+\frac{2}{N}$. Since

$$p_1\mu + q_1\lambda > \mu = 2-\gamma_1-(1-\gamma_1)-1+\mu = \frac{N}{2}\left(\frac{p_1-1}{r_1}+\frac{q_1}{r_2}\right) - (p_1-1)\alpha - q_1\beta - 1 + \mu,$$

which equivalently

$$\frac{N}{2}\left(\frac{p_1-1}{r_1}+\frac{q_1}{r_2}\right) - (p_1-1)[\alpha+\mu] - q_1[\beta+\lambda] < 1,$$

that is

$$\frac{p_1-1}{r_1} - \frac{2(p_1-1)}{N}[\alpha+\mu] + \frac{q_1}{r_2} - \frac{2q_1}{N}[\beta+\lambda] < \frac{2}{N}.$$

Namely $\frac{p_1-1}{\eta_1}+\frac{q_1}{w_1} < \frac{2}{N}$. From these facts, we can choose $w' \in (r_2, w_1)$, $\xi' \in (r_1, \eta_1)$

satisfying

$$\max\left\{\frac{1}{\eta_1},\frac{p_1}{\eta_1}+\frac{q_1}{w_1},\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}\right\}<\frac{p_1}{\xi'}+\frac{q_1}{w'}<\min\left\{\frac{2}{N}+\frac{1}{\eta_1},\frac{p_1}{r_1}+\frac{q_1}{r_2},1\right\}.$$

Note that

$$0<\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\xi'}+\frac{q_1}{w'}\right)\right)<1,\ 0<\frac{N}{2}\left(\frac{p_1}{\xi'}+\frac{q_1}{w'}-\frac{1}{\eta_1}\right)<1, \tag{5.44}$$

and

$$2-\gamma_1+\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)-\frac{N}{2}\left(\frac{p_1}{\xi'}+\frac{q_1}{w'}-\frac{1}{\eta_1}\right)-\frac{N}{2}\left(\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}\right)-\left(\frac{p_1}{\xi'}+\frac{q_1}{w'}\right)\right)=0.$$

From (5.4), (5.5), (5.43) and (5.44), we get

$$\begin{aligned}\left\|u(t)\right\|_{\eta_1} &\leq (t+1)^{-\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)}\left(\left\|u_0\right\|_{r_1}+\left\|u_0\right\|_{\eta_1}\right)\\ &\quad+\int_0^t (t-s)^{-\frac{N}{2}\left(\frac{p_1}{\xi'}+\frac{q_1}{w'}-\frac{1}{\eta_1}\right)}\int_0^s (s-\tau)^{-\gamma_1}\left\|u(\tau)\right\|_{\xi'}^{p_1}\left\|v(\tau)\right\|_{w'}^{q_1}d\tau ds\end{aligned}$$

$$\begin{aligned}&\leq (t+1)^{-\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)}\left(\left\|u_0\right\|_{r_1}+\left\|u_0\right\|_{\eta_1}\right)+C\left[\sup_{s\in(0,t)}\varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi(s)\right]^{q_1}\\ &\quad\times\int_0^t (t-s)^{-\frac{N}{2}\left(\frac{p_1}{\xi'}+\frac{q_1}{w'}-\frac{1}{\eta_1}\right)}\int_0^s (s-\tau)^{-\gamma_1}\tau^{-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\xi'}+\frac{q_1}{w'}\right)\right)}d\tau ds,\end{aligned}$$

which, yields

$$(t+1)^{\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)}\left\|u_1(t)\right\|_{\eta_1}\leq\left\|u_0\right\|_{r_1}+\left\|u_0\right\|_{\eta_1}+C\left[\sup_{s\in(0,t)}\varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi(s)\right]^{q_1}.$$

Estimate for $\|u(t)\|_\infty$. We have to distinguish two situations

Case a: Either $N\leq 2$ or ($\frac{p_1}{r_1}+\frac{q_1}{r_2}<\frac{4}{N}$ and $\frac{p_1}{\eta_1}+\frac{q_1}{w_1}<\frac{2}{N}$). From (5.35),(5.38) and (5.39), there exist $w^{''} \in (r_2, w_1)$, $\eta^{''} \in (r_1, \eta_1)$ such that

$$\max\left\{\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}, \frac{p_1}{\eta_1}+\frac{q_1}{w_1}\right\} < \left(\frac{p_1}{\eta^{''}}+\frac{q_1}{w^{''}}\right) < \min\left\{\frac{2}{N}, \frac{p_1}{r_1}+\frac{q_1}{r_2}, 1\right\}.$$

Since $w^{''} \in (r_2, w_1)$, $\eta^{''} \in (r_1, \eta_1)$, using interpolation inequality again as in (5.43), we obtain

$$\begin{aligned}\|u(t)\|_\infty \leq\ & (t+1)^{-\frac{N}{2}\frac{1}{r_1}}\left(\|u_0\|_{r_1}+\|u_0\|_\infty\right)+\int_0^t (t-s)^{-\frac{N}{2}\left(\frac{q_1}{w^{''}}+\frac{p_1}{\eta^{''}}\right)}\int_0^s (s-\tau)^{-\gamma_1}\\ & \times \|u(\tau)\|_{\eta^{''}}^{p_1}\|v(\tau)\|_{w^{''}}^{q_1}\, d\tau ds\end{aligned}$$

$$\begin{aligned}\|u(t)\|_\infty \leq\ & (t+1)^{-\frac{N}{2}\frac{1}{r_1}}\left(\|u_0\|_{r_1}+\|u_0\|_\infty\right)+C\left[\sup_{s\in(0,t)}\varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi(s)\right]^{q_1}\\ & \times\int_0^t (t-s)^{-\frac{N}{2}\left(\frac{q_1}{w^{''}}+\frac{p_1}{\eta^{''}}\right)}\int_0^s (s-\tau)^{-\gamma_1}(\tau+1)^{-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta^{''}}+\frac{q_1}{w^{''}}\right)\right)} d\tau ds\end{aligned}$$

Notice that $0 < \frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta^{''}}+\frac{q_1}{w^{''}}\right)\right) < 1$, $\frac{N}{2}\left(\frac{q_1}{w^{''}}+\frac{p_1}{\eta^{''}}\right) < 1$. On the other hand, since $\alpha - \frac{N}{2r_1} = \frac{p_2-q_1-1}{((p_1-1)(p_2-1)-q_1q_2)} < 0$, it follows that

$$\begin{aligned}& \alpha+2-\gamma_1-\frac{N}{2}\left(\frac{q_1}{w^{''}}+\frac{p_1}{\eta^{''}}\right)-\frac{N}{2}\left(\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}\right)-\left(\frac{p_1}{\eta^{''}}+\frac{q_1}{w^{''}}\right)\right)\\ < \ & 2-\gamma_1+\frac{N}{2r_1}-\frac{N}{2}\left[\frac{p_1}{r_1}+\frac{q_1}{r_2}\right]=0,\end{aligned}$$

which, together with (5.8), yields

$$(t+1)^\alpha\|u(t)\|_\infty \leq \left(\|u_0\|_{r_1}+\|u_0\|_\infty\right)+C\left[\sup_{s\in(0,t)}\varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi(s)\right]^{q_1}.$$

Or else, that is $N>2$ and ($\frac{p_1}{r_1}+\frac{q_1}{r_2} \geq \frac{4}{N}$, or $\frac{p_1}{\eta_1}+\frac{q_1}{w_1} \geq \frac{2}{N}$). From (5.35),(5.38) and (5.39), there exist $w'' \in (r_2, w_1)$, $\xi'' \in (r_1, \eta_1)$ such that

$$\max\left\{\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}, \frac{p_1}{\eta_1}+\frac{q_1}{w_1}, \frac{2}{N}\right\} < \left(\frac{p_1}{\xi''}+\frac{q_1}{w''}\right) < \min\left\{1, \frac{p_1}{r_1}+\frac{q_1}{r_2}\right\}.$$

Hence, the inequalities (5.5) and (5.43), give

$$\|u(t)\|_\infty \le (t+1)^{-\frac{N}{2}\frac{1}{r_1}}\left(\|u_0\|_{r_1}+\|u_0\|_\infty\right)$$
$$+\int_0^t (t-s+1)^{-\frac{N}{2}\left(\frac{p_1}{\xi''}+\frac{q_1}{w''}\right)}\int_0^s (s-\tau)^{-\gamma_1}\left(\|u(\tau)\|_{\xi''}^{p_1}\|v(\tau)\|_{w''}^{q_1}+\|u(\tau)\|_\infty^{p_1}\|v(\tau)\|_\infty^{q_1}\right)d\tau ds.$$

Therefore

$$\begin{aligned}\|u(t)\|_\infty \le\ & (t+1)^{-\frac{N}{2}\frac{1}{r_1}}\left(\|u_0\|_{r_1}+\|u_0\|_\infty\right)\\ &+\left[\sup_{s\in(0,t)}\varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi(s)\right]^{q_1}\int_0^t (t-s+1)^{-\frac{N}{2}\left(\frac{p_1}{\xi''}+\frac{q_1}{w''}\right)}\int_0^s (s-\tau)^{-\gamma_1}\\ &+\left[\sup_{s\in(0,t)}\varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi(s)\right]^{q_1}\int_0^t (t-s+1)^{-\frac{N}{2}\left(\frac{p_1}{\xi''}+\frac{q_1}{w''}\right)}\int_0^s (s-\tau)^{-\gamma_1}\\ &\times\left[(\tau+1)^{-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\xi''}+\frac{q_1}{w''}\right)\right)}+(\tau+1)^{-(\alpha p_1+\beta q_1)}\right]d\tau ds.\end{aligned}$$

Since $\alpha p_1+\beta q_1<1$, we observe that

$$\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N} < \frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}(\alpha p_1+q_1\beta).$$

It easy to see from (5.35), (5.39) and (5.40) that

$$\max\left\{\frac{p_1}{\eta_1}+\frac{q_1}{w_1}, \frac{2}{N}\right\} < \frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}(\alpha p_1+q_1\beta).$$

From these facts, we can assume that $\left(\frac{p_1}{\xi''}+\frac{q_1}{w''}\right) \le \frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}(\alpha p_1+q_1\beta)$. Thus

$$\alpha+1-\gamma_1-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}\right)+\frac{N}{2}\left(\frac{p_1}{\xi''}+\frac{q_1}{w''}\right) = \alpha p_1+q_1\beta-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\xi''}+\frac{q_1}{w''}\right)\right)\le 0,$$

and therefore, we conclude as in the previous case

$$(t+1)^{\alpha} \|u(t)\|_{\infty} \leq C\left(\|u_0\|_{r_1} + \|u_0\|_{\infty}\right) + C\left[\sup_{s\in(0,t)} \varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)} \psi(s)\right]^{q_1}.$$

From the previous estimates of $\|u_1(t)\|_{r_1}$, $\|u_1(t)\|_{\eta_1}$ and $\|u(t)\|_{\infty}$, we obtain

$$\varphi(t) \leq C\left(\|u_0\|_{r_1} + \|u_0\|_{\infty}\right) + C\left[\sup_{s\in(0,t)} \varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)} \psi(s)\right]^{q_1}.$$

By analogous computations, we get $\psi(t) \leq C\left(\|v_0\|_{r_1} + \|v_0\|_{\infty}\right) + C\left[\sup_{s\in(0,t)} \psi(s)\right]^{p_2}\left[\sup_{s\in(0,t)} \varphi(s)\right]^{q_2}$.
Denoting $f(t) = \sup_{s\in(0,t)} \varphi(s)$ and $g(t) = \sup_{s\in(0,t)} \psi(s)$, we get

$$f(t) \leq C\left(\|u_0\|_{r_1} + \|u_0\|_{\infty}\right) + Cf(t)^{p_1} g(t)^{q_1}, \tag{5.45}$$

$$g(t) \leq C\left(\|v_0\|_{r_2} + \|v_0\|_{\infty}\right) + Cg(t)^{p_2} f(t)^{q_2}. \tag{5.46}$$

Now, we define $h(t) := f(t) + g(t)$. Taking into account (5.45), (5.46) reads

$h(t) \leq C\left(\varepsilon + h^{p_1+q_1}(t) + h^{p_2+q_2}(t)\right), \forall t \in [0, T_{\max})$,

for some positive constant C independent of t and $A = \|u_0\|_{r_1} + \|u_0\|_{\infty} + \|v_0\|_{r_2} + \|v_0\|_{\infty} < \varepsilon$. We conclude by standard arguments for sufficiently small ε as in [57], it then follows that $h(t) \leq C\varepsilon$, $\forall\ t \in [0, T_{\max})$. Hence $f(t) \leq C\varepsilon$, $g(t) \leq C\varepsilon$, $\forall t \in [0, T_{\max})$.

5.5 Damped wave system

Similar considerations to the system for heat equations can be applied to the Cauchy problem (5.2) for the system of damped wave equations in low dimensional space.

5.5.1 Proof of theorem 5.2

We follow the same steps as in the proof of Theorem 5.1 with a slight modifications. So we maintain some notations used in the previous proof.

Let us define $\eta_1, w_1 > 0$ by

$$\frac{1}{\eta_1} = \frac{1}{r_1} - \frac{2}{N}(\alpha + \mu); \quad \frac{1}{w_1} = \frac{1}{r_2} - \frac{2}{N}(\beta + \lambda),$$

with $\mu, \lambda > 0$ satisfying

$$\mu < \min\left\{\gamma_1 - \alpha, \frac{N}{2r_1} - \alpha\right\}, \qquad \lambda < \min\left\{\gamma_2 - \beta, \frac{N}{2r_2} - \beta\right\},$$

$$\begin{aligned}
\frac{N}{2}\left(\frac{p_1}{r_1} + \frac{q_1}{r_2} - 1\right) - (\alpha p_1 + \beta q_1) &< (p_1\mu + q_1\lambda) < 1 - \alpha, \\
\frac{N}{2}\left(\frac{p_2}{r_2} + \frac{q_2}{r_1} - 1\right) - (\beta p_2 + \alpha q_2) &< (p_2\lambda + q_2\mu) < 1 - \beta,
\end{aligned}$$

with

$$\mu = \frac{(p_2 - q_1 - 1)}{p_1 - q_2 - 1}\lambda. \tag{5.47}$$

The existence of λ and μ are insured by the conditions (5.21)-(5.23).

Let $(u_i, v_i) \in \left\{W^{1-i,1}\left(\mathbb{R}^N\right) \times W^{1-i,\infty}\left(\mathbb{R}^N\right)\right\}^2$, $i = 0, 1$. From (5.34), we have $r_1 > 1$ and $r_2 > 1$.

Let (u, v) be a corresponding solution of (5.2) given by Proposition 5.2.

Our aim is to seek upper bound of solution in the functionals defined in (5.33), (5.34) each $t > 0$. It easy to check that all the requirements (5.38), (5.39) and (5.40) are fulfilled.

An estimate for u in $L^{r_1}\left(\mathbb{R}^N\right)$. From (5.40), we conclude that

$$\frac{p_1}{\eta_1} + \frac{q_1}{w_1} < \frac{p_1}{r_1} + \frac{q_1}{r_2} - \frac{2}{N}(\alpha p_1 + q_1\beta) < \frac{2}{N} + \frac{1}{r_1}.$$

So, we can select $w \in (r_2, w_1)$ and $\eta \in (r_1, \eta_1)$ such that

$$\max\left\{\frac{1}{r_1}, \frac{p_1}{\eta_1} + \frac{q_1}{w_1}, \frac{p_1}{r_1} + \frac{q_1}{r_2} - \frac{2}{N}\right\} < \frac{p_1}{\eta} + \frac{q_1}{w} < \min\left\{\frac{2}{N} + \frac{1}{r_1}, \frac{p_1}{r_1} + \frac{q_1}{r_2}, 1\right\}$$

Note that

$$0 \leq \frac{N}{2}\left(\frac{p_1}{\eta} + \frac{q_1}{w} - \frac{1}{r_1}\right) < 1,\ 0 \leq \frac{N}{2}\left(\frac{p_1}{r_1} + \frac{q_1}{r_2} - \left(\frac{p_1}{\eta} + \frac{q_1}{w}\right)\right) < 1.$$

and by (5.21)-(5.23), we get

$$\frac{1}{r_1} < \frac{p_1}{\eta_1} + \frac{q_1}{w_1},\ \frac{1}{\eta_1} < \frac{p_1}{\eta_1} + \frac{q_1}{w_1}, \frac{1}{w_1} < \frac{p_2}{\eta_1} + \frac{q_2}{w_1},\ \text{and}\ \frac{1}{\eta_1} < \frac{p_2}{\eta_1} + \frac{q_2}{w_1}.$$

From (5.20), we have

$$\begin{aligned}
\|u(t,.)\|_{r_1} \leq\ & \left\|\left(K_1(t) - e^{-\frac{t}{2}}W_1(t)\right)(\frac{1}{2}u_0 + u_1)\right\|_{r_1} + \left\|e^{-\frac{t}{2}}W_1(t)(\frac{1}{2}u_0 + u_1)\right\|_{r_1} \\
& + \left\|\left(K_0(t) - e^{-\frac{t}{2}}\left(W_0(t) + \frac{t}{8}W_1(t)\right)\right)u_0\right\|_{r_1} + \left\|e^{-\frac{t}{2}}\left(W_0(t) + \frac{t}{8}W_1(t)\right)u_0\right\|_{r_1} \\
& + \int_0^t \left\|\left(K_1(t-s) - e^{-\frac{t-s}{2}}W_1(t-s)\right)\int_0^s (s-\tau)^{-\gamma_1}|u(\tau)|^{p_1}|v(\tau)|^{q_1}\right\|_{r_1} d\tau ds \\
& + \int_0^t e^{-\frac{t-s}{2}}\left\|W_1(t-s)\int_0^s (s-\tau)^{-\gamma_1}|u(\tau)|^{p_1}|v(\tau)|^{q_1}\right\|_{r_1} d\tau ds.
\end{aligned}$$

Next, by the lemma 5.4, lemma 5.5, and (5.6) we get, for all $t \in [0, T_{max})$

$$\begin{aligned}
\|u(t,.)\|_{r_1} \leq\ & C\left(\|u_1\|_{L^{r_1}} + \|u_0\|_{W^{1,r_1}}\right) + \int_0^t (1+t-s)^{-\frac{N}{2}\left(\frac{p_1}{\eta} + \frac{q_1}{w} - \frac{1}{r_1}\right)} \\
& \times \int_0^s (s-\tau)^{-\gamma_1}\|u(\tau,.)\|_\eta^{p_1}\|v(\tau,.)\|_w^{q_1}\, d\tau ds \\
& + \int_0^t e^{-\frac{t-s}{2}}(t-s)\int_0^s (s-\tau)^{-\gamma_1}\||u(\tau)|^{p_1}|v(\tau)|^{q_1}\|_{r_1} d\tau ds.
\end{aligned}$$

By using the interpolation inequality to the last term, we get

$$\begin{aligned}\|u(t,.)\|_{r_1} &\leq C\left(\|u_1\|_{r_1}+\|u_0\|_{W^{1,r_1}}\right)\\&+\int_0^t (t-s+1)^{-\frac{N}{2}\left(\frac{p_1}{\eta}+\frac{q_1}{w}-\frac{1}{r_1}\right)}\int_0^s (s-\tau)^{-\gamma_1}\|u(\tau,.)\|_\eta^{p_1}\|v(\tau,.)\|_w^{q_1}\,d\tau ds\\&+\int_0^t e^{-\frac{t-s}{2}}(t-s)\int_0^s (s-\tau)^{-\gamma_1}\||u(\tau)|^{p_1}|v(\tau)|^{q_1}\|^{1-\theta}_{\left(\frac{p_1}{\eta_1}+\frac{q_1}{w_1}\right)^{-1}}\\&\times\||u(\tau)|^{p_1}|v(\tau)|^{q_1}\|_\infty^\theta\,d\tau ds,\end{aligned}\tag{5.48}$$

with $1-\theta=\frac{1}{r_1}\left(\frac{p_1}{\eta_1}+\frac{q_1}{w_1}\right)^{-1}$. Therefore

$$\begin{aligned}&\|u(t,.)\|_{r_1}\leq C\left(\|u_1\|_{r_1}+\|u_0\|_{W^{1,r_1}}\right)\\&+\int_0^t (t-s+1)^{-\frac{N}{2}\left(\frac{p_1}{\eta}+\frac{q_1}{w}-\frac{1}{r_1}\right)}\int_0^s (s-\tau)^{-\gamma_1}\|u(\tau,.)\|_\eta^{p_1}\|v(\tau,.)\|_w^{q_1}\,d\tau ds\\&+\int_0^t e^{-\frac{t-s}{2}}(t-s)\int_0^s (s-\tau)^{-\gamma_1}\|u(\tau,.)\|_{\eta_1}^{p_1(1-\theta)}\|v(\tau,.)\|_{w_1}^{q_1(1-\theta)}\|u(\tau,.)\|_\infty^{p_1\theta}\|v(\tau,.)\|_\infty^{q_1\theta}\,d\tau ds,\end{aligned}$$

From the definition of φ and ψ, we infer that

$$\begin{aligned}\|u(t,.)\|_{r_1} &\leq C\left(\|u_1\|_{r_1}+\|u_0\|_{W^{1,r_1}}\right)+\left[\sup_{s\in(0,t)}\varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi(s)\right]^{q_1}\\&\times\int_0^t (t-s+1)^{-\frac{N}{2}\left(\frac{p_1}{\eta}+\frac{q_1}{w}-\frac{1}{r_1}\right)}\int_0^s (s-\tau)^{-\gamma_1}\tau^{-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta}+\frac{q_1}{w}\right)\right)}d\tau ds\\&+\int_0^t e^{-\frac{t-s}{2}}(t-s)\int_0^s (s-\tau)^{-\gamma_1}(1+\tau)^{-(\alpha p_1+\beta q_1)\theta-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta_1}+\frac{q_1}{w_1}\right)\right)(1-\theta)}\,d\tau ds\\&\times\left[\sup_{s\in(0,t)}\varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi(s)\right]^{q_1},\end{aligned}$$

where we have have used the fact that (5.33) and (5.43) since $(u,v)\in X$.

By virtue of (5.40), we know that

$$\begin{aligned}&1-\gamma_1-(\alpha p_1-\beta q_1)\theta-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta_1}+\frac{q_1}{w_1}\right)\right)(1-\theta)\\=\;&1-\gamma_1-(\alpha p_1-\beta q_1)\theta-(p_1(\alpha+\mu)+q_1(\beta+\lambda))(1-\theta)\\=\;&-\alpha-(p_1\mu+q_1\lambda)(1-\theta)<0,\end{aligned}$$

and since $2-\gamma_1-\frac{N}{2}\left(\frac{p_1}{\eta}+\frac{q_1}{w}-\frac{1}{r_1}\right)-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta}+\frac{q_1}{w}\right)\right)=0$. Applying again Lemma

5.2, we obtain

$$\|u(t,.)\|_{r_1} \leq C\left(\|u_0\|_{W^{1,r_1}} + \|u_1\|_{r_1}\right) + C\left[\sup_{s\in(0,t)} \varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)} \psi(s)\right]^{q_1}.$$

An estimate for $\|u(t,.)\|_{\eta_1}$. Arguing in the same way as in the previous estimate

$$\begin{aligned}
\|u(t,.)\|_{\eta_1} \leq\ & \left\|\left(K_1(t) - e^{-\frac{t}{2}}W_1(t)\right)(\frac{1}{2}u_0 + u_1)\right\|_{\eta_1} + \left\|e^{-\frac{t}{2}}W_1(t)(\frac{1}{2}u_0 + u_1)\right\|_{\eta_1} \\
& + \left\|\left(K_0(t) - e^{-\frac{t}{2}}\left(W_0(t) + \frac{t}{8}W_1(t)\right)\right)u_0\right\|_{\eta_1} + \left\|e^{-\frac{t}{2}}\left(W_0(t) + \frac{t}{8}W_1(t)\right)u_0\right\|_{\eta_1} \\
& + \int_0^t \left\|\left(K_1(t-s) - e^{-\frac{t-s}{2}}W_1(t-s)\right)\int_0^s (s-\tau)^{-\gamma_1}|u(\tau)|^{p_1}|v(\tau)|^{q_1}\right\|_{\eta_1} d\tau ds \\
& + \int_0^t e^{-\frac{t-s}{2}}\left\|W_1(t-s)\int_0^s (s-\tau)^{-\gamma_1}|u(\tau)|^{p_1}|v(\tau)|^{q_1}\right\|_{\eta_1} d\tau ds.
\end{aligned}$$

From (5.23), exist w', η' satisfying

$$\max\left\{\frac{p_1}{r_1} + \frac{q_1}{r_2} - \frac{2}{N}, \frac{p_1}{\eta_1} + \frac{q_1}{w_1}\right\} < \frac{p_1}{\eta'} + \frac{q_1}{w'} < \min\left\{1, \frac{1}{\eta_1} + \frac{2}{N}, \frac{p_1}{r_1} + \frac{q_1}{r_2}\right\}$$

Note that $0 < \frac{N}{2}\left(\frac{p_1}{\eta'} + \frac{q_1}{w'} - \frac{1}{\eta_1}\right) < 1$, $0 < \frac{N}{2}\left(\left(\frac{p_1}{r_1} + \frac{q_1}{r_2}\right) - \left(\frac{p_1}{\eta'} + \frac{q_1}{w'}\right)\right) < 1$ and

$$2 - \gamma_1 + \frac{N}{2}\left(\frac{1}{r_1} - \frac{1}{\eta_1}\right) - \frac{N}{2}\left(\frac{p_1}{\eta'} + \frac{q_1}{w'} - \frac{1}{\eta_1}\right) - \frac{N}{2}\left(\left(\frac{p_1}{r_1} + \frac{q_1}{r_2}\right) - \left(\frac{p_1}{\eta'} + \frac{q_1}{w'}\right)\right) = 0.$$

From (5.20) (replacing w by w') and thanks to lemma 5.5, lemma 5.4 and (5.6), we have

$$\begin{aligned}
\|u(t,.)\|_{\eta_1} \leq\ & C(1+t)^{-\frac{N}{2}\left(\frac{1}{r_1} - \frac{1}{\eta_1}\right)}\left\|(\frac{1}{2}u_0 + u_1)\right\|_{r_1} \\
& + C(1+t)^{-\frac{N}{2}\left(\frac{1}{r_1} - \frac{1}{\eta_1}\right)}\left(\|u_1\|_{L^{\eta_1}} + \|u_0\|_{L^{\eta_1}}\right) + C(1+t)^{-\frac{N}{2}\left(\frac{1}{r_1} - \frac{1}{\eta_1}\right)}\|u_0\|_{W^{1,\eta_1}} \\
& + \int_0^t (1+t-s)^{-\frac{N}{2}\left(\frac{p_1}{\eta'} + \frac{q_1}{w'} - \frac{1}{\eta_1}\right)}\int_0^s (s-\tau)^{-\gamma_1}\|u(\tau,.)\|_{\eta'}^{p_1}\|v(\tau,.)\|_{w'}^{q_1} d\tau ds \\
& + \int_0^t e^{-\frac{t-s}{2}}(t-s)\int_0^s (s-\tau)^{-\gamma_1}\|u(\tau,.)\|_{\eta_1}^{p_1(1-\theta')}\|v(\tau,.)\|_{w_1}^{q_1(1-\theta')} \\
& \times \|u(\tau,.)\|_{\infty}^{p_1\theta'}\|v(\tau,.)\|_{\infty}^{q_1\theta'} d\tau ds.
\end{aligned}$$

with $1-\theta' = \frac{1}{\eta_1}\left(\frac{p_1}{\eta_1}+\frac{q_1}{w_1}\right)^{-1}$. Combining the last inequality with (5.33), (5.34) and (5.43),we obtain

$$\begin{aligned}
&\|u(t,.)\|_{\eta_1} \leq C\left(1+t\right)^{-\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)}\left(\|u_0\|_{r_1}+\|u_1)\|_{r_1}\right) \\
&+C\left(1+t\right)^{-\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)}\left(\|u_1\|_{L^{\eta_1}}+\|u_0\|_{L^{\eta_1}}\right)+C\left(1+t\right)^{-\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)}\|u_0\|_{W^{1,\eta_1}} \\
&+\left[\sup_{s\in(0,t)}\varphi\left(s\right)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi\left(s\right)\right]^{q_1} \\
&\times\int_0^t\left(1+t-s\right)^{-\frac{N}{2}\left(\frac{p_1}{\eta'}+\frac{q_1}{w'}-\frac{1}{\eta_1}\right)}\int_0^s\left(s-\tau\right)^{-\gamma_1}\left(1+\tau\right)^{-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta'}+\frac{q_1}{w'}\right)\right)}d\tau ds \\
&+\int_0^t e^{-\frac{t-s}{2}}\left(t-s\right)\int_0^s\left(s-\tau\right)^{-\gamma_1}\left(1+\tau\right)^{-(\alpha p_1+\beta q_1)\theta'-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta_1}+\frac{q_1}{w_1}\right)\right)(1-\theta')}d\tau ds \\
&\times\left[\sup_{s\in(0,t)}\varphi\left(s\right)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi\left(s\right)\right]^{q_1}.
\end{aligned} \tag{5.49}$$

Multiplying both sides of (5.49) by $(1+t)^{\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)}$, we find

$$\begin{aligned}
(1+t)^{\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)}\|u\left(t,.\right)\|_{\eta_1} \leq\; & C\left(\|u_0\|_{W^{1,\eta_1}}+\|u_0\|_{r_1}+\|u_1\|_{\eta_1}+\|u_1\|_{r_1}\right) \\
&+[\sup_{s\in(0,t)}\varphi\left(s\right)]^{p_1}[\sup_{s\in(0,t)}\psi\left(s\right)]^{q_1} \\
&\times(1+t)^{2-\gamma_1+\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)-\frac{N}{2}\left(\frac{p_1}{\eta'}+\frac{q_1}{w'}-\frac{1}{\eta_1}\right)-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{p_1}{\eta'}-\frac{q_1}{w'}\right)} \\
&+[\sup_{s\in(0,t)}\varphi\left(s\right)]^{p_1}[\sup_{s\in(0,t)}\psi\left(s\right)]^{q_1}\left(1+t\right)^{\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)} \\
&\times\int_0^t e^{-\frac{t-s}{2}}\left(t-s\right)\int_0^s\left(s-\tau\right)^{-\gamma_1}\left(1+\tau\right)^{-(\alpha p_1+\beta q_1)\theta'} \\
&\times(1+\tau)^{-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta_1}+\frac{q_1}{w_1}\right)\right)(1-\theta')}d\tau ds.
\end{aligned}$$

We have from the definition of η_1, w_1 that

$$-\left(\alpha p_1+\beta q_1\right)\theta'-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta_1}+\frac{q_1}{w_1}\right)\right)(1-\theta') = -\alpha-\left(p_1\mu+q_1\lambda\right)(1-\theta')+\gamma_1-1.$$

Using the fact $\mu = \lambda \frac{w_1}{\eta_1}$ and the condition $\mu < \gamma_1 - \alpha$, to obtain

$$-\alpha - (p_1\mu + q_1\lambda)(1-\theta') + \gamma_1 - 1 > -1.$$

It yields then by the Lemma 5.2 that

$$(t+1)^{\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)} \|u(t,.)\|_{\eta_1} \leq C\left(\|u_0\|_{W^{1,\eta_1}} + \|u_0\|_{L^{r_1}} + \|u_1\|_{r_1}\right) + C\left[\sup_{s\in(0,t)} \varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)} \psi(s)\right]^{q_1}$$

$$+C(1+t)^{\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)} \int_0^t e^{-\frac{t-s}{2}}(t-s)(1+s)^{-\alpha-(p_1\mu+q_1\lambda)(1-\theta')} ds$$
$$\times\left[\sup_{s\in(0,t)} \varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)} \psi(s)\right]^{q_1}.$$

From (5.47), we obtain $\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right) - \alpha - (p_1\mu + q_1\lambda)(1-\theta') = \mu - \lambda\frac{w_1}{\eta_1} = 0$. Therefore, by virtue of the Lemma 5.2 again, we conclude

$$(t+1)^{\frac{N}{2}\left(\frac{1}{r_1}-\frac{1}{\eta_1}\right)} \|u(t,.)\|_{\eta_1} \leq C\left(\|u_0\|_{W^{1,\eta_1}} + \|u_0\|_{L^{r_1}} + \|u_1\|_{r_1}\right) + C\left[\sup_{s\in(0,t)} \varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)} \psi(s)\right]^{q_1}.$$

Now, we estimate $\|u(t,.)\|_{\infty}$. We have to distinguish two situations.

Either: $N \leq 2$ or ($\frac{p_1}{r_1} + \frac{q_1}{r_2} < \frac{4}{N}$ and $\frac{p_1}{\eta_1} + \frac{q_1}{w_1} < \frac{2}{N}$). From (5.35),(5.38) and (5.39), there exists w'' such that

$$\max\left\{\frac{p_1}{r_1} + \frac{q_1}{r_2} - \frac{2}{N}, \frac{p_1}{\eta_1} + \frac{q_1}{w_1}\right\} < \frac{p_1}{\eta''} + \frac{q_1}{w''} < \min\left\{1, \frac{2}{N}, \frac{p_1}{r_1} + \frac{q_1}{r_2}\right\}.$$

Note that

$$0 < \frac{N}{2}\left(\frac{p_1}{\eta''} + \frac{q_1}{w''}\right) < 1, 0 < \frac{p_1}{\eta''} + \frac{q_1}{w''} < 1 \text{ and } 0 < \frac{N}{2}\left(\frac{p_1}{r_1} + \frac{q_1}{r_2} - \left(\frac{p_1}{\eta''} + \frac{q_1}{w''}\right)\right) < 1.$$

Since $w'' \in (r_2, w_1)$, using interpolation inequality as in (5.43), we obtain

$$\begin{aligned}\left\|u\left(t,.\right)\right\|_{\infty} \leq\ & (t+1)^{-\frac{N}{2r_1}}\left(\left\|u_0\right\|_{r_1}+\left\|u_1\right\|_{r_1}\right)+(t+1)^{-\frac{N}{2r_1}}\left(\left\|u_0\right\|_{W^{1,\infty}}+\left\|u_1\right\|_{\infty}\right)\\ &+\int_0^t (t-s+1)^{-\frac{N}{2}\left(\frac{p_1}{\eta''}+\frac{q_1}{w''}\right)}\int_0^s (s-\tau)^{-\gamma_1}\left\|u(\tau,.)\right\|_{\eta''}^{p_1}\left\|v(\tau,.)\right\|_{w''}^{q_1}d\tau ds\\ &+\int_0^t e^{-\frac{t-s}{2}}(t-s)\int_0^s (s-\tau)^{-\gamma_1}\left\|u(\tau,.)\right\|_{\infty}^{p_1}\left\|v(\tau,.)\right\|_{\infty}^{q_1}d\tau ds.\end{aligned}$$

Making use (5.33), (5.34) and (5.43), we get

$$\begin{aligned}\left\|u\left(t,.\right)\right\|_{\infty} \leq\ & (t+1)^{-\frac{N}{2r_1}}\left(\left\|u_0\right\|_{r_1}+\left\|u_1\right\|_{r_1}+\left\|u_0\right\|_{W^{1,\infty}}+\left\|u_1\right\|_{\infty}\right)\\ &+\left[\sup_{s\in(0,t)}\varphi\left(s\right)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi\left(s\right)\right]^{q_1}\int_0^t (1+t-s)^{-\frac{N}{2}\left(\frac{p_1}{\eta''}+\frac{q_1}{w''}\right)}\\ &\times\int_0^s (s-\tau)^{-\gamma_1}(1+\tau)^{-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta''}+\frac{q_1}{w''}\right)\right)}d\tau ds\\ &+\left[\sup_{s\in(0,t)}\varphi\left(s\right)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi\left(s\right)\right]^{q_1}\int_0^t e^{-\frac{t-s}{2}}(t-s)\\ &\times\int_0^s (s-\tau)^{-\gamma_1}(1+\tau)^{-(\alpha p_1+\beta q_1)}d\tau ds,\end{aligned}$$

and as $(\alpha p_1+\beta q_1)<1$,

$$\alpha-\frac{N}{2r_1}=\frac{p_2-q_1-1}{\left(q_1q_2-(p_1-1)\left(p_2-1\right)\right)}<0,\quad \alpha+1-\gamma_1-\alpha p_1-\beta q_1=0,$$

it follows from $\frac{N}{2}\left[\frac{(p_1-1)}{r_1}+\frac{q_1}{r_2}\right]=2-\gamma_1$ that

$$\begin{aligned}&\alpha+2-\gamma_1-\tfrac{N}{2}\left(\tfrac{p_1}{\eta''}+\tfrac{q_1}{w''}\right)-\tfrac{N}{2}\left(\tfrac{p_1}{r_1}+\tfrac{q_1}{r_2}\right)+\tfrac{N}{2}\left(\tfrac{p_1}{\eta''}+\tfrac{q_1}{w''}\right)\\ &=\alpha+2-\gamma_1-\tfrac{Np_1}{2r_1}-\tfrac{Nq_1}{2r_2}<\tfrac{N}{2r_1}+2-\gamma_1-\tfrac{Np_1}{2r_1}-\tfrac{Nq_1}{2r_2}=0.\end{aligned}$$

Therefore, we conclude from Lemma 5.2 that

$$(t+1)^{\alpha}\left\|u\left(t,.\right)\right\|_{\infty}\leq C(\left\|u_0\right\|_{r_1}+\left\|u_1\right\|_{r_1}+\left\|u_0\right\|_{W^{1,\infty}}+\left\|u_1\right\|_{\infty})+C\left[\sup_{s\in(0,t)}\varphi\left(s\right)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi\left(s\right)\right]^{q_1}.$$

Or else $N=3$ and $(\frac{p_1}{r_1}+\frac{q_1}{r_2}\geq\frac{4}{N}$ or $\frac{p_1}{\eta_1}+\frac{q_1}{w_1}\geq\frac{2}{N})$. Then, from (5.35), (5.39) and (5.40), we can choose w'' such that

$$\max\left\{\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N},\frac{p_1}{\eta_1}+\frac{q_1}{w_1},\frac{2}{N}\right\}<\frac{p_1}{\eta''}+\frac{q_1}{w''}<\min\left\{1,\frac{p_1}{r_1}+\frac{q_1}{r_2}\right\}.$$

Since $w''\in(r_2,w_1)$, $\eta''(r_1,\eta_1)$ by interpolation inequality, we get

$$\begin{aligned}
\|u(t,.)\|_\infty &\leq C(\|u_0\|_{r_1}+\|u_1\|_{r_1}+\|u_0\|_{W^{1,\infty}}+\|u_1\|_\infty)\\
&+\int_0^t(1+t-s)^{-\frac{N}{2}\left(\frac{p_1}{\eta''}+\frac{q_1}{w''}\right)}\int_0^s(s-\tau)^{-\gamma_1}\|u(\tau,.)\|_{\eta''}^{p_1}\|v(\tau,.)\|_{w''}^{q_1}d\tau ds\\
&+\int_0^t e^{-\frac{t-s}{2}}(t-s)\int_0^s(s-\tau)^{-\gamma_1}\|u(\tau,.)\|_\infty^{p_1}\|v(\tau,.)\|_\infty^{q_1}d\tau ds.\\
\|u(t,.)\|_\infty &\leq C(\|u_0\|_{r_1}+\|u_1\|_{r_1}+\|u_0\|_{W^{1,\infty}}+\|u_1\|_\infty)\\
&+\left[\sup_{s\in(0,t)}\varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi(s)\right]^{q_1}\int_0^t(1+t-s)^{-\frac{N}{2}\left(\frac{p_1}{\eta''}+\frac{q_1}{w''}\right)}\\
&\times\int_0^s(s-\tau)^{-\gamma_1}(\tau+1)^{-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta''}+\frac{q_1}{w''}\right)\right)}d\tau ds\\
&+\left[\sup_{s\in(0,t)}\varphi(s)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi(s)\right]^{q_1}\int_0^t e^{-\frac{t-s}{2}}(t-s)\\
&\times\int_0^s(s-\tau)^{-\gamma_1}(1+\tau)^{-(\alpha p_1+\beta q_1)}d\tau ds.
\end{aligned}$$

Since $\alpha p_1+\beta q_1<1$, we observe that

$$\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}<\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}(\alpha p_1+q_1\beta).$$

It easy to see from (5.35), (5.39) and (5.40) that

$$\max\left\{\frac{p_1}{\eta_1}+\frac{q_1}{w_1},\frac{2}{N}\right\}<\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}(\alpha p_1+q_1\beta).$$

From these facts, we can assume that $\left(\frac{p_1}{\eta''}+\frac{q_1}{w''}\right)\leq\frac{p_1}{r_1}+\frac{q_1}{r_2}-\frac{2}{N}(\alpha p_1+q_1\beta)$. Thus
$\alpha+1-\gamma_1-\frac{N}{2}\left(\frac{p_1}{r_1}-\frac{p_1}{\eta''}\right)-\frac{N}{2}\left(\frac{q_1}{r_2}-\frac{q_1}{w''}\right)=\alpha p_1+q_1\beta-\frac{N}{2}\left(\frac{p_1}{r_1}+\frac{q_1}{r_2}-\left(\frac{p_1}{\eta''}+\frac{q_1}{w''}\right)\right)\leq 0,$

and therefore, we conclude as in the previous case

$$(t+1)^{\alpha}\left\|u\left(t\right)\right\|_{\infty}\leq C\left(\left\|u_0\right\|_{r_1}+\left\|u_0\right\|_{\infty}\right)+C\left[\sup_{s\in(0,t)}\varphi\left(s\right)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi\left(s\right)\right]^{q_1}.$$

A combination of the above estimates yields the inequality

$$\varphi\left(t\right)\leq C\left(\left\|u_0\right\|_{r_1}+\left\|u_0\right\|_{\infty}\right)+C\left[\sup_{s\in(0,t)}\varphi\left(s\right)\right]^{p_1}\left[\sup_{s\in(0,t)}\psi\left(s\right)\right]^{q_1}.$$

By analogous computations, we get $\psi\left(t\right)\leq C\left(\left\|v_0\right\|_{r_1}+\left\|v_0\right\|_{\infty}\right)+C\left[\sup_{s\in(0,t)}\psi\left(s\right)\right]^{p_2}\left[\sup_{s\in(0,t)}\varphi\left(s\right)\right]^{q_2}$.
Denoting $f(t)=\sup_{s\in(0,t)}\varphi\left(t\right)$ and $g(t)=\sup_{s\in(0,t)}\psi\left(t\right)$, we get

$$\begin{aligned} f(t) &\leq C\left(\|u_0\|_{W^{1,r_1}\cap W^{1,\infty}}+\|u_1\|_{L^{r_1}\cap L^{\infty}}\right)+Cf(t)^{p_1}g(t)^{q_1}, \\ g(t) &\leq C\left(\|v_0\|_{W^{1,r_2}\cap W^{1,\infty}}+\|v_1\|_{L^{r_2}\cap L^{\infty}}\right)+Cg(t)^{p_2}f(t)^{q_2}. \end{aligned} \tag{5.50}$$

We now define $h\left(t\right):=f\left(t\right)+g\left(t\right)$. Taking into account (5.50) reads

$$\begin{aligned} h\left(t\right) &\leq C\left(A+h^{p_1+q_1}\left(t\right)+h^{p_2+q_2}\left(t\right)\right), \\ &\leq C\left(\varepsilon+h^{p_1+q_1}\left(t\right)+h^{p_2+q_2}\left(t\right)\right), \forall t\in\left[0,T_{\max}\right), \end{aligned}$$

where C is positive constant independent of t and $A=\|u_0\|_{W^{1,r_1}\cap W^{1,\infty}}+\|u_1\|_{L^{r_1}\cap L^{\infty}}+\|v_0\|_{W^{1,r_2}\cap W^{1,\infty}}+\|v_1\|_{L^{r_2}\cap L^{\infty}}$. By using this estimate and standard arguments as in [57], it then follows that $h(t)\leq C\varepsilon$, $\forall\ t\in[0,T_{\max})$. Hence $f(t)\leq C\varepsilon$, $g(t)\leq C\varepsilon$, $\forall t\in[0,T_{\max})$. Now, we show the global existence result in the energy space. We define $H(t)=\|u(t)\|_{L^1}+\|v(t)\|_{L^1}$ for all $t\in[0,T_{\max})$. Making use of Lemma 5.4 and Lemma 5.5, it is deduced from (5.20) that, for all $t\in[0,T_{\max})$,

$$\begin{cases} \|u(t)\|_{L^1}\leq C\left(\|u_0\|_{W^{1,1}}+\|u_1\|_{L^1}\right)+\displaystyle\int_0^t\int_0^s(s-\tau)^{-\gamma_1}\|u(\tau)\|_{\infty}^{p_1}\|v(\tau)\|_{q_1}^{q_1}\,d\tau ds, \\ \|v(t)\|_{L^1}\leq C\left(\|v_0\|_{W^{1,1}}+\|v_1\|_{L^1}\right)+\displaystyle\int_0^t\int_0^s(s-\tau)^{-\gamma_2}\|v(\tau)\|_{\infty}^{p_2}\|u(\tau)\|_{q_2}^{q_2}\,d\tau ds. \end{cases} \tag{5.51}$$

Adding the two inequalities of (5.51) and using the L^∞-estimates of u and v, we have that

$$\begin{aligned} H(t) &\leq C\left(\|u_0\|_{W^{1,1}} + \|u_1\|_{L^1} + \|v_0\|_{W^{1,1}} + \|v_1\|_{L^1}\right) \\ &+C\int_0^t\int_0^s (s-\tau)^{-\gamma_1}(1+\tau)^{-\alpha p_1-\beta(q_1-1)}\|v(\tau)\|_1 d\tau ds \\ &+C\int_0^t\int_0^s (s-\tau)^{-\gamma_2}(1+\tau)^{-\beta p_2-\alpha(q_2-1)}\|u(\tau)\|_1 d\tau dds \\ &\leq C\varepsilon + C\int_0^t\int_0^s (s-\tau)^{-\gamma_1}(1+\tau)^{-\alpha p_1-\beta(q_1-1)}H(\tau)d\tau ds \\ &+C\int_0^t\int_0^s (s-\tau)^{-\gamma_2}(1+\tau)^{-\beta p_2-\alpha(q_2-1)}H(\tau)d\tau ds \\ &\leq C\varepsilon + C\int_0^t\left((1+s)^{\beta-\alpha}+(1+s)^{\alpha-\beta}\right)\sup_{0\leq\tau\leq s}H(\tau)ds. \end{aligned} \tag{5.52}$$

From (5.52), and Gronwall's inequality, we get that $H(t) \leq C\varepsilon \exp C\left(t^{1+\beta-\alpha}+t^{1+\alpha-\beta}\right)$ for all $t\in[0,T_{\max})$. Therefore, we have that $(u,v)\in\left\{C([0,T_{\max});L^1(\mathbb{R}^N)\right\}^2$, $N=1,2,3,$ so for any $r\geq 1$, the solution (u,v) satisfies

$$\begin{aligned} \|u(t)\|_{L^r} &\leq C^{\frac{1}{r}}\varepsilon^{\frac{1}{r}}e^{C\left(t^{1+\beta-\alpha}+t^{1+\alpha-\beta}\right)/r}(1+t)^{-(r-1)\alpha/r}, \\ \|v(t)\|_{L^r} &\leq C^{\frac{1}{r}}\varepsilon^{\frac{1}{r}}e^{C\left(t^{1+\beta-\alpha}+t^{1+\alpha-\beta}\right)/r}(1+t)^{-(r-1)\beta/r}, \end{aligned} \tag{5.53}$$

for all $t\in[0,T_{\max})$.

Now, let $D=(\partial_t,\nabla_x)$,

$$u_L(t) = K_1(t)(\frac{1}{2}u_0+u_1)+K_0(t)u_0,\ v_L(t) = K_1(t)(\frac{1}{2}v_0+v_1)+K_0(t)v_0.$$

From Lemma 5.3 with $m=1$, we see that

$$\begin{aligned} \|Du_L(t)\| &\leq C(1+t)^{-N/2-1/2}I_{0,u},\ \|Dv_L(t)\|\leq C(1+t)^{-N/2-1/2}I_{0,v}, \\ \left\|DK_1(t-s)*J_{0|s}^{1-\gamma_1}(|v(s)|^p)\right\| &\leq C(1+t-s)^{-N/2-1/2}\left[J_{0|s}^{1-\gamma_1}(\||u(s)|^{p_1}|v(s)|^{q_1}\|_2)\right. \\ &\left.+J_{0|s}^{1-\gamma_1}(\||u(s)|^{p_1}|v(s)|^{q_1}\|_1)\right], \end{aligned} \tag{5.54}$$

$$\begin{aligned}\left\|DK_1(t-s)*J_{0|s}^{1-\gamma_2}\left(|u(s)|^q\right)\right\| \;\leq\; & C\left(1+t-s\right)^{-N/2-1/2}\left[J_{0|s}^{1-\gamma_2}\left(\left\||v(s)|^{p_2}\left|u(s)\right|^{q_2}\right\|_2\right)\right.\\ & \left.+J_{0|s}^{1-\gamma_2}\left(\left\||v(s)|^{p_2}\left|u(s)\right|^{q_2}\right\|_1\right)\right],\end{aligned}$$

where

$$\begin{cases} I_{0,u} := \|u_0\|_1 + \|u_0\|_{H^1} + \|u_1\|_1 + \|u_1\|_{H^1}, \\ I_{0,v} := \|v_0\|_1 + \|v_0\|_{H^1} + \|v_1\|_1 + \|v_1\|_{H^1}. \end{cases}$$

Then by use of (5.53) and (5.54) and choosing $\varepsilon \ll 1$, we deduce from (5.17) that

$$\begin{aligned}&\|Du\left(t\right)\| \leq \|Du_L\left(t\right)\| + \int_0^t \left\|DK_1(t-s)*J_{0|s}^{1-\gamma_1}\left(|v(s)|^p\right)\right\| ds\\ &\leq C\left(1+t\right)^{-N/2-1/2} I_{0,u} + C\int_0^t \left(1+t-s\right)^{-N/2-1/2}\left[J_{0|s}^{1-\gamma_1}\left(\left\||u(s)|^{p_1}\left|v(s)\right|^{q_1}\right\|_2\right)\right.\\ &\left.+J_{0|s}^{1-\gamma_1}\left(\left\||u(s)|^{p_1}\left|v(s)\right|^{q_1}\right\|_1\right)\right] ds\\ &\leq C\varepsilon + C\int_0^t \left(1+t-s\right)^{-N/2-1/2}\int_0^s \left(s-\tau\right)^{-\gamma_1}\|u\|_\infty^{p_1}\left(\|v(\tau)\|_{2q_1}^{q_1}+\|v(\tau)\|_{q_1}^{q_1}\right) d\tau ds\\ &\leq C\varepsilon + C\int_0^t \left(1+t-s\right)^{-N/2-1/2}\\ &\times\int_0^s \left(s-\tau\right)^{-\gamma_1}\left(\varepsilon^{\frac{1}{2}}e^{C\left(\tau^{1+\beta-\alpha}+\tau^{1+\alpha-\beta}\right)/2}\left(1+\tau\right)^{-(2p_1-1)\beta/2}+\varepsilon e^{C\left(\tau^{1+\beta-\alpha}+\tau^{1+\alpha-\beta}\right)}\left(1+\tau\right)^{-(p_1-1)\beta}\right) d\tau ds\\ &\leq C\left(1+e^{C\left(t^{1+\beta-\alpha}+t^{1+\alpha-\beta}\right)}\left(1+t\right)^{\beta}\right)\sqrt{\varepsilon},\ \forall t\in\left[0,T_{\max}\right),\end{aligned}$$

where C is positive constant independent of t. Similarly

$$\|Dv\left(t\right)\| \leq C\left(1+e^{C\left(t^{1+\beta-\alpha}+t^{1+\alpha-\beta}\right)}\left(1+t\right)^{\alpha}\right)\sqrt{\varepsilon},\ \forall t\in\left[0,T_{\max}\right),$$

which completes the proof of the Theorem.

Conclusion

As conclusion of this work, we note that the qualitative study of evolution problems such as the study of existence and uniqueness of local / global for heat and damped wave system with nonlinear source terms is interesting because it is a generalization of ordinary problem equation, that we always have the asymptotic behavior of solution for damped wave system is similar to the corresponding heat system in the infinity. We should also note the importance of studying the blow up problem in finite time because they are closest to reality because it is considered to be a model to many problems in many areas, for example in chemistry.

Many questions remain unresolved and deserve closer consideration, including

- the blow up case of the semilinear heat system and the corresponding system of damped wave system with nonlinear memory.
- The aim is to seek the necessary conditions between the parameters of the systems $p_1, p_2, q_1, q_2, \gamma_1, \gamma_2$,and the dimension of energy space N without the need for a scaling argument and without using the Fujita critical exponent, i-e the objective is to establish the blow up of weak solution by using the test method function.

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