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Option: **Probabilités et Statistiques**

Par

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Titre

Contrôle optimal des systèmes stochastiques partiellement observables

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By

Saliha Bougherara

Title

Optimal Control for Partially Observed Stochastic Systems

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2020.

Dedication

A special thanks to my family :

My parents,

My husband

My brothers and sisters,

for trusting and supporting me spiritually throughout writing this thesis.

I dedicate this work to my children : **Amine, Meriem** and **Mohamed.**

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Abstrac

The aim of this thesis is to study a stochastic partially observed optimal control problem, for systems of forward backward stochastic differential equations driven by both a family of Teugels martingales and an independent Brownian motion. By using Girsavov's theorem and a standard spike variational technique, we prove necessary conditions to characterize an optimal control under a partial observation, where the control domain is supposed to be convex.

Moreover, under some additional convexity conditions, we prove that these partially observed necessary conditions are sufficient. In fact, compared to the existing methods, we get the last achievement in two different cases according to the linearity or the nonlinearity of the terminal condition for the backward component. As an illustration of the general theory, an application to linear quadratic control problems is also investigated.

Noting that this kind of control problems have a powerful tool in the real world of applications. In such problems there is noise in the observation system and the controller is only able to observe partially the state via other variables. For example in financial models, one may observe the asset price but not completely its rate of return and/or its volatility, and the portfolio investment in this case is based only on the asset price information. This means that the controller is facing a partial observation control problem.
Keys words. Lévy process, stochastic maximum principle, partial information, partially observed, forward-backward stochastic systems, Teugels martingales.

Résumé

Les problèmes de contrôle stochastique des systèmes partiellement observables jouent un rôle important dans de nombreuses applications. Par exemple, dans les modèles financiers, on peut observer le prix de l'actif, mais pas complètement, son taux de rendement et / ou sa volatilité, et l'investissement de portefeuille est basé uniquement sur l'information sur le prix de l'actif.

En utilisant le théorème de Girsavov, nous prouvons les conditions nécessaires aux problèmes de contrôle stochastique partiellement observés pour les équations différentielles stochastiques dirigées par une famille de martingales de Teugels et un mouvement Brownien indépendant où le domaine de contrôle est convexe.

De plus, nous prouvons que ces conditions nécessaires obtenues sous l'observation partielle sont suffisantes sous certaines conditions supplémentaires de convexité. Comparé aux méthodes existantes, nous obtenons ces resultats dans deux cas différents en fonction de la linéarité ou de la non-linéarité de la condition terminale de l'EDSR. Comme illustration de la théorie générale, une application aux problèmes de contrôle linéaire quadratique est également étudiée.

Mots cles. Processus de Lévy, principe du maximum stochastique, information partielle, observation partielle, systèmes stochastiques progressives rétrogrades, Martingales de Teugels.

Symbols and Abbreviations

The following notation is frequently used in this thesis

a, e: almost everywhere.

a, s: almost surely.

$\stackrel{d}{=}$. is equality in distribution.

càdlàg: right continuous with left limits.

$e.g$: for example.

$\mathbb{R}$: real numbers.

$\mathbb{R}^n$: n-dimmensional real Euclidean space.

$\mathbb{R}^{n\times m}$: the set of all $(n \times m)$ real matrices.

$\bar{A}$: the closure of the set A.

1_A : the indicator function of the set A.

$(\Omega, \mathcal{F}, \mathbb{P})$: probability space.

$\{\mathcal{F}_t\}_{t\geq 0}$: filtration.

$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$: filtered probability space.

$\mathcal{N}$: the totality of the $\mathbb{P}$-negligible sets.

$\mathcal{G}_1 \vee \mathcal{G}_2$: the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$.

$\mathbb{E}[x]$: the expectation of the random variable x.

$\mathbb{E}[x \mid \mathcal{G}]$: conditional expectation.

$W = (W)_{t\in[0,T]}$: Brownian motion.

$L = (L)_{t\in[0,T]}$: Lévy process.

$H^{(i)}$: Teugels martingale.

l^2 : the Hilbert space of real-valued sequences $x = (x_n)_{n\geq 0}$ with norm

$$\|x\| = \left(\sum_{i=1}^{\infty} x_i\right)^{\frac{1}{2}} < \infty.$$

(a, b) : the inner product in $\mathbb{R}^n$, $\forall a, b \in \mathbb{R}^n$.

$|a| = \sqrt{(a, a)}$: the norm of $\mathbb{R}^n$, $\forall x \in \mathbb{R}^n$.

(A, B) : the inner product in $\mathbb{R}^{n\times d}$, $\forall x, y \in \mathbb{R}^{n\times d}$.

$|A| = \sqrt{(A, A)}$: the norm of $\mathbb{R}^{n\times d}$, $\forall A \in \mathbb{R}^{n\times d}$.

$l^2(\mathbb{R}^m)$: the space of $\mathbb{R}^m$-valued $\{f^i\}_{i\geq 0}$ such that

$$\left(\sum_{i=1}^{\infty} \left\|f^i\right\|_{\mathbb{R}^m}^2\right)^{\frac{1}{2}} < \infty.$$

$l^2_{\mathcal{F}}(0, T, \mathbb{R}^m)$: the Banach space of $l^2(\mathbb{R}^m)$-valued $\mathcal{F}_t$-predictable process such that

$$\left(\mathbb{E}\int_0^T \sum_{i=1}^{\infty} \left\|f^i(t)\right\|_{\mathbb{R}^m}^2\right)^{\frac{1}{2}} < \infty.$$

$\mathcal{L}^2_{\mathcal{F}}(0, T, \mathbb{R}^m)$: the Banach space of $\mathbb{R}^m$-valued $\mathcal{F}_t$-adapted process such that

$$\left(\mathbb{E}\int_0^T |f(t)|_{\mathbb{R}^m}^2\right)^{\frac{1}{2}} < \infty.$$

$\mathcal{S}^2_{\mathcal{F}}(0, T, \mathbb{R}^m)$: the Banach space of $\mathbb{R}^m$-valued $\mathcal{F}_t$-adapted and càdlàg process such that

$$\left(\mathbb{E}\sup_{0\leq t\leq T} |f(t)|^2\right)^{\frac{1}{2}} < \infty.$$

$L^2(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}^m)$: the Banach space of $\mathbb{R}^m$-valued, square integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

$SDEs$: Stochastic differential equations.

$BSDEs$: Backward stochastic differential equations.

$FBSDEs$: Forward-backward stochastic differential equations.

Contents

General introduction

The aim of stochastic optimal control theory is to handle a problem of finding a control variable for a given stochastic system in order to achieve certain optimality criterions. From mathematical point of view, based on the calculus of variations theory, it can be regarded as an optimization method for deriving control policies. We note that this theory have been widely developed due to the deterministic work of Lev Pontryagin and Richard Bellman in the 1950s. After that, the overwhelming majority of works have been extended to the stochastic cases. This theory has a potential tool in the real world of applications. Namely, one can mention several fields such as economic, biomedical, physical and electrical and aerospace engineering, can often be influenced by certain parameters or controls in order to optimize some properties or required results.

For a given a complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, one can describe the stochastic optimal control theory as the study of strategies, which in optimal sense, influence a stochastic differential system $X(\cdot)$ described by dynamics evolving over time. These strategies also called admissible controls are modeled by vectors of parameters, u, which takes its values in some set U (which supposed to be convex or non-convex). The object of optimal control problem is to seek some optimal controls in order to minimize a given cost functional (or maximize a reward functional) among the set of all admissible controls. The infimum of the cost functional is called the value function (as a function of the initial time and the state). This minimization problem is infinite dimensional, since we are minimizing a functional over the space of functions u_t, $t \in [0, T]$.

In sum, optimal control problems can be handled in two different approaches

The first is the Bellman dynamic programming principle, this method consists to find a solution of a non linear stochastic partial differential equation (SPDE for short), verified by the value function. It is called Hamilton-Jacobi-Bellman (HJB) equation. It is well known that the HJB equation does not necessarily admit smooth solution in general, we can give a meaning to this PDE with a concept of weak solution called viscosity solution.

The second is the maximum principle. This method which will be the center of our interest in this work, which consists to find an admissible control u^* that minimizes a cost functional subject to a stochastic differential equation on a finite time horizon. If u^* is an optimal control, the goal of the stochastic maximum principle is to derive a set of necessary and sufficient conditions that must be satisfied by this control. The first version of the stochastic maximum principle was extensively established in the 1970's by Bismut [5], Kushner [21], Bensoussan [3] and Haussmann [17].

In this Ph.D. thesis we deal with a partially observed as well as a partial information optimal control problem for stochastic systems with jumps. Introducing a functional cost which depends on the state and on the control variable, we are interested in minimizing its expected value over the set of all admissible controls which will be determined later. Let us emphasize the difference between partial information and partial observation models. Roughly speaking, the partial information problems discussed the case where the filtration describes the information flow is a sub-filtration of the complete information. While, the partial observation model describe the case where the information available to the controller at time t constitutes a noisy observation of the state. In such cases, the filtering theory can sometimes be used to transform the partial observation problem into a related problem with complete information. Therefore, one can say the partial information is rather general than that of the partial observation.

In the present work we focus on optimal control problems of stochastic differential equations driven by a family of Teugels martingales associated to some Lévy processes and an independent Brownian motion. Let us recall that this kind of martingales is introduced at first in Annular et al. [29]. Under the assumption that the Lévy measure has a finite exponential moments outside the origin, they construct a family consisting of countably many orthogonal square integrable martingales adapted to the filtration generated by a Lévy process. Then the system of iterated integrals generated by the orthogonalized Teugels martingales is introduced. Moreover they proved a very useful representation theorem. This theorem, which has been generalized by Bahlali et al. [4], states that every square integrable martingale adapted to the natural filtration of a Brownian motion and an independent Lévy process, can be written as a stochastic integral with respect to the Brownian motion and the sum of stochastic integrals with respect to the Teugels martingales associated to the Lévy process. In other words, this representation formula put Brownian motion and Lévy processes in a unified theory of square integrable martingales. See the excellent account by Davis [13] for further information in this subject.

It is worth mentioning that, the first fundamental result on stochastic optimal control theory of classical SDEs was obtained by Kushner [21], for classical regular or absolutely continuous controls. Since then, a huge literature has been produced on this subject. Optimal control problems with complete information, for a classical stochastic differential equation driven by a Brownian motion, have been studied extensively in the literature, see [2, 11, 17, 33]. One can refer to Yong et al. [45] or Fleming et al. [15] for a complete account on this subject and a complete list of references. Control problems for jump diffusions have been treated in [10, 16, 30, 40] and the case of partial information problems in [3, 8, 16, 20, 39].

In the past few years, the theory of optimal control for systems driven by Teugels martingales has developed rapidly. The first result in this direction which is devoted to the problem of stochastic optimal control for SDEs driven by Teugels martingales and an independent Brownian motion, has been proved by Q. X. Meng et al. [26]. In that paper the necessary and sufficient conditions for an optimal control in the case where the control domain is convex.

Starting from Nualart et al. [28] a new class of BSDEs driven by a family of Teugels martingales associated to some Lévy process has been introduced. They established the existence of a unique solutions to such equations, under the Lipschitz conditions. The results are important from a pure mathematical point of view and also in the finances world. Subsequently, K. Bahlali et al. [4] are studied the existence and uniqueness of solutions for BSDE driven by Teugels martingales and an independent Brownian motion in both globally and locally Lipschitz framework.

In 2010, optimal control of BSDEs driven by Teugels martingales has been addressed in Tang et al. [41], where necessary and sufficient conditions have been established. Motivated by all the aforementioned results, it is quite natural to extend the study of stochastic control theory to forward-backward stochastic differential equations, these equations consist of a forward stochastic differential equation of Ito's type and a backward stochastic differential equation, which makes the foundation of FBSDEs.

It is worth noting that in all previous control problems, the information available to the controllers is assumed to be completely observed. In fact, this is not always reasonable in the real world of applications because the controllers can only get partial information at most cases. This makes a motivation to study this kind of control problems. As so as we are aware, the first result treats a partial information optimal control problem for SDEs driven by both Teugels martingales and an independent Brownian motion is due Bahlali et al. [6]. Then, Hafayed et al. [18] extend the previous result to the mean-field-type partial information stochastic optimal control problem. Subsequently, Hafayed et al. [19] studied the partial information optimal control of mean-field forward-backward stochastic systems, driven by orthogonal Teugels martingales and an independent Brownian motion. Thereafter, Bougherara et al. [9] established a stochastic partially observed optimal control problem, for systems of forward backward stochastic differential equations which are driven by both a family of Teugels martingales and an independent Brownian motion. By using Girsavov's theorem and a standard spike variational technique, the authors proved a necessary conditions to characterize an optimal control under a partial observation, where the control domain is supposed to be convex. Moreover, under some additional convexity conditions, they proved that these partially observed necessary conditions are sufficient. Compared to the existing methods, they investigated the sufficient conditions in two different cases according to the linearity or the nonlinearity of the terminal condition for the backward component.

This work is structured as follows :

- Chapter 1, contains some preliminaries on Lévy processes and a brief introduction to Teugels martingales. Moreover, it study the existence and uniqueness of a solution to BSDEs and SDEs driven by a Lévy processes.

- Chapter 2, is devoted to necessary and sufficient conditions of optimality, under partial information for a system driven by both Teugels martingales and independent Brownian motion. The method used to prove the main result is based on some special perturbation of the optimal control and is inspired from the papers [8, 16].

- The main contribution of chapter 3 is to investigate a partially observed necessary as well as sufficient conditions of optimality for FBSDEs driven by Teugels martingales and an independent Brownian motion. To obtain the optimality necessary conditions, we use a convex perturbation method and differentiate the perturbed both the state equations and the cost functional, in order to get the adjoint process, which is a solution of a forward-backward SDE, driven by both a Brownian motion and a family of Teugels martingales, on top of the variational inequality between the

Hamiltonians. Moreover, an additional technical assumptions are required to prove that these partially observed necessary conditions are in fact sufficient.

Introduction générale

Le but de la théorie du contrôle optimal stochastique est de chercher une variable de contrôle vérifiant certains critères d'optimalité pour un système stochastique donné. Mathématiquement on peut le considéré comme une méthode d'optimisation. Cette méthode a été largement étudiée dans le cas déterministe par Lev Pontryagin et Richard Bellman dans les années 1950. Après cela, la plupart des travaux ont été généralisé au cas stochastique. Cette théorie est un outil important dans l'application, on peut par exemple mentionner plusieurs domaines tels que l'ingénierie, économique, biomédicale, physique, électrique et l'aérospatiale qui peuvent être influencés par certains paramètres afin d'optimiser quelques propriétés.

Etant donné un espace de probabilité filtré complet $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, la théorie de contrôle optimal stochastique a pour but d'étudier l'influence d'une variable de contrôle sur un système différentielle stochastique évoluant au cours de temps. Ces problèmes d'optimisations, sont modélisés par une variable de contrôle u qui prend ses valeurs dans un certains ensemble U (convexe ou non convexe selon le cas d'étude). L'objectif du contrôleur est de chercher des contrôles optimaux qui minimise une fonction de cout donnée (ou qui maximise une fonction de récompense) parmi l'ensemble de tous les contrôles admissibles. Ce problème d'optimisation est en dimension infinie, puisque nous minimisons une fonctionnelle sur l'espace des fonctions u_t, $t \in [0, T]$.

Généralement, les problèmes de contrôles optimaux peuvent être traités de deux manières différentes:

La première est le principe de la programmation dynamique de Bellman, cette méthode consiste à trouver une solution d'une équation différentielle aux dérivées partielles stochastique non linéaire (EDPS en abrégé). C'est l'équation de Hamilton-Jacobi-Bellman (HJB). Il est bien connu que l'équation de HJB n'admet pas nécessairement une solution régulière en général. Nous pouvons donner un sens à cette EDP avec un type de solution faible appelée solution de viscosité.

La seconde est le principe du maximum. Cette méthode, qui fera l'objet de ce travail consiste à trouver un contrôle admissible u^* qui minimise une fonction de coût dépendant de la solution d'un système différentiel stochastique contrôlé donné. Si u^* est un contrôle optimal, le principe de maximum stochastique a pour objectif de trouver l'ensemble des conditions nécessaires et suffisantes qui doit être satisfaites par ce contrôle. Rappelons que la première version du principe de maximum stochastique a été largement établie dans les années 1970 par Bismut [5], Kushner [21], Bensoussan [3] et Haussmann [17].

Dans cette thèse de doctorat, on commence par traiter un problème de contrôle optimal sous l'information partielle ensuit sous l'observation partielle pour les systèmes stochastiques présentant des sauts. En introduisant un coût fonctionnel qui dépend de l'état et de la variable de contrôle, nous souhaitons minimiser sa valeur par rapport à l'ensemble des contrôles admissibles qui sera déterminés ultérieurement. Dans la suit nous donnons la différence entre les modèles étudiés sous l'information partielle et l'observation partielle. Les problèmes sous l'information partielle décrivent le cas où la filtration qui présente le flux d'informations est une sous-filtration de l'information complète. Tandis que, le modèle sous l'observation partielle décrit le cas où l'information disponible au contrôleur à l'instant t constitue une observation de bruit dans l'état. Dans ce cas, la théorie de filtrage peut parfois être utilisée pour transformer le problème d'observation partielle en un problème sous l'information complète. Par conséquent, on peut dire que l'information partielle est plus générale que celle de l'observation partielle.

Dans notre travail, on s'intéresse aux problèmes de contrôle optimal pour les équations différentielles stochastiques dirigées à la fois par une famille de martingales de Teugels associe à un processus de Lévy et un Mouvement Brownien indépendant. Ce type de martingale est introduit premièrement dans Nualart et al. [29]. Sous l'hypothèse que la mesure de Lévy a un moment exponentiel en dehors de l'origine. Ils ont construit une famille dénombrable de martingales de carrées intégrables, orthogonal est adaptées à la filtration engendrée par le processus de Lévy. Ils ont introduit la théorie d'intégration contre les martingales orthogonales de Teugels. Ils ont aussi prouvés un théorème de représentation très important. Ce théorème, qui a été généralisé par Bahlali et al. **[4]**, montre que chaque martingale de carré intégrale adaptée à la filtration naturelle du mouvement Brownien et le processus de Lévy indépendant, peut être s'écrite comme la somme d'une intégrale stochastique par rapport à un mouvement Brownien et une somme d'intégrales stochastiques par rapport à une martingale de Teugels associées au processus de Lévy. Autrement dit, cette formule de représentation met le mouvement Brownien et le processus de Lévy dans une seule théorie de martingales de carrées intégrables. Pour plus de détail voir l'excellent document de Davis [13].

Le premier résultat concernant la théorie du contrôle optimal pour les EDS classiques est donné par Kushner [21]. Après cela, énormement articles dans la littérature ont étaient faites sur ce sujet. Les problèmes de contrôle optimal sous l'information complète ont étaient largement étudiés dans la littérature, voir par example [2, 11, 17, 33] pour plus d'information sur ce sujet. Les problèmes de contrôle pour des sytèmes présentant des sauts ont étaient traités dans [10, 16, 30, 40] et les problèmes sous l'information partielle ont étaient étudiés dans [3, 8, 16, 20, 39].

Dans les dernières années, la théorie du contrôle optimal pour des systèmes dirigées par les martingales de Teugels a été développé très rappidement. Le premier résultat dans ce sujet qui est le problème de contrôle optimale stochastique dirigée par une famille de martingales de Teugels et un mouvement Brownien indépendant est due à Q. X. Meng et al. [26]. Dans cet article, les conditions nécessaires et suffisantes d'optimalité sont obtenus dans le cas où le domaine de contrôle est convexe. A partir de Nualart et al. [28], une nouvelle classe des EDSR dirigées par une famille de martingales de Teugels a été introduite. Ils ont établi dans le cadre Lipschitzienne l'existence d'une solution unique pour ce type d'équations. Les résultats sont importants d'un point de vue purement mathématique et également dans le monde de finance. Ensuite, Bahlali et al. [4] étudient l'existence et l'unicité de solutions pour les EDSR globalement ou lecalement Lipschitiziènne dirigées par une famille de martingales de Teugels et un mouvement Brownien indépendant.

En 2010, le problème de contrôle optimal pour les EDSR dirigées par les martingales de Teugels a été étudié par Tang et al. [41], où les conditions nécessaires et suffisantes ont été établies. Motivé par tous les résultats mentionnés auparavant, il est tout à fait naturel d'étendre l'étude de la théorie du contrôle stochastique aux équations différentielles stochastiques progressives rétrogrades. Ces équations consistent en une équation différentielle stochastique progressive et une équation différentielle stochastique rétrograde.

Notons que dans tous les problèmes de contrôle précédents, les informations disponibles pour les contrôleurs sont supposées complètement observées. En réalité, cela n'est pas toujours raisonnable dans le monde réel des applications car les contrôleurs ne peuvent obtenir qu'une information partielle dans la plupart des cas. Cela donne une motivation pour étudier ce problème de contrôle. Le premier résultat qui traite le problème de contrôle optimal sous informations partielles pour les EDS dirigées par les martingales de Teugels et un mouvement Brownien indépendant est dû à Bahlali et al. [6]. Ensuite, Hafayed et al. [18] étendent le résultat précédent au problème de contrôle optimal stochastique sous l'informations partielles de type champ moyen. Par suite, Hafayed et al. [19] ont étudié le contrôle optimal sous l'information partielle pour des systèmes stochastiques à champ moyen, dirigés par des martingales de Teugels et un mouvement Brownien indépendant. Ensuite, Bougherara et al. [9] ont établi un problème de contrôle optimal partiellement observé, pour des systèmes d'équations différentielles stochastiques progressives rétrogrades qui sont dirigés à la fois par une famille de martingales de Teugels et un mouvement Brownien indépendant. En utilisant le théorème de Girsanov et une technique classique de calcul variationnel, les auteurs ont prouvé les conditions nécessaires d'optimalité sous l'observation partielle, où le domaine de contrôle est supposé convexe. De plus, sous certaines conditions supplémentaires de convexité, ils ont prouvé que ces conditions nécessaires partiellement observées sont suffisantes. Par rapport aux méthodes existantes, ils ont étudié les conditions suffisantes dans deux cas différents en fonction de la linéarité ou de la non linéarité de la condition terminale de l'EDSR.

Ce travail se décompose en trois chapitres :

Le chapitre 1 contient quelques notions préliminaires sur les processus de Lévy et une brève introduction aux ma

rtingales de Teugels. De plus, il étudie l'existence et l'unicité d'une solution pour les EDSR et les EDS dérigées par ceratin processus de Lévy.

Le chapitre 2 est consacré aux conditions nécessaires et suffisantes d'optimalité sous l'information partielle pour un système dirigé à la fois par une famille de martingales de Teugels et un mouvement Brownien indépendant. La méthode utilisée pour prouver les résultats principaux est inspirée des articles [8, 16], et basée sur une perturbation spéciale du contrôle optimal.

Le chapitre 3 étudie le problème de contrôle optimal pour les équations différentielles stochastiques dirigées par une famille de martingale de Teugels et un movement Brownien indépendant. Les contributions principaux de ce chapitre sont les conditions nécessaires et suffisantes partiellement observées, satisfaites par un contrôle optimal. Pour obtenir les conditions nécessaires d'optimalité, nous utilisons une méthode de perturbation convexe afin d'obtenir un processus adjoint et une inégalité variationnelle entre les Hamiltoniens. De plus, sous certains hypothèses supplémentaires, nous prouvons que ces conditions nécessaires partiellement observées sont en fait suffisantes.

Chapter 1

Lévy processes and Teugels martingales

Lévy processes are a class of stochastic processes with discontinuous paths, named after the French mathematician Paul Lévy, which include the Poisson process and Brownian motion as special cases. Lévy processes play a crucial role in several fields of science, such as a mathematical finance because they can describe the observed reality of financial markets in a more accurate way than models based on Brownian motion. They still provide prototypical examples such as semimartingales, Markov processes, and Teugels martingales. More precisely, these special kind of martingale are obtained as an orthogonalization of a compensated power jump processes associated to some Lévy processes.

This first chapter contains three sections. The first one, provides some definitions and properties of Lévy processes. The second, speaks of stochastic differential equations driven by Teugels martingales. The last one, is about Forward and Backward stochastic differential equations driven by Teugels martingales, this result was established by K.Bahlali et al. [4]. This chapter is motivated by ([12], [22], [35], and [38]).

1.1 Definitions and Properties of Lévy process

We assume as given a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$.

Definition 1.1.1 *An adapted process $L = \{L_t : t \geq 0\}$ on $\mathbb{R}^n$ is a Lévy process if it satisfies the following properties*

1. $L_0 = 0$ *a.s;*

2. *Independence of increments : For any $0 \leq t_1 \leq t_2 \leq ... \leq t_n < \infty$,*

$$L_{t_2} - L_{t_1},\ L_{t_3} - L_{t_2},\ L_{t_4} - L_{t_3}, ...,\ L_{t_n} - L_{t_{n-1}}$$

are independent;

3. *Stationary increments : For any $s < t$, $L_t - L_s$, is equal in distribution to L_{t-s}.*

4. *L_t is continuous in probability; that is, $\lim_{t\to s} L_t = L_s$, where the limit is taken in probability : i.e. for all $a > 0$ and for all $s \geq 0$:*

$$\lim_{t\to s} \mathbb{P}\left(|L_t - L_s| > a\right) = 0.$$

Remark 1.1.1 $L_0 = 0$ *a.s i.e* $\mathbb{P}\left(L_0 = 0\right) = 1$.

Remark 1.1.2 *If L is a Lévy process then one may construct a version of $t \mapsto L_t$ such that is almost surely right continuous with left limits.*

Remark 1.1.3 *According to the properties of stationary and independent increments, the Lévy process is a Markov process.*

Example 1.1.1 **1/** *The simplest Lévy process is the linear drift, a deterministic process.*

2/ *The Brownian motion is the only (non-deterministic) Lévy process with continuous sample paths.*

3/ *Other examples of Lévy processes are the Poisson and compound Poisson processes.*

We are now going to explore the relationship between Lévy processes and infinite divisibility.

Definition 1.1.2 *A random vector Φ is infinitely divisible if, for each $n \in \mathbb{N}$, there is an independent, identically distributed sequence $\Phi_{n,1}$, $\Phi_{n,2}$,..., $\Phi_{n,n}$, so that*

$$\Phi \stackrel{d}{=} \Phi_{n,1} + \Phi_{n,2} + ... + \Phi_{n,n}.$$

Proposition 1.1.1 *If L is Lévy process, then, L_t is an infinitely divisible for any $t \geq 0$.*

If we take the Fourier transform of each L_t we get a function $f(t,u) = f_t(u)$ given by

$$f_t(u) = \mathbb{E}\left[e^{iuL_t}\right],$$

where

a) $f_0(u) = \mathbb{E}\left[e^{iuL_0}\right] = 1.$

b) $f_{t+s}(u) = \mathbb{E}\left[e^{iuL_{t+s}}\right] = \mathbb{E}\left[e^{iu(L_{t+s}-L_s+L_s)}\right] = \mathbb{E}\left[e^{iu(L_{t+s}-L_s)}e^{iuL_s}\right] = f_t(u).f_s(u).$

c) $f_t(u) \neq 0$ for every (t,u).

d) $f_t(u)$ is continuous.

Remark 1.1.4 *Using the continuity in probability, we conclude*

$$f_t(u) = \mathbb{E}\left[e^{iuL_t}\right] = e^{-t\Psi(u)},$$

for some continuous function $\Psi(u)$ such that $\Psi(0) = 0$.

The jump process

The jump process $\triangle L = (\triangle L_t; t \geq 0)$ associated to the Lévy process L is defined, for each $t \geq 0$, via

$$\triangle L_t = L_t - L_{t-},$$

the jump at t with $L_{t-} = \lim_{s \to t} L_s$ (the left limit at t).

Remark 1.1.5 • *In the case of a Poisson process, all power jump processes will be the same, and equal to the original Poisson process.*

• *In the case of a Brownian motion, all power jump processes of order strictly greater than one will be equal to 0.*

Remark 1.1.6 *The condition of stochastic continuity of a Lévy process yields immediately that for any Lévy process L and and any fixed $t > 0$, then $\triangle L_t = 0, ...a.s$, so, a Lévy process has no fixed times of discontinuity.*

Definition 1.1.3 *We say that a Lévy processes L has bounded jumps if there exists $C > 0$ such that*

$$\sup_t |\triangle L_t| \leq C < \infty.$$

Proposition 1.1.2 *If L is càdlàg, let ΔL denote its associated jumps process. Then ΔL is not càdlàg.*

1.1.1 Lévy measure

Let Λ be a Borel set in $\mathbb{R}$ bounded away from 0 $\left(0 \notin \bar{\Lambda}, \bar{\Lambda}\right.$ is the closure of $\left.\Lambda\right)$, N_t^{Λ} is a counting process without an explosion :

$$N_t^{\Lambda} = \sum_{0<s\leq t} \mathbf{1}_{\Lambda} \left(\triangle L_s\right).$$

Definition 1.1.4 ***(Poisson random measure)*** *$N:\Lambda\rightarrow N_t^{\Lambda}$ is called the Poisson random measure of the Lévy process.*

Definition 1.1.5 *The measure ν defined by*

$$\nu\left(\Lambda\right)=\mathbb{E}\left\{N_1^{\Lambda}\right\}=\mathbb{E}\left\{\sum_{0<s\leq t}\mathbf{1}_{\Lambda}\left(\triangle L_s\right)\right\}$$

is called the Lévy measure of the Lévy process, where

$$\nu\left(\Lambda\right)=\mathbb{E}\left\{N_1^{\Lambda}\right\},$$

be the parameter of the Poisson process $N_t^{\Lambda}\left(\nu\left(\Lambda\right)\right)<\infty$.

Definition 1.1.6 *If ν is a Borel measure on $\mathbb{R}$. We say that ν is a Lévy measure if*

$$\begin{cases}\nu\left(\{0\}\right)=0,\\ \qquad\qquad \textit{and}\\ \displaystyle\int_{\mathbb{R}\backslash 0}\left(1\wedge z^2\right)\nu\left(dz\right)<+\infty.\end{cases}$$

Proposition 1.1.3 *Let L be a Lévy process, then*

a) *$\nu\left(\mathbb{R}\right)<\infty$, then almost all paths of L have a finite number of jumps on every compact interval. In that case, the Lévy process has finite activity.*

b) *$\nu\left(\mathbb{R}\right)=\infty$, then almost all paths of L have an infinite number of jumps on every compact interval. In that case, the Lévy process has infinite activity.*

Proposition 1.1.4 ***(The Lévy-khintchine formula)*** *Let (L_t) be a Lévy process on $\mathbb{R}^d$. Then, there exists a function $\varphi:\mathbb{R}^d\rightarrow\mathbb{R}$, is called the characteristic exponent where for* *each* $\theta \in \mathbb{R}$

$$\begin{aligned}(\phi(\theta))^t &= \mathbb{E}\left[e^{i\theta L_t}\right] = e^{t\varphi(\theta)} \\ &= \exp\left[ia\theta t - \frac{1}{2}\sigma^2\theta^2 t + t\int_{-\infty}^{+\infty}\left(e^{i\theta L} - 1 - i\theta I_{\{|L|<1\}}\right)\nu(dx)\right],\end{aligned}$$

where

- $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}\backslash\{0\}$ with $\displaystyle\int_{-\infty}^{+\infty}(1 \wedge x^2)\,\nu(dx) < \infty$.
- $\varphi(\theta) = \log\phi(\theta)$.

Theorem 1.1.1 *Let* Λ *be a Borel set of* $\mathbb{R}$, $0 \notin \bar{\Lambda}$, f *Borel and finite on* Λ*. Then*

$$\int_\Lambda f(x)N_t(\omega, dx) = \sum_{0<s\leq t} f(\Delta X_s)\mathbf{1}_\Lambda(\Delta X_s),$$

where $N_t(\omega, dx)$ *denote the random measure. Just as we showed that* N_t^Λ *has independent and stationary increments.*

Corollary 1.1.1 *Let* Λ *be a Borel set of* $\mathbb{R}$ *with* $0 \notin \bar{\Lambda}$ *and let* f *be Borel and finite on* Λ*. Then*

$$\sum_{0<s\leq t} f(\Delta X_s)\mathbf{1}_\Lambda(\Delta X_s)$$

is a Lévy process.

Theorem 1.1.2 *Let* Λ *be a Borel set of* $\mathbb{R}$ *with* $0 \notin \bar{\Lambda}$*. Then*

$$L_t - \sum_{0<s\leq t} f(\Delta X_s)\mathbf{1}_\Lambda(\Delta X_s)$$

is a Lévy process.

Semi-martingale

Definition 1.1.7 *A real random process $(M_t)_{t\geq 0}$ is called a martingale, if*

1. *M_t is integrable for each $t \geq 0$, that is $\mathbb{E}\,|M_t| < \infty$, $\forall t \geq 0$;*
2. *M_t is $\mathcal{F}_t$-adapted; that is, for each $t \geq 0$, M_t is $\mathcal{F}_t$-measurable;*
3. *$\mathbb{E}\,[M_t \mid \mathcal{F}_s] = M_s$, a.s. $\forall 0 \leq s \leq t$.*

Example 1.1.2 • *Standard Brownian motion is a martingale.*

• *If N_t is a Poisson process with rate λ then $N_t - \lambda t$ is a martingale.*

Remark 1.1.7 *Property (3) can also be written as : $\mathbb{E}\,[M_t - M_s \mid \mathcal{F}_s] = 0$.*

Remark 1.1.8 *If $(M_t)_{t\geq 0}$ is a martingale, the function $t \mapsto \mathbb{E}\,[M_t]$ is constant.*

Theorem 1.1.3 *M_t is a martingale if and only if $\mathbb{E}\,[M_r] = \mathbb{E}\,[M_0]$ for all stopping times r.*

Definition 1.1.8 1. *A martingale M is said to be an L^2-martingale or a square integrable martingale if*

$$\mathbb{E}\,[M_t^2] < \infty \quad \textit{for every} \quad t \geq 0.$$

2. *A process M is said to be uniformly integrable if and only if*

$$\sup_{t\geq 0} \mathbb{E}\left(|M_t|\, \mathbf{1}_{\{|M|\geq N\}}\right) \to 0 \quad \textit{as} \quad N \to \infty.$$

Proposition 1.1.5 *If $Z_t \in L^1$ and if Z_t is a process has independent increments, then*

$$M_t = Z_t - \mathbb{E}\,[Z_t]$$

is a martingale.

Definition 1.1.9 *An adapted, càdlàg process M is a local martingale if there exists a sequence of increasing stopping times r_n, with $\lim_{n\to\infty} r_n = \infty$ a.s. such that $M^{t\wedge r_n}$ is a uniformly integrable martingale for each n.*

Theorem 1.1.4 *Every bounded local martingale is a martingale.*

Theorem 1.1.5 *A process $H = (H_t)_{0\leq t\leq T}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a semi-martingale if it admits a representation*

$$H = H_0 + M + A,$$

where

- *M is a local martingale with $M_0 = 0$;*
- *A is a bounded variation process with $A_0 = 0$.*

Every càdlàg Lévy process is a semi-martingale if it can be decomposed as the sum of a local martingale and an adapted finite-variation process.

The follows two theorems combine to proof the following useful theorem, which is one of the fundamental theorem about Lévy processes and semi-martingale :

Theorem 1.1.6 *Let L be a Lévy process. Then $L_t = X + Y$, where X, Y are two Lévy processes, such that*

i) *X is a martingale with bounded jumps, $X_t \in L^p$ for all $p \geq 1$;*

ii) *Y has paths of finite variation on compacts.*

Proof. See Theorem 40 in Protter [35] (2004: P. 30). ■

Theorem 1.1.7 *A decomposable process is a semi-martingale.*

Proof. See [35]. ■

Theorem 1.1.8 *Every càdlàg Lévy process is a semi-martingale.*

Proof. By using the two former theorems, one can easily prove the above Theorem. ■

Theorem 1.1.9 *Let L be a Lévy process with jumps bounded by a. That is $\sup_s |\Delta L_s| \leq a$. Let*

$$Z_t = L_t - \mathbb{E}[L_t].$$

Then Z is a martingale and $Z_t = Z_t^c + Z_t^d$ where Z^c is a martingale with continuous paths, Z^d is a martingale,

$$Z_t^d = \int_{\{|x|\leq a\}} x(N_t(.,dx) - t\nu(dx)),$$

where

- *Z^c and Z^d are independent Lévy processes.*
- *$N_t(.,dx)$ denote the random measure.*

Proof. See Theorem 41 in Protter [35] (2004: P. 30-31). ■

1.1.2 Itô's formula

Theorem 1.1.10 *Let $L = \{L_t, t \in [0,T]\}$ be a càdlàg semi-martingale, with quadratic variation denoted by $[L] = \{[L]_t : t \in [0,T]\}$ and let F be a C^2 real valued function. Then $F(L)$ is also a semi-martingale and the following formula holds*

$$\begin{aligned} F(L_t) &= F(L_0) + \int_0^t F'(L_{s-})\,dL_s + \frac{1}{2}\int_0^t F''(L_{s-})\,d[L]_s^c \\ &\quad + \sum_{0<s\leq t} \{F(L_s) - F(L_{s-}) - F'(L_{s-})\Delta L_s\}, \end{aligned} \tag{1.1}$$

where $[L]^c$ (sometimes denoted by $\langle L \rangle$) is the continuous part of the quadratic variation $[L]$.

We also note that in the case where $F(x) = x^2$, the formula (1.1) takes the form

$$L_t^2 = L_0^2 + \int_0^t 2L_{s-} dL_s + \int_0^t d[L]_s .$$

Moreover if L and K are two càdlàg semimartingales then we have

$$L_t K_t = L_0 K_0 + \int_0^t L_{s-} dK_s + \int_0^t K_{s-} dL_s + \int_0^t d[L, K]_s ,$$

where $[L, K]$ stands for the quadratic covariation of L, K also called the bracket process.

1.1.3 Construction of Teugels martingales

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions.

We define the power jump processes

$$\begin{cases} L_t^{(1)} = L_t, \\ \qquad \text{and} \\ L_t^{(i)} = \sum\limits_{0<s\leq t} (\triangle L_s)^{(i)} \quad \text{for} \quad i \geq 2, \end{cases} \tag{1.2}$$

where $L_t^{(i)} = \left\{ L_t^{(i)}, t \geq 0 \right\}$, $i = 1, 2, ...$ are again Lévy processes.

Remark 1.1.9 • *If L is a Brownian motion, then $L_t^{(i)} = 0$ for all $i \geq 2$.*

• *If L is a Poisson, then $L_t^{(i)} = L$ for all $i \geq 1$.*

We define the normal martingale $Y_t^{(i)}$:

$$\begin{aligned} Y_t^{(i)} &= L_t^{(i)} - \mathbb{E}\left[L_t^{(i)}\right] \quad ,i \geq 1 \\ &= L_t^{(i)} - t\mathbb{E}\left[L_1^{(i)}\right] \\ &= L_t^{(i)} - m_i t, \end{aligned}$$

where

$$\begin{cases} \mathbb{E}\left[L_t\right] = \mathbb{E}\left[L_t^{(1)}\right] = tm_1 < \infty, \text{ with } m_1 = \mathbb{E}\left[L_1\right], \\ \qquad\qquad\qquad \text{and} \\ \mathbb{E}\left[L_t^{(i)}\right] = \mathbb{E}\left[\sum_{0<s\leq t} (\Delta L_s)^i\right] = t\int_{-\infty}^{+\infty} x^i \nu(dx) = m_i t < \infty \quad ,i \geq 2. \end{cases}$$

- The predictable quadratic covariation process of $Y^{(i)}$ and $Y^{(j)}$ is given by

$$\left\langle Y^{(i)}, Y^{(j)}\right\rangle_t = m_{i+j}t, \quad i,j \geq 2.$$

- The quadratic covariation of $Y^{(i)}$ and $Y^{(j)}$ is given by

$$\left[Y^{(i)}, Y^{(j)}\right]_t = L_t^{(i+j)} + 1_{\{i=j=1\}}\sigma^2 t, \quad i,j \geq 1.$$

Orthogonal martingales

◁ Square-integrable martingales

For M a martingale, write $M \in \mathcal{M}^2$ if M is L^2-bounded :

$$\sup_{t\geq 0} \mathbb{E}\left(M_t^2\right) < \infty.$$

Remark 1.1.10 *Further, if $M_0 = 0$, then $M \in \mathcal{M}^2$.*

Proposition 1.1.6 *If $M \in \mathcal{M}^2$, then* $\lim_{t \to +\infty} \mathbb{E}\left(M_t^2\right) = \mathbb{E}\left(M_\infty^2\right)$ *and* $M_t = \mathbb{E}\left(M_\infty \mid \mathcal{F}_t\right)$.

Definition 1.1.10 *A collection of $\{M_t : t \in T\}$ is said to be uniformly integrable (UI) if*

$$\sup_{t \in T} \mathbb{E}\left(|M_t| \mathbf{1}_{\{|M_t|>m\}}\right) \to 0, \text{ as } m \to \infty.$$

◁ **Orthogonality martingales in $\mathcal{M}_0^2$**

Definition 1.1.11 *Two square integrable martingales M and N are called orthogonal if their product is again a martingale.*

Definition 1.1.12 *Two martingales N, M are said to be strongly orthogonal, if NM is a uniformly integrable martingale. (see [35]).*

Definition 1.1.13 *Let N, $M \in \mathcal{M}_0^2$. Then N, M are strongly orthogonal if and only if we have that $[N, M]$ is a uniformly integrable martingale.*

Proposition 1.1.7 *If M and N are strongly orthogonal then*

$$\mathbb{E}\left(M_\infty N_\infty\right) = \mathbb{E}\left(M_0 N_0\right) = 0.$$

Definition 1.1.14 *Let $X, Y \in L^2$, we say that X and Y are weakly orthogonal, if*

$$\mathbb{E}\left(XY\right) = 0,$$

this relationship is denoted $X \perp Y$.

An orthonormalization procedure can be applied to the martingales $Y_t^{(i)}$ in order to obtain a set of pairwise strongly orthonormal martingales $\left\{H_t^{(i)}\right\}_{i=1}^{\infty}$ such that each $H^{(i)}$ is a linear combination of the $Y^{(j)}$, $j = 1, 2, ..., i$. (see [29])

Then the family of Teugels martingales $\left(H_t^{(i)}\right)_{i=1}^{\infty}$, is defined by

$$H_t^{(i)} = a_{i,i}Y_t^{(i)} + a_{i,i-1}Y_t^{(i-1)} + a_{i,i-2}Y_t^{(i-2)} + ... + a_{i,1}Y_t^{(1)}$$

Then, we get

$$\left[H^{(i)}, Y^{(j)}\right]_t = a_{i,i}L_t^{(i+j)} + a_{i,i-1}L_t^{(i+j-1)} + a_{i,i-2}L_t^{(i+j-2)} + ... + a_{i,1}L_t^{(1+j)} + a_{i,1}\sigma^2 t\mathbf{1}_{\{j=1\}},$$

and

$$\mathbb{E}\left(\left[H^{(i)}, Y^{(j)}\right]_t\right) = t(m_{i+j} + a_{i,i-1}m_{i+j-1} + ... + a_{i,1}m_{j+1} + a_{i,1}\sigma^2 t\mathbf{1}_{\{j=1\}}.$$

Remark 1.1.11 *We have that $\left[H^{(i)}, Y^{(j)}\right]$ is a martingale if and only if we have that*

$$\mathbb{E}\left(\left[H^{(i)}, Y^{(j)}\right]_1\right) = 0.$$

We consider two spaces :

- The first space S_1 is the space of all real polynomials on the positive real line. We endow this space with the scalar product $\langle ., . \rangle_1$; given by

$$\langle P(x), Q(x)\rangle_1 = \int_{-\infty}^{+\infty} P(x)Q(x)x^2\nu(dx) + \sigma^2 P(0)Q(0),$$

where $P(x), Q(x)$ are a polynomial of degree $n \in \mathbb{N}$.

Note that

$$\begin{aligned}\langle x^{i-1}, x^{j-1}\rangle_1 &= \int_{-\infty}^{+\infty} x^{i+j}\nu(dx) + \sigma^2\mathbf{1}_{\{i=j=1\}} \\ &= m_{i+j} + \sigma^2\mathbf{1}_{\{i=j=1\}}.\end{aligned}$$

- The second space S_2 is the space of the process of the form

$$S_2 = \left\{a_1Y^{(1)} + a_2Y^{(2)} + a_3Y^{(3)} + ... + a_nY^{(n)}\right\},$$

where $n \in \{1, 2, ...\}$, $a_i \in \mathbb{R}$, $i = 1, ..., n$. We endow this space with the scalar product $\langle ., . \rangle_2$, given by

$$\begin{aligned} \left\langle Y^{(i)}, Y^{(j)} \right\rangle_2 &= \mathbb{E}\left(\left[Y^{(i)}, Y^{(j)}\right]_1\right) = \mathbb{E}\left(L_1^{(i+j)}\right) + \sigma^2 \mathbf{1}_{\{i=j=1\}} \\ &= m_{i+j} + \sigma^2 \mathbf{1}_{\{i=j=1\}}, \ i, \ j = 1, 2, ... \end{aligned}$$

So it's clearly that $x^{i-1} \longleftrightarrow Y^{(i)}$ is an isometry between S_1 and S_2. It is therefore sufficient to orthogonalize the polynomials $\{1, x, x^2, ...\}$ in S_1 to obtain an orthogonalization of the martingale $\left\{Y^{(1)}, Y^{(1)}, ...\right\}$ in S_2.

Then, the coefficients a_{ij} correspond to the orthonormalization of the polynomials 1, x, x^2, ... with respect to the measure

$$\mu(dx) = x^2 \nu(dx) + \sigma^2 \delta_0(dx),$$

where $\delta_0(dx) = 1$ when $x = 0$ and zero otherwise, that is the polynomial defined by

$$q_{i-1} = a_{i,i}x^{i-1} + a_{i,i-1}x^{i-2} + a_{i,i-2}x^{i-3} + ... + a_{i,1} \quad \text{and} \quad q_{i-1}(0) = a_{i,1},$$

then $\{q_i(x)\}$ is the system of orthonormalized polynomial such that $q_{i-1}(x)$ corresponds to $H_t^{(i)}$. Also, we set

$$p_i(x) = xq_{i-1} = a_{i,i}x^i + a_{i,i-1}x^{i-1} + a_{i,i-2}x^{i-2} + ... + a_{i,1}x,$$

et

$$\begin{aligned} \tilde{p}_i(x) &= x(q_{i-1}(x) - q_{i-1}(0)) \\ &= a_{i,i}x^i + a_{i,i-1}x^{i-1} + a_{i,i-2}x^{i-2} + ... + a_{i,2}x^2. \end{aligned}$$

We use the power jump processes

$$\begin{aligned} H_t^{(i)} &= (a_{i,i}L_t^{(i)} + a_{i,i-1}L_t^{(i-1)} + ... + a_{i,2}L_t^{(2)}) + a_{i,1}L_t^{(1)} \\ &-t\mathbb{E}\left[a_{i,i}L_1^{(i)} + a_{i,i-1}L_1^{(i-1)} + ... + a_{i,2}L_1^{(2)}\right] - ta_{i,1}\mathbb{E}\left[L_t^{(1)}\right] \\ &= \sum_{0<s\leq t} (a_{i,i}\left(\triangle L_s\right)^{(i)} + a_{i,i-1}\left(\triangle L_s\right)^{(i-1)} + ... + a_{i,2}\left(\triangle L_s\right)^{(2)}) + a_{i,1}L_t^{(1)} \\ &-t\mathbb{E}\left[a_{i,i}L_1^{(i)} + a_{i,i-1}L_1^{(i-1)} + ... + a_{i,2}L_1^{(2)}\right] - ta_{i,1}\mathbb{E}\left[L_t^{(1)}\right] \qquad ,i \geq 1, \end{aligned}$$

thus

$$H_t^{(i)} = \sum_{0<s\leq t} \tilde{p}_i(\triangle L_s) - t\mathbb{E}\left[\sum_{0<s\leq 1} \tilde{p}_i(\triangle L_s)\right] + q_{i-1}\left(0\right)L_t - tq_{i-1}(0)\mathbb{E}\left[L_t^{(1)}\right] \quad ,i \geq 1,$$

where $\mathbb{E}\left[L_t\right] = \mathbb{E}\left[L_t^{(1)}\right] = tm_1 < \infty$, with $m_1 = \mathbb{E}\left[L_1\right]$.

Remark 1.1.12 *If $i = 1$, then $H_t^{(1)} = a_{1,1}(L_t - t\mathbb{E}\left[L_1\right])$ where*

- $a_{1,1} = \left[\int_{\mathbb{R}} (y^2)\,\nu\left(dy\right) + \sigma^2\right]^{\frac{-1}{2}}.$
- $\mathbb{E}\left[L_1\right] = a + \int_{\{|z|\geq 1\}} z\nu\left(dz\right).$
- *If $\int_{\mathbb{R}} |z|\,\nu\left(dz\right) < \infty$, assuming $a = \int_{\{|z|<1\}} z\nu\left(dz\right)$, we obtain*

$$\mathbb{E}\left[L_1\right] = \int_{\mathbb{R}} z\nu\left(dz\right).$$

Then $\left(H_t^{(i)}\right)_{i=1}^{\infty}$ is a family of strongly orthogonal martingales such that

$$\left\langle H_t^{(i)}, H^{(j)}\right\rangle_t = \delta_{ij}.t, \tag{1.3}$$

and that

$$\left[H^{(i)},H^{(j)}\right]_t-\left\langle H^{(i)},H^{(j)}\right\rangle_t, \tag{1.4}$$

is an $\mathcal{F}_t$-martingale, see [35]. We refer the reader to [4], [13], or [28] for the detailed proofs.

Proposition 1.1.8 *Let $\{M_t : t\in[0,T]\}$ be a square integrable martingale which is adapted to the filtration $\mathcal{F}_t$ defined above. Then, there exist $U\in\mathcal{P}^2$ and $Z\in\mathcal{P}^2\left(l^2_{\mathcal{F}}\right)$ such that*

$$M_t=\mathbb{E}\left[M_t\right]+\int_0^t U_s dW_s+\sum_{i=0}^{\infty}\int_0^t Z_s^{(i)}dH_s^{(i)}.$$

Proof. The Proof follows by combining the result of L∅kka [23] (Theorem 5) and that of Nualart et al. [29]. ■

1.2 Stochastic differential equations driven by Teugels martingales

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space and $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a filtered probability space where $\mathbb{F}=(\mathcal{F}_t)_{0\le t\le T}$ satisfies the usual conditions, a right continuous increasing family of compete sub σ-algebra of $\mathcal{F}$. Let $\{W_t\}_{t\ge0}$ be a d-standard Brownian motion and $\{L_t\}_{t\ge0}$ be a $\mathbb{R}^1$-valued Lévy process with a Lévy measure ν such that

$$\int_{\mathbb{R}}\left(1\wedge z^2\right)\nu\left(dz\right)<\infty,$$

which is independent of the Brownian motion W_t.

We denote by $\left(H^{(i)}\right)_{i\ge0}$ the Teugels martingales associated with the Lévy process $\{L_t : t\in[0,T]\}$. We assure that

$$\mathcal{F}_t=\sigma\left(L_s\right)\vee\sigma\left(W_s\right)\vee\mathcal{N},$$

where $\mathcal{N}$ denotes the totality of $\mathbb{P}$-null sets.

Considering the following stochastic differential equation

$$\begin{cases} dx_t & = b\left(t, x_t\right) dt + g\left(t, x_t\right) dW_t + \displaystyle\sum_{i=1}^{\infty} \sigma^i\left(t, x_{t-}\right) dH_t^{(i)}, \\ \\ x_0 & = x, \end{cases} \tag{1.5}$$

where

- x an $\mathcal{F}_0$-measurable random variable,
- and

$$b : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$g : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^{n \times d},$$

$$\sigma = \left(\sigma^i\right)_{i=1}^{\infty} : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n,$$

b, g and σ are progressively measurable maps and satisfying (M_1) :

i) $b(., 0) \in \mathcal{L}^2\left(0, T; \mathbb{R}^n\right)$, $g(., 0) \in \mathcal{L}^2\left(0, T; \mathbb{R}^{n \times d}\right)$, $\sigma(., 0) \in l^2_{\mathcal{F}}\left(0, T; \mathbb{R}^n\right)$.

ii) $\exists C > 0$ such as

$$\left|b\left(t, w, x\right) - b\left(t, w, y\right)\right| + \left|g\left(t, w, x\right) - g\left(t, y\right)\right|$$

$$+ \left\|\sigma\left(t, w, x\right) - \sigma\left(t, w, y\right)\right\|_{l^2(\mathbb{R}^n)} \leq C\left|x - y\right|,$$

for all x, $y \in \mathbb{R}^n$, $(t, \omega) \in [0, T] \times \Omega$.

Lemma 1.2.1 *Under the hypotheses* (M_1). *Then the SDE (1.5) has a unique solution. This solution belongs to* $S^2_{\mathcal{F}}\left(0, T; \mathbb{R}^n\right)$.

The above lemma can be proved similar to the case without Teugels martingales by the routine successive approximation argument and Burkholder-Davis-Gundy Inequality.

Lemma 1.2.2 *(Continuous dependence theory) Suppose that both the coefficients* (x, b, g, σ) *and* $(\overline{x}, \overline{b}, \overline{g}, \overline{\sigma})$. *Let* $x(.)$, $\overline{x}(.) \in S^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ *be the solutions of SDE (1.5) corresponding to* (x, b, g, σ) *and* $(\overline{x}, \overline{b}, \overline{g}, \overline{\sigma})$ *respectively. Then*

$$\begin{aligned}\mathbb{E} \sup_{0\le t\le T} |x_t - \overline{x}_t|^2 &\le K \Bigg[|x - \overline{x}|^2 + \mathbb{E}\int_0^T \left| b(s, \overline{x}_s) - \overline{b}(s, \overline{x}_s)\right|^2 ds \\ &+ \mathbb{E}\int_0^T |g(s, \overline{x}_s) - \overline{g}(s, \overline{x}_s)|^2 ds \\ &+ \mathbb{E}\int_0^T |\sigma(s, \overline{x}_s) - \bar{\sigma}(s, \overline{x}_s)|^2 ds.\end{aligned}$$

Particularly for $(\overline{x}, \overline{b}, \overline{g}, \overline{\sigma}) = (0, 0, 0, 0)$, *we have*

$$\mathbb{E} \sup_{0\le t\le T} |x_t|^2 \le K \left[|x|^2 + \mathbb{E}\int_0^T |b(s, 0)|^2 ds + \mathbb{E}\int_0^T |g(s, 0)|^2 ds + \mathbb{E}\int_0^T |\sigma(s, 0)|^2 ds \right] < +\infty.$$

Proof. Applying Ito's formula to $|x_t - \overline{x}_t|^2$, we obtain

$$\begin{aligned}|x_t - \overline{x}_t|^2 &= |x - \overline{x}|^2 + \int_0^t 2(x_s - \overline{x}_s, b(s, x_s) - \overline{b}(s, \overline{x}_s)\, ds \\ &+ \int_0^t 2(x_s - \overline{x}_s, g(s, x_s) - \overline{g}(s, \overline{x}_s)\, dW_s \\ &+ \sum_{i=1}^{\infty} \int_0^t 2(x_{s-} - \overline{x}_{s-}, \sigma^i(s, x_{s-}) - \bar{\sigma}^i(s, \overline{x}_{s-})\, dH^{(i)}_s \\ &+ \int_0^t |g(s, x_s) - \overline{g}(s, \overline{x}_s)|^2 ds \\ &+ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_0^t (\sigma^i(s, x_{s-}) - \bar{\sigma}^i(s, \overline{x}_{s-}), \\ &\sigma^j(s, x_s) - \bar{\sigma}^j(s, \overline{x}_{s-})\, d\left[H^{(i)}, H^{(j)}\right]_s,\end{aligned}$$

where $\left[H^{(i)}, H^{(j)}\right](.)$ denotes the quadratic variational process corresponding to $H^{(i)}(.)$

and $H^{(j)}(.)$, also called the bracket process.

We denote the predictable quadratic variational process corresponding to $H^{(i)}$ and $H^{(j)}$ by $\left\langle H^{(i)}, H^{(j)}\right\rangle(.)$ and have $\left[H^{(i)}, H^{(j)}\right]_t - \left\langle H^{(i)}, H^{(j)}\right\rangle_t$ is an $\mathcal{F}_t$-martingale.
Taking the expectation and using the fact that $\left\langle H^{(i)}, H^{(j)}\right\rangle(t) = \delta_{ij}t$ and Gronwall's inequality, we have

$$
\begin{aligned}
\mathbb{E}\left|x_t - \overline{x}_t\right|^2 \leq K\Bigg[& \mathbb{E}\left|x - \overline{x}\right|^2 + \mathbb{E}\int_0^T \left|b\left(s, \overline{x}_s\right) - \bar{b}\left(s, \overline{x}_s\right)\right|^2 ds \\
& + \mathbb{E}\int_0^T \left|g\left(s, \overline{x}_s\right) - \overline{g}\left(s, \overline{x}_s\right)\right|^2 ds \\
& + \mathbb{E}\int_0^T \left\|\sigma\left(s, \overline{x}_s\right) - \bar{\sigma}\left(s, \overline{x}_s\right)\right\|^2_{l^2(\mathbb{R}^n)} ds \Bigg],
\end{aligned}
$$

where the constant K depends only on the constants C and T. On the other hand, by the above inequality and B-D-G inequality, we get that

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq t\leq T}\left|x_t - \overline{x}_t\right|^2 \leq K\Bigg[& \left|x - \overline{x}\right|^2 + \mathbb{E}\int_0^T \left|b\left(s, \overline{x}_s\right) - \bar{b}\left(s, \overline{x}_s\right)\right|^2 ds \\
& + \mathbb{E}\int_0^T \left|g\left(s, \overline{x}_s\right) - \overline{g}\left(s, \overline{x}_s\right)\right|^2 ds \\
& + \mathbb{E}\int_0^T \left|\sigma\left(s, \overline{x}_s\right) - \bar{\sigma}\left(s, \overline{x}_s\right)\right|^2 ds,
\end{aligned}
$$

where the constant K is a generic constant whose values might change from line to line, and depends only on the constants C and T. The lemma is completed. ■

1.3 Backward stochastic differential equation driven by Teugels martingales

Let $W = (W_t)_{0 \leq t \leq T}$ be a standard d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F} = (\mathcal{F})_{0 \leq t \leq T}$ is the natural filtration of W, and T is a fixed finite horizon.

We denote by $S^2 [0.T]$ the set of real-valued progressively measurable processes y such that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |y_t|^2\right] < \infty,$$

and by $\mathcal{H}^2 [0.T]^d$ the set of $\mathbb{R}^d$-valued progressively measurable processes Z such that

$$\mathbb{E}\left[\int_0^T |z_t|^2 \, dt\right] < \infty.$$

We denote by $\mathcal{P}^2$ the subspace of H^2 formed by the predictable processes. Let l^2 be the space of real valued sequences $(x_n)_{n \geq 0}$ such that $\sum_{i=1}^{\infty} x_i^2$ is finite. We shall denote by $\mathcal{H}^2(l^2)$ and $\mathcal{P}^2 (l^2)$ the corresponding space of l^2-valued processes equipped with the norm

$$\|\Phi\|^2 = \sum_{i=1}^{\infty} \mathbb{E} \int_0^T \left|\Phi_t^{(i)}\right|^2_{\mathbb{R}^m}.$$

We are given a pair (ξ, f) called the terminal condition and generator (or driver), satisfying (M_2) :

- $\xi \in L^2 (\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^m)$.
- $f : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times l^2 (\mathbb{R}^n) \to \mathbb{R}^n$ is progressively measurable such that

(i) $f(., 0, 0, 0) \in \mathcal{L}^2_{\mathcal{F}} (0, T, \mathbb{R}^n)$.

(ii) f satisfies a uniform Lipschitz condition in (y, Z), i.e. there exists a constant C such that

$$\left|f\left(t,\omega,y,z,Z\right)-f\left(t,\omega,\overline{y},\overline{z},\bar{Z}\right)\right|\leq C\left(\left|y-\overline{y}\right|+\left|z-\overline{z}\right|+\left\|Z-\bar{Z}\right\|_{l^{2}(\mathbb{R}^{n})}\right),$$

for $\forall\left(y,z,Z\right),\ \left(\overline{y},\overline{z},\bar{Z}\right)\in\mathbb{R}^{n}\times\mathbb{R}^{n\times d}\times l^{2}\left(\mathbb{R}^{n}\right),\ (t,\omega)\in[0,T]\times\Omega.$

We consider the (unidimensional) backward stochastic differential equations (BSDE)

$$-dy_{t}=f\left(t,y_{t-},z_{t},Z_{t}\right)dt-z_{t}dW_{t}-\sum_{i=1}^{\infty}Z_{t}^{(i)}dH_{t}^{(i)},\tag{1.6}$$

has a unique solution

$$\left(y\left(.\right),z\left(.\right),Z\left(.\right)\right)\in\mathcal{S}_{\mathcal{F}}^{2}\left(0,T,\mathbb{R}^{n}\right)\times\mathcal{L}_{\mathcal{F}}^{2}\left(0,T,\mathbb{R}^{m}\right)\times l_{\mathcal{F}}^{2}\left(0,T,\mathbb{R}^{n}\right).$$

Theorem 1.3.1 *Given a pair* (ξ,f) *satisfying* (M_2)*, there exists a unique solution* $\left(y\left(.\right),z\left(.\right),Z\left(.\right)\right)$ *to the BSDE* (1.6)*.*

Proof. Uniqueness. Let $\left(y\left(.\right),z\left(.\right),Z\left(.\right)\right)$ and $\left(\overline{y}\left(.\right),\overline{z}\left(.\right),\overline{Z}\left(.\right)\right)$ be two solutions of equation (1.6), one can write

$$\begin{aligned}y_{t}-\overline{y}_{t}\ &=\int_{t}^{T}\left(f\left(t,y_{t-},z_{t},Z_{t}\right)-f\left(t,\overline{y}_{t-},\overline{z}_{t},\overline{Z}_{t}\right)\right)dt-\left(z_{t}-\bar{z}_{t}\right)dW_{t}\\&-\sum_{i=1}^{\infty}(Z_{t}^{(i)}-\bar{Z}_{t}^{(i)})dH_{t}^{(i)}.\end{aligned}$$

By using Itô's formula, we can fiend

$$\begin{aligned}d\left|y_{t}-\overline{y}_{t}\right|^{2}\ &=2\left(y_{t-}-\overline{y}_{t-}\right)\left[f\left(t,y_{t-},z_{t},Z_{t}\right)-f\left(t,\overline{y}_{t-},\overline{z}_{t},\overline{Z}_{t}\right)\right]dt\\&-2\left(y_{t-}-\overline{y}_{t-}\right)\left(z_{t}-\overline{z}_{t}\right)dW_{t}-2\sum_{i=1}^{\infty}\left(y_{t-}-\overline{y}_{t-}\right)(Z_{t}^{(i)}-\bar{Z}_{t}^{(i)})dH_{t}^{(i)}\\&-\left|z_{t}-\bar{z}_{t}\right|^{2}dt-\left\|Z_{t}-\overline{Z}_{t}\right\|^{2}dt.\end{aligned}$$

Taking expectation to obtain

$$\mathbb{E}\left|y_t-\overline{y}_t\right|^2+\mathbb{E}\int_t^T\left|z_s-\overline{z}_s\right|^2ds+\mathbb{E}\int_t^T\left\|Z_s-\overline{Z}_s\right\|^2ds$$

$$=2\mathbb{E}\int_t^T\left(y_{s-}-\overline{y}_{s-}\right)\left[f\left(s,y_{s-},z_s,Z_s\right)-f\left(s,\overline{y}_{s-},\overline{z}_s,\overline{Z}_s\right)\right]ds.$$

Because f is Lipschitz, we get

$$\mathbb{E}\left|y_t-\overline{y}_t\right|^2+\mathbb{E}\int_t^T\left|z_s-\overline{z}_s\right|^2ds+\mathbb{E}\int_t^T\left\|Z_s-\overline{Z}_s\right\|^2ds$$

$$\leq 2C\mathbb{E}\int_t^T\left(y_{s-}-\overline{y}_{s-}\right)\left(\left|y_s-\overline{y}_s\right|+\left|z_s-\overline{z}_s\right|+\left\|Z_s-\bar{Z}_s\right\|_{l^2(\mathbb{R}^n)}\right)ds.$$

By using Young's inequality, we have

$$\mathbb{E}\left|y_t-\overline{y}_t\right|^2\quad+\left(1-\frac{2L}{\beta^2}\right)\mathbb{E}\int_t^T\left|z_s-\overline{z}_s\right|^2ds+\left(1-\frac{2L}{\beta^2}\right)\mathbb{E}\int_t^T\left\|Z_s-\overline{Z}_s\right\|^2ds$$

$$\leq L\left(\beta^2+2\right)\mathbb{E}\int_t^T\left|y_s-\overline{y}_s\right|^2ds.$$

Here we have used the inequality $2xy\leq\beta^2x^2+\dfrac{y^2}{\beta^2}$. If we take $\dfrac{2L}{\beta^2}=\dfrac{1}{2}$, then we have

$$\mathbb{E}\left|y_t-\overline{y}_t\right|^2+\mathbb{E}\int_t^T\left|z_s-\overline{z}_s\right|^2ds+\mathbb{E}\int_t^T\left\|Z_s-\overline{Z}_s\right\|^2ds\leq C\mathbb{E}\int_t^T\left|y_s-\overline{y}_s\right|^2ds.$$

And by using Gronwall's lemma, we can follows uniqueness.

Existence. We can prove that the following BSDE

$$y_t=\xi+\int_t^Tf\left(s,0,0,0\right)ds-\int_t^Tz_sdW_s-\int_t^T\left\langle Z_s,dH_s\right\rangle,$$

has a solution by using the martingale representation theorem.

We define (y^n, z^n, Z^n) as follows :

1) $y^0 = z^0 = Z^0 = 0,$

2) $(y^{n+1}, z^{n+1}, Z^{n+1})$ is the unique solution to the BSDE :

$$y_t^{n+1} = \xi + \int_t^T f\left(s, y_{s-}^n, z_s^n, Z_s^n\right) ds - \int_t^T z_s^{n+1} dW_s - \int_t^T \left\langle Z_s^{n+1}, dH_s\right\rangle.$$

Now, we went to prove that (y^n, z^n, Z^n) is a Cauchy sequence in the Banach space ε. To simplify the notations, we put

$$\overline{y}_s^{n,m} := y_s^n - y_s^m \quad , \overline{z}_s^{n,m} := z_s^n - z_s^m \quad , \overline{Z}_s^{n,m} := Z_s^n - Z_s^m,$$

and

$$\overline{f}_s^{n,m} := f\left(s, y_{s-}^n, z_s^n, Z_s^n\right) - f\left(s, y_{s-}^m, z_s^m, Z_s^m\right).$$

By using Itô's formula, for every $n < m$, we can shows that

$$\begin{aligned}
e^{\alpha t}\left|\overline{y}_t^{n+1,m+1}\right|^2 \quad & + \int_t^T e^{\alpha s}\left|\overline{z}_s^{n+1,m+1}\right|^2 ds \\
& + \int_t^T e^{\alpha s}\left\|\overline{Z}_s^{n+1,m+1}\right\|^2 ds + \alpha \int_t^T e^{\alpha s}\left|\overline{y}_{s-}^{n+1,m+1}\right|^2 ds \\
& = 2\int_t^T e^{\alpha s}\overline{y}_{s-}^{n+1,m+1}\overline{f}_s^{n,m} ds - 2\int_t^T e^{\alpha s}\overline{y}_{s-}^{n+1,m+1}\overline{z}_s^{n,m} dW_s \\
& -2\int_t^T e^{\alpha s}\overline{y}_{s-}^{n+1,m+1}\left\langle \overline{Z}_s^{n,m}, dH_s\right\rangle - (N_T - N_t),
\end{aligned}$$

where $\{N_t : t \in [0,T]\}$ is a martingale, given by

$$N_t = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t e^{\alpha s}\overline{Z}_s^{n+1,m+1,(i)}\overline{Z}_s^{n+1,m+1,(j)}\left(d\left[H^{(i)},H^{(j)}\right]_s - d\left\langle H^{(i)},H^{(j)}\right\rangle_s\right).$$

By the formula $\left\langle H^{(i)},H^{(j)}\right\rangle = \delta_{i,j}t$, we get

$$\begin{aligned}
\mathbb{E}e^{\alpha t}\left|\overline{y}_t^{n+1,m+1}\right|^2 &+ \mathbb{E}\int_t^T e^{\alpha s}\left|\overline{z}_s^{n+1,m+1}\right|^2 ds \\
&+ \mathbb{E}\int_t^T e^{\alpha s}\left\|\overline{Z}_s^{n+1,m+1}\right\|^2 ds + \alpha\mathbb{E}\int_t^T e^{\alpha s}\left|\overline{y}_{s-}^{n+1,m+1}\right|^2 ds \\
&= 2\mathbb{E}\int_t^T e^{\alpha s}\overline{y}_{s-}^{n+1,m+1}\overline{f}_s^{n,m} ds.
\end{aligned}$$

Since f is L-Lipschitz, we get

$$\begin{aligned}
e^{\alpha t}\mathbb{E}\left|\overline{y}_t^{n+1,m+1}\right|^2 &+ \int_t^T e^{\alpha s}\mathbb{E}\left|\overline{z}_s^{n+1,m+1}\right|^2 ds \\
&+ \int_t^T e^{\alpha s}\mathbb{E}\left\|\overline{Z}_s^{n+1,m+1}\right\|^2 ds + \alpha\int_t^T e^{\alpha s}\mathbb{E}\left|\overline{y}_{s-}^{n+1,m+1}\right|^2 ds \\
&\leq 2L\mathbb{E}\int_t^T e^{\alpha s}\left|\overline{y}_{s-}^{n+1,m+1}\right|\left[\left|\overline{y}_{s-}^{n,m}\right| + \left|\overline{z}_s^{n,m}\right| + \left\|\overline{Z}_s^{n,m}\right\|\right] ds,
\end{aligned}$$

and

$$\begin{aligned}
e^{\alpha t}\mathbb{E}\left|\overline{y}_t^{n+1,m+1}\right|^2 &+ \int_t^T e^{\alpha s}\mathbb{E}\left|\overline{z}_s^{n+1,m+1}\right|^2 ds \\
&+ \int_t^T e^{\alpha s}\mathbb{E}\left\|\overline{Z}_s^{n+1,m+1}\right\|^2 ds \\
&+ (\alpha - L^2\beta^2)\int_t^T e^{\alpha s}\mathbb{E}\left|\overline{y}_{s-}^{n+1,m+1}\right|^2 ds \\
&\leq \frac{3}{\beta^2}\mathbb{E}\int_t^T e^{\alpha s}\left|\overline{y}_{s-}^{n+1,m+1}\right|\left[\left|\overline{y}_{s-}^{n,m}\right|^2 + \left|\overline{z}_s^{n,m}\right|^2 + \left\|\overline{Z}_s^{n,m}\right\|^2\right] ds.
\end{aligned}$$

If we take $\dfrac{3}{\beta^2} = \dfrac{1}{2}$ and $\alpha - 6L^2 = 1$, we get

$$e^{\alpha t}\mathbb{E}\left|\overline{y}_t^{n+1,m+1}\right|^2 \quad + \int_t^T e^{\alpha s}\mathbb{E}\left|\overline{z}_s^{n+1,m+1}\right|^2 ds + \int_t^T e^{\alpha s}\mathbb{E}\left\|\overline{Z}_s^{n+1,m+1}\right\|^2 ds$$

$$\leq \frac{1}{2}\mathbb{E}\int_t^T e^{\alpha s}\left|\overline{y}_{s-}^{n+1,m+1}\right|\left[\left|\overline{y}_{s-}^{n,m}\right|^2 + \left|\overline{z}_s^{n,m}\right|^2 + \left\|\overline{Z}_s^{n,m}\right\|^2\right] ds.$$

Which implied, for all $m > n$, that

$$e^{\alpha t}\mathbb{E}\left|\overline{y}_t^{n,m}\right|^2 + \int_t^T e^{\alpha s}\mathbb{E}\left|\overline{z}_s^{n,m}\right|^2 ds + \int_t^T e^{\alpha s}\mathbb{E}\left\|\overline{Z}_s^{n,m}\right\|^2 ds \leq \frac{C}{2^n}.$$

On can follows that there exists a universal constant C, by using Itô's formula and Doob's inequality,

$$\mathbb{E}\sup_{0\leq s\leq T}\left|\overline{y}_t^{n,m}\right|^2 + \int_t^T e^{\alpha s}\mathbb{E}\left|\overline{z}_s^{n,m}\right|^2 ds + \int_t^T e^{\alpha s}\mathbb{E}\left\|\overline{Z}_s^{n,m}\right\|^2 ds \leq \frac{C}{2^n}.$$

Finally, (y^n, z^n, Z^n) is a Cauchy sequence in the Banach space E. We can show that

$$(y, z, Z) = \lim_{n\to\infty}\left(y^n, z^n, Z^n\right),$$

solves our BSDE. $\blacksquare$

We consider the following forward-backward stochastic control system

$$\begin{cases} dx_t = b(t, x_t)\,dt + g(t, x_t)\,dW_t + \sum\limits_{i=1}^{\infty} \sigma^{(i)}(t, x_{t-})\,dH_t^{(i)}, \\ \\ x_0 = x, \\ \\ -dy_t = f(t, x_{t-}, y_{t-}, z_t, Z_t)\,dt - z_t dW_t - \sum\limits_{i=1}^{\infty} Z_t^{(i)} dH_t^{(i)}, \\ \\ y_T = \varphi(x_T), \end{cases}$$

where

$$b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$g : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d},$$

$$\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times l^2(\mathbb{R}^m) \to \mathbb{R}^m,$$

$$\varphi : \mathbb{R}^n \to \mathbb{R}^m.$$

Remark 1.3.1 *Suppose that the coefficients satisfy the assumption* (M_1) *and* (M_2)*, then by theorem* (1.3.1) *and lemma* (1.2.1)*, we have existence and uniqueness of our system.*

Chapter 2

Partial Information Optimality Conditions for Controlled SDEs driven by Teugels martingales

In this chapter, we consider a partial information stochastic control problem where the system is governed by a nonlinear stochastic differential equation driven by both Teugels martingales associated with some Lévy process and an independent Brownian motion, up to now, there is only one literature dealing with a partial information control problem for a system governed by SDEs driven by a both Teugels martingales and an independent Brownian motion. The control variable is allowed to enter into both coefficients and is assumed to be adapted to subfiltration which is possibly less than the whole one. We study the partial information necessary as well as sufficient conditions for optimality by using certain classical convex variational techniques.

In this second chapter, we will give the formulation of the stochastic control problem. Then, will prove optimality necessary conditions in the form of a maximum principle. These conditions turn out to be sufficient under some convexity assumptions. To illustrate the general results, some examples are solved. This chapter is inspired by ([6, 8]).

2.1 Formulations of stochastic optimal control problems

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a probability space equipped with a filtration satisfying the usual conditions, where W is a standard d-dimensional Brownian motion and $H_t = \left(H_t^{(i)}\right)_{i=1}^{\infty}$ are pairwise strongly orthonormal Teugels martingales, associated with some Lévy process. We assume that

$$\mathcal{F}_t = \sigma^W \vee \sigma^L \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of the $\mathbb{P}$-null set and $\mathcal{G}_1 \vee \mathcal{G}_2$ denotes the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$. Let T be a strictly positive real number.

Definition 2.1.1 *An admissible control is a measurable, adapted processes*

$$u : [0,T] \times \Omega \to U,$$

such that

$$\mathbb{E}\left[\int_0^T |u(s)|^2 \, ds\right] < \infty.$$

We consider the following nonlinear stochastic differential equation (SDE)

$$\begin{cases} dx_t &= b\left(t, x_t, u_t\right) dt + \sum_{i=1}^{d} g^i\left(t, x_t, u_t\right) dW_t^i + \sum_{i=1}^{\infty} \sigma^{(i)}\left(t, x_{t-}, u_t\right) dH_t^{(i)}, \\ x_0 &= x, \end{cases} \tag{2.1}$$

where b, g and $\sigma = \left(\sigma^{(i)}\right)_{i=1}^{\infty}$ are given deterministic functions and x is the initial data.

The control $u = (u\,(t))$ is required to be valued in some subset U of $\mathbb{R}^k$ and adapted to a subfiltration $(\mathcal{G}_t)_{t\geq 0}$ of $(\mathcal{F}_t)_{t\geq 0}$. We denote by $\mathcal{U} = \mathcal{U}_{\mathcal{G}}$ the class of all $\mathcal{G}_t$-adapted control processes.

The cost functional to be minimized, over the set $\mathcal{U}$, has the form

$$J(u(.)) = \mathbb{E}\left[\int_0^T l(t, x_t, u_t)\,dt + h(x_T)\right], \tag{2.2}$$

where, g and l are given maps and x_t is the trajectory of the system controlled by $u(t)$.

A control $\hat{u}(.) \in \mathcal{U}$ is called optimal, if it satisfies

$$J(\hat{u}(.)) = \inf_{u(.)\in\mathcal{U}} J(u(.)).$$

Partial information or incomplete information, means that the information available to the controller is possibly less than the whole information. That is, any admissible control is adapted to subfiltration $(\mathcal{G}_t)_{t\geq 0}$ of $(\mathcal{F}_t)_{t\geq 0}$. This kind of problems, which have potential applications in mathematical finance and mathematical economics, arise naturally, because it may fail to obtain an admissible control with full information in real world applications.

The following assumptions will be in force throughout this paper

$$b : [0,T] \times \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n,$$

$$g = (g^i)_{i=1}^d : [0,T] \times \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d},$$

$$\sigma = \left(\sigma^{(i)}\right)_{i=1}^{\infty} : [0,T] \times \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n,$$

$$h : \mathbb{R}^n \to \mathbb{R},$$

$$l : [0,T] \times \Omega \times \mathbb{R}^n \times U \to \mathbb{R}.$$

where

$(\mathbf{D}_1)$ The maps b, σ, g are measurable and

$$b(.,0,u_t) \in \mathcal{L}^2(0,T,\mathbb{R}^n), g(.,0,u_t) \in \mathcal{L}^2_{\mathcal{F}}\left(0,T,\mathbb{R}^{n\times d}\right),$$

$$\sigma(.,0,u_t) \in l^2_{\mathcal{F}}(0,T,\mathbb{R}^n).$$

$(\mathbf{D}_2)$

$i)$ b, σ, g, l and h are continuously differentiable with respect to (x,u).

$ii)$ The derivatives of b, σ, g are bounded.

$iii)$ l is bounded by $C\left(1+|x|^2+|u|^2\right)$ and their derivatives are

bounded by $C(1+|x|+|u|)$.

g is bounded by $C\left(1+|x|^2\right)$ and their derivatives are bounded by $C(1+|x|)$.

Under the assumptions $(\mathbf{D}_1)$ and $(\mathbf{D}_2)$, for every $u(.) \in \mathcal{U}$, equation (2.1) has a unique strong solution $x(.) \in \mathcal{S}^2_{\mathcal{F}}(0,T,\mathbb{R}^n)$ and the functional cost J is well defined from $\mathcal{U}$ into $\mathbb{R}$, see [26].

2.2 Partial information necessary optimality conditions

In this section, it establish optimality necessary conditions for our control problem, and it prove that if $\hat{u}(.)$ is a local optimal control for the control problem (2.1) and (2.2), then $\hat{u}(.)$ satisfies the necessary optimality conditions in some local form.

Let us assume the following.

$(\mathbf{D}_3)$: For all t, r such $0 \leq t \leq t + r \leq T$, all $i = 1, ..., k$ and all bounded $\mathcal{G}_t$-measurable $\alpha = \alpha(w)$, the control $\beta(s) := (0, ..., \beta_i(s), 0, ..., 0) \in U \subset \mathbb{R}^k$ with

$$\beta_i(s) = \alpha_i \chi_{[t,t+r]}(s) \quad , \; s \in [0, T].$$

belongs to $\mathcal{U}$.

$(\mathbf{D_4})$: For all $u(.), \beta \in \mathcal{U}$ with β bounded, there exist $\delta > 0$ such that $u(.) + y\beta \in \mathcal{U}$, for all $y \in (-\delta, \delta)$.

For given $u(.), \beta \in \mathcal{U}$ with β bounded, we define the process x_t by

$$x_t^1 = x_t^{(u(.),\beta)} = \frac{d}{dy} x_t^{(u(.),\beta)}.$$

Note that x_t^1 satisfies the following linear forward stochastic differential equation driven by both Teugels martingales and Brownian motion.

$$\begin{cases} dx_t^1 & = \left(b_x(t, x_t, u_t) x_t + b_u(t, x_t, u_t) \beta_t\right) dt \\ & + \sum_{i=1}^{d} \left(g_x^i(t, x_t, u_t) x_t + g_u^i(t, x_t, u_t) \beta_t\right) dW_t^i \\ & + \sum_{i=1}^{\infty} \left(\sigma_x^{(i)}(t, x_{t-}, u_t) x_t + \sigma_u^{(i)}(t, x_{t-}, u_t) \beta_t\right) dH_t^{(i)}, \\ x_0^1 & = 0. \end{cases} \tag{2.3}$$

We introduce the adjoint equation corresponding to the above variational equation, which is a linear backward stochastic differential equation (BSDE), given by

$$\begin{cases} -dp_t &= \left[b_x^* (t) \, p_t + \sum_{i=1}^{d} g_x^{i*} (t) \, q_t^i + \sum_{i=1}^{\infty} \sigma_x^{i*} (t) \, k_t^i + l_x^* (t) \right] dt \\ & \quad - \sum_{i=1}^{d} q_t^i dW_t - \sum_{i=1}^{\infty} k_t^{(i)} dH_t^{(i)}, \\ p_T &= h_x^* (x_T) , \end{cases} \tag{2.4}$$

where $\rho_x (t) = \rho_x (t, x_t, u_t)$ for $\rho = b,\ f,\ \sigma$ and l.

The Hamiltonian is defined by

$$\begin{aligned} H (t, x, u, p, q, k) := pb (t, x, u) + \sum_{i=1}^{d} q^i g^i (t, x, u) \\ + \sum_{i=1}^{\infty} k_t^{(i)} \sigma^{(i)} (t, x, u) + l (t, x, u) , \end{aligned} \tag{2.5}$$

where

$$H : [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times l^2 (\mathbb{R}^n) \to \mathbb{R}^n.$$

The adjoint equations can be rewritten in terms of the derivatives of the Hamiltonian as

$$\begin{cases} -dp_t &= H_x (t, x_t, u_t, p_t, q_t, k_t) \, dt - \sum_{i=1}^{d} q_t^i dW_t - \sum_{i=1}^{\infty} k_t^{(i)} dH_t^{(i)}, \\ p_T &= h_x^* (x_T) . \end{cases} \tag{2.6}$$

Applying the result of [28], it follows that under assumptions $(\mathbf{D}_1), (\mathbf{D}_2)$, the above BSDE admits one and only one $\mathcal{F}_t$-adapted solution

$$(p (.) , q (.) , k (.)) \in \mathcal{S}_{\mathcal{F}}^2 (0, T, \mathbb{R}^n) \times \mathcal{L}_{\mathcal{F}}^2 \left(0, T, \mathbb{R}^{n \times d}\right) \times l_{\mathcal{F}}^2 (0, T, \mathbb{R}^n) .$$

Then, we state and prove the main result of this section.

Theorem 2.2.1 *(Partial information necessary optimality conditions) Let $\hat{u}\left(.\right)$ be a local minimum for the cost J over $\mathcal{U}$ in the sense that for all $\beta \in \mathcal{U}$ with β bounded, there exists $\delta > 0$ such that $\hat{u}\left(.\right) + y\beta \in \mathcal{U}$ for all $y \in (-\delta, \delta)$ and*

$$\frac{d}{dy} J\left(\hat{u}\left(.\right) + y\beta\right) = 0, \tag{2.7}$$

and $\hat{x}\left(.\right)$ denotes the corresponding trajectory. Then, there exists a unique triplet of adapted processes

$$\left(\hat{p}\left(.\right), \hat{q}\left(.\right), \hat{k}\left(.\right)\right) \in \mathcal{S}^2_{\mathcal{F}}\left(0, T, \mathbb{R}^n\right) \times \mathcal{L}^2_{\mathcal{F}}\left(0, T, \mathbb{R}^{n\times d}\right) \times l^2_{\mathcal{F}}\left(0, T, \mathbb{R}^n\right),$$

which is solution of the backward stochastic differential equations (2.4) *such that $\hat{u}\left(.\right)$ is a stationary point for $\mathbb{E}\left[H \mid \mathcal{G}_t\right]$, in the sense that for almost all $t \in [0, T]$, we have*

$$\mathbb{E}\left[H_u\left(t, \hat{x}_t, \hat{u}_{t-}, \hat{p}_{t-}, \hat{q}_t, \hat{k}_t\right) \mid \mathcal{G}_t\right] = 0, \ \mathbb{P}\text{-}a.s. \tag{2.8}$$

Proof. By using (2.7), we get

$$\begin{aligned} 0 = \frac{d}{dy} J\left(\hat{u}\left(.\right) + y\beta\right) = \mathbb{E}\int_0^T l_x(t, \hat{x}_t, \hat{u}_t)\hat{x}_t dt \\ +\mathbb{E}\int_0^T l_u\left(t, \hat{x}_t, \hat{u}_t\right)\beta_t dt + \mathbb{E}\left(h_x\left(\hat{x}_T\right)\hat{x}_T\right). \end{aligned} \tag{2.9}$$

Applying Itô's formula to $(\hat{p}_t, \hat{x}_t)$ and using the fact that $\hat{p}_T = h_x^*(\hat{x}_T)$, we have

$$\begin{aligned}
&\mathbb{E}\left(h_x^*\left(\hat{x}_T\right)\hat{x}_T\right) \\
&= \mathbb{E}\int_0^T \hat{p}_t\left(b_x^*\left(t,\hat{x}_t,\hat{u}_t\right)\hat{x}_t + b_u^*\left(t,\hat{x}_t,\hat{u}_t\right)\beta_t\right)dt \\
&-\mathbb{E}\int_0^T \hat{x}_t H_x\left(t,\hat{x}_t,\hat{u}_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)dt \\
&+\mathbb{E}\int_0^T \sum_{i=1}^d \hat{q}_t^i\left(g_x^{i*}\left(t,\hat{x}_t,\hat{u}_t\right)\hat{x}_t + g_u^{i*}\left(t,\hat{x}_t,\hat{u}_t\right)\beta_t\right)dt \\
&+\mathbb{E}\int_0^T \sum_{i=1}^d \hat{k}_t^{(i)}\left(\sigma_x^{(i)*}\left(t,\hat{x}_t,\hat{u}_t\right)\hat{x}_t + \sigma_u^{i*}\left(t,\hat{x}_t,\hat{u}_t\right)\beta_t\right)dt.
\end{aligned} \tag{2.4}$$

Substituting (2.10) into (2.9), it follows immediately that,

$$\begin{aligned}
0 &= \mathbb{E}\int_0^T \hat{p}_t b_x^*\left(t,\hat{x}_t,\hat{u}_t\right)\hat{x}_t dt + \mathbb{E}\int_0^T \sum_{i=1}^d \hat{q}_t^i g_x^{i*}\left(t,\hat{x}_t,\hat{u}_t\right)\hat{x}_t dt \\
&+\mathbb{E}\int_0^T \sum_{i=1}^{+\infty} \hat{k}_t^{(i)}\sigma_x^{(i)*}\left(t,\hat{x}_t,\hat{u}_t\right)\hat{x}_t dt + \mathbb{E}\int_0^T l_x(t,\hat{x}_t,\hat{u}_t)\hat{x}_t dt \\
&-\mathbb{E}\int_0^T \hat{x}_t H_x\left(t,\hat{x}_t,\hat{u}_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)dt \\
&+\mathbb{E}\int_0^T \hat{p}_t b_u^*\left(t,\hat{x}_t,\hat{u}_t\right)\beta_t dt + \mathbb{E}\int_0^T \sum_{i=1}^d \hat{q}_t^i g_u^{i*}\left(t,\hat{x}_t,\hat{u}_t\right)\beta_t dt \\
&+\mathbb{E}\int_0^T \sum_{i=1}^{+\infty} \hat{k}_t^{(i)}\sigma_u^{(i)*}\left(t,\hat{x}_t,\hat{u}_t\right)\beta_t dt + \mathbb{E}\int_0^T l_u\left(t,\hat{x}_t,\hat{u}_t\right)\beta_t dt,
\end{aligned}$$

thus

$$\mathbb{E}\left[H_u\left(t,\hat{x}_t,\hat{u}_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)\beta_t\right]=0. \tag{2.11}$$

Fix $t\in[0,T]$ and apply the above to $\beta(s):=(0,...,\beta_i(s),0,...,0)$ with

$$\beta_i(s)=\alpha_i\chi_{[t,t+r]}(s),\ \ s\in[0,T],$$

where $t+r\leq T$ and $\alpha_i=\alpha_i(w)$ is bounded $\mathcal{G}_t$-measurable. Then, from (2.11), we get

$$\mathbb{E}\left[\int_t^{t+r}\frac{\partial}{\partial u_i}H\left(s,\hat{x}_s,\hat{u}_s,\hat{p}_s,\hat{q}_s,\hat{k}_s\right)\alpha_i ds\right]=0.$$

Dividing the above equality by r and sending r to 0, we conclude that

$$\mathbb{E}\left[\frac{\partial}{\partial u_i}H\left(s,\hat{x}_{s-},\hat{u}_s,\hat{p}_{s-},\hat{q}_s,\hat{k}_s\right)\alpha_i\right]=0. \tag{2.12}$$

Since (2.12) holds for all bounded $\mathcal{G}_t$-measurable α_i, we have

$$\mathbb{E}\left[H_u\left(t,\hat{x}_t,\hat{u}_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)\mid\mathcal{G}_t\right]=0,\quad \mathbb{P}\text{-}a.s,$$

this proves Theorem (2.2.1). ■

2.3 Partial information sufficient optimality condition

In this section, we prove that the partial information necessary optimality condition (2.8) is in fact sufficient, provided some convexity conditions on the Hamiltonian and the terminal cost are assumed.

Theorem 2.3.1 *(Partial information sufficient optimality condition). If we assume that the functions h and $H(t,.,u,p,q,k)$ are convex. Then, $\hat{u}(.)$ is a partial information optimal control if it satisfies* (2.8).

Proof. Let $\hat{u}(.)$ be an arbitrary element of $\mathcal{U}$ (candidate to be optimal), for any $u(.) \in \mathcal{U}$. It follows form the definition of the cost function (2.2) that

$$J(\hat{u}(.)) - J(u(.)) = \mathbb{E}\left[h(\hat{x}_T) - h(x_T)\right]$$

$$+\mathbb{E}\int_0^T \left(l(t, \hat{x}_t, \hat{u}_t) - l(t, x_t, u_t)\right) dt.$$

Since h is convex, we obtain

$$h(x_T) - h(\hat{x}_T) \geq h_x(\hat{x}_T)(x_T - \hat{x}_T),$$

thus,

$$h(\hat{x}_T) - h(x_T) \leq h_x(\hat{x}_T)(\hat{x}_T - x_T).$$

Hence,

$$J(\hat{u}(.)) - J(u(.)) \leq \mathbb{E}\left(h_x(\hat{x}_T)(\hat{x}_T - x_T)\right)$$

$$+\mathbb{E}\int_0^T \left(l(t, \hat{x}_t, \hat{u}_t) - l(t, x_t, u_t)\right) dt.$$

Using the fact that $\hat{p}_T = h_x^*(\hat{x}_T)$, then

$$\begin{aligned} J(\hat{u}(.)) - J(u(.)) &\leq \mathbb{E}\left(\hat{p}_T(\hat{x}_T - x_T)\right) \\ &\quad +\mathbb{E}\int_0^T \left(l(t, \hat{x}_t, \hat{u}_t) - l(t, x_t, u_t)\right) dt. \end{aligned}$$

On the other hand, Ito's formula applied to $(\hat{p}_t (x_t - \hat{x}_t))$ and taking expectations, we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\hat{p}_T\left(x_T-\hat{x}_T\right)\right) \\
&= -\mathbb{E}\int_0^T H_x\left(t,\hat{x}_t,\hat{u}_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)\left(\hat{x}_t-x_t\right)dt \\
&+\mathbb{E}\int_0^T \hat{p}_T\left(b\left(t,x_t,u_t\right)-b\left(t,\hat{x}_t,\hat{u}_t\right)\right)dt \\
&+\mathbb{E}\int_0^T \sum_{i=1}^{d}\hat{q}_t^i\left(g^i\left(t,x_t,u_t\right)-g^i\left(t,\hat{x}_t,\hat{u}_t\right)\right)dt \\
&\mathbb{E}\int_0^T \sum_{i=1}^{+\infty}\hat{k}_t^{(i)}\left(\sigma^{(i)}\left(t,x_t,u_t\right)+\sigma^{(i)}\left(t,\hat{x}_t,\hat{u}_t\right)\right)dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
J\left(\hat{u}\left(.\right)\right)-J\left(u\left(.\right)\right) \leq \mathbb{E}\int_0^T &\left(H\left(t,\hat{x}_t,\hat{u}_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)-H\left(t,x_t,u_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)\right)dt \\
-\mathbb{E}\int_0^T &H_x\left(t,\hat{x}_t,\hat{u}_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)\left(\hat{x}_t-x_t\right)dt.
\end{aligned}
\tag{2.13}
$$

By virtue the convexity of H in (x, u), one can get

$$
\begin{aligned}
&H\left(t,x_t,u_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)-H\left(t,\hat{x}_t,\hat{u}_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right) \\
&\geq H_x\left(t,\hat{x}_t,\hat{u}_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)\left(x_t-\hat{x}_t\right) \\
&+H_u\left(t,\hat{x}_t,\hat{u}_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)\left(u_t-\hat{u}_t\right).
\end{aligned}
\tag{2.14}
$$

Since $\mathbb{E}\left[H\left(t,\hat{x}_t,u_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)\mid\mathcal{G}_t\right]$ and u_t, $\hat{u}_t$ are $\mathcal{G}_t$-measurable, we can easily check that

$$
\begin{aligned}
&\left(\mathbb{E}\left[H_u\left(t,\hat{x}_t,u_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)\mid\mathcal{G}_t\right]\left(u_t-\hat{u}_t\right)\right) \\
&=\mathbb{E}\left[H_u\left(t,\hat{x}_t,u_t,\hat{p}_t,\hat{q}_t,\hat{k}_t\right)\left(u_t-\hat{u}_t\right)\mid\mathcal{G}_t\right]\geq 0.
\end{aligned}
\tag{2.15}
$$

Hence combining the condition (2.8), (2.14) and (2.15) , we get

$$\begin{aligned}
&H\left(t, x_t, u_t, \hat{p}_t, \hat{q}_t, \hat{k}_t\right) - H\left(t, \hat{x}_t, \hat{u}_t, \hat{p}_t, \hat{q}_t, \hat{k}_t\right) \\
&\geq H_x\left(t, \hat{x}_t, \hat{u}_t, \hat{p}_t, \hat{q}_t, \hat{k}_t\right)(x_t - \hat{x}_t).
\end{aligned}$$

Or equivalently

$$\begin{aligned}
&H\left(t, \hat{x}_t, \hat{u}_t, \hat{p}_t, \hat{q}_t, \hat{k}_t\right) - H\left(t, x_t, u_t, \hat{p}_t, \hat{q}_t, \hat{k}_t\right) - \\
&H_x\left(t, \hat{x}_t, \hat{u}_t, \hat{p}_t, \hat{q}_t, \hat{k}_t\right)(\hat{x}_t - x_t) \leq 0.
\end{aligned}$$

By the above inequality and (2.13), we have

$$J(\hat{u}(.)) - J(u(.)) \leq 0.$$

The theorem is proved. ■

2.3.1 Some particular cases

Throughout this subsection, we explain that the former study includes the following two particular cases according the definition of the Levy measure μ.

Case 1:

If $\nu = 0$; then $H_t^{(1)} = W_t$ is a standard Brownian motion and $H_t^{(i)} = 0$, for $i \geq 2$. In this case, we can reformulate the control problem as the following

Th controlled ant to minimize the cost (2.2) subject to

$$\begin{cases}
dx_t &= b(t, x_t, u_t)\,dt + \sum_{i=1}^{d} g^i(t, x_t, u_t)\,dW_t^i, \\
x_0 &= x.
\end{cases}$$

We can obtain the necessary and sufficient conditions of optimality by using the same method as in Bahlali et al [6], assuming that the Teugels martingales part is vanish.

Case 2:

In this case we assume that μ only has mass at 1, then $H_t^{(1)} = N_t - \lambda t$ is the compensated Poisson process with intensity λ and also $H_t^{(i)} = 0$, for $i \geq 2$. If we have for instance $\mu(dx) = \sum_{j=1}^{\infty} \alpha_j \delta_{\beta_j}(dx)$, where $\delta_{\beta_j}(dx)$ denotes the positive mass measure at $\beta_j \in \mathbb{R}$ of size 1.

Then, the process L takes the form

$$L_t = at + \sum_{j=1}^{\infty} \left(N_t^{(j)} + \alpha_j t \right),$$

where $\left\{ N_t^{(j)} \right\}_{j=1}^{\infty}$ denote the sequence of independent Poisson process with parameters $\{\alpha_t\}_{j=1}^{\infty}$. In this case

$$H_t^{(1)} = \sum_{j=1}^{\infty} \frac{\beta_1}{\sqrt{\alpha_j}} \left(N_t^{(j)} + \alpha_j t \right).$$

In this case, the control problem becomes:

Minimize the cost (2.2) subject to

$$\begin{cases} x_t & = b(t, x_t, u_t)dt + g(t, x_t, u_t)dW_t + \beta_1 \displaystyle\sum_{j=1}^{\infty} \frac{1}{\sqrt{\alpha_j}} d\left(N_t^{(j)} + \alpha_j t \right); \ t \in [0, T], \\ x_0 & = x \in \mathbb{R}^n. \end{cases}$$

The proof of the necessary and the sufficient condition of optimality under the partial information goes similar to Baghery et al ([8]).

2.4 Application

In this section, we give two examples to illustrate out the theoretical result of this chapter. The first one, treats the case of linear quadratic control problem under partial information. The second one deals with partial information mean-variance control problem.

2.4.1 Example 1

We apply theorem (2.2.1) and (2.3.1) to a stochastic linear quadratic optimal control problem, without terminal state constraints. Let the control domain be $U = \mathbb{R}^n$ and consider the following stochastic control problem

Minimize

$$J(u(.)) := \left[\mathbb{E}(Mx_T, x_T) + \int_0^T (Q_t x_t, x_t) + (N_t u_t, u_t)\, dt\right], \tag{2.16}$$

over the set of all admissible control $\mathcal{U}_{ad}$, subject to

$$\begin{cases} dx_t &= (A_t x_t + B_t u_t)\, dt + \sum_{i=1}^{d} (C_t^i x_t + D_t^i u_t)\, dW_t^i + \sum_{i=1}^{\infty} \left(E_t^{(i)} x_{t-} + F_t^{(i)} u_t\right) dH_t^{(i)}, \\ x_0 &= x, \end{cases} \tag{2.17}$$

where A, C^i, $E^{(i)}$, Q, $M \in \mathbb{R}^{n\times n}$, B, D^i, $F^{(i)}$ and $N \in \mathbb{R}^{n\times m}$ are bounded deterministic function and we assume that

$(\mathbf{M_5})$

$i)$ the matrix Q and M are symmetric and non negative.

$ii)$ the matrix N is uniformly positive.

To solve this problem we first write down the Hamiltonian

$$H : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times l^2(\mathbb{R}^n) \to \mathbb{R}^n,$$

by

$$\begin{aligned} H(t,x,u,p,q,k) &:= p(A_t x_t + B_t u_t) + \sum_{i=1}^{d} q^i \left(C_t^i x + D_t^i u\right) \\ &\quad + \sum_{i=1}^{\infty} k^{(i)} \left(E_t^{(i)} x + F_t^{(i)} u\right) + (Q_t x, x) + (N_t u, u). \end{aligned}$$

And the adjoint equation

$$\begin{cases} -dp_t & = \left[A_t^* p_t + \sum_{i=1}^{d} C_t^{i*} q_t^i + \sum_{i=1}^{\infty} E_t^{(i)*} k_t^{(i)} + Q_t x_t \right] dt \\ & -\sum_{i=1}^{d} q_t^i dW_t - \sum_{i=1}^{\infty} k_{t-}^{(i)} dH_t^{(i)}, \\ p_T & = M x_T, \end{cases} \tag{2.18}$$

or in the Hamiltonian form

$$\begin{cases} -dp_t & = H_x \left(t, x_t, u_t, p_t, q_t, k_t \right) dt - \sum_{i=1}^{d} q_t^i dW_t - \sum_{i=1}^{\infty} k_t^{(i)} dH_t^{(i)}, \\ p_T & = M x_T. \end{cases} \tag{2.19}$$

Let $\hat{u}\,(.)$ be a local optimal control of the partial information problem.

Then, from the theorem $(2.2.1)$, we have

$$\mathbb{E}\left[H_u \left(t, \hat{x}_t, \hat{u}_t, \hat{p}_t, \hat{q}_t, \hat{k}_t \right) \mid \mathcal{G}_t \right] = 0.$$

This gives, thanks to the definition of Hamiltonian function,

$$N_t \hat{u}_t + \mathbb{E}\left[B_t p_{t-} \mid \mathcal{G}_t \right] + \sum_{i=1}^{d} \mathbb{E}\left[D_t^{i*} q_t^i \mid \mathcal{G}_t \right] + \sum_{i=1}^{\infty} \mathbb{E}\left[F_t^{(i)} k_t^{(i)} \mid \mathcal{G}_t \right] = 0, \quad 0 \leq t \leq T \ a.s. \tag{2.5}$$

Conversely, for the sufficient part, let $\hat{u}\,(.) \in \mathcal{U}$ be a candidate to be optimal control and let $\hat{x}\,(.)$, $\left(\hat{p}_t, \hat{q}_t, \hat{k}_t \right)$ be respectively, the corresponding solution of (2.17) and (2.18).

Moreover, we suppose that the pair $(u\,(.)\,, x\,(.))$ is an arbitrary admissible control. Then, we can easily check that

$$\begin{aligned}
&J\left(u\left(.\right)\right)-J\left(\hat{u}\left(.\right)\right)= \\
&\mathbb{E}\int_0^T\left[\left(Q_t\left(x_t-\hat{x}_t\right),x_t-\hat{x}_t\right)+\left(N_t\left(u_t-\hat{u}_t\right),u_t-\hat{u}_t\right)\right. \\
&\left.+2\left(Q_t\hat{x}_t,x_t-\hat{x}_t\right)+\left(N_t\hat{u}_t,u_t-\hat{u}_t\right)\right]dt \\
&+\mathbb{E}\left[\left(M\left(x_T-\hat{x}_T\right),x_T-\hat{x}_T\right)\right. \\
&\left.+\ 2\left(M\hat{x}_T,x_T-\hat{x}_T\right)\right].
\end{aligned} \tag{2.21}$$

On the other hand, by applying Itô's formula to $(\hat{p}_t, x_t-\hat{x}_t)$ and and by taking expectations, we obtain

$$\begin{aligned}
&\mathbb{E}\left(M\hat{x}_T,x_T-\hat{x}_T\right) \\
&=\mathbb{E}\int_0^T\left[-\left(Q_t,x_t-\hat{x}_t\right)+\left(B_t^*\hat{p}_t,u_t-\hat{u}_t\right)\right. \\
&\left.+\sum_{i=1}^{d}\left(D_t^iq_t^i,u_t-\hat{u}_t\right)+\sum_{i=1}^{\infty}\left(F_t^{(i)}k_t^{(i)},u_t-\hat{u}_t\right)\right]dt,
\end{aligned} \tag{2.22}$$

where we have used the fact that $\hat{p}_T = M\hat{x}_T$.

Combining (2.21) and (2.22), we obtain

$$\begin{aligned}
&J\left(u\left(.\right)\right)-J\left(\hat{u}\left(.\right)\right) \\
&\geq\mathbb{E}\int_0^T\left[N_t\hat{u}_t+B_t^*p_t+\sum_{i=1}^{d}D_t^{i*}q_t+\sum_{i=1}^{\infty}F_t^{(i)}k_t^{(i)}\right]dt.
\end{aligned}$$

By using the above inequality and (2.8) we get

$$J\left(u\left(.\right)\right)-J\left(\hat{u}\left(.\right)\right)\geq 0.$$

Then $\hat{u}\left(.\right)$ is a partial information optimal control. Finally from (2.20) we get

$$\begin{aligned}\hat{u}_t = -N_t^{-1}&\left[\mathbb{E}\left[B_t^* p_{t-}\mid \mathcal{G}_t\right]+\sum_{i=1}^{d}\mathbb{E}\left[D_t^{i*}q_t^i\mid \mathcal{G}_t\right]\right.\\ &\left.+\sum_{i=1}^{\infty}\mathbb{E}\left[F_t^{(i)}k_t^{(i)}\mid \mathcal{G}_t\right]\right],\ 0\leq t\leq T\ a.s.\end{aligned}$$

Chapter 3

Partially Observed stochastic control problem for FBSDEs driven by Teugels martingales

This chapter studies the partially observed optimal control problem for a forward backward stochastic differential equation driven by both a Brownian motion and independent family of Teugels martingales. We first break down the difference between partial information and partial observation models. Broadly speaking the partial information case means that, the information available to the controller at time t is a subfiltration of the full information. While one can describe the partial observation model by assuming that the flow of information at time t constitutes a noisy observation of the state. Noting that, we can convert the partial observed control problem to a related a problem under a full information by using the so called the filtering theory. To wrap up, the partial information can be summarized by any subfiltration free of specific observation structures; thus it includes the partial observation models. Especially, the white noise observation models. This type of stochastic control problem is the subject of the current study.

This chapter will be organized as follows. In section 1, are devoted to the proof of the partially observed necessary and sufficient condition of optimality. In section 3, we will take an example.

3.1 Preliminaries and problem formulation

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a given filtered probability space satisfying the usual condition. We are interested in partially observed optimal control of systems driven by a forward backward stochastic differential equation of the type

$$\begin{cases} dx_t = b\left(t, x_t, u_t\right) dt + g\left(t, x_t, u_t\right) dW_t + \sum_{i=1}^{\infty} \sigma^{(i)}\left(t, x_{t-}, u_t\right) dH_t^{(i)}, \\ x_0 = x, \\ -dy_t = f\left(t, x_{t-}, y_{t-}, z_t, Z_t, u_t\right) dt - z_t dW_t - \sum_{i=1}^{\infty} Z_t^{(i)} dH_t^{(i)}, \\ y_T = \varphi\left(x_T\right), \end{cases} \tag{3.1}$$

where W is a standard d-dimensional Brownian motion and $H\left(t\right) = \left(H_t^{(i)}\right)_{i=1}^{\infty}$ are pairwise strongly orthonormal Teugels martingales, associated with some Lévy process, which is independent from W_t. b, f and $\sigma = \left(\sigma^i\right)_{i=1}^{\infty}$ are given deterministic functions and x is the initial data.

The control problem consists in minimizing the following cost functional.

$$J\left(u\right) = \mathbb{E}\left[h\left(y_0\right) + M\left(x_T\right) + \int_0^T l\left(t, x_t, y_t, z_t, Z_t, u_t\right) dt\right],$$

over a partially observed class of admissible controls to be specified later.

The two above settings imply that the random variable L_t have moments in all orders.

We also assume that

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^Y \vee \mathcal{F}_t^L \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of the $\mathbb{P}$-null set and $\mathcal{F}_t^W$, $\mathcal{F}_t^Y$ and $\mathcal{F}_t^L$ denotes the $\mathbb{P}$-completed natural filtration generated by W, Y and L respectively with two mutually independent standard Brownian motions W and Y valued in $\mathbb{R}^d$ and $\mathbb{R}^r$, respectively and an independent $\mathbb{R}^m$-valued Lévy process.

3.1.1 Formulation of the control problem

Let T be a strictly positive real number. An admissible control is an $\mathcal{F}_t^Y$- predictable process $u = (u_t)$ with values in some convex subset U of $\mathbb{R}^k$ *and satisfies*

$$\mathbb{E}\left[\int_0^T |u_t|^2 \, dt\right] < \infty.$$

We denote the set of all admissible controls by $\mathcal{U}$. The control u is called partially observable. Let us also assume that the coefficient of the controlled FBSDE (3.1) are defined as follows

$$b : [0,T] \times \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n,$$

$$g : [0,T] \times \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d},$$

$$\sigma : [0,T] \times \Omega \times \mathbb{R}^n \times U \to l^2\left(\mathbb{R}^n\right),$$

$$f : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times l^2\left(\mathbb{R}^m\right) \times U \to \mathbb{R}^m,$$

$$\varphi : \mathbb{R}^n \times \Omega \to \mathbb{R}^m.$$

We assume that the state processes (x, y, z, Z) cannot be observed directly, but the controllers can observe a related noisy process Y, called the observation process, which is described by

$$dY_t = \xi\left(t, x_t^{\upsilon}, y_t^{\upsilon}, z_t^{\upsilon}, Z_t^{\upsilon}, \upsilon_t\right) dt + dW_t^{\upsilon} \quad , \; Y_0 = 0, \tag{3.2}$$

where

$$\xi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times l^2\left(\mathbb{R}^m\right) \times U \to \mathbb{R}^n,$$

and W^{υ} is an $\mathbb{R}^r$-valued stochastic processes depending on the control υ. Define $d\mathbb{P}^{\upsilon} = \Gamma^{\upsilon} d\mathbb{P}$, where

$$\Gamma_t^{\upsilon} := \exp\left\{\int_0^t \left(\xi\left(s, x_s, y_s, z_s, Z_s, \upsilon_s\right), dY_s\right) - \frac{1}{2}\int_0^t \left|\xi\left(s, x_s, y_s, z_s, Z_s, \upsilon_s\right)\right|^2 ds\right\}.$$

Obviously Γ^{υ} is the unique $\mathcal{F}_t^Y$-adapted solution of the following SDE

$$d\Gamma_t^{\upsilon} = \Gamma_t^{\upsilon}\left(\xi\left(t, x_t, y_t, z_t, Z_t, \upsilon_t\right), dY_t\right) \quad , \; \Gamma_0^{\upsilon} = 1. \tag{3.3}$$

Then Girsanov's theorem shows that

$$dW_t^{\upsilon} = dY_t - \int_0^t \xi\left(s, x_s^{\upsilon}, y_s^{\upsilon}, z_s^{\upsilon}, Z_s^{\upsilon}, \upsilon_s\right) ds,$$

is an $\mathbb{R}^r$-valued Brownian motion and $(H_t^{(i)})_{i=1}^{\infty}$ is still a Teugels martingale under the probability measure $\mathbb{P}^{\upsilon}$.

The objective is to characterize an admissible controls which minimize the following cost functional.

$$J\left(u\right) = \mathbb{E}^u\left[h\left(y_0\right) + M\left(x_T\right) + \int_0^T l\left(t, x_t, y_t, z_t, Z_t, u_t\right) dt\right], \tag{3.4}$$

where $\mathbb{E}^u$ denotes the expectation with respect to the probability measure space $\mathbb{P}^u$ and

$$\begin{aligned}
M &: \mathbb{R}^n \times \Omega \to \mathbb{R}, \\
h &: \mathbb{R}^m \times \Omega \to \mathbb{R}, \\
l &: [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times l^2\left(\mathbb{R}^m\right) \times U \to \mathbb{R}.
\end{aligned}$$

It is obvious that the cost functional (3.4) can be rewritten as the following

$$J(u) = \mathbb{E}\left[h(y_0) + \Gamma_T M(x_T) + \int_0^T \Gamma_t l(t, x_t, y_t, z_t, Z_t, u_t)\, dt\right]. \tag{3.5}$$

Now we can state our partially observed control problem.

Problem A. Minimize (3.5) over $u \in \mathcal{U}$, subject to (3.1) and (3.3).

A control is said to be partially observed if the control is a non-anticipative functional of the observation Y. A set of controls is said to be partially observed if its every element is partially observed. Hence, the set of admissible controls $\mathcal{U}$ is partially observed.

An admissible control $\hat{u}$ is called a partially observed optimal if it attains the minimum of $J(u)$ over $\mathcal{U}$. The equations (3.1) and (3.2) are called respectively the state and the observation equations, and the solution $\left(\hat{x}, \hat{y}, \hat{z}, \hat{Z}\right)$ corresponding to $\hat{u}$ is called an optimal trajectory.

Throughout this paper, we shall make the following assumptions

(A_1)

- The random mappings b, g, σ and φ are measurable with $b(.,0,u) \in \mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R}^n)$, $g(.,0,u) \in \mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R}^n)$, $\sigma(.,0,u) \in l^2_{\mathcal{F}}(0,T,\mathbb{R}^m)$ and $\varphi(0) \in L^2(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}^m)$.

- b, g, σ and φ are continuously differentiable in (x,u). They are bounded by $(1+|x|+|u|)$ and their derivatives in (x,u) are continuous and uniformly bounded.

- The random mapping f is measurable with $f(.,0,0,0,0) \in \mathcal{L}_{\mathcal{F}}^{2}(0,T,\mathbb{R}^{m})$, f is continuous and continuously differentiable with respect to (x,y,z,Z,u). Moreover it is bounded by $(1+|x|+|y|+|z|+|Z|+|u|)$ and their derivatives are uniformly bounded.

(A_2)

- l is continuously differentiable with respect to (x,y,z,Z,u) and bounded by

$\left(1+|x|^2+|y|^2+|z|^2+|Z|^2+|u|^2\right)$. Furthermore, their derivatives are uniformly bounded.

- M is continuously differentiable in x and h is continuously differentiable in y. Moreover, for almost all $(t,\omega) \in [0,T] \times \Omega$, there exists a constant C, for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$|M_x| \leq C(1+|x|) \quad \text{and} \quad |h_y| \leq C(1+|y|).$$

(A_3) ξ is continuously differentiable in (x,y,z,Z,u) and their derivatives in (x,y,z,Z,u) are uniformly bounded.

Following [29], it holds that under assumptions (A_1), there is a unique solution

$$(x,y,z,Z) \in \mathcal{S}_{\mathcal{F}}^{2}(0,T,\mathbb{R}^{n}) \times \mathcal{S}_{\mathcal{F}}^{2}(0,T,\mathbb{R}^{m}) \times \mathcal{L}_{\mathcal{F}}^{2}\left(0,T,\mathbb{R}^{m\times d}\right) \times l_{\mathcal{F}}^{2}(0,T,\mathbb{R}^{m}),$$

which solves the state equation (3.1).

Let $(x_t^1, y_t^1, z_t^1, Z_t^1)$ and Γ_t^1 be the solutions at time$t$ of the following linear FBSDE and SDE, respectively,

$$\begin{cases} dx_t^1 & = \left(b_x(t)\, x_t^1 + b_u(t)\,(\upsilon_t - u_t)\right) dt + \left(g_x(t)\, x_t^1 + g_u(t)\,(\upsilon_t - u_t)\right) dW_t \\ & + \sum_{i=1}^{\infty} \left(\sigma_x^{(i)}(t)\, x_{t-}^1 + \sigma_u^{(i)}(t)\,(\upsilon_t - u_t)\right) dH_t^{(i)}, \\ -dy_t^1 & = [f_x(t)\, x_t^1 + f_y(t)\, y_t^1 + f_z(t)\, z_t^1 + f_Z(t)\, Z_t^1 \\ & + f_u(t)\,(\upsilon_t - u_t)]\, dt - z_t^1 dW_t - \sum_{i=1}^{\infty} Z_{t-}^{(i),1} dH_t^{(i)}, \\ x_0^1 & = 0,\ y_T^1 = \varphi_x(x_T)\, x_T^1, \end{cases} \tag{3.6}$$

and

$$\begin{cases} d\Gamma_t^1 & = \left[\Gamma_t^1 \xi^*(t) + \Gamma_t\left(\xi_x(t)\, x_t^1\right)^* + \Gamma_t\left(\xi_y(t)\, y_t^1\right)^* + \Gamma_t\left(\xi_z(t) z_t^1\right)^* \right. \\ & \left. + \Gamma_t\left(\xi_Z(t)\, Z_t^1\right)^* + \Gamma_t\left(\xi_u(t)\,(\upsilon_t - u_t)\right)^*\right] dY_t, \\ \Gamma_0^1 & = 0, \end{cases} \tag{3.7}$$

where

$$b_\rho(t) = b_\rho(t, x_t, u_t) \qquad \text{for} \quad \rho = x, u \text{ and } b = b, g, \sigma,$$

$$f_\rho(t) = f_\rho(t, x_t, y_t, z_t, Z_t, u_t) \quad \text{for} \quad \rho = x, y, z, Z, u \text{ and } f = f, \xi.$$

Set $\vartheta_t = \Gamma^{-1}\Gamma^1$ satisfies the following dynamics

$$\begin{cases} d\vartheta_t & = \left(\xi_x x_t^1 + \xi_y y_t^1 + \xi_z z_t^1 + \xi_Z Z_t^1 + \xi_u(\upsilon_t - u_t)\right) d\tilde{W}, \\ \vartheta_0 & = 0. \end{cases} \tag{3.8}$$

For any $u \in \mathcal{U}$ and the corresponding state trajectory (x, y, z, Z), we introduce the following system of forward backward SDE, called the adjoint equations :

$$\begin{cases} -dp_t & = \left(b_x^*\left(t\right)p_t + f_x^*\left(t\right)q_t + g_x^*\left(t\right)k_t + \xi_x^*(t)\Xi_t + \sum_{i=1}^{\infty}\sigma_x^{(i)*}\left(t\right)Q_t + l_x\left(t\right)\right)dt \\ & -k_t dW_t - \sum_{i=1}^{\infty}Q_t^{(i)}dH_t^{(i)}, \\ dq_t & = \left(f_y^*\left(t\right)q_t + \xi_y^*(t)\Xi_t + l_y\left(t\right)\right)dt + \left(f_z^*\left(t\right)q_t + \xi_z^*(t)\Xi_t + l_z\left(t\right)\right)dW_t \\ & +\sum_{i=1}^{\infty}\left(f_{Z^{(i)}}^*\left(t\right)q_t + \xi_{Z^{(i)}}^*(t)\Xi_t + l_{Z^{(i)}}\left(t\right)\right)dH_t^{(i)}, \\ p_T & = M_x\left(x_T\right) + \varphi_x^*\left(x_T\right)q_T,\ q_0 = h_y\left(y_0\right). \end{cases} \tag{3.9}$$

It is clear that (p,k,Q) is the adjoint process corresponding to the forward part of our system (3.1) and q is corresponding to the backward part. Manifestly, the above FBSDE admit a unique solution

$$(p,k,Q,q) \in \mathcal{S}_{\mathcal{F}}^2\left(0,T,\mathbb{R}^n\right) \times \mathcal{L}_{\mathcal{F}}^2\left(0,T,\mathbb{R}^{n\times d}\right) \times l_{\mathcal{F}}^2\left(0,T,\mathbb{R}^n\right) \times \mathcal{S}_{\mathcal{F}}^2\left(0,T,\mathbb{R}^m\right).$$

under the assumptions (A_1). We further introduce the following auxiliary BSDE, which also admit a unique solution under the assumptions (A_1),

$$-dP_t = l(t,x_t,y_t,z_t,Z_t,\upsilon_t)dt - \Xi_t d\tilde{W}_t \quad,\ P_T = M(x_T). \tag{3.10}$$

Let us now, define the Hamiltonian function

$$H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times l^2\left(\mathbb{R}^m\right) \times \mathcal{U} \times \mathbb{R}^n$$

$$\times \mathbb{R}^m \times \mathbb{R}^{n\times d} \times l^2\left(\mathbb{R}^n\right) \times \mathbb{R}^n \rightarrow \mathbb{R},$$

by

$$H\left(t,x,y,z,Z,u,p,q,P,Q,\Xi\right):=\left(p,b\left(t,x,u\right)\right)+\left(q,f\left(t,x,y,z,Z,u\right)\right)+\left(\Xi,\xi\right)$$
$$+\sum_{i=1}^{d}\left(k^{(i)},g^{(i)}\left(t,x,u\right)\right)+\sum_{i=1}^{\infty}\left(Q^{(i)},\sigma^{(i)}\left(t,x,u\right)\right)+l\left(t,x,y,z,Z,u\right).$$

3.2 A partial information necessary conditions for optimality

In this section we derive a partially observed necessary conditions for optimality for our control problem under the previous assumptions. The main objective is to solve the problem A.

3.2.1 Some auxiliary results

Let υ be an arbitrary element of $\mathcal{U}$, then for a sufficiently small $\theta > 0$ and for each $t \in [0,T]$, we define a perturbed control as follows

$$u_t^\theta = u_t + \theta\left(\upsilon_t - u_t\right).$$

Since the action space being convex, it is clear that u_t^θ is an admissible control. Let us now, pointing out that we need the following two lemmas to state and prove the main result of this section. In fact, they play a crucial role in the sequel.

Lemma 3.2.1 *If the assumptions (A_1) and (A_3) hold true, then we have the following estimates*

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|x_t^\theta - x_t\right|^2\right]=0, \tag{3.11}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|y_t^\theta - y_t\right|^2+\int_0^T\left(\left|z_t^\theta - z_t\right|^2+\left\|Z_t^\theta - Z_t\right\|^2_{l^2(\mathbb{R}^m)}\right)ds\right]=0, \tag{3.12}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\Gamma_t^\theta-\Gamma_t\right|^2\right]=0. \tag{3.13}$$

Proof. We first prove (3.11). Applying Itô's formula to $\left|x_t^\theta-x_t\right|^2$, taking expectations and using the relations

$\left\langle H^{(i)},H^{(j)}\right\rangle_s=\delta_{i,j}.t$ and $\left[H^{(i)},H^{(j)}\right]_t-\left\langle H^{(i)},H^{(j)}\right\rangle_t$ is an $\mathcal{F}_t$-martingale together with the fact that b, σ, g are uniformly Lipschitz in (x,u), one can get

$$\begin{aligned}\mathbb{E}\left|x_t^\theta-x_t\right|^2 &\leq C\mathbb{E}\int_0^t\left|x_s^\theta-x_s\right|^2ds+C\mathbb{E}\int_0^t\left|u_s^\theta-u_s\right|^2\\ &\leq C\mathbb{E}\int_0^t\left|x_s^\theta-x_s\right|^2ds+C\theta^2.\end{aligned}$$

Thus (3.11) follows immediately, by using Gronwall's lemma and letting θ goes to 0.

Let us now prove (3.12). Applying Itô's formula to $\left|y_t^\theta-y_t\right|^2$ and taking expectation to obtain

$$\mathbb{E}\left|y_t^\theta-y_t\right|^2+\mathbb{E}\int_t^T\left|z_s^\theta-z_s\right|^2ds+\mathbb{E}\int_t^T\left\|Z_s^\theta-Z_s\right\|_{l^2(\mathbb{R}^m)}^2ds=\mathbb{E}\left|\varphi\left(x_T^\theta\right)-\varphi\left(x_T\right)\right|^2$$

$$+2\mathbb{E}\int_t^T\left(y_{s-}^\theta-y_{s-}\right)\left[f\left(s,x_s^\theta,y_{s-}^\theta,z_s^\theta,Z_s^\theta,u_s^\theta\right)-f\left(s,x_s,y_{s-},z_s,Z_s,u_s\right)\right]ds.$$

From Young's inequality, for each $\varepsilon>0$, we have

$$\begin{aligned}&\mathbb{E}\left|y_t^\theta-y_t\right|^2+\mathbb{E}\int_t^T\left|z_s^\theta-z_s\right|^2ds+\mathbb{E}\int_t^T\left\|Z_s^\theta-Z_s\right\|_{l^2(\mathbb{R}^m)}^2ds\\ &\leq\mathbb{E}\left|\varphi\left(x_T^\theta\right)-\varphi\left(x_T\right)\right|^2+\frac{1}{\varepsilon}\mathbb{E}\int_t^T\left|y_s^\theta-y_s\right|^2ds\\ &+\varepsilon\mathbb{E}\int_t^T\left|f\left(s,x_s^\theta,y_{s-}^\theta,z_s^\theta,Z_s^\theta,u_s^\theta\right)-f\left(s,x_s,y_{s-},z_s,Z_s,u_s\right)\right|^2ds.\end{aligned}$$

Then,

$$
\begin{aligned}
&\mathbb{E}\left|y_t^\theta-y_t\right|^2+\mathbb{E}\int_t^T\left|z_s^\theta-z_s\right|^2 ds+\mathbb{E}\int_t^T\left\|Z_s^\theta-Z_s\right\|_{l^2(\mathbb{R}^m)}^2 ds\\
&\leq \mathbb{E}\left|\varphi\left(x_T^\theta\right)-\varphi\left(x_T\right)\right|^2+\frac{1}{\varepsilon}\mathbb{E}\int_t^T\left|y_s^\theta-y_s\right|^2 ds\\
&+C\varepsilon\mathbb{E}\int_t^T\left|f\left(s,x_s^\theta,y_s^\theta,z_s^\theta,Z_s^\theta,u_s^\theta\right)-f\left(s,x_s,y_s,z_s,Z_s,u_s^\theta\right)\right|^2 ds\\
&+C\varepsilon\mathbb{E}\int_t^T\left|f\left(s,x_s,y_s,z_s,Z_s,u_s^\theta\right)-f\left(s,x_s,y_s,z_s,Z_s,u_s\right)\right|^2 ds.
\end{aligned}
$$

Due the fact that φ and f are uniformly Lipschitz with respect to x, y, z, Z and u, one can get

$$
\begin{aligned}
&\mathbb{E}\left|y_t^\theta-y_t\right|^2+\mathbb{E}\textstyle\int_t^T\left|z_s^\theta-z_s\right|^2 ds+\mathbb{E}\displaystyle\int_t^T\left\|Z_s^\theta-Z_s\right\|_{l^2(\mathbb{R}^m)}^2 ds\\
&\leq\left(\frac{1}{\varepsilon}+C\varepsilon\right)\mathbb{E}\int_t^T\left|y_s^\theta-y_s\right|^2 ds\\
&+C\varepsilon\mathbb{E}\textstyle\int_t^T\left|z_s^\theta-z_s\right|^2 ds+C\varepsilon\mathbb{E}\displaystyle\int_t^T\left\|Z_s^\theta-Z_s\right\|_{l^2(\mathbb{R}^m)}^2 ds+\alpha_t^\theta,
\end{aligned}
\tag{3.14}
$$

where α_t^θ is given by

$$
\alpha_t^\theta=\mathbb{E}\left|x_T^\theta-x_T\right|^2+C\varepsilon\mathbb{E}\int_t^T\left|x_s^\theta-x_s\right|^2 ds+C\varepsilon\theta^2.
$$

By invoking (3.11) and sending θ to 0, we have

$$
\lim_{\theta\to 0}\alpha_t^\theta=0. \tag{3.15}
$$

We now pick up $\varepsilon = \dfrac{1}{2C}$, and replacing its value in (3.14) to obtain

$$\mathbb{E}\left|y_t^\theta - y_t\right|^2 + \frac{1}{2}\mathbb{E}\int_t^T \left|z_s^\theta - z_s\right|^2 ds + \frac{1}{2}\mathbb{E}\int_t^T \left\|Z_s^\theta - Z_s\right\|^2_{l^2(\mathbb{R}^m)} ds$$

$$\leq \left(2C + \frac{1}{2}\right)\mathbb{E}\int_t^T \left|y_s^\theta - y_s\right|^2 ds + \alpha_t^\theta.$$

Consequently, we obtain the desired result (3.12), by using Gronwall's lemma and letting θ goes to 0. We now proceed to prove (3.13). Itô's formula applied to $\left|\Gamma_t^\theta - \Gamma_t^u\right|^2$ yields

$$\mathbb{E}\left|\Gamma_t^\theta - \Gamma_t^u\right|^2 \quad \leq C\mathbb{E}\int_0^t \left|\Gamma_s^\theta - \Gamma_s^u)\right|^2 ds + C\beta_t^\theta. \tag{3.16}$$

Here β_t^θ is given by the following equality

$$\beta_t^\theta = \mathbb{E}^u \int_0^t \left|\xi\left(s, x_s^\theta, y_s^\theta, z_s^\theta, Z_s^\theta, u_s^\theta\right) - \xi\left(s, x_s, y_s, z_s, Z_s, u_s\right)\right|^2 ds.$$

Keeping in mind that ξ is continuous in (x, y, z, Z, u), it is not difficult to see that

$$\lim_{\theta\to 0} \beta_t^\theta = 0.$$

Hence, we obtain (3.13) by using Gronwall's lemma and by sending θ to 0. ■

Before we state and prove the next lemma, let us introduce the following short hand notations,

$$\tilde{\rho}_t^\theta = \theta^{-1}\left(\rho_t^\theta - \rho_t\right) - \rho_t^1, \quad \text{for} \quad \rho = x, y, z, Z \quad \text{and} \quad \Gamma. \tag{3.17}$$

Lemma 3.2.2 *Assume that* (A_1) *and* (A_3) *are in force. Then, we have the following convergence results*

$$\lim_{\theta\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T}\left|\tilde{x}_t^\theta\right|^2\right] = 0, \tag{3.18}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\tilde{y}_t^\theta\right|^2+\int_0^T\left(\left|\tilde{z}_t^\theta\right|^2+\left\|\tilde{Z}_t^\theta\right\|^2_{l^2(\mathbb{R}^m)}\right)dt\right]=0, \tag{3.19}$$

$$\mathbb{E}\int_0^T\left|\tilde{\Gamma}_t^\theta\right|^2dt=0 \tag{3.20}$$

Proof. First, we start by giving the proof of (3.18). By the notation (3.17) and the first order expansion, it is easy to check that $\tilde{x}_t^\theta$ satisfies the following SDE

$$\begin{cases} d\tilde{x}_t^\theta & =\left(b_t^x\tilde{x}_t^\theta dt+\alpha_t^\theta\right)dt+\left(g_t^x\tilde{x}_t^\theta dt+\beta_t^\theta\right)dW_t \\ & +\sum_{i=1}^{\infty}\left(\sigma_t^{(i),x}\tilde{x}_t^\theta+\gamma_t^{(i),\theta}\right)dH_t^{(i)}, \\ \tilde{x}_0^\theta & =0, \end{cases} \tag{3.21}$$

where

$$b_t^x=\int_0^1 b_x\left(t,x_t+\lambda\theta\left(\tilde{x}_t^\theta+x_t^1\right),u_t^\theta\right)d\lambda,\quad \text{for}\quad b=b,g,\sigma.$$

and

$$\begin{aligned}\alpha_t^\theta &=\int_0^1\left[b_x\left(t,x_t+\lambda\theta\left(\tilde{x}_t^\theta+x_t^1\right),u_t+\lambda\theta\left(\upsilon_t-u_t\right)\right)-b_x\left(t,x_t,u_t\right)\right]d\lambda x_t^1 \\ &+\int_0^1\left[b_u\left(t,x_t+\lambda\theta\left(\tilde{x}_t^\theta+x_t^1\right),u_t+\lambda\theta\left(\upsilon_t-u_t\right)\right)-b_u\left(t,x_t,u_t\right)\right]d\lambda\left(\upsilon_t-u_t\right), \\ \beta_t^\theta &=\int_0^1\left[g_x\left(t,x_t+\lambda\theta\left(\tilde{x}_t^\theta+x_t^1\right),u_t+\lambda\theta\left(\upsilon_t-u_t\right)\right)-g_x\left(t,x_t,u_t\right)\right]d\lambda x_t^1 \\ &+\int_0^1\left[g_u\left(t,x_t+\lambda\theta\left(\tilde{x}_t^\theta+x_t^1\right),u_t+\lambda\theta\left(\upsilon_t-u_t\right)\right)-g_u\left(t,x_t,u_t\right)\right]d\lambda\left(\upsilon_t-u_t\right),\end{aligned}$$

and

$$\begin{aligned}\gamma_t^{(i),\theta} &= \int_0^1 \left[\sigma_x^{(i)}\left(t, x_t + \lambda\theta\left(\tilde{x}_t^\theta + x_t^1\right), u_t + \lambda\theta\left(\upsilon_t - u_t\right)\right) - \sigma_x^{(i)}\left(t, x_t, u_t\right)\right] d\lambda x_t^1 \\ &+ \int_0^1 \left[\sigma_u^{(i)}\left(t, x_t + \lambda\theta\left(\tilde{x}_t^\theta + x_t^1\right), u_t + \lambda\theta\left(\upsilon_t - u_t\right)\right) - \sigma_u^{(i)}\left(t, x_t, u_t\right)\right] d\lambda\left(\upsilon_t - u_t\right).\end{aligned}$$

Since b_x, b_u, g_x, g_u and σ_x, σ_u are continuous in (x, u), it is not difficult to see that

$$\lim_{\theta\to 0}\mathbb{E}\left(\left|\alpha_t^\theta\right|^2 + \left|\beta_t^\theta\right|^2 + \left|\gamma_t^{(i),\theta}\right|^2\right) = 0. \tag{3.22}$$

Applying Itô's formula to $\left(\tilde{x}_t^\theta\right)^2$, we get

$$\begin{aligned}\mathbb{E}\left|\tilde{x}_t^\theta\right|^2 &= 2\mathbb{E}\int_0^t \tilde{x}_s^\theta\left(b_s^x\tilde{x}_s^\theta + \alpha_s^\theta\right) ds + \mathbb{E}\int_0^t \left|\left(g_s^x\tilde{x}_s^\theta + \beta_s^\theta\right)\right|^2 ds \\ &+ \sum_{i=1}^\infty \mathbb{E}\int_0^t \left|\sigma_s^{(i),x}\tilde{x}_s^\theta + \gamma_s^{(i),\theta}\right|^2 ds.\end{aligned}$$

Using the inequality $2ab \leq a^2 + b^2$, seeing that b_s^x, g_s^x and σ_s^x are bounded, to obtain

$$\mathbb{E}\left|\tilde{x}_t^\theta\right|^2 \leq (1 + 2C)\,\mathbb{E}\int_0^t \left|\tilde{x}_s^\theta\right|^2 ds + \mathbb{E}\int_0^t \left(\left|\alpha_s^\theta\right|^2 + \left|\beta_s^\theta\right|^2 + \left|\gamma_s^{(i),\theta}\right|^2\right) ds.$$

Finally, by using Gronwall's lemma and (3.22), we obtain (3.18).

We now turn out to prove (3.19). Again, in view of the notations (3.17), one can easily show that

$\left(\tilde{y}_t^\theta, \tilde{z}_t^\theta, \tilde{Z}_t^\theta\right)$ satisfies the following BSDE

$$\begin{cases} d\tilde{y}_t^\theta = \left(f_t^x\tilde{x}_t^\theta + f_t^y\tilde{y}_t^\theta + f_t^z\tilde{z}_t^\theta + f_t^Z\tilde{Z}_t^\theta + \chi_t^\theta\right) dt + \tilde{z}_t^\theta dW_t + \sum_{i=1}^\infty \tilde{Z}_t^\theta dH_t^{(i)}, \\ \tilde{y}_T^\theta = \theta^{-1}\left(\varphi\left(x_T^\theta\right) - \varphi\left(x_T\right)\right) - \varphi_x\left(x_T\right) x_T^1, \end{cases}$$

where $\tilde{x}_t^\theta$ is the solution to the SDE (3.21) and

$$f_t^x = -\int_0^1 f_x\left(\Lambda_t^\theta\left(u_t\right)\right) d\lambda, \quad \text{for} \quad x = x, y, z, Z,$$

and

$$\begin{aligned} \chi_t^\theta \ &= \int_0^1 \left(f_x\left(\Lambda_s^\theta\left(u_t\right)\right) - f_x\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda \ x_t^1 \\ &+ \int_0^1 \left(f_y\left(\Lambda_s^\theta\left(u_t\right)\right) - f_y\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda \ y_t^1 \\ &+ \int_0^1 \left(f_z\left(\Lambda_s^\theta\left(u_t\right)\right) - f_z\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda \ z_t^1 \\ &+ \int_0^1 \left(f_Z\left(\Lambda_s^\theta\left(u_t\right)\right) - f_Z\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda \ Z_t^1 \\ &+ \int_0^1 \left(f_u\left(\Lambda_s^\theta\left(u_t\right)\right) - f_u\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda \ \left(\upsilon_t - u_t\right), \end{aligned}$$

and

$$\begin{aligned} \Lambda_t^\theta\left(u\right) = \Big(&t, x_t + \lambda\theta\left(\tilde{x}_t^\theta + x_t^1\right), y_t + \lambda\theta\left(\tilde{y}_t^\theta + y_t^1\right), \\ &z_t + \lambda\theta\left(\tilde{z}_t^\theta + z_t^1\right), Z_t + \lambda\theta\left(\tilde{Z}_t^\theta + Z_t^1\right), u_t + \lambda\theta\left(\upsilon_t - u_t\right)\Big). \end{aligned}$$

Due the fact that f_x, f_y, f_z and f_Z are continuous, we have

$$\lim_{\theta\to 0}\mathbb{E}\left|\chi_t^\theta\right|^2 = 0. \tag{3.23}$$

Again, Itô's formula applied to $\left|\tilde{y}_t^\theta\right|^2$ leads to the following equality

$$\begin{aligned} &\mathbb{E}\left|\tilde{y}_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|\tilde{z}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left\|\tilde{Z}_s^\theta\right\|_{l^2(\mathbb{R}^m)}^2 \\ &= \mathbb{E}\left|\tilde{y}_T^\theta\right|^2 + 2\mathbb{E}\int_t^T \tilde{y}_s^\theta\left(f_s^x\tilde{x}_s^\theta + f_s^y\tilde{y}_s^\theta + f_s^z\tilde{z}_s^\theta + f_s^Z\tilde{Z}_s^\theta + \chi_s^\theta\right) ds. \end{aligned}$$

By using Young's inequality, for each $\varepsilon > 0$, we obtain

$$\begin{aligned}
&\mathbb{E}\left|\tilde{y}_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|\tilde{z}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left\|\tilde{Z}_s^\theta\right\|_{l^2(\mathbb{R}^m)}^2 ds \\
&\leq \mathbb{E}\left|\tilde{y}_T^\theta\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|\tilde{y}_s^\theta\right|^2 ds + \varepsilon\mathbb{E}\int_t^T \left|\left(f_s^x\tilde{x}_s^\theta + f_s^y\tilde{y}_s^\theta + f_s^z\tilde{z}_s^\theta + f_s^Z\tilde{Z}_s^\theta + \chi_s^\theta\right)\right|^2 ds \\
&\leq \mathbb{E}\left|\tilde{y}_T^\theta\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|\tilde{y}_s^\theta\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left|f_s^x\tilde{x}_s^\theta\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left|f_s^y\tilde{y}_s^\theta\right|^2 ds \\
&+C\varepsilon\mathbb{E}\int_t^T \left|f_s^z\tilde{z}_s^\theta\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left|f_s^Z\tilde{Z}_s^\theta\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left|\chi_s^\theta\right|^2 ds
\end{aligned}$$

It follows that, in view of the boundedness of f_t^x, f_t^y, f_t^z and f_t^Z,

$$\begin{aligned}
&\mathbb{E}\left|\tilde{y}_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|\tilde{z}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left\|\tilde{Z}_s^\theta\right\|_{l^2(\mathbb{R}^m)}^2 ds \\
&\leq \left(\frac{1}{\varepsilon} + C\varepsilon\right)\mathbb{E}\int_t^T \left|\tilde{y}_s^\theta\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left|\tilde{z}_s^\theta\right|^2 ds \\
&+C\varepsilon\mathbb{E}\int_t^T \left\|\tilde{Z}_s^\theta\right\|_{l^2(\mathbb{R}^m)}^2 ds + \mathbb{E}\left|\tilde{y}_T^\theta\right|^2 + C\varepsilon\eta_t^\theta,
\end{aligned}$$

where

$$\eta_t^\theta = \mathbb{E}\int_t^T \left|f_s^x\tilde{x}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left|\chi_s^\theta\right|^2 ds.$$

Hence, in view of (3.18), the fact that φ_x, f_s^x are continuous and bounded, we get

$$\lim_{\theta\to 0}\mathbb{E}\left|\tilde{y}_T^\theta\right|^2 = 0. \tag{3.24}$$

and

$$\lim_{\theta\to 0}\mathbb{E}\int_t^T \left|f_s^x\tilde{x}_s^\theta\right|^2 ds = 0. \tag{3.25}$$

Furthermore, From (3.23) and (3.25), we deduce that

$$\lim_{\theta\to 0}\eta_t^\theta = 0. \tag{3.26}$$

If we choose $\varepsilon = \dfrac{1}{2C}$, it holds that,

$$\mathbb{E}\left|\tilde{y}_t^{\theta}\right|^2 + \frac{1}{2}\mathbb{E}\int_t^T \left|\tilde{z}_s^{\theta}\right|^2 ds + \frac{1}{2}\mathbb{E}\int_t^T \left\|\tilde{Z}_s^{\theta}\right\|_{l^2(\mathbb{R}^m)}^2 ds$$

$$\leq \left(2C + \frac{1}{2}\right)\mathbb{E}\int_t^T \left|\tilde{y}_s^{\theta}\right|^2 ds + \frac{1}{2}\eta_t^{\theta}.$$

The estimates (3.19) follow from an application of Gronwall's lemma together with (3.23) and (3.26).

Now we proceed to prove (3.20). From (3.17), it is plain to check that $\tilde{\Gamma}^{\theta}$ satisfies the following

equality,

$$\begin{aligned} d\tilde{\Gamma}^{\theta} &= \left[\tilde{\Gamma}_t^{\theta}\xi\left(t, x_t^{\theta}, y_t^{\theta}, z_t^{\theta}, Z_t^{\theta}, u_t^{\theta}\right) + \bar{\chi}_t^{\theta}\right] dY_t \\ &\quad + \Gamma_t\left\{\xi_t^x\tilde{x}_t^{\theta} + \xi_t^y\tilde{y}_t^{\theta} + \xi_t^z\tilde{z}_t^{\theta} + \xi_t^Z\tilde{Z}_t^{\theta}\right\} dY_t, \end{aligned}$$

where

$$\xi_t^x = \int_0^1 \xi_x\left(\Lambda_t^{\theta}\left(u_t\right)\right) d\lambda, \quad \text{for} \quad x = x, y, z, Z,$$

and $\bar{\chi}_t^{\theta}$ is given by

$$\begin{aligned} \bar{\chi}_t^{\theta} &= \Gamma_t\left[\int_0^1 \left(\xi_x\left(\Lambda_s^{\theta}\left(u_t\right)\right) - \xi_x\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda\ x_t^1\right. \\ &\quad + \int_0^1 \left(\xi_y\left(\Lambda_s^{\theta}\left(u_t\right)\right) - \xi_y\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda\ y_t^1 \\ &\quad + \int_0^1 \left(\xi_z\left(\Lambda_s^{\theta}\left(u_t\right)\right) - \xi_z\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda\ z_t^1 \\ &\quad + \int_0^1 \left(\xi_Z\left(\Lambda_s^{\theta}\left(u_t\right)\right) - \xi_Z\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda\ Z_t^1 \\ &\quad \left. + \int_0^1 \left(\xi_u\left(\Lambda_s^{\theta}\left(u_t\right)\right) - \xi_u\left(t, x_t, y_t, z_t, Z_t, u_t\right)\right) d\lambda\ \left(v_t - u_t\right)\right] \\ &\quad + \Gamma_t^1\left[\xi\left(t, x_t^{\theta}, y_t^{\theta}, z_t^{\theta}, Z_t^{\theta}, u_t^{\theta}\right) - \xi(t)\right], \end{aligned}$$

We deduce, taking into account the fact that ξ_x, ξ_y, ξ_z and ξ_Z are continuous,

$$\lim_{\theta\to 0}\mathbb{E}\left|\bar{\chi}_t^\theta\right|^2 = 0. \tag{3.27}$$

Applying Itô's formula to $\left|\tilde{\Gamma}_t^\theta\right|^2$, taking expectation, and using the fact that ξ, ξ_t^x, ξ_t^y, ξ_t^z and ξ_t^Z are bounded, to obtain

$$\begin{aligned}\mathbb{E}\left|\tilde{\Gamma}_t^\theta\right|^2 &\leq C\mathbb{E}\textstyle\int_0^T\left|\tilde{\Gamma}_t^\theta\right|^2 dt + C\mathbb{E}\displaystyle\int_0^T\left|\tilde{x}_t^\theta\right|^2 dt\\ &+C\mathbb{E}\int_0^T\left|\tilde{y}_t^\theta\right|^2 dt + C\mathbb{E}\int_0^T\left|\tilde{z}_t^\theta\right|^2 dt\\ &+C\mathbb{E}\int_0^T\left\|\tilde{Z}_t^\theta\right\|^2 dt + C\mathbb{E}\int_0^T\left|\bar{\chi}_t^\theta\right|^2 dt.\end{aligned}$$

Keeping in mind the relations (3.18) and (3.27), we deduce that, using Gronwall's inequality,

$$\lim_{\theta\to 0}\sup_{0\leq t\leq T}\mathbb{E}\left|\tilde{\Gamma}_t^\theta\right|^2 = 0.$$

■

3.2.2 Variational inequality and optimality necessary conditions

Since u is an optimal control, then, with the fact that $\theta^{-1}\left[J\left(u_t^\theta\right) - J\left(u_t\right)\right] \geq 0$, we have the following lemma.

Lemma 3.2.3 *Suppose that the assumptions (A_1), (A_2) and (A_3) are satisfied. Then the following variational inequality holds*

$$\begin{aligned}&0 \leq \mathbb{E}\left[\Gamma_T M_x\left(x_T\right)x_T^1 + h_y\left(y_0\right)y_0^1 + \Gamma_T^1 M(x_T)\right]\\ &+\mathbb{E}\int_0^T\left(\Gamma_t^1 l(t) + \Gamma_t\left(l_x\left(t\right)x_t^1 + l_y\left(t\right)y_t^1 + l_z\left(t\right)z_t^1 + l_Z\left(t\right)Z_t^1 + l_u\left(t\right)\left(\upsilon_t - u_t\right)\right)\right)dt,\end{aligned} \tag{3.28}$$

where $l_\rho(t) = l_\rho(t, x_t, y_t, z_t, Z_t, u_t)$ *for* $\rho = x,\ y,\ z,\ Z$.

Proof. From the definition of the cost functional and by using the first order development, one can get

$$\begin{aligned}
0 \leq \theta^{-1}\left[J\left(u_t^\theta\right) - J\left(u_t\right)\right] &= \theta^{-1}\mathbb{E}\left[\left(\Gamma_T^\theta - \Gamma_T\right) M\left(x_T^\theta\right)\right] \\
&+\theta^{-1}\mathbb{E}\left[\Gamma_T \int_0^1 M_x\left(x_T + \lambda\left(x_T^\theta - x_T\right)\right)\left(x_T^\theta - x_T\right) d\lambda\right] \\
&+\theta^{-1}\mathbb{E}\left[\int_0^1 h_y\left(y_0 + \lambda\left(y_0^\theta - y_0\right)\right)\left(y_0^\theta - y_0\right) d\lambda\right] \\
&+\theta^{-1}\mathbb{E}\left[\int_0^T l\left(t, x_t^\theta, y_t^\theta, z_t^\theta, Z_t^\theta, u_t^\theta\right)\left(\Gamma_t^\theta - \Gamma_t\right) dt\right] \\
&+\theta^{-1}\mathbb{E}\left[\int_0^T \Gamma_t \left(\int_0^1 \left(l_x\left(\Lambda_t^\theta(u)\right)\left(x_t^\theta - x_t\right)\right.\right.\right. \\
&+l_y\left(\Lambda_t^\theta(u)\right)\left(y_t^\theta - y_t\right) + l_z\left(\Lambda_t^\theta(u)\right)\left(z_t^\theta - z_t\right) \\
&\left.\left.\left.+l_Z\left(\Lambda_t^\theta(u)\right)\left(Z_t^\theta - Z_t\right) + l_u\left(\Lambda_t^\theta(u)\right)\left(u_t^\theta - u_t\right)\right) d\lambda\right) dt\right].
\end{aligned}$$

Finally by using (3.18), (3.19), (3.20) and letting θ goes to 0, we obtain (3.28). In view of (3.8), the variational inequality (3.28) can be rewritten as

$$\begin{aligned}
&0 \leq \mathbb{E}^u\left[M_x\left(x_T\right) x_T^1\right] + \mathbb{E}^u\left[h_y\left(y_0\right) y_0^1\right] + \mathbb{E}^u\left[\vartheta_T M(x_T)\right] \\
&+\mathbb{E}^u \int_0^T \left(\vartheta_t l(t) + \left(l_x(t) x_t^1 + l_y(t) y_t^1 + l_z(t) z_t^1 + l_Z(t) Z_t^1 + l_u(t)\left(\upsilon_t - u_t\right)\right)\right) dt,
\end{aligned} \tag{3.29}$$

The main result of this section can be stated us follows. ■

Theorem 3.2.1 *(Partial information maximum principle) Suppose (A_1), (A_2) and (A_3) hold. Let (x, y, z, Z, u) be an optimal solution of the control problem A. There are 4-typle (p, q, k, Q) and a pair (P, Ξ) of $\mathcal{F}_t$-adapted processes which satisfy (3.9) and (3.10) respectively, such that the following maximum principle holds true,*

$$\mathbb{E}^u\left[\left(H_v\left(t, x_t, y_t, z_t, Z_t, u_t, p_t, q_t, k_t, Q_t, \Xi_t\right), (v_t - u_t)\right) \mid \mathcal{F}_t^Y\right] \geq 0,\ \forall v \in \mathcal{U},\ a.\mathbb{E},\ a.s. \tag{3.30}$$

Proof. By applying Itô's formula to (p_t, x_t^1) and (q_t, y_t^1) and using the fact that $q_0 = h_y(y_0)$ and $p_T = M_x\left(x_T\right) + \varphi_x\left(x_T\right) q_T$, we have

$$\begin{aligned}
\mathbb{E}^u\left[M_x\left(x_T\right) x_T^1\right] + \mathbb{E}^u\left[\varphi_x\left(x_T\right) q_T x_T^1\right] &= -\mathbb{E}^u \int_0^T f_x\left(t, x_t, y_t, z_t, Z_t, u_t\right) q_t x_t^1 dt \\
&-\mathbb{E}^u \int_0^T l_x\left(t, x_t, y_t, z_t, Z_t, u_t\right) x_t^1 dt \\
&-\mathbb{E}^u \int_0^T \xi_x\left(t, x_t, y_t, z_t, Z_t, u_t\right) \Xi_t x_t^1 dt \\
&+\mathbb{E}^u \int_0^T b_u\left(t, x_t, u_t\right)\left(v_t - u_t\right) p_t dt \\
&+\mathbb{E}^u \int_0^T g_u\left(t, x_t, u_t\right)\left(v_t - u_t\right) k_t dt \\
&+\sum_{i=1}^{\infty} \mathbb{E}^u \int_0^T \sigma_u^{(i)}\left(t, x_t, u_t\right)\left(v_t - u_t\right) Q_t^{(i)} dt,
\end{aligned} \tag{3.31}$$

and

$$
\begin{aligned}
-\mathbb{E}^u\left[\varphi_x\left(x_T\right)q_T x_T^1\right]+\mathbb{E}^u\left[h_y(y_0)y_0^1\right] &= \mathbb{E}^u\int_0^T f_x\left(t,x_t,y_t,z_t,Z_t,u_t\right)q_t x_t^1 dt \\
&+\mathbb{E}^u\int_0^T f_v\left(t,x_t,y_t,z_t,Z_t,u_t\right)\left(\upsilon_t-u_t\right)q_t dt \\
&-\mathbb{E}^u\int_0^T \left(l_y\left(t\right)y_t^1+l_z\left(t\right)z_t^1+\sum_{i=1}^{\infty} l_{Z^{(i)}}\left(t\right)Z_t^{(i)1}\right)dt \\
&-\mathbb{E}^u\int_0^T \Xi_t\left(\xi_y\left(t\right)y_t^1+\xi_z\left(t\right)z_t^1+\sum_{i=1}^{\infty}\xi_{Z^{(i)}}\left(t\right)Z_t^{(i)1}\right)dt.
\end{aligned} \tag{3.32}
$$

On the other hand, Itô's formula applied to (ϑ_t, P_t), gives us

$$
\begin{aligned}
&\mathbb{E}^u\left(\vartheta_T M\left(x_T\right)\right) \\
&= -\mathbb{E}^u\int_0^T \vartheta_t l(t)dt+\mathbb{E}^u\int_0^T \Xi_t\left(\xi_x x_t^1+\xi_y y_t^1+\xi_z z_t^1+\xi_Z Z_t^1+\xi_v\left(\upsilon_t-u_t\right)\right)dt.
\end{aligned} \tag{3.33}
$$

Consequently, From (3.31), (3.32) and (3.33), we infer that

$$
\begin{aligned}
&\mathbb{E}^u\left[M_x\left(x_T\right)x_T^1\right]+\mathbb{E}^u\left[h_y(y_0)y_0^1\right]+\mathbb{E}^u\left[\vartheta_T M\left(x_T\right)\right] \\
&= \mathbb{E}^u\int_0^T \left(b_v\left(t,x_t,u_t\right)p_t\left(\upsilon_t-u_t\right)+gv\left(t,x_t,u_t\right)k_t\left(\upsilon_t-u_t\right)+l_v\left(t\right)\left(\upsilon_t-u_t\right)\right. \\
&\left.+\, f_v\left(t\right)\left(\upsilon_t-u_t\right)q_t+\Xi_t\xi_v\left(\upsilon_t-u_t\right)\right)+\sum_{i=1}^{\infty}\sigma_u^{(i)}\left(t,x_t,u_t\right)Q_t^{(i)}\left(\upsilon_t-u_t\right)\Big)dt \\
&-\mathbb{E}^u\int_0^T \left(\vartheta_t l(t)+l_x\left(t\right)x_t^1+l_y\left(t\right)y_t^1+l_z\left(t\right)z_t^1+\sum_{i=1}^{\infty} l_{Z^{(i)}}\left(t\right)Z_t^{(i)1}+l_v\left(t\right)\left(\upsilon_t-u_t\right)\right)dt,
\end{aligned}
$$

thus

$$\mathbb{E}^{u}\left[M_{x}\left(x_{T}\right)x_{T}^{1}\right]+\mathbb{E}^{u}\left[h_{y}(y_{0})y_{0}^{1}\right]+\mathbb{E}^{u}\left[\vartheta_{T}M\left(x_{T}\right)\right]$$

$$=\mathbb{E}^{u}\int_{0}^{T}H_{v}\left(t,x_{t},y_{t},z_{t},Z_{t},u_{t},p_{t},q_{t},k_{t},Q_{t}\right)\left(\upsilon_{t}-u_{t}\right)dt$$

$$-\mathbb{E}^{u}\int_{0}^{T}\left(l_{x}\left(t\right)x_{t}^{1}+l_{y}\left(t\right)x_{y}^{1}+l_{z}\left(t\right)z_{t}^{1}+\sum_{i=1}^{\infty}l_{Z^{(i)}}\left(t\right)Z_{t}^{(i)1}+l_{u}\left(t\right)\left(\upsilon_{t}-u_{t}\right)\right)dt,$$

This together with the variational inequality(3.29) imply (3.30), which achieve the proof.

■

3.3 A partial information sufficient conditions of optimality

In this section, we will prove that the partial information maximum principle condition for the Hamiltonian function is in fact sufficient under additional convexity assumptions. It should be noted that we shall prove our result in two different cases. In the first case, we are going to prove the sufficient condition without assuming the linearity of the terminal condition for the backward part of the state equation. To this end, we restrict ourselves to the one dimensional case $n=m=1$ and we state now the main result of this section.

Theorem 5.5 Suppose (A_1), (A_2) and (A_3) hold. Assume further that the functions φ, M and $H\left(t,.,.,.,.,p_t,q_t,k_t,Q_t,\Xi_t\right)$ are convex, h is convex function and increasing. If the following maximum condition holds

$$\mathbb{E}^{u}\left(H_{v}\left(t,x_{t},y_{t},z_{t},Z_{t},u_{t},p_{t},q_{t},k_{t},Q_{t},\Xi_{t}\right),\left(\upsilon_{t}-u_{t}\right)\mid\mathcal{F}_{t}^{Y}\right)\geq0, \tag{3.34}$$

$\forall\upsilon_t\in\mathcal{U}$, *a.e, a.s, then u is an optimal control in the sense that* $J\left(u\right)\leq\inf_{\upsilon\in\mathcal{U}}J\left(\upsilon\right).$

Proof. Let u be an arbitrary element of $\mathcal{U}$ (candidate to be optimal) and (x^u, y^u, z^u, Z^u) is the corresponding trajectory. For any $\upsilon \in \mathcal{U}$ and its corresponding trajectory $(x^\upsilon, y^\upsilon, z^\upsilon, Z^\upsilon)$, by the definition of the cost function (3.5), one can write

$$J(\upsilon) - J(u) = \mathbb{E}\left[\Gamma_T^\upsilon M(x_T^\upsilon) - \Gamma_T^u M(x_T^u)\right] + \mathbb{E}\left[h(y_0^\upsilon) h(y_0^u) - h(y_0^u)\right]$$

$$+\mathbb{E}\int_0^T \left(\Gamma_t^\upsilon l(t, x_t^\upsilon, y_t^\upsilon, z_t^\upsilon, Z_t^\upsilon, \upsilon_t) - \Gamma_t^u l(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t)\right) dt.$$

Since h and M are convex

$$\mathbb{E}\left[h(y_0^\upsilon) - h(y_0^u)\right] \geq \mathbb{E}\left(h_y(y_0^u)(y_0^\upsilon - y_0^u)\right),$$

$$\text{and}$$

$$\mathbb{E}\left(\Gamma_T^\upsilon M(x_T^\upsilon) - \Gamma_T^u M(x_T^u)\right) \geq \tag{3.35}$$

$$\mathbb{E}\left[\left(\Gamma_T^\upsilon - \Gamma_T^u\right) M(x_T^u)\right] + \mathbb{E}^u\left[M_x(x_T^u)(x_T^\upsilon - x_T^u)\right].$$

And

$$\mathbb{E}\int_0^T \left(\Gamma_t^\upsilon l(t, x_t^\upsilon, y_t^\upsilon, z_t^\upsilon, Z_t^\upsilon, \upsilon_t) - \Gamma_t^u l(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t)\right) dt$$

$$= \mathbb{E}\int_0^T \Gamma_t^\upsilon \left(l(t, x_t^\upsilon, y_t^\upsilon, z_t^\upsilon, Z_t^\upsilon, \upsilon_t) - l(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t)\right) dt \tag{3.36}$$

$$+\mathbb{E}\int_0^T \left(\Gamma_t^\upsilon - \Gamma_t^u\right) l(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t)\, dt.$$

Thus

$$
\begin{aligned}
& J(v) - J(u) \\
& \geq \mathbb{E}^u \left[M_x \left(x_T^u \right) \left(x_T^v - x_T^u \right) \right] + \mathbb{E} \left[h_y \left(y_0^u \right) \left(y_0^v - y_0^u \right) \right] \\
& + \mathbb{E}^u \int_0^T \left(l \left(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t \right) - l \left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t \right) \right) dt \\
& + \mathbb{E} \left[\left(\Gamma_T^v - \Gamma_T^u \right) \left(\int_0^T l \left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t \right) dt + M \left(x_T^u \right) \right) \right].
\end{aligned}
$$

Noting that

$$
\begin{aligned}
p_T &= M_x \left(x_T \right) + \varphi_x^* \left(x_T \right) q_T, \\
q_0 &= h_y \left(y_0 \right),
\end{aligned}
$$

then, we have

$$
\begin{aligned}
& J(v) - J(u) \geq \mathbb{E}^u \left[p_T^u \left(x_T^v - x_T^u \right) \right] - \mathbb{E}^u \left[q_T^u \varphi_x \left(x_T \right) \left(x_T^v - x_T^u \right) \right] + \mathbb{E} \left[q_0^u \left(y_0^v - y_0^u \right) \right] \\
& + \mathbb{E}^u \int_0^T \left(l \left(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t \right) - l \left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t \right) \right) dt \\
& + \mathbb{E} \left[\left(\Gamma_T^v - \Gamma_T^u \right) \left(\int_0^T l \left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t \right) dt + M \left(x_T^u \right) \right) \right],
\end{aligned}
$$

by using the fact that h is convex function and increasing, we can write

$$
\begin{aligned}
& J(v) - J(u) \geq \mathbb{E}^u \left[p_T^u \left(x_T^v - x_T^u \right) \right] - \mathbb{E}^u \left[q_T^u \left(y_T^v - y_T^u \right) \right] + \mathbb{E} \left[q_0^u \left(y_0^v - y_0^u \right) \right] \\
& + \mathbb{E}^u \int_0^T \left(l \left(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t \right) - l \left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t \right) \right) dt \\
& + \mathbb{E} \left[\left(\Gamma_T^v - \Gamma_T^u \right) \left(\int_0^T l \left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t \right) dt + M \left(x_T^u \right) \right) \right].
\end{aligned}
$$

On other hand, by applying Ito's formula respectively to $p_t^u \left(x_t^v - x_t^u \right)$, $q_t^u \left(y_t^v - y_t^u \right)$ and

$P_t^u\left(\Gamma_t^v-\Gamma_t^u\right)$, and by taking expectations to the previous inequality, we get

$$
\begin{aligned}
& J(v)-J(u) \\
& \geq \mathbb{E}^u \int_0^T\left(H\left(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\right. \\
& \left.-H\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\right) d t \\
& -\mathbb{E}^u \int_0^T H_x\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(x^v-x^u\right) d t \\
& -\mathbb{E} \int_0^T H_y\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(y^v-y^u\right) d t \\
& -\mathbb{E}^u \int_0^T H_z\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(z^v-z^u\right) d t \\
& -\sum_{i=0}^{+\infty} \mathbb{E}^u \int_0^T H_{Z^{(i)}}\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(Z^{(i) v}-Z^{(i) u}\right) d t .
\end{aligned}
\tag{3.37}
$$

By using the fact H is convex in (x, y, z, Z, u), we get

$$
\begin{aligned}
& \left(H\left(t, x_t^v, y_t^v, z_t^v, Z_t^v, v_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\right. \\
& \left.-H\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\right) \\
& \geq H_x\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(x_t^v-x_t^u\right) \\
& +H_y\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(y_t^v-y_t^u\right) \\
& +H_z\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(z_t^v-z_t^u\right) \\
& +H_Z\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(Z_t^v-Z_t^u\right) \\
& +H_v\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(v_t-u_t\right) .
\end{aligned}
\tag{3.38}
$$

Substituting (3.38) into (3.37), we have

$$
J(v)-J(u) \quad \geq \mathbb{E}^u \int_0^T H_v\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(v_t-u_t\right) d t,
$$

and thus

$$
J(v)-J(u) \geq \mathbb{E} \int_0^T \Gamma_t^u \mathbb{E}\left[H_v\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k, Q_t^u, \Xi_t^u\right)\left(u_t-v_t\right) \mid \mathcal{F}_t^Y\right] d t,
$$

in view of the condition (3.34) above and keeping in mind that $\Gamma_t^v > 0$, one can get $J(u) - J(v) \leq 0$, which achieve the proof. ■

Before we treat the second result of this section, it is worth to pointing out that we can prove a partial observed sufficient conditions of optimality without assuming neither that x and y need to be in the dimension one, nor that the function φ needs to be negative and decreasing.

Assume that $\varphi(x) = Nx$, where N is a nonzero constant matrix with order $m \times n$. Then, by using similar arguments developed above, we can easily state and prove the following theorem which illustrate the second case.

Theorem 3.3.1 *Assume that* (A_1), (A_2) *and* (A_3) *are in force. Assume that the functions* $h(.)$, $M(.)$ *and* $H(t,.,.,.,.,p_t,q_t,k_t,Q_t,\Xi_t)$ *are convex with* $\varphi(x) = Nx$. *If further the maximum condition* (3.34) *holds true, then* u *is an optimal control in the sense that*

$$J(u) \leq \inf_{v \in \mathcal{U}} J(v). \tag{3.39}$$

Proof. Let v be an arbitrary element of $\mathcal{U}$ and (x^v, y^v, z^v, Z^v) is its corresponding trajectory. By using the definition of cost functional (3.5), taking under consideration the convexity property of h and M, a simple computation gives us

$$\begin{aligned}
&J(v) - J(u)\\
&\geq \mathbb{E}^u\left[M_x\left(x_T^u\right)\left(x_T^v - x_T^u\right)\right] + \mathbb{E}\left[h_y\left(y_0^u\right)\left(y_0^v - y_0^u\right)\right]\\
&+\mathbb{E}^u\int_0^T \left(l\left(t,x_t^v,y_t^v,z_t^v,Z_t^v,v_t\right) - l\left(t,x_t^u,y_t^u,z_t^u,Z_t^u,u_t\right)\right)dt\\
&+\mathbb{E}\left[\left(\Gamma_T^v - \Gamma_T^u\right)\left(\int_0^T l\left(t,x_t^u,y_t^u,z_t^u,Z_t^u,u_t\right)dt + M\left(x_T^u\right)\right)\right].
\end{aligned}$$

On the other hand, in view of $\varphi(x) = Nx$, we apply Ito's formula to $p_t^u\left(x_t^v - x_t^u\right)$, $q_t^u\left(y_t^v - y_t^u\right)$ and $P_t^u\left(\Gamma_t^v - \Gamma_t^u\right)$, respectively, then by combining their results together with the above inequality one can get

$$
\begin{aligned}
& J(\upsilon) - J(u) \\
& \geq \mathbb{E}^u \int_0^T \left(H\left(t, x_t^\upsilon, y_t^\upsilon, z_t^\upsilon, Z_t^\upsilon, \upsilon_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\right. \\
& \left.-H\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\right) dt \\
& -\mathbb{E}^u \int_0^T H_x\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(x^\upsilon - x^u\right) dt \\
& -\mathbb{E}^u \int_0^T H_y\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(y^\upsilon - y^u\right) dt \\
& -\mathbb{E}^u \int_0^T H_z\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(z^\upsilon - z^u\right) dt \\
& -\sum_{i=0}^{+\infty} \mathbb{E}^u \int_0^T H_{Z^{(i)}}\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(Z^{(i)\upsilon} - Z^{(i)u}\right) dt.
\end{aligned}
$$

Since H is convex with respect to (x, y, z, Z, u) for almost all $(t, w) \in [0, T] \times \Omega$,

$$
J(u) - J(\upsilon) \leq -\mathbb{E}^u \int_0^T H_\upsilon\left(t, x_t^u, y_t^u, z_t^u, Z_t^u, u_t, p_t^u, q_t^u, k_t^u, Q_t^u, \Xi_t^u\right)\left(\upsilon_t - u_t\right) dt, .
$$

It turns out, using the condition (3.34) taking into account the fact that $\Gamma_t^\upsilon > 0$,

$$
J(u) - J(\upsilon) \leq 0
$$

This means that u is an optimal partially observed control process and (x^u, y^u, z^u, Z^u) is an optimal 4-typle.The proof is complete. ■

3.4 Application

In this section, we consider a partial observed linear quadratic control problem as a particular case of our control problem A. We find an explicit expression of the corresponding optimal control by applying the necessary and sufficient conditions of optimality. Consider the following control problem :
Minimize the expected quadratic cost function

$$\begin{aligned} J(u) \; &:= \mathbb{E}^{v}\left[M_1(x_T, x_T) + M_2(y_0, y_0)\right] \\ &+\mathbb{E}^{v}\int_0^T \left[K_t(x_t, x_t) + L_t(y_t, y_t) + \digamma_t(z_t, z_t)\right. \\ &\left.+\sum_{i=1}^{\infty} G_t\left(Z_t^{(i)}, Z_t^{(i)}\right) + R_t(u_t, u_t)\right] dt, \end{aligned} \tag{3.40}$$

where $(.,.)$: the inner product in $\mathbb{R}^n$. Subject to

$$\begin{cases} dx_t &= \left[(A_t^1, x_t) + (A_t^2, u_t)\right] dt + \left[(A_t^3, x_t) + (A_t^4, u_t)\right] dW_t \\ &+\sum_{i=1}^{\infty}\left[\left(A_t^{5,(i)}, x_t\right) + \left(A_t^{6,(i)}, u_t\right)\right] dH_t^{(i)}, \\ dy_t &= -\left[(B_t^1, x_t) + (B_t^2, y_t) + (B_t^3, z_t) + \sum_{i=1}^{\infty}\left(B_t^{4,(i)}, Z_t^{(i)}\right) + (B_t^5, u_t)\right] dt \\ &+z_t dW_t + \sum_{i=1}^{\infty} Z_t^{(i)} dH_t^{(i)}, \\ x_0 &= 0,\; y_T = \zeta, \end{cases} \tag{3.41}$$

where the observation state is given by the following SDE,

$$dY_t = \Lambda_t dt + d\tilde{W}_t \quad , Y_0 = 0. \tag{3.42}$$

Define $d\mathbb{P}^{v}=\Gamma^{v}d\mathbb{P}$ and we denote by Γ^{v} the unique $\mathcal{F}_{t}^{Y}$ adapted solution of

$$d\Gamma_{t}^{v}=\Gamma_{t}^{v}\left(D\left(t\right),dY_{t}\right)\quad,\ \Gamma_{0}^{v}=1. \tag{3.43}$$

Here $K\left(.\right)>0,\ L\left(.\right)>0,\ \digamma\left(.\right)>0,\ G\left(.\right)>0,\ R\left(.\right)>0,\ M_{1}\geq0,\ M_{2}\geq0,\ A^{i}\left(.\right),\ B^{j}\left(.\right)$ and $D\left(.\right)$ are bounded and deterministic, for $i=1,...,6$, and $j=1,...,5$.

To overcome this problem, we first write down the Hamiltonian function

$$\begin{aligned}
&H\left(t,x,y,z,Z,u,p,q,k,Q,\Xi\right):=\\
&(p_{t},(A_{t}^{1},x_{t})+(A_{t}^{2},u_{t}))+(k_{t},(A_{t}^{3},x_{t})+(A_{t}^{4},u_{t}))+\Lambda_{t}\Xi_{t}\\
&+(q_{t},(B_{t}^{1},x_{t})+(B_{t}^{2},y_{t})+(B_{t}^{3},z_{t})+\sum_{i=1}^{\infty}\left(B_{t}^{4,(i)},Z_{t}^{(i)}\right)+(B_{t}^{5},u_{t}))\\
&+\sum_{i=1}^{\infty}\left(Q_{t}^{(i)},\left(A_{t}^{5,(i)},x_{t}\right)+\left(A_{t}^{6,(i)},u_{t}\right)\right)+[K_{t}\left(x_{t},x_{t}\right)+L_{t}(y_{t},y_{t})\\
&+\digamma_{t}(z_{t},z_{t})+\sum_{i=1}^{\infty}G_{t}\left(Z_{t}^{(i)},Z_{t}^{(i)}\right)+R_{t}\left(u_{t},u_{t}\right)],
\end{aligned} \tag{3.44}$$

and the adjoint equations associated to the system $(3.41)-(3.43)$ are given by

$$\begin{cases}
-dp_{t} & =[(p,A_{t}^{1})+(k_{t},A_{t}^{3})+(q,B_{t}^{1})+\sum_{i=1}^{\infty}\left(Q_{t}^{(i)},A_{t}^{5,(i)}\right)\\
& \quad+2x_{t}K_{t}]\,dt-k_{t}dW_{t}-\sum_{i=1}^{\infty}Q_{t}^{(i)}dH_{t}^{(i)},\\
p_{T} & =2M_{1}x_{T},\\
dq_{t} & =[(q_{t},B_{t}^{2})+2L_{t}y_{t}]\,dt+[(q_{t},B_{t}^{3})+2\digamma_{t}z_{t}]\,dW_{t}\\
& \quad+\sum_{i=1}^{\infty}\left[\left(q_{t},B_{t}^{4,(i)}\right)+2G_{t}Z_{t}^{(i)}\right]dH_{t}^{(i)},\\
q_{0} & =2M_{2}y_{0}.
\end{cases} \tag{3.45}$$

and

$$\begin{cases} -dP_t = \left(K_t\left(x_t, x_t\right) + L_t\left(y_t, y_t\right) + \digamma_t\left(z_t, z_t\right)\right. \\ \left. +\sum_{i=1}^{\infty} G_t\left(Z_t^{(i)}, Z_t^{(i)}\right) + R_t\left(u_t, u_t\right)\right) dt - \Xi_t d\tilde{W}_t \\ P_T = M_1\left(x_T, x_T\right). \end{cases} \tag{3.46}$$

According to Theorem (4.4), if $\hat{u}$ is a partial observed optimal control, then it satisfies

$$\begin{aligned} 2\hat{u}_t = R_t^{-1} & \left(-A_t^2 \mathbb{E}\left[\hat{p}_t \mid \mathcal{F}_t^Y\right] - B_t^5 \mathbb{E}\left[\hat{q}_t \mid \mathcal{F}_t^Y\right]\right. \\ & \left. -A_t^4 \mathbb{E}\left[\hat{k}_t \mid \mathcal{F}_t^Y\right] - \sum_{i=1}^{\infty} A_t^{6,(i)} \mathbb{E}\left[\hat{Q}_t^{(i)} \mid \mathcal{F}_t^Y\right]\right). \end{aligned} \tag{3.47}$$

• Conversely, for the sufficient part, let $\hat{u} \in \mathcal{U}$ be a candidate to be optimal control and let $\left(\hat{x}, \hat{y}, \hat{z}, \hat{Z}\right)$ be the solution to the FBSDE (3.41) corresponding to $\hat{u}$ and (p, k, Q, q), (P, Ξ) are the solution to the corresponding solution to (3.45) and (3.46). It is straight forward to check that the functional H is convex in (x, y, z, Z, u). Thus, If $\hat{u}$ satisfies (3.47) and the partially observed maximum principle condition (3.30) above. Then by applying Theorem (5.6), one can easily check that $\hat{u}$ is an optimal control of our partially observed control problem.

Conclusion

In this thesis we are interested in two aspects of stochastic control problem which are, the partial information and the partially observed stochastic control problem for systems driven by Teugels martingales. Our main results within this work were the stochastic maximum principle which consists to find an admissible control u that minimizes a given cost functional subject to a stochastic differential equation on a finite time horizon. In fact, we have established the necessary as well as sufficient optimality conditions for a partially observed stochastic control problem for systems described by forward-backward stochastic differential equations driven by Teugels martingales associated with some Lévy processes and an independent Brownian motion.

The method of demonstration is based on the convex perturbation method. By differentiating the perturbed both the state equations and the cost functional, we get the adjoint process, which is a solution of a forward-backward SDE, driven by both a Brownian motion and a family of Teugels martingales, on top of the variational inequality between the Hamiltonian. Moreover, under some additional convexity conditions, we have proved that these partially observed necessary optimality conditions are also sufficient. Compared to the existing methods, we have investigated the sufficient optimality conditions in two different ways. We first proved our result assuming that the terminal condition for the backward component of our multidimensional FBSDE is linear with respect to the forward component. Then, we have proved the same result assuming that our FBSDE is one-dimensional and the terminal condition of the BSDE need not to be linear.

Note that we can extend our results to the case where the system is governed by a forward-backward doubly stochastic differential equations driven by Teugels martingales and an independent Brownian motion, this will be our main concern in the future relevant work.

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