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Etude qualitative des quelques systèmes amortis pour les équations d'ondes

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Abstract

The present thesis is devoted to the study of well-posedness and asymptotic behaviour in time of solution for damped systems. This work consists of four chapters. In chapter 1, we recall of some fundamental inequalities. In chapter 2, we consider a very important problem from the point of view of application in sciences and engineering. A system of three wave equations having a different damping effects in an unbounded domain with strong external forces. Using the Faedo-Galerkin method and some energy estimates, we will prove the existence of global solution in $\mathbb{R}^n$ owing to to the weighted function. By imposing a new appropriate conditions, which are not used in the literature, with the help of some special estimates and generalized Poincaré's inequality, we obtain an unusual decay rate for the energy function. In chapter 3, we will concerned with a problem for coupled nonlinear viscoelastic wave equation with distributed delay and strong damping and source terms, under suitable conditions we prove a blow up/growth results of solutions. In chapter 4, we consider one-dimensional porous-elastic system with nonlinear damping, infinite memory and distributed delay terms. We show the well posedness of solution by the semigroup theory and that the solution energy has an explicit and optimal decay, for the cases of equal and nonequal speeds of wave propagation.

Keywords and phrases: Viscoelastic wave equation, Strong nonlinear system, Global solution, Faedo-Galerkin approximation, Decay rate, Blow up, Strong damping, Distributed delay, Porous-elastic system.

AMS Subject Classification: 35L05, 58J45, 35L80, 35B40, 35L20, 58G16, 35B40, 35L70.

Résumé

La présente these est consacrée à l'étude de l'existence, l'unicité et du comportement asymptotique en temps de la solution pour des quelques systèmes amortis. Cette these se compose de quatre chapitres. Au chapitre 1, nous rappelons quelques résultats et inégalités fondamentales. Dans le chapitre 2, nous considérons un problème trés important du point de vue de l'application en sciences et en ingénierie. Un système de trois équations d'onde ayant des effets d'amortissement différents dans un domaine illimité avec une forces externes. En utilisant la méthode de Faedo-Galerkin et quelques estimations d'énergie, nous prouverons l'existence d'une solution globale dans $\mathbb{R}^n$ grace à la fonction pondérée. En imposant de nouvelles conditions appropriées, qui ne sont pas utilisées dans la littérature, à l'aide de quelques estimations spéciales et de l'inégalité de Poincaré généralisée, nous obtenons un taux de décroissance inhabituel pour la fonction énergétique. Dans le chapitre 3, nous traiterons un système couple d'équation d'onde viscoélastique non linéaire avec un retard distribué et un amortissement et des termes sources, dans des conditions appropriées, nous prouvons un résultat d'explosion/croissance des solutions. Dans le chapitre 4, nous considérons un système de poreux-élastique unidimensionnel avec amortissement non linéaire, mémoire infinie et termes de retard distribué. Nous montrons que la solution est bien posée par la théorie des semi-groupes et que l'énergie de la solution a une décroissance explicite et optimale, pour les cas de vitesses de propagation des ondes égales et non égales.

Mots-clés et phrases : Équation d'onde viscoélastique, Système non linéaire fort, Solution globale, Approximation de Faedo-Galerkin, Taux de décroissance, Blow up, Fort amortissement, Retard distribué, Système de poreux-élastique.

AMS Subject Classification: 35L05, 58J45, 35L80, 35B40, 35L20, 58G16, 35B40, 35L70.

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Introduction

Stabilization of evolution problems

Problems of global existence and stability in time of Partial Differential Equations are subject, recently, of many works. In this thesis we are interested in the study of the global existence and the stabilization of some evolution equations. The purpose of the stabilization is to attenuate the vibrations by feedback, thus consists in guaranteeing the decrease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation.

More precisely, the problem of stabilization consists in determining the asymptotic behavior of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero.

This problem has been studied by many authors for various systems. They are several type of stabilization,

(1) Strong stabilization: $E(t) \longrightarrow 0$, as $t \longrightarrow \infty$.

(2) Logarithmic stabilization: $E(t) \leq c(\log t)^{-\delta}, \forall t > 0, (c, \delta > 0)$.

(3) polynomial stabilization: $E(t) \leq ct^{-\delta}, \forall t > 0, (c, \delta > 0)$.

(4) Uniform stabilization: $E(t) \leq ce^{-\delta t}, \forall t > 0, (c, \delta > 0)$.

For wave equation with dissipation of the form

$$u'' - \Delta_x u + g\left(u'\right) = 0,$$

stabilization problems have been investigated by many authors:

When $g : \mathbb{R} \to \mathbb{R}$ is continuous and increasing function such that $g(0) = 0$, global existence of solutions is known for all initial conditions (u_0, u_1) given in $H_0^1(\Omega) \times L^2(\Omega)$ .This result is, for a consequence of the general theory of nonlinear semi-groups of contractions generated by a maximal monotone operator.

Moreover, if we impose on the control the condition, $\forall\lambda \neq 0$, $g(\lambda) \neq 0$, then strong asymptotic stability of solutions occurs in $H_0^1(\Omega) \times L^2(\Omega)$, *i.e.*,

$$\left(u, u^{'}\right) \to (0,0) \quad \text{strongly in } H_0^1(\Omega) \times L^2(\Omega),$$

without speed of convergence. These results follow, from the invariance principle of Lasalle. If the solution goes to 0 as time goes to ∞, how to get energy decay rates?

Dafermos has written in 1978 "Another advantage of this approach is that it is so simplistic that it requires only quite weak assumptions on the dissipative mechanism. The corresponding drawback is that the deduced information is also weak, never yielding, for example, decay rates of solutions."

Many authors have worked since then on energy decay rates. First results were obtained for linear stabilization, then for polynomial stabilization (see A. Haraux [28], V. Komornik [34], and E. Zuazua [59]) and then extended to arbitrary growing feedbacks (close to 0). In the same time, geometrical aspects were considered.

By combining the multiplier method with the techniques of micro-local analysis, Lasiecka et *al* [35, 18], have investigated different dissipative systems of partial differential equations (with Dirichlet and Neumann boundary conditions) under general geometrical conditions with nonlinear feedback without any growth restrictions near the origin or at infinity. The computation of decay rates is reduced to solving an appropriate explicitly given ordinary differential equation of monotone type. More precisely, the following explicit decay estimate of the energy is obtained:

$$E(t) \leq h\left(\frac{t}{t_0} - 1\right), \forall t \geq 0.$$

where $t_0 > 0$ and h is the solution of the following differential equation:

$$h^{'}(t) + q(h(t)) = 0, t \geq 0 \text{ and } h(0) = E(0),$$

and the function q is determined entirely from the behavior at the origin of the nonlinear feedback by proving that E satisfies

$$(Id - q)^{-1}(E((m+1)t_0)) \leq E(mt_0), \ \forall m \in \mathbb{N}.$$

System of nonlinear wave equations

To enrich this topic, it is necessary to talk about previous works regarding the nonlinear coupled system of wave equations, from a qualitative and quantitative study.Let us beginning with the single wave equation treated in [37], where the aim goal was mainely on the system

$$\begin{cases} u_{tt} + \mu u_t - \Delta u - \omega \Delta u_t = u \ln |u|, \ (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \end{cases} \tag{0.0.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$. The author firstly constructed a local existence of weak solution by using contraction mapping principle and of course showed the global existence, decay rate and infinite time blow up of the solution with condition on initial energy.

In $m-$equations, paper in [2] considered a system

$$u_{itt} + \gamma u_{it} - \Delta u_i + u_i = \sum_{i,j=1, i\neq j}^{m} |u_j|^{p_j} |u_i|^{p_i} u_i, \ i = 1, 2, \ldots, m, \tag{0.0.2}$$

where the absence of global solutions with positive initial energy was investigated. Next, a nonexistence of global solutions for system of three semilinear hyperbolic equations was introduced in [3]. A coupled system semilinear hyperbolic equations was investigated by many authors and a different results were obtained with the nonlinearities in the form $f_1 = |u|^{p-1}|v|^{q+1}u, f_2 = |v|^{p-1}|u|^{q+1}v$.

In the case of non-bounded domain $\mathbb{R}^n$, we mention the paper recently published by T. Miyasita and Kh. Zennir in [39], where the considered equation as follows

$$u_{tt} + a u_t - \phi(x)\Delta \left(u + \omega u_t - \int_0^t g(t-s)u(s)\, ds \right) = u|u|^{p-1}, \tag{0.0.3}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x). \end{cases} \tag{0.0.4}$$

The authors was successful in highlighting the existence of unique local solution and they continued to extend it to be global in time. The rate of the decay for solution was the main result by considering the relaxation function is strictly convex, for more results related to decay rate of solution of this type of problems, please see [51, 52, 58].

Regarding the study of the coupled system of two nonlinear wave equations, it is worth recalling some of the work recently published. Baowei Feng and *al.* considered in [24], a coupled system for viscoelastic wave equations with nonlinear sources in bounded domain ($(x,t) \in \Omega \times (0,\infty)$) with smooth boundary as follows

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + u_t = f_1(u,v) \\ \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + v_t = f_2(u,v). \end{cases} \tag{0.0.5}$$

Here, the authors concerned with a system in $\mathbb{R}^n (n = 1, 2, 3)$. Under appropriate hypotheses, they established a general decay result by multiplication techniques to extends some existing results for a single equation to the case of a coupled system.

It is worth noting here that there are several studies in this field and we particularly refer to the generalization that Shun and *all.* made in studying a complicate non-linear case with degenerate damping term in [50]. The IBVP for a system of nonlinear viscoelastic wave equations in a bounded domain was considered in the problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + (|u|^k + |v|^q)|u_t|^{m-1}u_t = f_1(u,v), \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + (|v|^\theta + |u|^\rho)|v_t|^{r-1}v_t = f_2(u,v), \\ u(x,t) = v(x,t) = 0, x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x) \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), \end{cases} \tag{0.0.6}$$

where Ω is a bounded domain with a smooth boundary. Given certain conditions on the kernel functions, degenerate damping and nonlinear source terms, they got a decay rate of the energy function for some initial data.

Damped porous-elastic system

As introduced in [11], the one-dimensional porous-elastic model constitute a system of two partial differential equations with unknown (u, φ) given by

$$\begin{aligned} &\rho_0 u_{tt} = \mu u_{xx} + \beta\varphi_x, \text{ in } (0,l) \times (0,L), \\ &\rho_0 k \varphi_{tt} = \alpha\varphi_{xx} - \beta u_x - \tau\varphi_t - \xi\varphi, \text{ in } (0,l) \times (0,L), \end{aligned} \tag{0.0.7}$$

where $l, L > 0$ the constant ρ is the mass density, κ is the equilibrated inertia and the constants $\mu, \alpha, \beta, \tau, \xi$ are assumed satisfy an appropriate conditions. This type of problem has been studied by many authors and a lot of results have been showed (Please see [21, 20, 22, 10, 5, 42, 43, 56]). The pioneer contribution was obtained by [47] for the problem (0.0.7). The basic evolution equations for one-dimensional theories of porous materials with memory effect are given by

$$\rho u_{tt} = T_x, J\phi_{tt} = H_x + G, \tag{0.0.8}$$

where T is the stress tensor, H is the equilibrated stress vector and G is the equilibrated body force. The variables u and ϕ are the displacement of the solid elastic material and the volume fraction, respectively. The constitutive equations are

$$T = \mu u_x + b\phi, H = \delta\phi_x - \int_0^t g(t-s)\,\phi_x(s)\,ds, G = -bu_x - \xi\phi. \tag{0.0.9}$$

A porous-elastic system was considered by [4] in the system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \text{ in } (0,1)\times(0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \displaystyle\int_0^t g(t-s)\,\phi_{xx}(x,s)\,ds = 0, \text{ in } (0,1)\times(0,\infty). \end{cases} \tag{0.0.10}$$

System (0.0.10) subjected Neumann-Dirichlet boundary conditions where g is the relaxation function, the authors obtained a general decay result for the case of equal speeds of wave propagation (See [27, 57]). In [25], the authors improved to the case of non-equal speed of wave propagation. In [20] the authors considered the following system with memory and distributed delay terms

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \displaystyle\int_0^t g(s)\phi_{xx}(t-s)ds \\ \qquad + \mu_1\phi_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\phi_t(x, t-\varrho)d\varrho = 0. \end{cases} \tag{0.0.11}$$

The exponential stability results of systems with memory and distributed delay terms for the case of equal speeds of wave propagation under a suitable assumptions are proved. In [32], the following system was considered

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + a\phi + \displaystyle\int_0^\infty g(s)\phi_{xx}(t-s)ds + \alpha(t)f(\phi_t) = 0. \end{cases} \tag{0.0.12}$$

The authors proved the global well posedness and stability results of (0.0.12) which has been extended in [33] for the case of nonequal speeds of wave propagation. Very recently, one-dimensional equations of an homogeneous and isotropic porous-elastic solid with interior time-dependent delay term feedbacks has been treated by E. Borges Filho and M. L. Santos in [11].

Chapter 1

Preliminary

1- Continuous function spaces

2- L^p Spaces

3- Sobolev Spaces

4- Semigroups of bounded linear operators

5- Lyapunov stability theory

6- Problems with delay

In this preliminary we shall introduce and state some necessary notations needed in the proof of our results, and some the basic results which concerning the semi-groupe theory and Layponov functionals and other theorems. The knowledge of all these notations and results are important for our study, see, e.g., ([12, 13, 23, 15, 44])

1.1 Continuous function spaces

We start this work by giving some useful notations and conventions.

Let $x = (x_1, x_2, ..., x_n)$ denote the generic point of an open set Ω of $\mathbb{R}^n$. Let u be a function defined from Ω to $\mathbb{R}^n$, we designate by $D_i u(x) = u_i(x) = \dfrac{\partial u(x)}{\partial x_i}$ the partial derivative of u with respect to x_i $(1 \leq i \leq n)$. Let's also define the gradient and the p-Laplacian from u, respectively as following

$$\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n})^{\ T} \quad \text{and} \quad |\nabla u|^2 = \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2$$

$$\Delta_p u(x) = div\left(|\nabla u|^{p-2} \nabla u \right)(x).$$

Note by $C(\Omega)$ the space of continuous functions from Ω to $\mathbb{R}$, $C(\Omega, \mathbb{R}^m)$ the space of continuous functions from Ω to $\mathbb{R}^m$ and $C_b\left(\overline{\Omega}\right)$ the space of all continuous and bounded functions on $\overline{\Omega}$, it is equipped with the norm $\|.\|_\infty$;

$$\|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)|$$

For $k \geq 1$ integer, $C^k(\Omega)$ is the space of functions u which are k times derivable and whose derivation of order k is continuous on Ω. $C_c^k(\Omega)$ is the set of functions of $C^k(\Omega)$ whose support is compact and contained in Ω.

We also define $C^k\left(\overline{\Omega}\right)$ as the set of restrictions to $\overline{\Omega}$ of elements from $C^k(\mathbb{R}^n)$ or as being the set of functions of $C^k(\Omega)$, such that for all $0 \leq j \leq k$, and for all $x_0 \in \partial\Omega$, the limit $\lim\limits_{x \to x_0} D_j u(x)$ exists and depends only on x_0.

$C_0^\infty(\Omega)$ or $\mathfrak{D}(\Omega)$, is the space of the infinitely differentiable functions, with compact supports called test function space.

The Hölder space $C^{k,\alpha}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$ and $k \geq 0$ an integer, $0 < \alpha \leq 1$, consists of those real or complex-valued k-times continuously differentiable functions f on Ω verifying

$$|f^{\beta}(x) - f^{\beta}(y)| \leq C\|x-y\|^{\alpha}$$

where $C > 0$, $|\beta| \leq k$.

1.2 L^p Spaces

Let Ω be an open set of $\mathbb{R}^n$, equipped with the Lebesgue measure dx. We denote by $L^1(\Omega)$ the space of integrable functions on Ω with values in $\mathbb{R}$, it is provided with the norm

$$\|u\|_{L^1} = \int_{\Omega} |u(x)|\, dx.$$

Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, we define the space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R},\ f \text{ measurable and } \int_{\Omega} |f(x)|^p\, dx < +\infty \right\}$$

equipped with norm

$$\|u\|_{L^p} = \left(\int_{\Omega} |u(x)|^p\, dx \right)^{\frac{1}{p}}.$$

We also define the space $L^{\infty}(\Omega)$

$$L^{\infty}(\Omega) = \{ f : \Omega \to \mathbb{R},\ f \text{ measurable},\ \exists c > 0, \text{ so that } |f(x)| \leq c \quad \text{a.e. on } \Omega \},$$

it will be equipped with the essential-sup norm

$$\|u\|_{L^{\infty}} = ess\sup_{x \in \Omega} |u(x)| = \inf\{ c\ ;\ |u(x)| \leq c \quad \text{a.e. on } \Omega \}.$$

We say that a function $f : \Omega \to \mathbb{R}$ belongs to $L^p_{loc}(\Omega)$ if $\mathbf{1}_K f \in L^p(\Omega)$ for any compact $K \subset \Omega$.

Theorem 1. *(Dominated convergence Theorem)*

Let $\{f_n\}_{n\geq 1}$ be a series of functions of $L^1(\Omega)$ converging almost everywhere to a measurable function f. It is assumed that there exists $g \in L^1(\Omega)$ such that for all $n \geq 1$, we get

$$|f_n| \leq g \quad a.e \text{ on } \Omega.$$

Then $f \in L^1(\Omega)$ *and*

$$\lim_{n\to+\infty} \|f_n - f\|_{L^1} = 0, \text{ and } \int_\Omega f(x)\,dx = \lim_{n\to+\infty} \int_\Omega f_n(x)\,dx.$$

1.3 Sobolev spaces

Definition 1. *Let Ω be an open set of $\mathbb{R}$, and $1 \le i \le n$. A function $u \in L^1_{loc}(\Omega)$ has an i^{th} weak derivative in $L^1_{loc}(\Omega)$ if there exists $f_i \in L^1_{loc}(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega u(x)\,\partial_i\varphi(x)\,dx = -\int_\Omega f_i(x)\,\varphi(x)\,dx.$$

This leads to say that the i^{th} derivative within the meaning of distributions of u belongs to $L^1_{loc}(\Omega)$, we write

$$\partial_i u = \frac{\partial u}{\partial x_i} = f_i$$

1.3.1 $W^{1,p}(\Omega)$ spaces

Let Ω be a bounded or unbounded open set of $\mathbb{R}^n$, and $p \in \mathbb{R}$, $1 \le p \le +\infty$, the space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega);\ \text{such that } \partial_i u \in L^p(\Omega), 1 \le i \le n\}$$

where $\partial_i u$ is the i^{th} weak derivative of $u \in L^1_{loc}(\Omega)$.

For $1 \le p < +\infty$ we define the space $W_0^{1,p}(\Omega)$ as being the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$, and we write

$$W_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,p}}.$$

Theorem 2. *(Poincaré's inequality)*

Assume Ω is a bounded open subset of $\mathbb{R}^n$, $u \in W_0^{1,p}(\Omega)$ for some $1 \le p < n$. Then we have the estimate

$$\|u\|_{L^q(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}$$

for each $q \in [1, p^*]$, where $p^* = \dfrac{np}{n-p}$ and the constant C depends only on q, p, n and Ω.

Remark 1. *In view of this Poincaré's inequality, if* Ω *is bounded, then on* $W_0^{1,p}(\Omega)$ *the norm* $\|u\|_{W^{1,p}(\Omega)}$ *is equivalent to* $\|\nabla u\|_{L^p(\Omega)}$.

Theorem 3. *(Rellich-Kondrachov compactness theorem) [13]*

Assume Ω *is a bounded open subset of* $\mathbb{R}^n$ *with* C^1 *boundary, and* $1 \leq p < n$. *Then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

for each $1 \leq q < p^*$.

1.3.2 $W^{m,p}(\Omega)$ Spaces

Let Ω be an open set of $\mathbb{R}^n, m \geq 2$ integer number and p real number such that $1 \leq p \leq +\infty$, we define the space $W^{m,p}(\Omega)$ as following

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega), \text{ such that } \partial^\alpha u \in L^p(\Omega), \ \forall \alpha, |\alpha| \leq m\}$$

where $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + ... + \alpha_n$ the length of α and $\partial^\alpha u = \partial_1^{\alpha_1}...\partial_n^{\alpha_n}$ is the weak derivative of a function $u \in L^1_{loc}(\Omega)$ in the sense of definition (1).

The space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \|u\|_{L^p} + \sum_{0<|\alpha|\leq m} \|\partial^\alpha u\|_{L^p}.$$

For $p = 2$, the space $W^{m,2}(\Omega)$ is noted $H^m(\Omega)$.

1.4 Semigroups of bounded linear operators

The goal of this section is to prove Lumer-Phillips' theorem (see Theorems 1.4.3 and 1.4.6 of [44]) in a Hilbert space setting. For that purpose, we first recall the notion of m-dissipative operators.

Definition 2. *Let* $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$ *be a (unbounded) linear operator.* $\mathcal{A}$ *is called dissipative if* $\Re(\mathcal{A}v, v)_x \leq 0, \forall v \in D(\mathcal{A})$. *The dissipative operator* $\mathcal{A}$ *is called* m*-dissipative if* $(\lambda I - \mathcal{A})$ *is surjective for some* $\lambda > 0$.

Theorem 4. *A linear operator $\mathcal{A}$ is dissipative if and only if*

$$\left\|(\lambda I - \mathcal{A})_x\right\|_X \geq \lambda \left\|x\right\|_X, \forall x \in D(\mathcal{A}), \lambda > 0, \tag{1.4.1}$$

Proof. Assume that $\mathcal{A}$ is dissipative and fix $x \in D(\mathcal{A})$ and $\lambda > 0$. Then

$$\lambda \left\|x\right\|_X^2 \leq \mathfrak{R}\left((\lambda - \mathcal{A})x, x\right)_X$$

and by Cauchy-Schwarz's inequality we conclude that

$$\lambda \left\|x\right\|_X^2 \leq \left\|(\lambda - \mathcal{A})x\right\|_X \left\|x\right\|_X,$$

that directly leads to (1.4.1). Conversely assume that (1.4.1) holds and fix $x \in D(\mathcal{A})$, then for all $\lambda > 0$, one has

$$\lambda^2 \left\|x\right\|_X^2 \leq \lambda \left\|x\right\|_X^2 - 2\lambda\mathfrak{R}\left(\mathcal{A}x, x\right)_x + \left\|\mathcal{A}x\right\|_X^2.$$

Dividing this inequality by 2λ, we get equivalently

$$\mathfrak{R}\left(\mathcal{A}x, x\right)_x \leq \frac{1}{2\lambda}\left\|\mathcal{A}x\right\|_X^2, \lambda > 0.$$

Passing to the limit as λ goes to infinity yields the dissipatedness of $\mathcal{A}$. Now we can prove some useful properties of m-dissipative operators. □

Theorem 5. *Let $\mathcal{A}$ be a m-dissipative operator. Then the next properties hold.*

1. *$\mathcal{A}$ is closed.*
2. *For all $\lambda > 0$, the operator $\lambda I - \mathcal{A}$ is an isomorphism from $D(\mathcal{A})$ onto X. Moreover $(\lambda I - \mathcal{A})^{-1}$is a linear bounded operator such that*

$$\left\|(\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

3. *$D(\mathcal{A})$ is dense in X.*

Proof. Let us start with point 1. As $\mathcal{A}$ is a m-dissipative operator, there exists $\lambda_0 > 0$ such that $R(\lambda_0 I - \mathcal{A}) = X$, hence by (1.4.1) it follows that $\lambda_0 I - \mathcal{A}$ has a bounded inverse. As $(\lambda_0 I - \mathcal{A})^{-1}$ is bounded, it is also closed. Then $\lambda_0 I - \mathcal{A}$ is closed and therefore $\mathcal{A}$ as well. To prove point 2 it suffices to prove that $R(\lambda I - \mathcal{A}) = X$ for all $\lambda > 0$. For that purpose, we introduce the set

$$\Lambda = \left\{\lambda \in (0, \infty) \text{ such that } R(\lambda I - \mathcal{A}) = X \right\}.$$

First Λ is open. Indeed (1.4.1) implies that Λ is a subset of the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$. As $\rho(\mathcal{A})$ is open, for every $\lambda \in \Lambda$, there exists a neighborhood of λ included in $\rho(\mathcal{A})$. The intersection of this neighborhood with the real line is clearly included into Λ, which proves that Λ is open. Let us also show that Λ is closed. Let a sequence. $(\lambda_n)_n$ of elements of Λ such that

$$\lambda_n \longrightarrow \lambda > 0 \text{ as } n \longrightarrow \infty$$

Then for an arbitrary element $y \in X$, and any n, there exists $x_n \in D(\mathcal{A})$ such that

$$(\lambda_n I - \mathcal{A})_{x_n} = y \tag{1.4.2}$$

Owing to (1.4.1), it follows that

$$\|x_n\|_X \leq \lambda_n^{-1} \|y\|_X \tag{1.4.3}$$

and therefore the sequence $(x_n)_n$ is bounded. Now we apply (1.4.1) with $x_n - x_m$ and λ_m to obtain

$$\lambda_m \|x_n - x_m\|_X \leq \|\lambda_m (x_n - x_m) - \mathcal{A}(x_n - x_m)\|_X ,$$

and by using (1.4.2) we deduce that

$$\lambda_m \|x_n - x_m\|_X \leq |\lambda_m - \lambda_n| \|x_n\|_X .$$

and by (1.4.3), we deduce that there exists $x \in X$ such that x_n converges to x in X. But (1.4.2) then implies that $\mathcal{A}x_n$ converges to $\lambda x - y$ and since $\mathcal{A}$ is closed, we conclude that $x \in D(\mathcal{A})$ with $\lambda x - \mathcal{A}x = y$. This shows that λ belongs to Λ and the closeness of Λ is proved. In conclusion Λ is a closed, open and non empty subset of $(0, \infty)$ and therefore it coincides with $(0, \infty)$.

Let us finish with point 3. Let $y \in X$ be such that

$$(y, x)_X = 0, x \in D(\mathcal{A}) \tag{1.4.4}$$

If we show that

$$(y, \mathcal{A}x)_X = 0, x \in D(\mathcal{A}) \tag{1.4.5}$$

then we will obtain that

$$(y, x - \mathcal{A}x)_X = 0, x \in D(\mathcal{A})$$

and since $R(I-\mathcal{A}) = X$, we deduce that $y = 0$.

It then remains to show (1.4.5). Let $x \in D(\mathcal{A})$ be fixed, then by point 2, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in D(\mathcal{A})$ and

$$x = x_n - \frac{1}{n}\mathcal{A}x_n, \forall n \in \mathbb{N}. \tag{1.4.6}$$

This implies that

$$\mathcal{A}x_n = n(x_n - x)$$

and from the regularity $x, x_n \in D(\mathcal{A})$, we deduce that x_n belongs to $D(\mathcal{A}^2)$ and that the next identity holds

$$\mathcal{A}x = \mathcal{A}\left(I - \frac{1}{n}\mathcal{A}\right)x_n.$$

or equivalently

$$\mathcal{A}x_n = \mathcal{A}\left(I - \frac{1}{n}\mathcal{A}\right)^{-1}\mathcal{A}x.$$

From point 2, we know that

$$\left\|\left(I - \frac{1}{n}\mathcal{A}\right)^{-1}\right\|_{\mathcal{L}(X)} \leq 1$$

and therefore

$$\|\mathcal{A}x_n\|_X \leq \|\mathcal{A}x\|_X .$$

Moreover as X is a Hilbert space, there exists a subsequence $(\mathcal{A}x_{nk})$ of $(\mathcal{A}x_n)_n$ and $z \in X$ such that $\mathcal{A}x_{nk}$ converges weakly to z This implies that the sequence of pairs $((x_{nk}, \mathcal{A}x_{nk}))_k$ converges weakly to(x, z) in $X \times X$.Hence by Mazur's Lemma there exists another sequence $((\widetilde{x}_l, z_l))_l$ made of convex combinations of $(x_{nj}, \mathcal{A}x_{nj})$ (that then guarantees that $z_l = \mathcal{A}\widetilde{x}_l$) such that $(\widetilde{x}_l, z_l) = (\widetilde{x}_l, \mathcal{A}\widetilde{x}_l)$ converges strongly to(x, z) in $X \times X$ as l goes to ∞. As $\mathcal{A}$ is closed, we deduce that $z = \mathcal{A}x$.

Finally by (1.4.6) and (1.4.4) we have

$$(y, \mathcal{A}x_{nk})_X = n_k(y, x_{nk} - x) = 0$$

and passing to the limit in k, we find that (1.4.5) holds.

Let us now go on with the notion of linear semigroups. □

Definition 3. *A one parameter family* $(S(t))_{t\geq 0}$ *of* $\mathcal{L}(X)$ *is a semigroup of bounded linear operators on* X *if*

1. $$S(0) = Id_x,$$

2. $$S(t+s) = S(t)S(s), \ \forall t, s \geq 0.$$

The linear operator $\mathcal{A}$ *defined by:*

$$D(\mathcal{A}) = \left\{ z \in X; \lim_{t \longrightarrow 0^+} \frac{S(t)z - z}{t} \text{ exists} \right\}$$

and

$$\mathcal{A}z = \lim_{t \longrightarrow 0^+} \frac{S(t)z - z}{t}, \forall z \in D(\mathcal{A})$$

is called the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$ *and* $D(\mathcal{A})$ *is called the domain of* $\mathcal{A}$.

A semigroup $(S(t))_{t\geq 0}$ *of bounded linear operators is called a strongly continuous (or a* C_0-*semigroup)if*

$$\lim_{t \longrightarrow 0^+} S(t)z = z, \ \forall z \in X. \tag{1.4.7}$$

A strongly continuous $(S(t))_{t\geq 0}$ *on* X *satisfying*

$$\|S(t)\|_{\mathcal{L}(X)} \leq 1, \ \forall t \geq 0,$$

is called a C_0*-semigroup of contractions.*

Let us now prove some useful properties of C_0- semigroups of contractions.

Theorem 6. *Let* $(S(t))_{t\geq 0}$ *be a* C_0-*semigroup of contractions on* X. *Then*

1. *For all* $x \in X$, *the mapping* $t \longrightarrow S(t)x$ *is a continuous function from* $[0, \infty)$ *into* X.
2. *For all* $x \in X$ *and all* $t \geq 0$,
$$\lim_{h \longrightarrow 0} \frac{1}{h} \int_t^{t+h} S(s)xds = S(s)x. \tag{1.4.8}$$
3. *For all* $x \in X$ *and all* $t > 0$, *the element* $\int_0^t S(s)xds$ *belongs to* $D(\mathcal{A})$, *and*
$$\mathcal{A}\left(\int_0^t S(s)xds\right) = S(t)x - x \tag{1.4.9}$$

4. For all $x \in D(\mathcal{A})$ and all $t > 0$, the element $S(t)x$ belongs to $D(\mathcal{A})$, and the mapping $t \longrightarrow S(t)x$ is a continuous differentiable function from $(0, \infty)$ into X and

$$\frac{d}{dt}S(t)x = \mathcal{A}S(t)x = S(t)\mathcal{A}x,\ \forall t \geq 0. \tag{1.4.10}$$

5. For all $x \in D(\mathcal{A})$ and all $t > s \geq 0$, we have

$$S(t)x - S(s)x = \int_s^t S(u)\mathcal{A}xdu = \int_s^t \mathcal{A}S(u)xdu.$$

Proof. For point 1, by (1.4.7), the continuity property trivially holds at $t = 0$. Now fix $x \in X$ and take an arbitrary $t > 0$ then for $h \geq 0$, we may write

$$S(t+h)x - S(t)x = S(t)(S(h)x - x),$$

and consequently

$$\|S(t+h)x - S(t)x\|_X \leq \|S(h)x - x\|_X,$$

On the other hand for $h < 0$ such that $t + h > 0$, we have,

$$S(t+h)x - S(t)x = S(t+h)(x - S(-h)x).$$

In both cases, by (1.4.7) we find that $S(t+h)x - S(t)x$ goes to zero as h goes to zero. Point 2 directly follows from point 1.

To prove point 3, fix $x \in X$ and $h > 0$. then we clearly have

$$\begin{aligned}\frac{S(h)-I}{h}\int_0^t S(s)xds &= \frac{1}{h}\int_0^t (S(s+h)x - S(s)x)ds \\ &= \frac{1}{h}\int_0^{t+h} S(s)xds - \frac{1}{h}\int_0^t S(s)xds\end{aligned}$$

Hence by (1.4.8), we deduce that the right-hand side tends to $S(t)x - x$ as h goes to zero.By the definition of A this proves the assertions. For point 4, let $x \in D(\mathcal{A})$ and $t, h > 0$,then by the semigroup property

$$\frac{S(h)-I}{h}S(t)x = S(t)\left(\frac{S(h)-I}{h}\right)x.$$

Hence by the definition of $\mathcal{A}$ and the continuity of the semigroup, we get

$$\lim_{h \longrightarrow 0^+}\frac{S(h)-I}{h}S(t)x = S(t)\lim_{h \longrightarrow 0^+}\left(\frac{S(h)-I}{h}\right)x = S(t)\mathcal{A}x.$$

This shows that $S(t)x$ belongs to $D(\mathcal{A})$, that $\mathcal{A}S(t)x = S(t)\mathcal{A}x$ and that the right derivative of $S(t)x$ exists with

$$\frac{d^+}{dt}S(t)x = \mathcal{A}S(t)x = S(t)\mathcal{A}x$$

For the left derivative, for $0 < h < t$ we write

$$\begin{aligned}\frac{S(t)x - S(t-h)x}{h} - S(t)\mathcal{A}x &= S(t-h)\left(\frac{S(h)x - x}{h} - \mathcal{A}x\right) \\ &\quad + \left(S(t-h)\mathcal{A}x - S(t)\mathcal{A}x\right).\end{aligned}$$

□

1.5 Lyapunov Stability Theory

The investigation of stability for hereditary systems is often related to the construction of Lyapunov functionals. The general method of Lyapunov functionals construction which was proposed by V. Kolmanovskii and L. Shaikhet [19] and successfully used already for functional differential equations, for difference equations with discrete time, for difference equations with continuous time, is used here to investigate the stability of delay evolution equations, in particular, partial differential equations.

1.5.1 Notations and definitions

Let U and H be two real separable Hilbert spaces such that $U \subset H \equiv H^* \subset U^*$, where the injections are continuous and dense. Let $\|\|, ||$ and $\|\|_*$ be the norms in U, H and H^*respectively, $((\cdot,\cdot))$ and $(\cdot,\cdot)$ be the scalar products in U and H respectively, and $\langle .,. \rangle$ the duality product between U and $U*$. We assume that

$$|u| \le \beta \|u\|, u \in U \tag{1.5.1}$$

Let $C(-h, 0, H)$ be the Banach space of all continuous functions from $[-h, 0]$ to H, $x_t \in C(-h, 0, H)$ for each $t \in [0, \infty)$, be the function defined by $x_t(s) = x(t+s)$ for all $s \in [-h, 0]$. The space $C(-h, 0, U)$ is similarly defined. Let $A(t,\cdot) : U \to U^*$, $f_1(t,\cdot) : C(-h, 0, H) \to U*$ and

$f_2(t,\cdot) : C(-h,0,U) \to U*$ be three families of nonlinear operators defined for $t > 0, A(t,0) = 0, f_1(t,0) = 0, \quad f_2(t,0) = 0$.

Consider the equation

$$\frac{du(t)}{dt} = A\left(t, u(t)\right) + f_1(t, u_t) + f_1(t, u_t), t > 0 \tag{1.5.2}$$

$$u(s) = \psi(s), s \in [-h, 0]$$

Let us denote by $u(\cdot;\psi)$ the solution of Eq. (1.5.2) corresponding to the initial condition ψ.

Definition 4. *The trivial solution of Eq. (1.5.2) is said to be stable if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|u\left(t;\psi\right)| < \varepsilon \text{ for all } t \geq 0, \text{ if } |\psi|_{C_H} = \sup_{s\in[-h,0]} |\psi\left(s\right)| < \delta.$$

Definition 5. *The trivial solution of Eq. (1.5.2) is said to be exponentially stable if it is stable and there exists a positive constant* λ *such that for any* $\psi \in C(-h,0,U)$ *there exists* C*(which may depend on* ψ*) such that* $|u\left(t;\psi\right)| \leq Ce^{-\lambda t}$ *for* $t > 0$.

1.5.2 Lyapunov type stability theorem

Let us now prove a theorem which will be crucial in our stability investigation.

Theorem 7. *Assume that there exists a functional* $V(t, u_t)$ *such that the following conditions hold for some positive numbers* c_1, c_2 *and* λ*:*

$$|u\left(t;u_t\right)| \leq c_1 e^{\lambda t} \left|u(t)\right|^2, t \geq 0, \tag{1.5.3}$$

$$|u\left(0;u_0\right)| \leq c_2 \left|\psi\right|^2_{C_H}, \tag{1.5.4}$$

$$\frac{d}{dt} V\left(t, u_t\right) \leq 0, t \geq 0. \tag{1.5.5}$$

Then the trivial solution of Eq. (1.5.2) is exponentially stable.

Note that Theorem 7 implies that the stability investigation of Eq. (1.5.2) can be reduced to the construction of appropriate Lyapunov functionals. A formal procedure to construct Lyapunov functionals is described below.

1.5.3 Procedure of Lyapunov functionals construction

The procedure consists of four steps.

Step 1.

To transform Eq. (1.5.2) into the form

$$\frac{dz(t,u_t)}{dt}A_1\left(t,u(t)\right)+A_2\left(t,u_t\right) \tag{1.5.6}$$

where $z(t,\cdot)$ and $A_2(t,\cdot)$ are families of nonlinear operators, $z(t,0)=0, A_2(t,0)=0$,operator $A_1(t,\cdot)$ only depends on t and $u(t)$, but does not depend on the previous values $u(t+s), s<0$.

Step 2.

Assume that the trivial solution of the auxiliary equation without memory

$$\frac{dy(t)}{dt}=A_1\left(t,y(t)\right) \tag{1.5.7}$$

is exponentially stable and therefore there exists a Lyapunov function $v(t,y(t))$, which satisfies the conditions of Theorem 7.

Step 3.

A Lyapunov functional $V(t,ut)$ for Eq. (1.5.6) is constructed in the form $V=V1+V2$, where $V_1(t,ut)=v(t,z(t,u_t))$. Here the argument y of the function $v(t,y)$ is replaced on the

functional $z(t,x_t)$ from the left-hand part of Eq. (1.5.6).

Step 4.

Usually, the functional $V_1(t,u_t)$ almost satisfies the conditions of Theorem 7. In order to fully satisfy these conditions, it is necessary to calculate $\frac{d}{dt}V_1(t,u_t)$ and estimate it. Then, the additional functional $V_2(t,u_t)$ can be chosen in a standard way.

Note that the representation (1.5.6) is not unique. This fact allows, using different representations type of (1.5.6) or different ways of estimating $\frac{d}{dt}V_1(t,u_t)$, to construct different Lyapunov functionals and, as a result, to get different sufficient conditions of exponential stability.

1.6 Problems with a delay

In this section we introduce a large number of problems, both old and new, which are treated using the general theory of differential equations. We attempt to give sufficient description concerning the derivation, solution, and properties of solutions so that the reader will be able to appreciate some of the flavor of the problem. In none of the cases do we give a complete treatment of the problem, but offer references for further study.

Economics models

The following problem is copied from an elementary text on differential equations by Boyce and DiPrima [12]: "A young person with no initial capital invests k dollars per year at an annual interest rate τ. Assume that investments are made continuously and that interest is compounded continuously. If $\tau = 7.5\%$, determine k so that one million dollars will be available at the end of forty years."

It is solved by writing

$$S' = 0.075S + k, \; S(0) = 0,$$

and solving for $S(40)$. Several things are idealized in the problem, but still it is a fair model. It is noted there that in certain contexts continuous investment yields roughly the same as daily investment and it allows the student the opportunity to see the power of differential equations in giving a simple solution to an otherwise tedious problem.

Now the forty years is up and for computational convenience instead of the one million dollars let us say that the person has $\$900,000$ to invest and to live off the proceeds. During times of low interest rates a financial advisor may recommend bank certificates of deposit of 90-day maturity, automatically renewed at the existing interest rate, but lettered so that $\$10,000$ of the total matures every day and both principal and interest are reinvested. This enables the investor to quickly take advantage of rising rates and to lock in high interest long-term instruments if they become available. We imagine that this is changed to continuous reinvestment, just as the elementary problem imagined continuous investment of k dollars per year. If the total value is again $S(t)$, then from just the investment we would have

$$S'(t) = b(t)S(t - (1/4)).$$

The $b(t)$ represents a product. One factor is the fraction of the total amount of $S(t - 1/4)$ which was invested three months earlier and matured today. The other factor is the interest being offered at that time. In addition, the person withdraws a percentage of the total $S(t)$ continuously for living expenses, resulting in an equation

$$S'(t) = -a(t)S(t) + b(t)S(t - 1/4),\ S(t) = \psi(t) \text{ for } -1/4 \le t \le t_0.$$

Here, the initial condition is an initial function $\psi : [-1/4, 0] \to \mathbb{R}$ with $\psi(t)$ being exactly that amount $S(t)$ which was invested at time t.

We can draw several conclusions of the following type. First, if the solutions are bounded, then times are likely to become difficult since inflation will eat away at the value and medical bills will increase with time; at this time, some studies have shown that those retiring with income sufficient to meet three times their current need approach desperate conditions within fifteen years. Next, we can ask if solutions will tend to zero. If they do, the person will be destined for the poor farm. At a minimum, the retiree must adjust the withdrawals so that the conditions of our theorem are not met.

Clearly, in this example it will make sense for both $a(t)$ and $b(t)$ to vary; $a(t)$ can be negative the day the income tax refund check arrives, and $b(t)$ can be negative when the bank fails and the FDIC assumes control see [14].

Controlling a ship

Minorsky (1962) designed an automatic steering device for the battleship New Mexico. The following is a sketch of the problem see [15].

Let the rudder of the ship have angular position $x(t)$ and suppose there is a friction force proportional to the velocity, say $-cx'(t)$. There is a direction indicating instrument which points in the actual direction of motion and there is an instrument pointing in the desired direction. These two are connected by a device which activates an electric motor producing a certain force to move the rudder so as to bring the ship onto the desired course. There is a time lag of amount $h > 0$ between the time the ship gets off course and the time the electric motor activates the restoring force. The equation for $x(t)$ is

$$x''(t) + cx'(t) + g(x(t - h)) = 0, \tag{1.6.1}$$

where $xg(x) > 0$ if $x \neq 0$ and c is a positive constant. The object is to give conditions ensuring that $x(t)$ will stay near zero so that the ship closely follows its proper course.

Epidemics (Cooke and Yorke)

In the work of Cooke and Yorke (1973) the Lotka assumption is changed so that the number of births per unit time is a function only of the population size, not of the age distribution see [15]. Under this assumption, we let $x(t)$ be the population size and let the number of births be $B(t) = g(x(t))$. Assume each individual has life span L so that the number of deaths per unit time is $g(x(t - L))$. Then the population size is described by

$$x'(t) = g(x(t)) - g(x(t - L)), \tag{1.6.2}$$

where g is some differentiable function. We note that every constant function is a solution of (1.6.2).

The following model for the spread of gonorrhea is considered by Cooke and Yorke (1973). The population is divided into two classes:

(a) $S(t)$ =the number of susceptibles, and

(b) $x(t)$ =the number of infectious.

The rate of new infection depends only on contacts between susceptible and infectious individuals. Since Set) equals the constant total population minus $x(t)$, the rate is some function $g(x(t))$. Assume that an exposed individual is immediately infectious and stays infectious for a period L (the time for treatment and cure). Then x also satisfies (1.6.2) holds. Now, at any time t, $x(t)$ equals the sum of capital produced over the period $[t - L, t]$ plus a constant c denoting the value of nondepreciating assets. Thus,

$$\begin{aligned} x(t) &= \int_0^L P(s)g[x(t-s)]ds + c \\ &= \int_{t-L}^t P(t-u)g[x(u)]du + c. \end{aligned} \tag{1.6.3}$$

Some models of war and peace

L. F. Richardson (1881-1953, see [15]), a British Quaker, observed two world wars and was concerned about them (cf. Richardson, 1960; Jacobson, 1984). He speculated that wars begin where arms races end and he felt that international dynamics could be modeled mathematically because of human motivations. He claimed that men are guided by "their traditions, which are fixed, and their instincts which are mechanical"; thus, on a grand scale they are incapable of good and evil. He sought to develop a theory of international dynamics to guide statesmen with domestic and foreign policy, much as dynamics guides machine design.

Let X and Y be nations suspicious of each other. Suppose X and Y create stocks of arms x and y, respectively; more generally, x and y represent "threats minus cooperation" so that negative values have meaning. At least three things affect the arms buildup of X;

(a) Economic burden;

(b) Terror at the sight of $y(t)$ (or national pride);

(c) Grievances and suspicions of y.

The same will, of course, apply to Y.

Richardson assumed that each side had complete and instantaneous knowledge of the arms of the other side and that each side could react instantaneously. He reasoned from (a) that

$$dx/dt = -a_1 x,$$

because the burden is proportional to the size x, and he argued from (b) that

$$dx/dt = -a_1 x + b_1 y,$$

because the terror is proportional to the size y. Finally, Richardson assumed constant standing grievances, say g_t so that the complete system is

$$\begin{aligned} x' &= -a_1 x + b_1 y + g_1, \\ y' &= -a_2 y + b_2 x + g_2, \end{aligned} \tag{1.6.4}$$

with a_i, b_i, and g_i, $i = 1, 2$ being positive constants. Domestic and foreign policy will set the a_i and b_i, although Richardson maintained a more mechanical view.

Hill (1978) recognized deficiencies in Richardson's model. He reasoned that it takes time to respond to an observed situation and, therefore, proposed the model

$$\begin{aligned} x' &= -a_1 x(t-T) + b_1 y(t-T) + g_1, \\ y' &= -a_2 y(t-T) + b_2 x(t-T) + g_2. \end{aligned}$$

where T is a positive constant.

Prey-predator population models (Lotka-Voltera)

Let $x(t)$ be the population at time t of some species of animal called prey and let $y(t)$ be the population of a predator species which lives off these prey. We assume that $x(t)$ would increase at a rate proportional to $x(t)$ if the prey were left alone, i.e., we would have $x'(t) = a_1 x(t)$, where $a_1 > 0$. However the predators are hungry, and the rate at which each of them eats prey is limited only by his ability to find prey. (This seems like a reasonable assumption as long as there are not too many prey available.) Thus we shall assume that the activities of the predators reduce the growth rate of $x(t)$ by an amount proportional to the product $x(t)y(t)$, i.e.,

$$x'(t) = a_1 x(t) - b_1 x(t) y(t),$$

where b_1 is another positive constant.

Now let us also assume that the predators are completely dependent on the prey as their food supply. If there were no prey, we assume $y'(t) = -a_2 y(t)$, where $a_2 > 0$, i.e., the predator species would die out exponentially. However, given food the predators breed at a rate proportional to their number and to the amount of food available to them. Thus we consider the pair of equations

$$\begin{aligned} x'(t) &= a_1 x(t) - b_1 x(t) y(t), \\ y'(t) &= -a_2 y(t) + b_2 x(t) y(t), \end{aligned} \tag{1.6.5}$$

where a_1, a_2, b_1, and b_2 are positive constants. This well-known model was invented and studied by Lotka [1920], [1925] and Volterra [1928], [1931].

Vito Volterra was trying to understand the observed fluctuations in the sizes of populations $x(t)$ of commercially desirable fish and $y(t)$ of larger fish which fed on the smaller ones in the Adriatic Sea in the decade from 1914 to 1923 see [23].

The sunflower equation

Somolinos (1978) has considered the equation

$$x'' + (a/r)x' + (b/r)\sin x(t-r) = 0,$$

and has obtained interesting results on the existence of periodic solutions. The study of this problem goes back to the early 1800s and has attracted much attention. It involves the motion of a sunflower plant see [15].

Chapter 2

The effect of damping terms on decay rate for system of three nonlinear wave equations with memories

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1- Position of problem and preliminaries
2- Main results and Proofs
3- Conclusion

Here, we consider a system of three wave equations having a different damping effects in an unbounded domain with strong external forces. Using the Faedo-Galerkin method and some energy estimates, we will prove the existence of global solution in $\mathbb{R}^n$ owing to to the weighted function. By imposing a new appropriate conditions, which are not used in the literature, with the help of some special estimates and generalized Poincaré's inequality, we obtain an unusual decay rate for the energy function.

2.1 Position of problem and preliminaries

We consider, for $x \in \mathbb{R}^n,\ t > 0$, the following system

$$\begin{cases} \left(|u_t|^{\kappa-2}u_t\right)_t + au_t = \Theta(x)\Delta\left(u + \omega u_t - \int_0^t \varpi_1(t-s)u(s)\,ds\right) + f_1(u,v,w) \\ \left(|v_t|^{\kappa-2}v_t\right)_t + av_t = \Theta(x)\Delta\left(v + \omega v_t - \int_0^t \varpi_2(t-s)v(s)\,ds\right) + f_2(u,v,w) \\ \left(|w_t|^{\kappa-2}w_t\right)_t + aw_t = \Theta(x)\Delta\left(w + \omega w_t - \int_0^t \varpi_3(t-s)w(s)\,ds\right) + f_3(u,v,w) \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x) \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), w_t(x,0) = w_1(x), \end{cases} \tag{2.1.1}$$

where $a \in \mathbb{R}$, $\omega > 0$, $n \geq 3, \kappa \geq 2$, the functions $f(.,.,.) \in (\mathbb{R}^3, \mathbb{R}), i = 1, 2, 3$ are given by

$$f_1(u,v,w) = (p+1)\Big[d|u+v+w|^{(p-1)}(u+v+w) + e|u|^{(p-3)/2}u|v|^{(p+1)/2}\Big],$$

$$f_2(u,v,w) = (p+1)\Big[d|u+v+w|^{(p-1)}(u+v+w) + e|v|^{(p-3)/2}v|w|^{(p+1)/2}\Big],$$

$$f_3(u,v,w) = (p+1)\Big[d|u+v+w|^{(p-1)}(u+v+w) + e|w|^{(p-3)/2}w|u|^{(p+1)/2}\Big],$$

with $d, e > 0, p > 3$. The function $\Theta(x) > 0$ for all $x \in \mathbb{R}^n$ is a density and $(\Theta)^{-1} = 1/\Theta(x) \equiv \theta(x)$ such that

$$\theta \in L^\tau(\mathbb{R}^n) \quad \text{with} \quad \tau = \frac{2n}{2n - rn + 2r} \quad \text{for} \quad 2 \leq r \leq \frac{2n}{n-2}. \tag{2.1.2}$$

It is note hard to see that there exists a function $\mathcal{F} \in C^1(\mathbb{R}^3, \mathbb{R})$ such that

$$uf_1(u,v,w) + vf_2(u,v,w) + wf_3(u,v,w) = (p+1)\mathcal{F}(u,v,w),\ \forall (u,v,w) \in \mathbb{R}^3. \tag{2.1.3}$$

satisfies

$$(p+1)\mathcal{F}(u,v,w) = |u+v+w|^{p+1} + 2|uv|^{(p+1)/2} + 2|vw|^{(p+1)/2} + 2|wu|^{(p+1)/2}. \tag{2.1.4}$$

We define the function spaces $\mathcal{H}$ as the closure of $C_0^\infty(\mathbb{R}^n)$, as in [45], we have

$$\mathcal{H} = \{v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \mid \nabla v \in L^2(\mathbb{R}^n)^n\},$$

with respect to the norm $\|v\|_{\mathcal{H}} = (v,v)_{\mathcal{H}}^{1/2}$ for the inner product

$$(v,w)_{\mathcal{H}} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx,$$

and $L^2_\theta(\mathbb{R}^n)$ as that to the norm $\|v\|_{L^2_\theta} = (v,v)^{1/2}_{L^2_\theta}$ for

$$(v,w)_{L^2_\theta} = \int_{\mathbb{R}^n} \theta v w \, dx.$$

For general $r \in [1, +\infty)$

$$\|v\|_{L^r_\theta} = \left(\int_{\mathbb{R}^n} \theta \left| v \right|^r \, dx \right)^{\frac{1}{r}}.$$

is the norm of the weighted space $L^r_\theta(\mathbb{R}^n)$.

The lack of existence (Blow up) is considered one of the most important qualitative studies that must be spoken of, given its importance in terms of application in various applied sciences. Concerning the nonexistence of solution for a more degenerate case for coupled system of wave equations with different damping, we mention the papers [46, 53].

We introduce a very useful Sobolev embedding and generalized Poincaré inequalities.

Lemma 1. *[39] Let θ satisfy (2.1.2). For a positive constants $C_\tau > 0$ and $C_P > 0$ depending only on θ and n, we have*

$$\|v\|_{\frac{2n}{n-2}} \leq C_\tau \|v\|_{\mathcal{H}},$$

and

$$\|v\|_{L^2_\theta} \leq C_P \|v\|_{\mathcal{H}},$$

for $v \in \mathcal{H}$.

Lemma 2. *[31] Let θ satisfy (2.1.2), then the estimates*

$$\|v\|_{L^r_\theta} \leq C_r \|v\|_{\mathcal{H}},$$

and

$$C_r = C_\tau \|\theta\|^{\frac{1}{r}}_{\tau},$$

hold for $v \in \mathcal{H}$. Here $\tau = 2n/(2n - rn + 2r)$ for $1 \leq r \leq 2n/(n-2)$.

We assume that the kernel functions $\varpi_1, \varpi_2, \varpi_3 \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$1 - \overline{\varpi_1} = l > 0 \quad \text{for} \quad \overline{\varpi_1} = \int_0^{+\infty} \varpi_1(s)\, ds, \;\; \varpi_1'(t) \leq 0, \tag{2.1.5}$$

$$1-\overline{\varpi_2}=m>0 \quad \text{for} \quad \overline{\varpi_2}=\int_0^{+\infty}\varpi_2(s)\,ds,\ \varpi_2'(t)\leq 0, \tag{2.1.6}$$

$$1-\overline{\varpi_3}=\nu>0 \quad \text{for} \quad \overline{\varpi_3}=\int_0^{+\infty}\varpi_3(s)\,ds,\ \varpi_3'(t)\leq 0, \tag{2.1.7}$$

we mean by $\mathbb{R}^+$ the set $\{\tau \mid \tau\geq 0\}$. Noting by

$$\mu(t)=\max_{t\geq 0}\Big\{\varpi_1(t),\varpi_2(t),\varpi_3(t)\Big\}, \tag{2.1.8}$$

and

$$\mu_0(t)=\min_{t\geq 0}\Big\{\int_0^t\varpi_1(s)ds,\int_0^t\varpi_2(s)ds,\int_0^t\varpi_3(s)ds\Big\}. \tag{2.1.9}$$

We assume that there is a function $\chi\in C^1(\mathbb{R}^+,\mathbb{R}^+)$ such that

$$\varpi_i'(t)+\chi(\varpi_i(t))\leq 0,\quad \chi(0)=0,\quad \chi'(0)>0 \quad \text{and} \quad \chi''(\xi)\geq 0,\ i=1,2,3, \tag{2.1.10}$$

for any $\xi\geq 0$.

Holder and Young's inequalities give

$$\begin{aligned}\|uv\|_{L_\theta^{(p+1)/2}}^{(p+1)/2} &\leq \left(\|u\|_{L_\theta^{(p+1)}}^2+\|v\|_{L_\theta^{(p+1)}}^2\right)^{(p+1)/2}\\ &\leq \left(l\|u\|_{\mathcal{H}}^2+m\|v\|_{\mathcal{H}}^2\right)^{(p+1)/2},\end{aligned} \tag{2.1.11}$$

and

$$\|vw\|_{L_\theta^{(p+1)/2}}^{(p+1)/2}\leq \left(m\|v\|_{\mathcal{H}}^2+\nu\|w\|_{\mathcal{H}}^2\right)^{(p+1)/2}, \tag{2.1.12}$$

and

$$\|wu\|_{L_\theta^{(p+1)/2}}^{(p+1)/2}\leq \left(\nu\|w\|_{\mathcal{H}}^2+l\|u\|_{\mathcal{H}}^2\right)^{(p+1)/2}. \tag{2.1.13}$$

Thanks to Minkowski's inequality to give

$$\begin{aligned}\|u+v+w\|_{L_\theta^{(p+1)}}^{(p+1)} &\leq c\left(\|u\|_{L_\theta^{(p+1)}}^2+\|v\|_{L_\theta^{(p+1)}}^2+\|w\|_{L_\theta^{(p+1)}}^2\right)^{(p+1)/2}\\ &\leq c\left(\|u\|_{\mathcal{H}}^2+\|v\|_{\mathcal{H}}^2+\|w\|_{\mathcal{H}}^2\right)^{(p+1)/2}.\end{aligned}$$

Then there exist $\eta>0$ such that

$$\begin{aligned}&\|u+v+w\|_{L_\theta^{(p+1)}}^{(p+1)}+2\,\|uv\|_{L_\theta^{(p+1)/2}}^{(p+1)/2}+2\,\|vw\|_{L_\theta^{(p+1)/2}}^{(p+1)/2}+2\,\|wu\|_{L_\theta^{(p+1)/2}}^{(p+1)/2}\\ &\leq \eta\left(l\|u\|_{\mathcal{H}}^2+m\|v\|_{\mathcal{H}}^2+\nu\|w\|_{\mathcal{H}}^2\right)^{(p+1)/2}.\end{aligned} \tag{2.1.14}$$

We need to define positive constants λ_0 and $\mathcal{E}_0$ by

$$\lambda_0 \equiv \eta^{-1/(p-1)} \quad \text{and} \quad \mathcal{E}_0 = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\eta^{-2/(p-1)}. \tag{2.1.15}$$

The mainely aim here is to obtain a novel decay rate of solution from the convexity property of the function χ given in Theorem 10.

We denote an eigenpair $\{(\lambda_i, e_i)\}_{i\in\mathbb{N}} \subset \mathbb{R} \times \mathcal{H}$ of

$$-\Theta(x)\Delta e_i = \lambda_i e_i \quad x \in \mathbb{R}^n,$$

for any $i \in \mathbb{N}$. Then

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots \uparrow +\infty,$$

holds and $\{e_i\}$ is a complete orthonormal system in $\mathcal{H}$.

Definition 6. *The triplet functions (u, v, w) is said a weak solution to (2.1.1) on $[0, T]$ if satisfies for $x \in \mathbb{R}^n$,*

$$\begin{cases} \int_{\mathbb{R}^n} \Big(|u_t|^{\kappa-2}u_t\Big)_t \varphi dx + a\int_{\mathbb{R}^n} u_t\varphi dx = \int_{\mathbb{R}^n} \Theta(x)\Delta\Big(u + \omega u_t - \int_0^t \varpi_1(t-s)u(s)\,ds\Big)\varphi dx \\ \qquad + \int_{\mathbb{R}^n} f_1(u,v,w)\varphi dx, \\ \int_{\mathbb{R}^n} \Big(|v_t|^{\kappa-2}v_t\Big)_t \psi dx + a\int_{\mathbb{R}^n} v_t\psi dx = \int_{\mathbb{R}^n} \Theta(x)\Delta\Big(v + \omega v_t - \int_0^t \varpi_2(t-s)v(s)\,ds\Big)\psi dx \\ \qquad + \int_{\mathbb{R}^n} f_2(u,v,w)\psi dx, \\ \int_{\mathbb{R}^n} \Big(|w_t|^{\kappa-2}w_t\Big)_t \Psi dx + a\int_{\mathbb{R}^n} w_t\Psi dx = \int_{\mathbb{R}^n} \Theta(x)\Delta\Big(w + \omega w_t - \int_0^t \varpi_3(t-s)w(s)\,ds\Big)\Psi dx \\ \qquad + \int_{\mathbb{R}^n} f_3(u,v,w)\Psi dx, \end{cases} \tag{2.1.16}$$

for all test functions $\varphi, \psi, \Psi \in \mathcal{H}$ for almost all $t \in [0, T]$.

2.2 Main results and Proofs

2.2.1 Main results

The next theorem is concerned on the local solution (in time $[0,T]$).

Theorem 8. *(Local existence) Assume that*

$$1 < p \leq \frac{n+2}{n-2} \quad \text{and that} \quad n \geq 3. \tag{2.2.1}$$

Let $(u_0, v_0, w_0) \in \mathcal{H}^3$ *and* $(u_1, v_1, w_3) \in L^\kappa_\theta(\mathbb{R}^n) \times L^\kappa_\theta(\mathbb{R}^n) \times L^\kappa_\theta(\mathbb{R}^n)$. *Under the assumptions (2.1.2)-(2.1.13) and (2.1.5)-(2.1.10), suppose that*

$$a + \lambda_1 \omega > 0. \tag{2.2.2}$$

Then (2.1.1) admits a unique local solution (u, v, w) *such that*

$$(u, v, w) \in \mathcal{X}^3_T,\ \ \mathcal{X}_T \equiv C\big([0,T]; \mathcal{H}\big) \cap C^1\big([0,T]; L^\kappa_\theta(\mathbb{R}^n)\big),$$

for sufficiently small $T > 0$.

Remark 2. *The constant* λ_1 *introduced in (2.2.2) being the first eigenvalue of the operator* $-\Delta$.

We will show now the global solution in time established in Theorem 9. Let us introduce the potential energy $J : \mathcal{H}^3 \to \mathbb{R}$ defined by

$$\begin{aligned} J(u, v, w) = & \left(1 - \int_0^t \varpi_1(s)\, ds\right) \|u\|^2_{\mathcal{H}} + (\varpi_1 \circ u) \\ & + \left(1 - \int_0^t \varpi_2(s)\, ds\right) \|v\|^2_{\mathcal{H}} + (\varpi_2 \circ v) \\ & + \left(1 - \int_0^t \varpi_3(s)\, ds\right) \|w\|^2_{\mathcal{H}} + (\varpi_3 \circ w)\,. \end{aligned} \tag{2.2.3}$$

The modified energy is defined by

$$\mathcal{E}(t) = \frac{\kappa - 1}{\kappa}\Big(\|u_t\|^\kappa_{L^\kappa_\theta} + \|v_t\|^\kappa_{L^\kappa_\theta} + \|w_t\|^\kappa_{L^\kappa_\theta} \Big) + \frac{1}{2} J(u, v, w) - \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u, v, w) dx, \tag{2.2.4}$$

here

$$(\varpi_j \circ w)(t) = \int_0^t \varpi_j(t - s) \|w(t) - w(s)\|^2_{\mathcal{H}}\, ds,$$

for any $w \in L^2(\mathbb{R}^n)$, $j = 1, 2, 3$.

Theorem 9. *(Global existence) Let (2.1.2)-(2.1.13) and (2.1.5)-(2.1.10) hold. Under (2.2.1), (2.2.2) and for sufficiently small* $(u_0, u_1), (v_0, v_1), (w_0, w_1) \in \mathcal{H} \times L^{\kappa}_{\theta}(\mathbb{R}^n)$*, problem (2.1.1) admits a unique global solution* (u, v, w) *such that*

$$(u, v, w) \in \mathcal{X}^3,\ \ \mathcal{X} \equiv C\big([0, +\infty); \mathcal{H}\big) \cap C^1\big([0, +\infty); L^{\kappa}_{\theta}(\mathbb{R}^n)\big). \tag{2.2.5}$$

The nonclassical decay rate for solution is given in the next Theorem

Theorem 10. *(Decay of solution) Let (2.1.2)-(2.1.13) and (2.1.5)-(2.1.10) hold. Under conditions (2.2.1), (2.2.2) and*

$$\gamma = \eta\Big(\frac{2(p+1)}{p-1}\mathcal{E}(0)\Big)^{(p-1)/2} < 1, \tag{2.2.6}$$

there exists $t_0 > 0$ *depending only on* $\varpi_1, \varpi_2, \varpi_3$*,* a*,* ω*,* λ_1 *and* $H'(0)$ *such that*

$$0 \leq \mathcal{E}(t) < \mathcal{E}(t_0) \exp\left(-\int_{t_0}^{t} \frac{\mu(s)}{1-\mu_0(t)}\right), \tag{2.2.7}$$

holds for all $t \geq t_0$*.*

In particular, by the positivity of μ in (2.1.8), we have, as in [38],

$$0 \leq \mathcal{E}(t) < \mathcal{E}(t_0) \exp\left(-\int_{t_0}^{t} \mu(s)\, ds\right),$$

for a single wave equation. Condition (2.1.10) is imposed to make a different from [38] and [58], it leads $(\mu' + \nu\mu) \circ u$, here $\nu \in \mathbb{R}$.

The next, Lemma will play an important role in the sequel.

Lemma 3. *For* $(u, v, w) \in \mathcal{X}^3_T$*, the functional* $\mathcal{E}(t)$ *associated with problem (2.1.1) is a decreasing energy.*

Proof. For $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
&\mathcal{E}(t_2) - \mathcal{E}(t_1) \\
= &\int_{t_1}^{t_2} \frac{d}{dt} E(t)\, dt \\
= &-\int_{t_1}^{t_2} \left(a \left\| u_t \right\|^2_{L^2_\theta} + \omega \left\| u_t \right\|^2_{\mathcal{H}} + \frac{1}{2}\varpi_1(t) \left\| u \right\|^2_{\mathcal{H}} - \frac{1}{2}(\varpi_1' \circ u) \right) dt \\
- &\int_{t_1}^{t_2} \left(a \left\| v_t \right\|^2_{L^2_\theta} + \omega \left\| v_t \right\|^2_{\mathcal{H}} + \frac{1}{2}\varpi_2(t) \left\| v \right\|^2_{\mathcal{H}} - \frac{1}{2}(\varpi_2' \circ v) \right) dt \\
- &\int_{t_1}^{t_2} \left(a \left\| w_t \right\|^2_{L^2_\theta} + \omega \left\| w_t \right\|^2_{\mathcal{H}} + \frac{1}{2}\varpi_3(t) \left\| w \right\|^2_{\mathcal{H}} - \frac{1}{2}(\varpi_3' \circ w) \right) dt \\
\leq &\; 0,
\end{aligned}$$

owing to (2.1.5)-(2.1.10). □

The inner product is given as

$$(v, w)_* = \omega \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx + a \int_{\mathbb{R}^n} \theta v w \, dx,$$

and the associated norm is given by

$$\left\| v \right\|_* = \sqrt{(v, v)_*},$$

$\forall v, w \in \mathcal{H}$. By (2.2.2), we get

$$(v, v)_* = \omega \int_{\mathbb{R}^n} \left| \nabla v \right|^2 \, dx + a \int_{\mathbb{R}^n} \theta v^2 \, dx \geq (\omega \lambda_1 + a) \int_{\mathbb{R}^n} \theta v^2 \, dx \geq 0.$$

The following Lemma yields.

Lemma 4. *Let θ satisfy (2.1.2). Under condition (2.2.2), we get*

$$\sqrt{\omega} \left\| v \right\|_{\mathcal{H}} \leq \left\| v \right\|_* \leq \sqrt{\omega + C_P^2} \left\| v \right\|_{\mathcal{H}},$$

for $v \in \mathcal{H}$.

2.2.2 Proofs

We sketch here the outline of the proof for local solution by a standard procedure(See [58]).

Proof. (Of Theorem 8.) Let $(u_0,u_1),(v_0,v_1),(w_0,w_1)\in\mathcal{H}\times L^{\kappa}_{\theta}(\mathbb{R}^n)$. For any $(u,v,w)\in\mathcal{X}_T^3$, we can obtain a weak solution of the related system

$$\begin{cases}\left(|z_t|^{\kappa-2}z_t\right)_t+az_t-\Theta(x)\Delta\left(z+\omega z_t\right)=-\Theta(x)\Delta\int_0^t\varpi_1(t-s)u(s)\,ds+f_1(u,v,w)\\ \left(|y_t|^{\kappa-2}y_t\right)_t+ay_t-\Theta(x)\Delta\left(y+\omega y_t\right)=-\Theta(x)\Delta\int_0^t\varpi_2(t-s)v(s)\,ds+f_2(u,v,w)\\ \left(|\zeta_t|^{\kappa-2}\zeta_t\right)_t+a\zeta_t-\Theta(x)\Delta\left(\zeta+\omega\zeta_t\right)=-\Theta(x)\Delta\int_0^t\varpi_3(t-s)w(s)\,ds+f_3(u,v,w)\\ z(x,0)=u_0(x),y(x,0)=v_0(x),\zeta(x,0)=w_0(x)\\ z_t(x,0)=u_1(x),y_t(x,0)=v_1(x),\zeta_t(x,0)=w_1(x).\end{cases}\tag{2.2.8}$$

We reduces problem (2.2.8) to Cauchy problem for system of ODE by using the Faedo-Galerkin approximation. We then find a solution map $\top:(u,v,w)\mapsto(z,y,\zeta)$ from $\mathcal{X}_T^3$ to $\mathcal{X}_T^3$. We are now ready to show that $\top$ is a contraction mapping in an appropriate subset of $\mathcal{X}_T^3$ for a small $T>0$. Hence $\top$ has a fixed point $\top(u,v,w)=(u,v,w)$, which gives a unique solution in $\mathcal{X}_T^3$. □

We will show the global solution. By using conditions on functions $\varpi_1,\varpi_2,\varpi_3$, we have

$$\begin{aligned}\mathcal{E}(t)&\geq\frac{1}{2}J(u,v,w)-\int_{\mathbb{R}^n}\theta(x)\mathcal{F}(u,v,w)dx\\ &\geq\frac{1}{2}J(u,v,w)-\frac{1}{p+1}\|u+v+w\|^{(p+1)}_{L_\theta^{(p+1)}}-\frac{2}{p+1}\Big(\|uv\|^{(p+1)/2}_{L_\theta^{(p+1)/2}}+\|vw\|^{(p+1)/2}_{L_\theta^{(p+1)/2}}+\|wu\|^{(p+1)/2}_{L_\theta^{(p+1)/2}}\Big)\\ &\geq\frac{1}{2}J(u,v,w)-\frac{\eta}{p+1}\left[l\|u\|^2_{\mathcal{H}}+m\|v\|^2_{\mathcal{H}}+\nu\|w\|^2_{\mathcal{H}}\right]^{(p+1)/2}\\ &\geq\frac{1}{2}J(u,v,w)-\frac{\eta}{p+1}\Big(J(u,v,w)\Big)^{(p+1)/2}\\ &=G(\beta),\end{aligned}\tag{2.2.9}$$

here $\beta^2=J(u,v,w)$, for $t\in[0,T)$, where

$$G(\xi)=\frac{1}{2}\xi^2-\frac{\eta}{p+1}\xi^{(p+1)}.$$

Noting that $\mathcal{E}_0=G(\lambda_0)$, given in (2.1.15). Then

$$\begin{cases}G(\xi)>0\quad in\quad \xi\in[0,\lambda_0]\\ G(\xi)<0\quad in\quad \xi\geq\lambda_0.\end{cases}\tag{2.2.10}$$

Moreover, $\lim_{\xi\to+\infty}G(\xi)\to-\infty$. Then, we have the following Lemma

Lemma 5. *Let* $0 \leq \mathcal{E}(0) < \mathcal{E}_0$.

(i) If $\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 + \|w_0\|_{\mathcal{H}}^2 < \lambda_0^2$, *then local solution of (2.1.1) satisfies*

$$J(u, v, w) < \lambda_0^2, \ \forall t \in [0, T).$$

(ii) If $\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 + \|w_0\|_{\mathcal{H}}^2 > \lambda_0^2$, *then local solution of (2.1.1) satisfies*

$$\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2 > \lambda_1^2, \ \forall t \in [0, T), \lambda_1 > \lambda_0.$$

Proof. Since $0 \leq \mathcal{E}(0) < \mathcal{E}_0 = G(\lambda_0)$, there exist ξ_1 and ξ_2 such that $G(\xi_1) = G(\xi_2) = \mathcal{E}(0)$ with $0 < \xi_1 < \lambda_0 < \xi_2$.

The case (i)**.** By (2.2.9), we have

$$G(J(u_0, v_0, w_0)) \leq \mathcal{E}(0) = G(\xi_1),$$

which implies that $J(u_0, v_0, w_0) \leq \xi_1^2$. Then we claim that $J(u, v, w) \leq \xi_1^2, \ \forall t \in [0, T)$. Moreover, there exists $t_0 \in (0, T)$ such that

$$\xi_1^2 < J(u(t_0), v(t_0), w(t_0)) < \xi_2^2.$$

Then

$$G(J(u(t_0), v(t_0), w(t_0))) > \mathcal{E}(0) \geq \mathcal{E}(t_0),$$

by Lemma 3, which contradicts (2.2.9). Hence we have

$$J(u, v, w) \leq \xi_1^2 < \lambda_0^2, \ \forall t \in [0, T).$$

The case (ii)**.** We can now show that $\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 + \|w_0\|_{\mathcal{H}}^2 \geq \xi_2^2$ and that $\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2 \geq \xi_2^2 > \lambda_0^2$ in the same way as (i). □

Proof. (*Of Theorem 9.*) Let $(u_0, u_1), (v_0, v_1), (w_0, w_1) \in \mathcal{H} \times L_\theta^\kappa(\mathbb{R}^n)$ satisfy both $0 \leq \mathcal{E}(0) < \mathcal{E}_0$

and $\|u_0\|^2_{\mathcal{H}} + \|v_0\|^2_{\mathcal{H}} + \|w_0\|^2_{\mathcal{H}} < \lambda_0^2$. By Lemma 3 and Lemma 5, we have

$$\begin{aligned}
&\frac{2(\kappa-1)}{\kappa}\Big(\|u_t\|^\kappa_{L^\kappa_\theta} + \|v_t\|^\kappa_{L^\kappa_\theta} + \|w_t\|^\kappa_{L^\kappa_\theta}\Big) + l\,\|u\|^2_{\mathcal{H}} + m\,\|v\|^2_{\mathcal{H}} + \nu\,\|w\|^2_{\mathcal{H}} \\
&\le \frac{2(\kappa-1)}{\kappa}\Big(\|u_t\|^\kappa_{L^\kappa_\theta} + \|v_t\|^\kappa_{L^\kappa_\theta} + \|w_t\|^\kappa_{L^\kappa_\theta}\Big) + \left(1 - \int_0^t \varpi_1(s)\,ds\right)\|u\|^2_{\mathcal{H}} + (\varpi_1 \circ u) \\
&+ \left(1 - \int_0^t \varpi_2(s)\,ds\right)\|u\|^2_{\mathcal{H}} + (\varpi_2 \circ v) + \left(1 - \int_0^t \varpi_3(s)\,ds\right)\|w\|^2_{\mathcal{H}} + (\varpi_3 \circ w) \\
&\le 2\mathcal{E}(t) + \frac{2\eta}{p+1}\Big[l\,\|u\|^2_{\mathcal{H}} + m\,\|u\|^2_{\mathcal{H}} + \nu\,\|w\|^2_{\mathcal{H}}\Big]^{(p+1)/2} \\
&\le 2\mathcal{E}(0) + \frac{2\eta}{p+1}\Big(J(u,v,w)\Big)^{(p+1)/2} \\
&\le 2\mathcal{E}_0 + \frac{2\eta}{p+1}\lambda_0^{p+1} \\
&= \eta^{-2/(p-1)}.
\end{aligned} \tag{2.2.11}$$

This completes the proof. □

Let

$$\begin{aligned}
\Lambda(u,v,w) &= \frac{1}{2}\left(1 - \int_0^t \varpi_1(s)\,ds\right)\|u\|^2_{\mathcal{H}} + \frac{1}{2}(\varpi_1 \circ u) \\
&+ \frac{1}{2}\left(1 - \int_0^t \varpi_2(s)\,ds\right)\|v\|^2_{\mathcal{H}} + \frac{1}{2}(\varpi_2 \circ v) \\
&+ \frac{1}{2}\left(1 - \int_0^t \varpi_3(s)\,ds\right)\|w\|^2_{\mathcal{H}} + \frac{1}{2}(\varpi_3 \circ w) - \int_{\mathbb{R}^n} \theta(x)\mathcal{F}(u,v,w)dx,
\end{aligned} \tag{2.2.12}$$

$$\begin{aligned}
\Pi(u,v,w) &= \left(1 - \int_0^t \varpi_1(s)\,ds\right)\|u\|^2_{\mathcal{H}} + (\varpi_1 \circ u) \\
&+ \left(1 - \int_0^t \varpi_2(s)\,ds\right)\|v\|^2_{\mathcal{H}} + (\varpi_2 \circ v) \\
&+ \left(1 - \int_0^t \varpi_3(s)\,ds\right)\|w\|^2_{\mathcal{H}} + (\varpi_3 \circ w) - (p+1)\int_{\mathbb{R}^n} \theta(x)\mathcal{F}(u,v,w)dx.
\end{aligned} \tag{2.2.13}$$

Lemma 6. *Let (u,v,w) be the solution of problem (2.1.1). If*

$$\|u_0\|^2_{\mathcal{H}} + \|v_0\|^2_{\mathcal{H}} + \|w_0\|^2_{\mathcal{H}} - (p+1)\int_{\mathbb{R}^n} \theta(x)\mathcal{F}(u_0,v_0,w_0)dx > 0. \tag{2.2.14}$$

Then under condition (2.2.6), the functional $\Pi(u,v,w) > 0, \ \forall t > 0$.

Proof. By (2.2.14) and continuity, there exists a time $t_1 > 0$ such that

$$\Pi(u,v,w) \geq 0, \forall t < t_1.$$

Let

$$Y = \{(u,v,w) \mid \Pi(u(t_0),v(t_0),w(t_0)) = 0,\ \Pi(u,v,w) > 0, \forall t \in [0,t_0)\}. \tag{2.2.15}$$

Then, by (2.2.12), (2.2.13), we have for all $(u,v,w) \in Y$,

$$\begin{aligned}
&\Lambda(u,v,w) \\
&= \frac{p-1}{2(p+1)}\Big[\Big(1-\int_0^t \varpi_1(s)\,ds\Big)\|u\|_{\mathcal{H}}^2 + \Big(1-\int_0^t \varpi_2(s)\,ds\Big)\|v\|_{\mathcal{H}}^2 + \Big(1-\int_0^t \varpi_3(s)\,ds\Big)\|w\|_{\mathcal{H}}^2\Big] \\
&+\frac{p-1}{2(p+1)}\Big[(\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)\Big] + \frac{1}{p+1}\Pi(u,v,w) \\
&\geq \frac{p-1}{2(p+1)}\Big[l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 + \nu\,\|w\|_{\mathcal{H}}^2 + (\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)\Big].
\end{aligned}$$

Owing to (2.2.4), it follows for $(u,v,w) \in Y$

$$l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 + \nu\,\|w\|_{\mathcal{H}}^2 \leq \frac{2(p+1)}{p-1}\Lambda(u,v,w) \leq \frac{2(p+1)}{p-1}\mathcal{E}(t) \leq \frac{2(p+1)}{p-1}\mathcal{E}(0). \tag{2.2.16}$$

By (2.1.14), (2.2.6) we have

$$\begin{aligned}
(p+1)\int_{\mathbb{R}^n} \mathcal{F}(u(t_0),v(t_0),w(t_0)) &\leq \eta\left(l\|u(t_0)\|_{\mathcal{H}}^2 + m\|v(t_0)\|_{\mathcal{H}}^2 + \nu\,\|w(t_0)\|_{\mathcal{H}}^2\right)^{(p+1)/2} \\
&\leq \eta\Big(\frac{2(p+1)}{p-1}E(0)\Big)^{(p-1)/2}(l\,\|u(t_0)\|_{\mathcal{H}}^2 + m\,\|v(t_0)\|_{\mathcal{H}}^2 + \nu\,\|w(t_0)\|_{\mathcal{H}}^2) \\
&\leq \gamma(l\,\|u(t_0)\|_{\mathcal{H}}^2 + m\,\|v(t_0)\|_{\mathcal{H}}^2 + \nu\,\|w(t_0)\|_{\mathcal{H}}^2) \\
&< \Big(1-\int_0^{t_0}\varpi_1(s)ds\Big)\|u(t_0)\|_{\mathcal{H}}^2 + \Big(1-\int_0^{t_0}\varpi_2(s)ds\Big)\|v(t_0)\|_{\mathcal{H}}^2 \\
&+ \Big(1-\int_0^{t_0}\varpi_3(s)ds\Big)\|w(t_0)\|_{\mathcal{H}}^2 \\
&< \Big(1-\int_0^{t_0}\varpi_1(s)ds\Big)\|u(t_0)\|_{\mathcal{H}}^2 + \Big(1-\int_0^{t_0}\varpi_2(s)ds\Big)\|v(t_0)\|_{\mathcal{H}}^2 \\
&+ \Big(1-\int_0^{t_0}\varpi_3(s)ds\Big)\|w(t_0)\|_{\mathcal{H}}^2 \\
&+ (\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w),
\end{aligned} \tag{2.2.17}$$

hence $\Pi(u(t_0),v(t_0),w(t_0)) > 0$ on Y, which contradicts the definition of Y since $\Pi(u(t_0),v(t_0),w(t_0)) = 0$. Thus $\Pi(u,v,w) > 0,\ \forall t > 0$. □

We are ready to prove the decay rate.

Proof. (*Of Theorem 10.*) By (2.1.14) and (2.2.16), we have for $t \geq 0$

$$0 < l \|u\|_{\mathcal{H}}^2 + m \|v\|_{\mathcal{H}}^2 + \nu \|w\|_{\mathcal{H}}^2 \leq \frac{2(p+1)}{p-1}\mathcal{E}(t). \tag{2.2.18}$$

Let

$$I(t) = \frac{\mu(t)}{1-\mu_0(t)},$$

where μ and μ_0 defined in (2.1.8) and (2.1.9).

Noting that $\lim\limits_{t\to+\infty} \mu(t) = 0$ by (2.1.5)-(2.1.9), we have

$$\lim_{t\to+\infty} I(t) = 0, \quad I(t) > 0, \ \forall t \geq 0.$$

Then we take $t_0 > 0$ such that

$$0 < \frac{2(\kappa-1)}{\kappa} I(t) < \min\left\{2\left(\omega\lambda_1 + a\right), \chi'(0)\right\},$$

with (2.1.10) for all $t > t_0$. Due to (2.2.4), we have

$$\begin{aligned}
\mathcal{E}(t) \ &\leq \ \frac{(\kappa-1)}{\kappa}\Big(\|u_t\|_{L_\theta^\kappa}^\kappa + \|v_t\|_{L_\theta^\kappa}^\kappa + \|w_t\|_{L_\theta^\kappa}^\kappa \Big) + \frac{1}{2}[(\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)] \\
&+ \ \frac{1}{2}\left(1 - \int_0^t \varpi_1(s)\, ds\right) \|u\|_{\mathcal{H}}^2 + \frac{1}{2}\left(1 - \int_0^t \varpi_2(s)\, ds\right) \|v\|_{\mathcal{H}}^2 + \frac{1}{2}\left(1 - \int_0^t \varpi_3(s)\, ds\right) \|w\|_{\mathcal{H}}^2 \\
&\leq \ \frac{(\kappa-1)}{\kappa}\Big(\|u_t\|_{L_\theta^\kappa}^\kappa + \|v_t\|_{L_\theta^\kappa}^\kappa + \|w_t\|_{L_\theta^\kappa}^\kappa \Big) + \frac{1}{2}[(\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)] \\
&+ \ \frac{1}{2}(1-\mu_0(t))[\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2].
\end{aligned}$$

Then by definition of $I(t)$, we have

$$\begin{aligned}
I(t)\mathcal{E}(t) \ &\leq \ \frac{(\kappa-1)}{\kappa} I(t)\Big(\|u_t\|_{L_\theta^\kappa}^\kappa + \|v_t\|_{L_\theta^\kappa}^\kappa + \|w_t\|_{L_\theta^\kappa}^\kappa \Big) + \frac{1}{2}\mu(t)[\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2] \\
&+ \ \frac{1}{2} I(t)[(\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)],
\end{aligned} \tag{2.2.19}$$

and Lemma 3, we have for all $t_1, t_2 \geq 0$

$$\begin{aligned}
&\mathcal{E}(t_2) - \mathcal{E}(t_1) \\
&\quad \leq \ -\int_{t_1}^{t_2} \left(a\|w_t\|_{L_\theta^2}^2 + a\|u_t\|_{L_\theta^2}^2 + \omega\|u_t\|_{\mathcal{H}}^2 + \frac{1}{2}\mu(t)[\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2] \right) dt \\
&\quad - \ \int_{t_1}^{t_2} \left(a\|v_t\|_{L_\theta^2}^2 + \omega\|v_t\|_{\mathcal{H}}^2 + \omega\|w_t\|_{\mathcal{H}}^2 - \frac{1}{2}(\varpi_1' \circ u) - \frac{1}{2}(\varpi_2' \circ v) - \frac{1}{2}(\varpi_3' \circ w) \right) dt
\end{aligned}$$

then, by generalized Poincaré's inequalities, we get

$$\begin{aligned}
\mathcal{E}'(t) &\leq -(\omega\lambda_1 + a)\,[\|u_t\|^2_{L^2_\theta} + \|v_t\|^2_{L^2_\theta} + \|w_t\|^2_{L^2_\theta}] \\
&- \frac{1}{2}\mu(t)[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}} + \|w\|^2_{\mathcal{H}}] \\
&+ \frac{1}{2}[(\varpi_1' \circ u) + (\varpi_2' \circ v) + (\varpi_3' \circ w)],
\end{aligned}$$

Finally, $\forall t \geq t_0$, we have

$$\begin{aligned}
&\mathcal{E}'(t) + I(t)\mathcal{E}(t) \\
&\leq \left\{\frac{(\kappa-1)}{\kappa}I(t) - (\omega\lambda_1 + a)\right\}\left(\|u_t\|^2_{L^2_\theta} + \|v_t\|^2_{L^2_\theta} + \|w_t\|^2_{L^2_\theta}\right) \\
&+ \frac{1}{2}[(\varpi_1' \circ u) + (\varpi_2' \circ v) + (\varpi_3' \circ w)] + \frac{1}{2}I(t)((\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)) \\
&\leq \frac{1}{2}\int_0^t \{\varpi_1'(t-\tau) + I(t)\varpi_2(t-\tau)\}\,\|u(t) - u(\tau)\|^2_{\mathcal{H}}\,d\tau \\
&+ \frac{1}{2}\int_0^t \{\varpi_2'(t-\tau) + I(t)\varpi_2(t-\tau)\}\,\|v(t) - v(\tau)\|^2_{\mathcal{H}}\,d\tau \\
&+ \frac{1}{2}\int_0^t \{\varpi_3'(t-\tau) + I(t)\varpi_3(t-\tau)\}\,\|w(t) - w(\tau)\|^2_{\mathcal{H}}\,d\tau \\
&\leq \frac{1}{2}\int_0^t \{\varpi_1'(\tau) + I(t)\varpi_1(\tau)\}\,\|u(t) - u(t-\tau)\|^2_{\mathcal{H}}\,d\tau \\
&+ \frac{1}{2}\int_0^t \{\varpi_2'(\tau) + I(t)\varpi_2(\tau)\}\,\|v(t) - v(t-\tau)\|^2_{\mathcal{H}}\,d\tau \\
&+ \frac{1}{2}\int_0^t \{\varpi_3'(\tau) + I(t)\varpi_3(\tau)\}\,\|w(t) - w(t-\tau)\|^2_{\mathcal{H}}\,d\tau \\
&\leq \frac{1}{2}\int_0^t \left\{-\chi\Big(\varpi_1(\tau)\Big) + \chi'(0)\varpi_1(\tau)\right\}\|u(t) - u(t-\tau)\|^2_{\mathcal{H}}\,d\tau \\
&+ \frac{1}{2}\int_0^t \left\{-\chi\Big(\varpi_2(\tau)\Big) + \chi'(0)\varpi_2(\tau)\right\}\|v(t) - v(t-\tau)\|^2_{\mathcal{H}}\,d\tau \\
&+ \frac{1}{2}\int_0^t \left\{-\chi\Big(\varpi_3(\tau)\Big) + \chi'(0)\varpi_3(\tau)\right\}\|w(t) - w(t-\tau)\|^2_{\mathcal{H}}\,d\tau \\
&\leq 0,
\end{aligned}$$

by the convexity of χ and (2.1.10), we have

$$\chi(\xi) \geq \chi(0) + \chi'(0)\xi = \chi'(0)\xi.$$

Then

$$\mathcal{E}(t) \leq \mathcal{E}(t_0)\exp\left(-\int_{t_0}^t I(s)ds\right),$$

which completes the proof.

□

Chapter 3

Coupled nonlinear viscoelastic wave equation with distributed delay and strong damping and source terms

1- Introduction
2- Blow up in finite time
3- Growth of solutions to system

3.1 Introduction

In this section, we are concerned with a problem for a coupled nonlinear viscoelastic wave equation with distributed delay and strong damping and source terms, under suitable assumptions we will prove the time blow up result and show the exponential growth of solutions. Namely, we consider the following problem

$$\begin{cases} u_{tt} - \Delta u - \omega_1 \Delta u_t + \displaystyle\int_0^t g(t-s)\Delta u(s)ds \\ \quad +\mu_1 u_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u_t(x, t-\varrho) d\varrho = f_1(u,v), \ \ (x,t) \in \Omega \times \mathbb{R}_+ \\ v_{tt} - \Delta v - \omega_2 \Delta v_t + \displaystyle\int_0^t h(t-s)\Delta v(s)ds \\ \quad +\mu_3 v_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v_t(x, t-\varrho) d\varrho = f_2(u,v), \ \ (x,t) \in \Omega \times \mathbb{R}_+ \\ u\left(x,t\right) = 0, \ \ v\left(x,t\right) = 0, \ \ x \in \partial\Omega \\ u_t\left(x,-t\right) = f_0\left(x,t\right), \ \ v_t\left(x,-t\right) = k_0\left(x,t\right), \ \ (x,t) \in \Omega \times (0, \tau_2) \\ u\left(x,0\right) = u_0\left(x\right), u_t\left(x,0\right) = u_1\left(x\right), x \in \Omega \\ v\left(x,0\right) = v_0\left(x\right), v_t, \left(x,0\right) = v_1\left(x\right), x \in \Omega, \end{cases} \tag{3.1.1}$$

where

$$\begin{cases} f_1(u,v) = a_1|u+v|^{2(p+1)}(u+v) + b_1|u|^p u|v|^{p+2} \\ f_2(u,v) = a_1|u+v|^{2(p+1)}(u+v) + b_1|v|^p v|u|^{p+2}, \end{cases} \tag{3.1.2}$$

and $\omega_1, \omega_2, \mu_1, \mu_3, a_1, b_1 > 0$, and τ_1, τ_2 are the time delay with $0 \leq \tau_1 < \tau_2$, and μ_2, μ_4 are a L^∞ functions, and g, h are a differentiable functions.

Viscous materials are the opposite of elastic materials that possess the ability to store and dissipate mechanical energy. As the mechanical properties of these viscous substances are of great importance when they appear in many applications of natural sciences. Many authors have given attention and attention to this problem since the beginning of the new millennium. In the case of only one equation and if $w_1 = 0$, that is for absence of Δu_t, and $\mu_1 = \mu_2 = 0$. Our problem (3.1.1) has been studied by Berrimi and Messaoudi [7]. By using the Galerkin method they established the local existence result. Also, they showed the local solution is global in time under a suitable conditions, and with the same rate of decaying (polynomial or exponential) of the kernel g. they proved that the dissipation given by the viscoelastic integral term is strong enough to stabilize the oscillations of the solution. Also their result has been obtained under weaker conditions than those used by Cavalcanti et al [16].

In [17] the authors are considered the following problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^\gamma u = 0, \tag{3.1.3}$$

the authors proved the exponential decay result. This later result has been improved by Berrimi et al [7], in which they showed that the viscoelastic dissipation alone is strong enough to stabilize the problem even with an exponential rate.

In many works on this field, under assumptions of the kernel g. For the problem (3.1.1) and with $\mu_1 \neq 0$, for example in [30] Kafini et al proved a blow up result for the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^{\infty} g(t-s)\Delta u(s)ds + u_t = |u|^{p-2}u, \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \end{cases} \tag{3.1.4}$$

where g satisfies $\int_0^{\infty} g(s)ds < (2p-4)/(2p-3)$, Initial data was supported with negative energy like that $\int u_0 u_1 dx > 0$.

If $(w > 0)$. In [49], Song et al considered with the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^{\infty} g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2}.u, \quad (x,t) \in \Omega \times (0,\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x). \end{cases} \tag{3.1.5}$$

Under suitable assumptions on g, that there were solutions of (3.1.5) with initial energy, they showed the blow up in a finite time. For the same problem (3.1.5), in [48], Song et al proved that there were solutions of (3.1.5) with positive initial energy that blow up in finite time. In [54], Zennir studied the following problem

$$\begin{cases} u_{tt} - \Delta u - \omega\Delta u_t + \int_0^{t} g(t-s)\Delta u(s)ds \\ +a|u_t|^{m-2}u_t = |u|^{p-2}u, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \quad x \in \Omega \\ u(x,t) = 0, \quad x \in \partial\Omega, \end{cases} \tag{3.1.6}$$

the author proved the exponential growth result under suitable assumptions.

In [36] the authors considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^{\infty} g(s)\Delta u(t-s)ds - \varepsilon_1\Delta u_t + \varepsilon_2 u_t|u_t|^{m-2} = \varepsilon_3 u|u|^{p-2} \\ u(x,t) = 0, \quad x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{3.1.7}$$

they showed a blow up result if $p > m$, and established the global existence.

In the case of coupled of equations, in [1], the authors are studied the following system of equations

$$\begin{cases} u_{tt} - \Delta u + u_t|u_t|^{m-2} = f_1(u,v) \\ v_{tt} - \Delta v + v_t|v_t|^{r-2} = f_2(u,v), \end{cases} \tag{3.1.8}$$

with nonlinear functions f_1 and f_2 satisfying appropriate conditions. Under certain restrictions imposed on the parameters and the initial data, they obtained numerous results on the existence of weak solutions. They also showed that any weak solution with negative initial energy blows up for a finite period of time by using the same techniques as in [29]. And in [6], the authors considered the system:

$$\begin{cases} u_{tt} - \Delta u + (a|u|^k + b|v|^l)u_t|u_t|^{m-2} = f_1(u,v) \\ v_{tt} - \Delta v + (a|u|^\theta + b|v|^\vartheta)v_t|v_t|^{r-2} = f_2(u,v), \end{cases} \tag{3.1.9}$$

they stated and proved the blows up in finite time of solution, under some restrictions on the initial data and (with positive initial energy) for some conditions on the functions f_1 and f_2. In [41], the authors extended the result of [6], are considered the following nonlinear viscoelastic system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(t-s)ds + (a|u|^k + b|v|^l)u_t|u_t|^{m-2} = f_1(u,v) \\ v_{tt} - \Delta v + \int_0^\infty h(s)\Delta v(t-s)ds + (a|u|^\theta + b|v|^\varrho)v_t|v_t|^{r-2} = f_2(u,v), \end{cases} \tag{3.1.10}$$

they proved that the solutions of a system of wave equations with viscoelastic term, degenerate damping, and strong nonlinear sources acting in both equations at the same time are globally nonexisting provided that the initial data are sufficiently large in a bounded domain of Ω.

A complement to these works, we are working to prove the blow-up result with distributed delay of the problem (3.1.1), under appropriate assumptions and we prove these results using the energy method. In the following, let $c, c_i > 0, i = 1, ..., 12$.

We prove the blow-up result under the following suitable assumptions.

(**A1**) $g, h : \mathbb{R}_+ \to \mathbb{R}_+$ are a differentiable and decreasing functions such that

$$\begin{aligned} g(t) \geq 0 \quad , \quad 1 - \int_0^\infty g(s)\,ds = l_1 > 0, \\ h(t) \geq 0 \quad , \quad 1 - \int_0^\infty h(s)\,ds = l_2 > 0. \end{aligned} \tag{3.1.11}$$

(**A2**) There exists a constants $\xi_1, \xi_2 > 0$ such that

$$g'(t) \leq -\xi_1 g(t) \quad , \quad t \geq 0,$$
$$h'(t) \leq -\xi_2 h(t) \quad , \quad t \geq 0. \tag{3.1.12}$$

(**A3**) $\mu_2, \mu_4 : [\tau_1, \tau_2] \to \mathbb{R}$ are a L^∞ functions so that

$$\left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho < \mu_1 \quad , \quad \delta > \frac{1}{2},$$
$$\left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| d\varrho < \mu_3 \quad , \quad \delta > \frac{1}{2}. \tag{3.1.13}$$

3.2 Blow up in finite time

In this subsection, we prove the blow up result of solution of problem (3.1.1).

First, as in [40], we introduce the new variables

$$\begin{cases} y(x, \rho, \varrho, t) = u_t(x, t - \varrho\rho) \\ z(x, \rho, \varrho, t) = v_t(x, t - \varrho\rho), \end{cases} \tag{3.2.1}$$

then we obtain

$$\begin{cases} \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0 \\ y(x, 0, \varrho, t) = u_t(x, t), \end{cases} \tag{3.2.2}$$

and

$$\begin{cases} \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0 \\ z(x, 0, \varrho, t) = v_t(x, t), \end{cases} \tag{3.2.3}$$

Let us denote by

$$gou = \int_\Omega \int_0^t g(t-s)|u(t) - u(s)|^2 ds dx. \tag{3.2.4}$$

Therefore, problem (3.1.1) takes the form

$$\begin{cases} u_{tt} - \Delta u - \omega_1 \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds \\ \quad +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t)d\varrho = f_1(u,v), \quad x \in \Omega, t \geq 0 \\ v_{tt} - \Delta v - \omega_2 \Delta v_t + \int_0^t h(t-s)\Delta v(s)ds \\ \quad +\mu_3 v_t + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z(x,1,\varrho,t)d\varrho = f_2(u,v), \quad x \in \Omega, t \geq 0 \\ \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0 \\ \varrho z_t(x,\rho,\varrho,t) + z_\rho(x,\rho,\varrho,t) = 0, \end{cases} \tag{3.2.5}$$

with initial and boundary conditions

$$\begin{cases} u(x,t) = 0, \ \ v(x,t) = 0 \quad x \in \partial\Omega, \\ y(x,\rho,\varrho,0) = f_0(x,\varrho\rho), \ \ z(x,\rho,\varrho,0) = k_0(x,\varrho\rho) \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) \\ v(x,0) = v_0(x), \ \ v_t(x,0) = v_1(x), \end{cases} \tag{3.2.6}$$

where

$$(x,\rho,\varrho,t) \in \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

Theorem 11. *Assume (3.1.11),(3.1.12), and (3.1.13) holds. Let*

$$\begin{cases} -1 < p < \dfrac{4-n}{n-2}, \quad n \geq 3 \\ p \geq -1, \qquad n = 1,2. \end{cases} \tag{3.2.7}$$

Then for any initial data

$$(u_0, u_1, v_0, v_1, f_0, k_0) \in \mathcal{H},$$

where

$$\begin{aligned} \mathcal{H} &= H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1) \times (\tau_1,\tau_2)) \\ &\quad \times L^2(\Omega \times (0,1) \times (\tau_1,\tau_2)). \end{aligned}$$

the problem (3.2.5) has a unique solution

$$u \in C([0,T];\mathcal{H}),$$

for some $T > 0$.

Lemma 7. *There exists a function $F(u,v)$ such that*

$$\begin{aligned} F(u,v) &= \frac{1}{2(\rho+2)}\left[uf_1(u,v)+vf_2(u,v)\right] \\ &= \frac{1}{2(\rho+2)}\left[a_1|u+v|^{2(p+2)}+2b_1|uv|^{p+2}\right]\geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u}=f_1(u,v),\quad \frac{\partial F}{\partial v}=f_2(u,v).$$

we take $a_1=b_1=1$ for convenience.

Lemma 8. *[41] There exist two positive constants c_0 and c_1 such that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(p+2)}+|v|^{2(p+2)}\right)\leq F(u,v)\leq \frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)}+|v|^{2(p+2)}\right). \tag{3.2.8}$$

We define the energy functional

Lemma 9. *Assume (3.1.11),(3.1.12),(3.1.13), and (3.2.7) hold, let (u,v,y,z) be a solution of (3.2.5), then $E(t)$ is non-increasing, that is*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|v_t\|_2^2+\frac{1}{2}l_1\|\nabla u\|_2^2+\frac{1}{2}l_2\|\nabla v\|_2^2 \\ &+\frac{1}{2}(go\nabla u)+\frac{1}{2}(ho\nabla v)+\frac{1}{2}K(y,z)-\int_\Omega F(u,v)dx, \end{aligned} \tag{3.2.9}$$

satisfies

$$\begin{aligned} E'(t) &\leq -c_3\{\|u_t\|_2^2+\|v_t\|_2^2+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\ &+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\}\ \leq 0, \end{aligned} \tag{3.2.10}$$

where

$$K(y,z)=\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho\{|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)+|\mu_4(\varrho)|z^2(x,\rho,\varrho,t)\}d\varrho d\rho dx. \tag{3.2.11}$$

Proof. By multiplying $(3.2.5)_1, (3.2.5)_2$ by u_t, v_t and integrating over Ω, we get

$$\begin{aligned}
\frac{d}{dt} & \{\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}l_1\|\nabla u\|_2^2 + \frac{1}{2}l_2\|\nabla v\|_2^2 + \frac{1}{2}(go\nabla u) \\
& + \frac{1}{2}(ho\nabla v) - \int_\Omega F(u,v)dx\} \\
= & -\mu_1\|u_t\|_2^2 - \int_\Omega u_t \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho dx \\
& -\mu_3\|v_t\|_2^2 - \int_\Omega v_t \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|z(x,1,\varrho,t)d\varrho dx \\
& + \frac{1}{2}(g'o\nabla u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \omega_1\|\nabla u_t\|_2^2 \\
& + \frac{1}{2}(h'o\nabla v) - \frac{1}{2}h(t)\|\nabla v\|_2^2 - \omega_2\|\nabla v_t\|_2^2, \qquad (3.2.12)
\end{aligned}$$

and, from $(3.2.5)_3, (3.2.5)_4$ we have

$$\begin{aligned}
& \frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx \\
& = -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} 2|\mu_2(\varrho)|yy_\rho d\varrho d\rho dx \\
= & +\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,0,\varrho,t)d\varrho dx \\
& -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\
= & \frac{1}{2}(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)d\varrho)\|u_t\|_2^2 \\
& -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx,
\end{aligned}$$

$$\begin{aligned}
& \frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_4(\varrho)|z^2(x,\rho,\varrho,t)d\varrho d\rho dx \\
& = -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} 2|\mu_4(\varrho)|zz_\rho d\varrho d\rho dx \\
= & +\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|z^2(x,0,\varrho,t)d\varrho dx \\
& -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx \\
= & \frac{1}{2}(\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)d\varrho)\|v_t\|_2^2 \\
& -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx,
\end{aligned}$$

then, we get

$$\begin{aligned}\frac{d}{dt}E(t) &= -\mu_1\|u_t\|_2^2 - \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|u_t y(x,1,\varrho,t)d\varrho dx + \frac{1}{2}(g'o\nabla u)\\ &-\frac{1}{2}g(t)\|\nabla u\|_2^2 - \omega_1\|\nabla u_t\|_2^2 + \frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)d\varrho)\|u_t\|_2^2\\ &-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\\ &-\mu_3\|v_t\|_2^2 - \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|v_t z(x,1,\varrho,t)d\varrho dx + \frac{1}{2}(h'o\nabla v)\\ &-\frac{1}{2}h(t)\|\nabla v\|_2^2 - \omega_2\|\nabla v_t\|_2^2 + \frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)d\varrho)\|v_t\|_2^2\\ &-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx. \end{aligned} \tag{3.2.13}$$

By (3.2.12)-(3.2.13), we get (3.2.9). By using Young's inequality, (3.1.11),(3.1.12) and (3.1.13) in (3.2.13), we obtain (3.2.10). □

Now we define the functional

$$\begin{aligned}\mathbb{H}(t) = -E(t) &= -\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|v_t\|_2^2 - \frac{1}{2}l_1\|\nabla u\|_2^2 - \frac{1}{2}l_2\|\nabla v\|_2^2\\ &-\frac{1}{2}(go\nabla u) - \frac{1}{2}(ho\nabla v) - \frac{1}{2}K(y,z)\\ &+\frac{1}{2(p+2)}[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]. \end{aligned} \tag{3.2.14}$$

Theorem 12. *Assume (3.1.11)-(3.1.13), and (3.2.7) hold. Assume further that $E(0) < 0$, then the solution of problem (3.2.5) blow up in finite time.*

Proof. From (3.2.9), we have

$$E(t) \leq E(0) \leq 0. \tag{3.2.15}$$

Therefore

$$\begin{aligned}\mathbb{H}'(t) = -E'(t) &\geq c_3(\|u_t\|_2^2 + \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\\ &+\|v_t\|_2^2 + \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx), \end{aligned} \tag{3.2.16}$$

hence

$$\begin{aligned} \mathbb{H}'(t) &\geq c_3 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx \geq 0 \\ \mathbb{H}'(t) &\geq c_3 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| z^2(x,1,\varrho,t) d\varrho dx \geq 0, \end{aligned} \tag{3.2.17}$$

and

$$\begin{aligned} 0 \leq \mathbb{H}(0) \leq \mathbb{H}(t) &\leq \frac{1}{2(p+2)} [\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}] \\ &\leq \frac{c_1}{2(p+2)} [\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}]. \end{aligned} \tag{3.2.18}$$

We set

$$\begin{aligned} \mathcal{K}(t) &= \mathbb{H}^{1-\alpha} + \varepsilon \int_\Omega (uu_t + vv_t) dx + \frac{\varepsilon}{2} \int_\Omega (\mu_1 u^2 + \mu_3 v^2) dx \\ &\quad + \frac{\varepsilon}{2} \int_\Omega (\omega_1 (\nabla u)^2 + \omega_2 (\nabla v)^2) dx, \end{aligned} \tag{3.2.19}$$

where $\varepsilon > 0$ to be assigned later and

$$0 < \alpha < \frac{2p+2}{4(p+2)} < 1. \tag{3.2.20}$$

By multiplying $(3.2.5)_1, (3.2.5)_2$ by u, v and with a derivative of (3.2.19), we get

$$\begin{aligned} \mathcal{K}'(t) &= (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) - \varepsilon(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) \\ &\quad + \varepsilon \int_\Omega \nabla u \int_0^t g(t-s) \nabla u(s) ds dx + \varepsilon \int_\Omega \nabla v \int_0^t h(t-s) \nabla v(s) ds dx \\ &\quad - \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u y(x,1,\varrho,t) d\varrho dx - \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v z(x,1,\varrho,t) d\varrho dx \\ &\quad + \varepsilon [\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]. \end{aligned} \tag{3.2.21}$$

Using Young's inequality, we get

$$\begin{aligned} \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u y(x,1,\varrho,t) d\varrho dx &\leq \varepsilon \{ \delta_1 (\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho) \|u\|_2^2 \\ &\quad + \frac{1}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx \}, \end{aligned} \tag{3.2.22}$$

$$\begin{aligned}\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|vz(x,1,\varrho,t)d\varrho dx &\leq \varepsilon\{\delta_2(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho)\|v\|_2^2\\ &\quad+\frac{1}{4\delta_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\},\end{aligned} \tag{3.2.23}$$

and, we have

$$\begin{aligned}\varepsilon\int_0^t g(t-s)ds\int_\Omega\nabla u.\nabla u(s)dxds &= \varepsilon\int_0^t g(t-s)ds\int_\Omega\nabla u.(\nabla u(s)-\nabla u(t))dxds\\ &\quad+\varepsilon\int_0^t g(s)ds\|\nabla u\|_2^2\\ &\geq \frac{\varepsilon}{2}\int_0^t g(s)ds\|\nabla u\|_2^2-\frac{\varepsilon}{2}(go\nabla u),\end{aligned} \tag{3.2.24}$$

$$\begin{aligned}\varepsilon\int_0^t h(t-s)ds\int_\Omega\nabla v.\nabla v(s)dxds &= \varepsilon\int_0^t h(t-s)ds\int_\Omega\nabla v.(\nabla v(s)-\nabla v(t))dxds\\ &\quad+\varepsilon\int_0^t h(s)ds\|\nabla v\|_2^2\\ &\geq \frac{\varepsilon}{2}\int_0^t h(s)ds\|\nabla v\|_2^2-\frac{\varepsilon}{2}(ho\nabla v),\end{aligned} \tag{3.2.25}$$

we obtain, from (3.2.21),

$$\begin{aligned}\mathcal{K}'(t) &\geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon(\|u_t\|_2^2+\|u_t\|_2^2)\\ &\quad-\varepsilon((1-\frac{1}{2}\int_0^t g(s)ds)\|\nabla u\|_2^2+(1-\frac{1}{2}\int_0^t h(s)ds)\|\nabla v\|_2^2)\\ &\quad-\varepsilon\delta_1(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\|u\|_2^2-\varepsilon\delta_2(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho)\|v\|_2^2\\ &\quad-\frac{\varepsilon}{2}(go\nabla u)-\frac{\varepsilon}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\\ &\quad-\frac{\varepsilon}{2}(ho\nabla v)-\frac{\varepsilon}{4\delta_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\\ &\quad+\varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}].\end{aligned} \tag{3.2.26}$$

Therefore, using (3.2.17) and by setting δ_1,δ_1 so that, $\dfrac{1}{4\delta_1 c_3}=\dfrac{\kappa\mathbb{H}^{-\alpha}(t)}{2}$, and $\dfrac{1}{4\delta_2 c_3} = \dfrac{\kappa \mathbb{H}^{-\alpha}(t)}{2}$, substituting in (3.2.26), we get

$$\begin{aligned}
\mathcal{K}'(t) \geq\ & [(1-\alpha) - \varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) \\
& -\varepsilon[(1-\frac{1}{2}\int_0^t g(s)ds)]\|\nabla u\|_2^2 - \varepsilon[(1-\frac{1}{2}\int_0^t h(s)ds)]\|\nabla v\|_2^2 \\
& -\varepsilon\frac{\mathbb{H}^{\alpha}(t)}{2c_3\kappa}(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|d\varrho)\|u\|_2^2 - \frac{\varepsilon}{2}(go\nabla u) \\
& -\varepsilon\frac{\mathbb{H}^{\alpha}(t)}{2c_3\kappa}(\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|d\varrho)\|v\|_2^2 - \frac{\varepsilon}{2}(ho\nabla v) \\
& +\varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]. \qquad (3.2.27)
\end{aligned}$$

For $0 < a < 1$, from (3.2.14)

$$\begin{aligned}
\varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}] =\ & \varepsilon a[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}] \\
& +\varepsilon 2(p+2)(1-a)\mathbb{H}(t) \\
& +\varepsilon(p+2)(1-a)(\|u_t\|_2^2 + \|v_t\|_2^2) \\
& +\varepsilon(p+2)(1-a)(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 \\
& +\varepsilon(p+2)(1-a)(1-\int_0^t h(s)ds)\|\nabla v\|_2^2 \\
& -\varepsilon(p+2)(1-a)(go\nabla u) \\
& -\varepsilon(p+2)(1-a)(ho\nabla v) \\
& +\varepsilon(p+2)(1-a)K(y,z). \qquad (3.2.28)
\end{aligned}$$

Substituting in (3.2.27), we get

$$\begin{aligned}
\mathcal{K}'(t) \geq\ & [(1-\alpha) - \varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon[(p+2)(1-a)+1]\Big(\|u_t\|_2^2 + \|v_t\|_2^2\Big) \\
& +\varepsilon[(p+2)(1-a)\Big(1-\int_0^t g(s)ds\Big) - \Big(1-\frac{1}{2}\int_0^t g(s)ds\Big)]\|\nabla u\|_2^2 \\
& +\varepsilon[(p+2)(1-a)(1-\int_0^t h(s)ds) - \Big(1-\frac{1}{2}\int_0^t h(s)ds\Big)]\|\nabla v\|_2^2 \\
& -\varepsilon\frac{\mathbb{H}^{\alpha}(t)}{2c_3\kappa}\Big(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|d\varrho\Big)\|u\|_2^2 - \varepsilon\frac{\mathbb{H}^{\alpha}(t)}{2c_3\kappa}\Big(\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|d\varrho\Big)\|v\|_2^2 \\
& +\varepsilon(p+2)(1-a)K(y,z) + \varepsilon[(p+2)(1-a) - \frac{1}{2}](go\nabla u + ho\nabla v) \\
& +\varepsilon a[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}] + \varepsilon 2(p+2)(1-a)\mathbb{H}(t).
\end{aligned}$$

(3.2.29)

Since (3.2.7) hold, we obtain by using (3.2.18) and (3.2.20)

$$\begin{aligned} \mathbb{H}^{\alpha}(t)\|u\|_2^2 &\leq c_4(\|u\|_{2(p+2)}^{2\alpha(p+2)+2} + \|v\|_{2(p+2)}^{2\alpha(p+2)}\|u\|_2^2), \\ \mathbb{H}^{\alpha}(t)\|v\|_2^2 &\leq c_5(\|v\|_{2(p+2)}^{2\alpha(p+2)+2} + \|u\|_{2(p+2)}^{2\alpha(p+2)}\|v\|_2^2), \end{aligned} \tag{3.2.30}$$

for some positive constants c_4, c_5. By using (3.2.20) and the algebraic inequality

$$B^{\theta} \leq (B+1) \leq (1+\frac{1}{b})(B+b), \ \ \forall B > 0, \ \ 0 < \theta < 1, \ \ b > 0.$$

We have, $\forall t > 0$

$$\begin{aligned} \|u\|_{2(p+2)}^{2\alpha(p+2)+2} &\leq d(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)) \leq d(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \\ \|v\|_{2(p+2)}^{2\alpha(p+2)+2} &\leq d(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)) \leq d(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \end{aligned} \tag{3.2.31}$$

where $d = 1 + \frac{1}{\mathbb{H}(0)}$. Also, since

$$(X+Y)^{\gamma} \leq C(X^{\gamma} + Y^{\gamma}), \ \ X, Y > 0, \ \ \gamma > 0. \tag{3.2.32}$$

We conclude

$$\begin{aligned} \|v\|_{2(p+2)}^{2\alpha(p+2)}\|u\|_2^2 &\leq c_6(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_2^{2(p+2)}) \leq c_7(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}), \\ \|u\|_{2(p+2)}^{2\alpha(p+2)}\|v\|_2^2 &\leq c_8(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_2^{2(p+2)}) \leq c_9(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}). \end{aligned} \tag{3.2.33}$$

Substituting (3.2.31) and (3.2.33) in (3.2.30), we get

$$\begin{aligned} \mathbb{H}^{\alpha}(t)\|u\|_2^2 &\leq c_{10}(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}) + c_{10}\mathbb{H}(t), \\ \mathbb{H}^{\alpha}(t)\|v\|_2^2 &\leq c_{11}(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}) + c_{11}\mathbb{H}(t). \end{aligned} \tag{3.2.34}$$

Combining (3.2.29) and (3.2.34), using (3.2.8), we get

$$
\begin{aligned}
\mathcal{K}'(t) \geq\ & [(1-\alpha)-\varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon[(p+2)(1-a)+1](\|u_t\|_2^2+\|v_t\|_2^2)\\
&+\varepsilon\{[(p+2)(1-a)-1]-(\int_0^t g(s)ds)[(p+2)(1-a)-\frac{1}{2}]\}\|\nabla u\|_2^2\\
&+\varepsilon\{[(p+2)(1-a)-1]-(\int_0^t h(s)ds)[(p+2)(1-a)-\frac{1}{2}]\}\|\nabla v\|_2^2\\
&+\varepsilon(p+2)(1-a)K(y,z)+\varepsilon[(p+2)(1-a)-\frac{1}{2}](go\nabla u+ho\nabla v)\\
&+\varepsilon(c_0a-\frac{\lambda_1+\lambda_2}{2c_3\kappa})[\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}]\\
&+\varepsilon(2(p+2)(1-a)-\frac{\lambda_1+\lambda_2}{2c_3\kappa})\mathbb{H}(t),
\end{aligned}
\tag{3.2.35}
$$

where $\lambda_1 = c_{10}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho,\ \ \lambda_2 = c_{11}\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho.$

In this stage, we take $a>0$ small enough so that

$$\alpha_1 = (p+2)(1-a)-1>0,$$

and we assume

$$\max\{\int_0^\infty g(s)ds, \int_0^\infty h(s)ds\} < \frac{(p+2)(1-a)-1}{((p+2)(1-a)-\frac{1}{2})} = \frac{2\alpha_1}{2\alpha_1+1}, \tag{3.2.36}$$

we have

$$
\begin{aligned}
\alpha_2 &= \{(p+2)(1-a)-1)-\int_0^t g(s)ds((p+2)(1-a)-\frac{1}{2})\}>0\\
\alpha_3 &= \{(p+2)(1-a)-1)-\int_0^t h(s)ds((p+2)(1-a)-\frac{1}{2})\}>0
\end{aligned}
$$

then we choose κ so large that

$$
\begin{aligned}
\alpha_4 &= ac_0-\frac{\lambda_1+\lambda_2}{2c_3\kappa}>0,\\
\alpha_5 &= 2(p+2)(1-a)-\frac{\lambda_1+\lambda_2}{2c_3\kappa}>0.
\end{aligned}
$$

We fixed κ and a, we appoint ε small enough so that

$$\alpha_6 = (1-\alpha)-\varepsilon\kappa>0.$$

Thus, for some $\beta > 0$, estimate (3.2.35) becomes

$$\begin{aligned}\mathcal{K}'(t) \geq\ & \beta\{\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \\ & +(go\nabla u) + (ho\nabla v) + K(y,z) \\ & +[\|u\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}]\}. \end{aligned} \tag{3.2.37}$$

By (3.2.8), for some $\beta_1 > 0$, we obtain

$$\begin{aligned}\mathcal{K}'(t) \geq\ & \beta_1\{\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \\ & +(go\nabla u) + (ho\nabla v) + K(y,z) \\ & +[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]\}, \end{aligned} \tag{3.2.38}$$

and

$$\mathcal{K}(t) \geq \mathcal{K}(0) > 0, \quad t > 0. \tag{3.2.39}$$

Next, using Holder's and Young's inequalities, we have

$$\begin{aligned}\left|\int_\Omega (uu_t + vv_t)dx\right|^{\frac{1}{1-\alpha}} \geq\ & C[\|u\|_{2(p+2)}^{\frac{\theta}{1-\alpha}} + \|u_t\|_2^{\frac{\mu}{1-\alpha}} \\ & +\|v\|_{2(p+2)}^{\frac{\theta}{1-\alpha}} + \|v_t\|_2^{\frac{\mu}{1-\alpha}}], \end{aligned} \tag{3.2.40}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$.

We take $\theta = 2(1-\alpha)$, to get

$$\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq 2(p+2).$$

Subsequently, for $s = \frac{2}{(1-2\alpha)}$ and by using (3.2.14), we obtain

$$\begin{aligned}\|u\|_{2(p+2)}^{\frac{2}{1-2\alpha}} &\leq d(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \\ \|v\|_{2(p+2)}^{\frac{2}{1-2\alpha}} &\leq d(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \ \ \forall t \geq 0. \end{aligned}$$

Therefore,

$$\left|\int_\Omega (uu_t + vv_t)dx\right|^{\frac{1}{1-\alpha}} \geq c_{12}[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + \|u_t\|_2^2 + \|v_t\|_2^2 + \mathbb{H}(t)]. \tag{3.2.41}$$

Subsequently,

$$\begin{aligned}
\mathcal{K}^{\frac{1}{1-\alpha}}(t) &= (\mathbb{H}^{1-\alpha} + \varepsilon \int_{\Omega} (uu_t + vv_t)dx + \frac{\varepsilon}{2} \int_{\Omega} (\mu_1 u^2 + \mu_3 v^2)dx \\
&\quad + \frac{\varepsilon}{2} \int_{\Omega} (\omega_1 \nabla u^2 + \omega_2 \nabla v^2)dx)^{\frac{1}{1-\alpha}} \\
&\leq c\{\mathbb{H}(t) + |\int_{\Omega} (uu_t + vv_t)dx|^{\frac{1}{1-\alpha}} + \|u\|_2^{\frac{2}{1-\alpha}} + \|\nabla u\|_2^{\frac{2}{1-\alpha}} \\
&\quad + \|v\|_2^{\frac{2}{1-\alpha}} + \|\nabla v\|_2^{\frac{2}{1-\alpha}}\} \\
&\leq c[\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (go\nabla u) \\
&\quad + (ho\nabla v) + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}]. \qquad (3.2.42)
\end{aligned}$$

From (3.3.15) and (3.3.18), gives

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}^{\frac{1}{1-\alpha}}(t), \qquad (3.2.43)$$

where $\lambda > 0$, this depends only on β and c.

by integration of (3.3.19), we obtain

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0) - \lambda \frac{\alpha}{(1-\alpha)} t}.$$

Hence, $\mathcal{K}(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\lambda \alpha \mathcal{K}^{\alpha/(1-\alpha)}(0)}.$$

Then the proof is completed. □

3.3 Growth of solutions to system

In this section, we prove the growth result of solution of problem (3.1.1).

First, as in [40], we introduce the new variables (3.2.1) then we obtain (3.2.2) and (3.2.3). Therefore, problem (3.1.1) takes the form (3.2.5) with initial and boundary conditions (3.2.6). Contrary to the previous paragraph in the last section, in the next theorem we give the global existence result, its proof based on the potential well depth method in which the concept of so-called stable set appears, where we show that if we restrict our initial data in the stable set, then our local solution obtained is global in time.

Theorem 13. *Suppose that (3.1.11),(3.1.12),(3.1.13), and (3.2.7) holds. If* $u_0, v_0 \in W$*,* $u_1, v_1 \in H_0^1(\Omega)$ *and* $y, z \in L^2(\Omega \times (0,1) \times (\tau_1, \tau_2))$

$$\frac{bC_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{3.3.1}$$

where C_* *is the best Poincare's constant. Then the local solution* (u, v, y, z) *is global in time.*

To achieve our goal, we need to use Lemma 7, Lemma 8 and Lemma 9 and then define the functional $\mathbb{H}(t)$ as in (3.2.14).

Theorem 14. *Assume (3.1.11)-(3.1.13), and (3.2.7) hold. Assume further that* $E(0) < 0$*, then the solution of problem (3.2.5) growths exponentially.*

Proof. From (3.2.9), we follow a similar calculation as in section 3.2. Until this step, we set

$$\begin{aligned} \mathcal{K}(t) &= \mathbb{H} + \varepsilon \int_\Omega (uu_t + vv_t)dx + \frac{\varepsilon}{2}\int_\Omega (\mu_1 u^2 + \mu_3 v^2)dx \\ &+ \frac{\varepsilon}{2}\int_\Omega (\omega_1(\nabla u)^2 + \omega_2(\nabla v)^2)dx. \end{aligned} \tag{3.3.2}$$

where $\varepsilon > 0$ to be assigned later.

By multiplying $(3.2.5)_1, (3.2.5)_2$ by u, v and with a derivative of (3.3.2), we get

$$\begin{aligned} \mathcal{K}'(t) &= \mathbb{H}'(t) + \varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) - \varepsilon(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) \\ &+ \varepsilon\int_\Omega \nabla u \int_0^t g(t-s)\nabla u(s)dsdx + \varepsilon\int_\Omega \nabla v\int_0^t h(t-s)\nabla v(s)dsdx \\ &- \varepsilon\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|uy(x,1,\varrho,t)d\varrho dx - \varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|vz(x,1,\varrho,t)d\varrho dx \\ &+ \varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]. \end{aligned} \tag{3.3.3}$$

Using Young's inequality, we get

$$\begin{aligned}
\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|uy(x,1,\varrho,t)d\varrho dx &\le \varepsilon\{\delta_1(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\|u\|_2^2 \\
&\quad +\frac{1}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\}.
\end{aligned} \tag{3.3.4}$$

$$\begin{aligned}
\varepsilon\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|vz(x,1,\varrho,t)d\varrho dx &\le \varepsilon\{\delta_2(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho)\|v\|_2^2 \\
&\quad +\frac{1}{4\delta_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\}.
\end{aligned} \tag{3.3.5}$$

and, we have

$$\begin{aligned}
\varepsilon\int_0^t g(t-s)ds\int_\Omega \nabla u.\nabla u(s)dxds &= \varepsilon\int_0^t g(t-s)ds\int_\Omega \nabla u.(\nabla u(s)-\nabla u(t))dxds \\
&\quad +\varepsilon\int_0^t g(s)ds\|\nabla u\|_2^2 \\
&\ge \frac{\varepsilon}{2}\int_0^t g(s)ds\|\nabla u\|_2^2-\frac{\varepsilon}{2}(go\nabla u).
\end{aligned} \tag{3.3.6}$$

$$\begin{aligned}
\varepsilon\int_0^t h(t-s)ds\int_\Omega \nabla v.\nabla v(s)dxds &= \varepsilon\int_0^t h(t-s)ds\int_\Omega \nabla v.(\nabla v(s)-\nabla v(t))dxds \\
&\quad +\varepsilon\int_0^t h(s)ds\|\nabla v\|_2^2 \\
&\ge \frac{\varepsilon}{2}\int_0^t h(s)ds\|\nabla v\|_2^2-\frac{\varepsilon}{2}(ho\nabla v).
\end{aligned} \tag{3.3.7}$$

we obtain, from (3.3.3),

$$\begin{aligned}
\mathcal{K}'(t) \ge\ & \mathbb{H}'(t)+\varepsilon(\|u_t\|_2^2+\|u_t\|_2^2) \\
& -\varepsilon((1-\frac{1}{2}\int_0^t g(s)ds)\|\nabla u\|_2^2+(1-\frac{1}{2}\int_0^t h(s)ds)\|\nabla v\|_2^2) \\
& -\varepsilon\delta_1(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\|u\|_2^2-\varepsilon\delta_2(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho)\|v\|_2^2 \\
& -\frac{\varepsilon}{2}(go\nabla u)-\frac{\varepsilon}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\
& -\frac{\varepsilon}{2}(ho\nabla v)-\frac{\varepsilon}{4\delta_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx \\
& +\varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}].
\end{aligned} \tag{3.3.8}$$

Therefore, using (3.2.17) and by setting δ_1, δ_1 so that, $\dfrac{1}{4\delta_1 c_3} = \dfrac{\kappa}{2}$,
and $\dfrac{1}{4\delta_2 c_3} = \dfrac{\kappa}{2}$, substituting in (3.3.8), we get

$$\begin{aligned}
\mathcal{K}'(t) \quad \geq \quad & [1-\varepsilon\kappa]\mathbb{H}'(t) + \varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) \\
& -\varepsilon[(1-\frac{1}{2}\int_0^t g(s)ds)]\|\nabla u\|_2^2 - \varepsilon[(1-\frac{1}{2}\int_0^t h(s)ds)]\|\nabla v\|_2^2 \\
& -\varepsilon\frac{1}{2c_3\kappa}(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|d\varrho)\|u\|_2^2 - \frac{\varepsilon}{2}(go\nabla u) \\
& -\varepsilon\frac{1}{2c_3\kappa}(\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|d\varrho)\|v\|_2^2 - \frac{\varepsilon}{2}(ho\nabla v) \\
& +\varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]. \qquad (3.3.9)
\end{aligned}$$

For $0 < a < 1$, from (3.2.14)

$$\begin{aligned}
\varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}] \quad = \quad & \varepsilon a[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}] \\
& +\varepsilon 2(p+2)(1-a)\mathbb{H}(t) \\
& +\varepsilon(p+2)(1-a)(\|u_t\|_2^2 + \|v_t\|_2^2) \\
& +\varepsilon(p+2)(1-a)(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 \\
& +\varepsilon(p+2)(1-a)(1-\int_0^t h(s)ds)\|\nabla v\|_2^2 \\
& -\varepsilon(p+2)(1-a)(go\nabla u) \\
& -\varepsilon(p+2)(1-a)(ho\nabla v) \\
& +\varepsilon(p+2)(1-a)K(y,z). \qquad (3.3.10)
\end{aligned}$$

substituting in (3.3.9), we get

$$\begin{aligned}
\mathcal{K}'(t) \geq\ & [1-\varepsilon\kappa]\mathbb{H}'(t) + \varepsilon[(p+2)(1-a)+1](\|u_t\|_2^2 + \|v_t\|_2^2) \\
& +\varepsilon[(p+2)(1-a)(1-\int_0^t g(s)ds) - (1-\frac{1}{2}\int_0^t g(s)ds)]\|\nabla u\|_2^2 \\
& +\varepsilon[(p+2)(1-a)(1-\int_0^t h(s)ds) - (1-\frac{1}{2}\int_0^t h(s)ds)]\|\nabla v\|_2^2 \\
& -\varepsilon\frac{1}{2c_3\kappa}(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|d\varrho)\|u\|_2^2 - \varepsilon\frac{1}{2c_3\kappa}(\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|d\varrho)\|v\|_2^2 \\
& +\varepsilon(p+2)(1-a)K(y,z) + \varepsilon[(p+2)(1-a) - \frac{1}{2}](go\nabla u + ho\nabla v) \\
& +\varepsilon a[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}] + \varepsilon 2(p+2)(1-a)\mathbb{H}(t)
\end{aligned} \tag{3.3.11}$$

Using Poincare's inequality, we obtain

$$\begin{aligned}
\mathcal{K}'(t) \geq\ & [1-\varepsilon\kappa]\mathbb{H}'(t) + \varepsilon[(p+2)(1-a)+1](\|u_t\|_2^2 + \|v_t\|_2^2) \\
& +\varepsilon\{[(p+2)(1-a)-1] - (\int_0^t g(s)ds)[(p+2)(1-a) - \frac{1}{2}] \\
& -\frac{c}{2\kappa}(\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds)\}\|\nabla u\|_2^2 \\
& +\varepsilon\{[(p+2)(1-a)-1] - (\int_0^t h(s)ds)[(p+2)(1-a) - \frac{1}{2}] \\
& -\frac{c}{2\kappa}(\int_{\tau_1}^{\tau_2} |\mu_4(s)|ds)\}\|\nabla v\|_2^2 \\
& +\varepsilon(p+2)(1-a)K(y,z) + \varepsilon[(p+2)(1-a) - \frac{1}{2}](go\nabla u + ho\nabla v) \\
& +\varepsilon c_0 a[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}] \\
& +\varepsilon 2(p+2)(1-a)\mathbb{H}(t)
\end{aligned} \tag{3.3.12}$$

In this stage, we take $a > 0$ small enough so that

$$\alpha_1 = (p+2)(1-a) - 1 > 0$$

and we assume

$$\max\{\int_0^\infty g(s)ds, \int_0^\infty h(s)ds\} < \frac{(p+2)(1-a)-1}{((p+2)(1-a) - \frac{1}{2})} = \frac{2\alpha_1}{2\alpha_1 + 1} \tag{3.3.13}$$

then we choose κ so large that

$$\begin{aligned}\alpha_2 &= \{(p+2)(1-a)-1)-\int_0^t g(s)ds((p+2)(1-a)-\frac{1}{2}) \\ &\quad -\frac{c}{2\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds)\}>0 \\ \alpha_3 &= \{(p+2)(1-a)-1)-\int_0^t h(s)ds((p+2)(1-a)-\frac{1}{2}) \\ &\quad -\frac{c}{2\kappa}(\int_{\tau_1}^{\tau_2}|\mu_4(s)|ds)\}>0\end{aligned}$$

we fixed κ and a, we appoint ε small enough so that

$$\alpha_4 = 1-\varepsilon\kappa > 0$$

and, from (3.2.19)

$$\begin{aligned}\mathcal{K}(t) &\leq \frac{1}{2(p+2)}[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}] \\ &\leq \frac{c_1}{2(p+2)}[\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}],\end{aligned} \tag{3.3.14}$$

Thus, for some $\beta>0$, estimate (3.3.12) becomes

$$\begin{aligned}\mathcal{K}'(t) &\geq \beta\{\mathbb{H}(t)+\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2 \\ &\quad +(go\nabla u)+(ho\nabla v)+K(y,z) \\ &\quad +[\|u\|_{2(p+2)}^{2(p+2)}+\|u\|_{2(p+2)}^{2(p+2)}]\}.\end{aligned} \tag{3.3.15}$$

By (3.2.8), for some $\beta_1>0$, we obtain

$$\begin{aligned}\mathcal{K}'(t) &\geq \beta_1\{\mathbb{H}(t)+\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2 \\ &\quad +(go\nabla u)+(ho\nabla v)+K(y,z) \\ &\quad +[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}]\}.\end{aligned} \tag{3.3.16}$$

and

$$\mathcal{K}(t)\geq\mathcal{K}(0)>0, \quad t>0. \tag{3.3.17}$$

Next, using Young's and Poincare's inequalities, from (3.2.19) we have

$$\begin{aligned}
\mathcal{K}(t) &= (\mathbb{H}^{1-\alpha} + \varepsilon \int_\Omega (uu_t + vv_t)dx + \frac{\varepsilon}{2}\int_\Omega (\mu_1 u^2 + \mu_3 v^2)dx \\
&\quad + \frac{\varepsilon}{2}\int_\Omega (\omega_1 \nabla u^2 + \omega_2 \nabla v^2)dx) \\
&\leq c\{\mathbb{H}(t) + |\int_\Omega (uu_t + vv_t)dx| + \|u\|_2 + \|\nabla u\|_2 \\
&\quad + \|v\|_2 + \|\nabla v\|_2\} \\
&\leq c[\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \\
&\leq c[\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (go\nabla u) \\
&\quad + (ho\nabla v) + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}].
\end{aligned} \tag{3.3.18}$$

for some $c > 0$. From inequalities (3.3.15) and (3.3.18) we obtain the differential inequality

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}(t), \tag{3.3.19}$$

where $\lambda > 0$, depending only on β and c.

a simple integration of (3.3.19), we obtain

$$\mathcal{K}(t) \geq \mathcal{K}(0)e^{(\lambda t)}, \forall t > 0 \tag{3.3.20}$$

From (3.2.19) and (4.2.12), we have

$$\mathcal{K}(t) \leq \frac{c_1}{2(p+2)}[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}]. \tag{3.3.21}$$

By (3.3.20) and (3.3.21), we have

$$\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \geq Ce^{(\lambda t)}, \forall t > 0$$

Therefore, we conclude that the solution is growths exponentially. This completes the proof.

□

Chapter 4

Well-posedness and stability result for a nonlinear damped porous-elastic system with infinite memory and distributed delay terms

In the present chapter, we consider one-dimensional porous-elastic system with nonlinear damping, infinite memory and distributed delay terms. A new minimal conditions on the nonlinear term and the relationship between the weights of the different damping mechanism are used to show the well posedness of solution by the semigroup theory and that the solution energy has an explicit and optimal decay, for the cases of equal and nonequal speeds of wave propagation.

4.1 Introduction

We investigate the well-posedness and stability results with distributed delay for the cases of equal and nonequal speeds of wave propagation, under an additional conditions, of the following system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \displaystyle\int_0^\infty g(p)\phi_{xx}(t-p)dp \\ \qquad + \mu_1\phi_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\phi_t(x,t-\varrho)d\varrho + \alpha(t)f(\phi_t) = 0, \end{cases} \tag{4.1.1}$$

where

$$(x, \varrho, t) \in (0,1) \times (\tau_1, \tau_2) \times (0, \infty),$$

with the Neumann-Dirichlet boundary conditions

$$u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, \quad t \geq 0, \tag{4.1.2}$$

and the initial data

$$\begin{aligned} &u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in (0,1) \\ &\phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), \quad x \in (0,1) \\ &\phi_t(x,-t) = f_0(x,t), \quad (x,t) \in (0,1) \times (0,\tau_2). \end{aligned} \tag{4.1.3}$$

Here $\rho, \mu, J, b, \delta, \xi$ and μ_1 are positive constants satisfying $\mu\xi > b^2$, the term $\alpha(t)f(\phi_t)$, where the functions α and f are specified later, represent the nonlinear damping term. The term $\displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\phi_t(x,t-\varrho)d\varrho$ is a distributed delay that acts only on the porous equation and τ_1, τ_2 are two real numbers with $0 \leq \tau_1 \leq \tau_2$, where μ_2 is an L^∞function and the function g is called the relaxation function. We first state the following assumptions:

(**H1**) $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfying

$$g(0) > 0, \;\; \delta - \int_0^\infty g(p)dp = l > 0, \;\; \int_0^\infty g(p)dp = g_0. \tag{4.1.4}$$

(**H2**) There exists a non-increasing differentiable function $\alpha, \eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \leq -\eta(t)g(t), \quad t \geq 0, \tag{4.1.5}$$

and

$$\lim_{t\to\infty} \frac{-\alpha'(t)}{\alpha(t)} = 0. \tag{4.1.6}$$

(**H3**) $f \in C^0(\mathbb{R}, \mathbb{R})$ is a non-decreasing such that there exist $v_1, v_2, \varepsilon > 0$ and a strictly increasing function $G \in C^1([0, \infty))$, with $G(0) = 0$ and G is a linear or strictly convex C^2-function on $(0, \varepsilon]$ such that

$$\begin{aligned} s^2 + f^2(s) \leq sf(s), \quad \forall |s| < \varepsilon \\ v_1|s| \leq |f(s)| \leq v_2|s|, \ \forall |s| \geq \varepsilon. \end{aligned} \tag{4.1.7}$$

which implies that $sf(s) > 0$ for all $s \neq 0$. The function f satisfies

$$|f(\psi_2) - f(\psi_1)| \leq k_0(|\psi_2|^\beta + |\psi_1|^\beta)|\psi_2 - \psi_1|, \ \psi_1, \psi_2 \in \mathbb{R}, \tag{4.1.8}$$

where $k_0, \beta > 0$.

(**H4**) The bounded function $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho < \mu_1. \tag{4.1.9}$$

Now, as in [40], taking the following new variable

$$y(x, \rho, \varrho, t) = \phi_t(x, t - \varrho\rho),$$

then we obtain

$$\begin{cases} \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0 \\ y(x, 0, \varrho, t) = \phi_t(x, t). \end{cases}$$

As in [26], we introduce now the following new variable

$$\eta^t(x, s) = \phi(x, t) - \phi(x, t - s), \ (x, t, s) \in (0, 1) \times \mathbb{R}_+ \times \mathbb{R}_+,$$

where η^t is the relative history of ϕ satisfies

$$\eta_t^t + \eta_s^t = \phi_t(x, t), \ (x, t, s) \in (0, 1) \times (0, 1) \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Consequently, the problem (4.1.1) is equivalent to

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 \\ J\phi_{tt} - l\phi_{xx} + bu_x + \xi\phi + \displaystyle\int_0^\infty g(p)\eta_{xx}^t(p)dp \\ \qquad + \mu_1\phi_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x, 1, \varrho, t) d\varrho + \alpha(t) f(\phi_t) = 0 \\ \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0 \\ \eta_t^t + \eta_s^t = \phi_t(x, t), \end{cases} \tag{4.1.10}$$

where

$$(x,\rho,\varrho,t)\in(0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty),$$

with the following boundary and initial conditions

$$u_x(0,t)=u_x(1,t)=\phi(0,t)=\phi(1,t)=0, t\geq 0, \tag{4.1.11}$$

$$\begin{aligned}
&u(x,0)=u_0(x), u_t(x,0)=u_1(x), \quad x\in(0,1)\\
&\phi(x,0)=\phi_0(x), \phi_t(x,0)=\phi_1(x), \quad x\in(0,1)\\
&y(x,\rho,\varrho,0)=f_0(x,\rho\varrho), \quad x\in(0,1), \rho\in(0,1), \varrho\in(0,\tau_2)\\
&\eta^t(x,0)=0, \eta^0(x,s)=\eta_0(x,s), \quad (x,s)\in(0,1)\times\mathbb{R}_+.
\end{aligned}$$

Meanwhile, from $(4.1.1)_1$ and (4.1.3), it follows that

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)\,dx=0. \tag{4.1.12}$$

So, by solving (4.1.12) and using the initial data of u, we get

$$\int_0^1 u(x,t)\,dx=t\int_0^1 u_1(x)\,dx+\int_0^1 u_0(x)\,dx.$$

Consequently, if we let

$$\bar{u}(x,t)=u(x,t)-t\int_0^1 u_1(x)\,dx-\int_0^1 u_0(x)\,dx, \tag{4.1.13}$$

we get

$$\int_0^1 \bar{u}(x,t)\,dx=0, \quad \forall t\geq 0.$$

Therefore, the use of Poincare's inequality for $\bar{u}$ is justified. In addition, simple substitution shows that $(\bar{u},\phi,y,\eta^t)$ satisfies system (4.1.1). Henceforth, we work with $\bar{u}$ instead of u but write u for simplicity of notation.

By imposing a new appropriate conditions (H3), with the help of some special results, we obtain an unusual weaker decay result using Lyaponov functiona, extending some earlier results known in the existing literature. The main results in this manuscript are the following. Theorem 2.1 for the existence and uniqueness of solution and Theorem 3.7 for the general stability estimates.

4.2 Well-posedness

In this section, we prove the existence and uniqueness result of the system (4.1.10)-(4.1.12) using the semigroup theory. To achieve our goal, we first introduce the vector function

$$U = (u, u_t, \phi, \phi_t, y, \eta^t)^T,$$

and the new dependent variables $v = u_t, \psi = \phi_t, \varphi = \eta^t$, then the system (4.1.10) can be written as follows

$$\begin{cases} U_t = \mathcal{A}U + \Gamma(U) \\ U(0) = U_0 = (u_0, u_1, \phi_0, \phi_1, f_0, \eta_0)^T, \end{cases} \tag{4.2.1}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} :\to \mathcal{H}$ is the linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \frac{\mu}{\rho} u_{xx} + \frac{b}{\rho}\phi_x \\ \psi \\ \frac{l}{J}\psi_{xx} + \frac{b}{J}u_x - \frac{\xi}{J}\phi_x + \frac{1}{J}\int_0^\infty g(p)\varphi_{xx}(p)dp \\ \qquad - \frac{\mu_1}{J}\psi - \frac{1}{J}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x, 1, \varrho, t) d\varrho \\ -\frac{1}{\varrho} y_\rho \\ -\varphi_s + \psi \end{pmatrix}, \tag{4.2.2}$$

and

$$\Gamma(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\alpha(t)}{J} f(\psi) \\ 0 \\ 0 \end{pmatrix}, \tag{4.2.3}$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H} \;=\; H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)) \times L_g(0,1),$$

where

$$\begin{aligned} L_*^2(0,1) &= \{\Phi \in L^2(0,1) \;/\; \int_0^1 \Phi(x)dx = 0\}, \\ H_*^1(0,1) &= H^1(0,1) \cap L_*^2(0,1), \\ L_g(0,1) &= \{\Phi : \mathbb{R}_+ \to H_0^1(0,1), \;\; \int_0^1 \int_0^\infty g(s)\Phi_x^2(p)dp < \infty\}, \end{aligned}$$

where the space $L_g(0,1)$ is endowed with the following inner product

$$\langle \Phi_1, \Phi_2 \rangle_{L_g(0,1)} = \int_0^1 \int_0^\infty g(p)\Phi_{1x}(p)\Phi_{2x}(p)dp.$$

For any

$$U = (u, v, \phi, \psi, y, \varphi)^T \in \mathcal{H}, \quad \widehat{U} = (\widehat{u}, \widehat{v}, \widehat{\phi}, \widehat{\psi}, \widehat{y}, \widehat{\varphi})^T \in \mathcal{H}.$$

The space $\mathcal{H}$ equipped with the inner product defined by

$$\begin{aligned} \langle U, \widehat{U} \rangle_{\mathcal{H}} = \rho \int_0^1 v\widehat{v}dx + \mu \int_0^1 u_x \widehat{u}_x dx + J \int_0^1 \psi\widehat{\psi}dx \\ + \xi \int_0^1 \phi\widehat{\phi}dx + l \int_0^1 \phi_x \widehat{\phi}_x dx + b \int_0^1 (u_x \widehat{\phi} + \widehat{u}_x \phi)dx \\ + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y\widehat{y} d\varrho d\rho dx + \langle \varphi, \widehat{\varphi} \rangle_{L_g(0,1)}. \end{aligned} \tag{4.2.4}$$

The domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} U \in \mathcal{H} \;/\; u \in H_*^2 \cap H_*^1, \;\; \phi \in H^2 \cap H_0^1, \\ v \in H_*^1, \;\; \psi \in H_0^1(0,1), \;\; \varphi \in L_g(0,1), \\ y, y_\rho \in L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)), \;\; y(x, 0, \varrho, t) = \psi, \end{array} \right\}$$

where

$$H_*^2(0,1) = \left\{\Phi \in H^2(0,1) \;/\; \Phi_x(1) = \Phi_x(0) = 0\right\}.$$

Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Now, we can state and prove the existence result.

Theorem 15. *Let $U_0 \in \mathcal{H}$ and assume that (4.1.4)-(4.1.9) hold. Then, there exists a unique solution $U \in \mathcal{C}(\mathbb{R}_+, \mathcal{H})$ of problem (4.2.1). Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then*

$$U \in \mathcal{C}(\mathbb{R}_+, \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+, \mathcal{H}).$$

Proof. First, we prove that the operator $\mathcal{A}$ is dissipative. For any $U_0 \in \mathcal{D}(\mathcal{A})$ and by using (4.2.4), we have

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = &-\mu_1 \int_0^1 \psi^2 dx - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\psi y(x,1,\varrho,t) d\varrho dx \\ &- \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y_\rho y d\varrho d\rho dx - \int_0^1 \int_0^\infty g(p)\varphi_{xp}(p)\varphi_x(p) dp dx.\end{aligned} \tag{4.2.5}$$

For the third term of the RHS of (4.2.5), we have

$$\begin{aligned}- \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y_\rho y d\varrho d\rho dx &= -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_2(\varrho)| \frac{d}{d\rho} y^2 d\rho d\varrho dx \\ &= -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx \\ &\quad + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,0,\varrho,t) d\varrho dx.\end{aligned} \tag{4.2.6}$$

By using Young's inequality, we get

$$\begin{aligned}- \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| \psi y(x,1,\varrho,t) d\varrho dx \le &\frac{1}{2} (\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho) \int_0^1 \psi^2 dx \\ &+ \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx.\end{aligned} \tag{4.2.7}$$

By integration the last term of the right-hand side of (4.2.5), we have

$$- \int_0^1 \int_0^\infty g(p)\varphi_{xp}(p)\varphi_x(p) dp dx = \frac{1}{2} \int_0^1 \int_0^\infty g'(p)\varphi_x^2(p) dp dx. \tag{4.2.8}$$

Substituting (4.2.6), (4.2.7) and (4.2.8) into (4.2.5), using the fact that $y(x,0,\varrho,t) = \psi(x,t)$ and (4.1.9), we obtained

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &\le -\Big(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho\Big) \int_0^1 \psi^2 dx + \frac{1}{2} \int_0^1 \int_0^\infty g'(p)\varphi_x^2(p) dp dx \\ &\le 0.\end{aligned} \tag{4.2.9}$$

Hence, the operator $\mathcal{A}$ is dissipative.

Next, we prove that the operator $\mathcal{A}$ is maximal. It's enough to show that the operator $(\lambda I - \mathcal{A})$ is surjective. Indeed, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$, we prove that there exists a unique $V = (u, v, \phi, \psi, y, \varphi) \in \mathcal{D}(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})V = F. \tag{4.2.10}$$

That is

$$\begin{cases} \lambda u - v = f_1 \in H^1_*(0,1) \\ \rho\lambda v - \mu u_{xx} - b\phi_x = \rho f_2 \in L^2_*(0,1) \\ \lambda\phi - \psi = f_3 \in H^1_0(0,1) \\ J\lambda\psi - l\phi_{xx} + bu_x + \xi\phi - \int_0^\infty g(p)\varphi_{xx}(p)dp \\ \qquad + \mu_1\psi + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t)d\varrho = Jf_4 \in L^2(0,1) \\ \lambda\varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = \varrho f_5 \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \\ \lambda\varphi + \varphi_s - \psi = f_6 \in L_g(0,1). \end{cases} \tag{4.2.11}$$

We note that the equation $(4.2.11)_5$ with $y(x,0,\varrho,t) = \psi(x,t)$ has a unique solution given by

$$y(x,\rho,\varrho,t) = e^{-\lambda\rho\varrho}\psi + \varrho e^{\lambda\varrho\rho}\int_0^\rho e^{\lambda\varrho\sigma} f_5(x,\sigma,\varrho,t)d\sigma, \tag{4.2.12}$$

then

$$y(x,1,\varrho,t) = e^{-\lambda\varrho}\psi + \varrho e^{\lambda\varrho}\int_0^1 e^{\lambda\varrho\sigma} f_5(x,\sigma,\varrho,t)d\sigma, \tag{4.2.13}$$

and we infer from $(4.2.11)_6$ that

$$\varphi = e^{\lambda s}\int_0^s e^{\tau}(\psi + f_6(\tau))d\tau, \tag{4.2.14}$$

and we have

$$v = \lambda u - f_1, \;\; \psi = \lambda\phi - f_3. \tag{4.2.15}$$

Inserting (4.2.13), (4.2.14) and (4.2.15) in $(4.2.11)_2$ and $(4.2.11)_4$, we get

$$\begin{cases} \rho\lambda^2 u - \mu u_{xx} - b\phi_x = h_1 \in L^2_*(0,1) \\ \mu_3\phi - \mu_4\phi_{xx} + bu_x = h_2 \in L^2(0,1), \end{cases} \tag{4.2.16}$$

where

$$\begin{cases} \mu_3 = J\lambda^2 + \xi + \lambda\mu_1 + \lambda \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| e^{-\lambda\varrho} d\varrho \\ \mu_4 = l + \displaystyle\int_0^{\infty} g(p)(1 - e^{\lambda p}) dp, \\ h_1 = \rho(\lambda f_1 + f_2) \\ h_2 = (J\lambda + \mu_1 + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| e^{-\lambda\varrho} d\varrho) f_3 + J f_4 \\ \qquad - \displaystyle\int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| e^{\lambda\varrho} \int_0^1 e^{\lambda\varrho\sigma} f_5(x,\sigma,\varrho,t) d\sigma d\varrho \\ \qquad + \displaystyle\int_0^{\infty} g(p) e^{\lambda p} \int_0^p e^{\tau} (\psi + f_6(\tau))_{xx} d\tau dp. \end{cases} \tag{4.2.17}$$

We multiply (4.2.16) by $\widehat{u}, \widehat{\phi}$, respectively and integrate their sum over $(0,1)$ to get the following variational formulation

$$B((u,\phi),(\widehat{u},\widehat{\phi})) = \Upsilon(\widehat{u},\widehat{\phi}), \tag{4.2.18}$$

where

$$B : (H_*^1(0,1) \times H_0^1(0,1))^2 \to \mathbb{R},$$

is the bilinear form defined by

$$\begin{aligned} B((u,\phi),(\widehat{u},\widehat{\phi})) &= \lambda^2\rho \int_0^1 u\widehat{u}dx + \mu_3 \int_0^1 \phi\widehat{\phi}dx + \mu \int_0^1 u_x\widehat{u}_x dx \\ &\quad + \mu_4 \int_0^1 \phi_x\widehat{\phi}_x dx + b \int_0^1 (u_x\widehat{\phi} + \phi\widehat{u}_x) dx, \end{aligned} \tag{4.2.19}$$

and

$$\Upsilon : (H_*^1(0,1) \times H_0^1(0,1)) \to \mathbb{R},$$

is the linear functional given by

$$\Upsilon(\widehat{u},\widehat{\phi}) = \int_0^1 h_1\widehat{u}dx + \int_0^1 h_2\widehat{\phi}dx \tag{4.2.20}$$

Now, for $V = H_*^1(0,1) \times H_0^1(0,1)$, equipped with the norm

$$\|(u,\phi)\|_V^2 = \|u\|_2^2 + \|\phi\|_2^2 + \|u_x\|_2^2 + \|\phi_x\|_2^2,$$

we have

$$\begin{aligned} B((u,\phi),(u,\phi)) &= \lambda^2\rho \int_0^1 u^2 dx + \mu_3 \int_0^1 \phi^2 dx + \mu \int_0^1 u_x^2 dx \\ &\quad + 2b \int_0^1 u_x\phi dx + \mu_4 \int_0^1 \phi_x^2 dx. \end{aligned} \tag{4.2.21}$$

On the other hand, we can write

$$\mu u_x^2 + 2bu_x\phi + \mu_3\phi^2 \quad = \quad \frac{1}{2}\Big[\mu\Big(u_x + \frac{b}{\mu}\phi\Big)^2 + \mu_3\Big(\phi + \frac{b}{\mu_3}u_x\Big)^2 + \Big(\mu - \frac{b^2}{\mu_3}\Big)u_x^2 + \Big(\mu_3 - \frac{b^2}{\mu}\Big)\phi^2\Big].$$

Since $\mu\xi > b^2$, we deduce that

$$\mu u_x^2 + 2bu_x\phi + \mu_3\phi^2 > \frac{1}{2}\Big[\Big(\mu - \frac{b^2}{\mu_3}\Big)u_x^2 + \Big(\mu_3 - \frac{b^2}{\mu}\Big)\phi^2\Big],$$

then, for some $M_0 > 0$

$$B((u,\phi),(u,\phi)) \geq M_0\|(u,\phi)\|_V^2. \tag{4.2.22}$$

Thus B is coercive, similarly,

$$\Upsilon(\widehat{u},\widehat{\phi}) \geq M_1\|(\widehat{u},\widehat{\phi})\|_V^2. \tag{4.2.23}$$

Consequently, using Lax-Milgram theorem, we conclude that (4.1.10) has a unique solution

$$(u,\phi) \in H_*^1(0,1 \times H_0^1(0,1).$$

Substituting u, ϕ into (4.2.13), (4.2.14) and (4.2.15), respectively, we have

$$\begin{aligned} &v \in H_*^1(0,1), \quad \psi \in H_0^1(0,1), \quad \varphi \in L_g(0,1) \\ &y, y_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)). \end{aligned} \tag{4.2.24}$$

Moreover, if we take $\widehat{u} = 0 \in H_*^1(0,1)$ in (4.2.18) to obtain

$$\mu_3\int_0^1 \phi\widehat{\phi}dx + b\int_0^1 u_x\widehat{\phi}dx + \mu_4\int_0^1 \phi_x\widehat{\phi}_x dx = \int_0^1 h_2\widehat{\phi}d, \tag{4.2.25}$$

we get

$$\mu_4\int_0^1 \phi_x\widehat{\phi}_x dx = \int_0^1 (h_2 - \mu_3\phi - bu_x)\widehat{\phi}dx, \;\; \forall\widehat{\phi} \in H_0^1(0,1), \tag{4.2.26}$$

which yields

$$\mu_4\phi_{xx} = (h_2 - \mu_3\phi - bu_x) \in L^2(0,1). \tag{4.2.27}$$

Thus

$$\phi \in H^2(0,1) \cap H_0^1(0,1). \tag{4.2.28}$$

Consequently, (4.2.26) takes the following form

$$\int_0^1 (-\mu_4\phi_{xx} - h_2 + \mu_3\phi + bu_x)\widehat{\phi}dx = 0, \;\; \forall\widehat{\phi} \in H_0^1(0,1).$$

Hence, we get

$$-\mu_4\phi_{xx} + \mu_3\phi + bu_x = h_2.$$

This give $(4.2.16)_2$. Similarly, if we take $\widehat{\phi} = 0 \in H_0^1(0,1)$ in (4.2.18) to obtain

$$\mu\int_0^1 u_x\widehat{u}_x dx + b\int_0^1 \phi\widehat{u}_x dx + \lambda^2\rho\int_0^1 u\widehat{u}dx = \int_0^1 h_1\widehat{u}dx,$$

we get

$$\mu\int_0^1 u_x\widehat{u}_x dx = \int_0^1 (h_1 + b\phi_x - \lambda^2\rho u)\widehat{u}dx, \;\; \forall\widehat{u} \in H_*^1(0,1), \tag{4.2.29}$$

which yields

$$-\mu u_{xx} = (h_1 + b\phi_x - \lambda^2\rho u) \in L_*^2(0,1). \tag{4.2.30}$$

Consequently, (4.2.29) takes the following form

$$\int_0^1 (-\mu u_{xx} - h_1 - b\phi_x + \lambda^2\rho u)\widehat{u}dx = 0, \;\; \forall\widehat{u} \in H_*^1(0,1).$$

Hence, we get

$$-\mu u_{xx} - b\phi_x + \lambda^2\rho u = h_1.$$

This give $(4.2.16)_1$.

Moreover, (4.2.29) also holds for any $\Phi \in C^1([0,1])$. Then, by using integration by parts, we obtain

$$\mu\int_0^1 u_x\Phi_x dx + \int_0^1 (-h_1 - b\phi_x + \lambda^2\rho u)\Phi dx = 0, \;\; \forall\Phi \in C^1([0,1]). \tag{4.2.31}$$

Then, we get for any $\Phi \in C^1([0,1])$

$$u_x(1)\Phi(1) - u_x(0)\Phi(0) = 0. \tag{4.2.32}$$

Since Φ is arbitrary, we get that $u_x(0) = u_x(1) = 0$. Hence, $u \in H_*^2(0,1) \cap H_*^1(0,1)$. Therefore, the application of regularity theory for the linear elliptic equations guarantees the existence of unique $U \in \mathcal{D}(\mathcal{A})$ such that (4.2.10) is satisfied. Consequently, we conclude that $\mathcal{A}$ is a maximal dissipative operator. Now, we prove that the operator Γ defined in (4.2.3) is locally Lipschitz in $\mathcal{H}$. Let

$$U = (u, v, \phi, \psi, y, \varphi)^T \in \mathcal{H}, \widehat{U} = (\widehat{u}, \widehat{v}, \widehat{\phi}, \widehat{\psi}, \widehat{y}, \widehat{\varphi})^T \in \mathcal{H}.$$

Then, we have

$$\|\Gamma(U) - \Gamma(\widehat{U})\|_{\mathcal{H}} \leq M_3 \|f(\psi) - f(\widehat{\psi})\|_{L^2(01)}.$$

By using (4.1.8) and Holder and Poincare's inequalities, we can get

$$\begin{aligned} \|f(\psi) - f(\widehat{\psi})\|_{L^2(01)} &\leq k_0(\|\psi\|_{2\beta}^{\beta} + \|\widehat{\psi}\|_{2\beta}^{\beta})\|\psi - \widehat{\psi}\| \\ &\leq k_1 \|\psi_x - \widehat{\psi}_x\|_{L^2(01)}, \end{aligned}$$

which gives us

$$\|\Gamma(U) - \Gamma(\widehat{U})\|_{\mathcal{H}} \leq M_4 \|U - \widehat{U}\|_{\mathcal{H}}.$$

Then, the operator Γ is locally Lipschitz in $\mathcal{H}$. Consequently, the well-posedness result follows from the Hille-Yosida theorem. The proof is completed. □

4.3 Stability Result

In this section, we state and prove our decay result for the energy of the system (4.1.10)-(4.1.12) using the multiplier technique. We need the following Lemmas.

Lemma 10. *The energy functional $\mathcal{E}$, defined by*

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2}\int_0^1 \left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + l\phi_x^2 + \xi\phi^2 + 2bu_x\phi\right] dx \\ &+ \frac{1}{2}\int_0^1 \int_0^{\infty} g(p)\varphi_x^2(p) dp dx \\ &+ \frac{1}{2}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| y^2(x, \rho, \varrho, t)\, d\varrho d\rho dx, \end{aligned} \tag{4.3.1}$$

satisfies

$$\begin{aligned} \mathcal{E}'(t) &\leq -\eta_0 \int_0^1 \phi_t^2 dx + \frac{1}{2}\int_0^1 \int_0^{\infty} g'(p)\varphi_x^2(p) dp dx + \alpha(t)\int_0^1 \phi_t f(\phi_t) dx \\ &\leq 0, \end{aligned} \tag{4.3.2}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \geq 0$ and $\varphi(s) = \eta^t = \phi(x,t) - \phi(x, t-p)$.

Proof. Multiplying $(4.1.10)_1$ by u_t and $(4.1.10)_2$ by ϕ_t, then integration by parts over $(0,1)$ and using (4.1.11), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho u_t^2+\mu u_x^2+J\phi_t^2+\delta\phi_x^2+\xi\phi^2+2bu_x\phi\right]dx\\
&-\int_0^1 \phi_{xt}\int_0^\infty g(p)\varphi_x(p)dpdx+\mu_1\int_0^1 \phi_t^2dx\\
&+\int_0^1 \phi_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y\left(x,1,\varrho,t\right)d\varrho dx+\alpha(t)\int_0^1\phi_t f(\phi_t)dx=0. \qquad (4.3.3)
\end{aligned}$$

The last term in the LHS of (4.3.3) is estimated as follows

$$\begin{aligned}
\int_0^1 \phi_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y\left(x,1,\varrho,t\right)d\varrho dx \;\le\; &\frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\Big)\int_0^1\phi_t^2dx \qquad (4.3.4)\\
&+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2\left(x,1,\varrho,t\right)d\varrho dx,
\end{aligned}$$

and

$$\begin{aligned}
-\int_0^1\phi_{xt}\int_0^\infty g(p)\varphi_x\left(p\right)dpdx \;\le\; &\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^\infty g(p)\varphi_x^2\left(p\right)dpdx\\
&-\frac{1}{2}\int_0^1\int_0^\infty g'(p)\varphi_x^2\left(p\right)dpdx. \qquad (4.3.5)
\end{aligned}$$

Now, multiplying the equation $((4.1.10))_3$ by $y|\mu_2(\varrho)|$ and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned}
&\frac{d}{dt}\;\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx\\
&= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|yy_\rho\left(x,\rho,\varrho,t\right)d\varrho d\rho dx\\
&= -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\frac{d}{d\rho}y^2\left(x,\rho,\varrho,t\right)d\varrho d\rho dx\\
&= \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|(y^2\left(x,0,\varrho,t\right)-y^2(x,1,\varrho,t))d\varrho dx\\
&= \frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\int_0^1\phi_t^2dx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2\left(x,1,\varrho,t\right)d\varrho dx. \qquad (4.3.6)
\end{aligned}$$

Now, using (4.3.3), (4.3.4), (4.3.5) and (4.3.6), we have

$$\begin{aligned}
\mathcal{E}'(t) \;\le\; &-\Big(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\Big)\int_0^1\phi_t^2dx+\frac{1}{2}\int_0^1\int_0^\infty g'(p)\varphi_x^2\left(p\right)dpdx\\
&-\alpha(t)\int_0^1\phi_t f(\phi_t)dx, \qquad (4.3.7)
\end{aligned}$$

then, by (4.1.4), there exists a positive constant η_0 such that

$$\mathcal{E}'(t) \leq -\eta_0 \int_0^1 \phi_t^2 dx + \frac{1}{2}\int_0^1 \int_0^\infty g'(p)\varphi_x^2(p)\, dpdx - \alpha(t)\int_0^1 \phi_t f(\phi_t)dx, \tag{4.3.8}$$

hence, by (4.1.5) – (4.1.9) we obtain $\mathcal{E}$ is a non-increasing function. □

Remark 3. *Using ($\mu\xi > b^2$), we conclude that the energy $\mathcal{E}(t)$definie by (4.3.1) satisfies*

$$\begin{aligned}\mathcal{E}(t) \quad > \quad & \frac{1}{2}\int_0^1 \left[\rho u_t^2 + \widehat{\mu} u_x^2 + J\phi_t^2 + l\phi_x^2 + \widehat{\xi}\phi^2\right] dx \\ & + \frac{1}{2}\int_0^1 \int_0^\infty g(p)\varphi_x^2(p)dpdx \\ & + \frac{1}{2}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx,\end{aligned} \tag{4.3.9}$$

where

$$\widehat{\mu} = \frac{1}{2}(\mu - \frac{b^2}{\xi}) > 0, \quad \widehat{\xi} = \frac{1}{2}(\xi - \frac{b^2}{\mu}) > 0,$$

then $\mathcal{E}(t)$ is positive function.

Lemma 11. *The functional*

$$D_1(t) := J\int_0^1 \phi_t\phi dx + \frac{b\rho}{\mu}\int_0^1 \phi\int_0^x u_t(y)\, dydx + \frac{\mu_1}{2}\int_0^1 \phi^2 dx, \tag{4.3.10}$$

satisfies

$$\begin{aligned}D_1'(t) \quad \leq \quad & -\frac{l}{2}\int_0^1 \phi_x^2 dx - \widehat{\mu}\int_0^1 \phi^2 dx + \varepsilon_1\int_0^1 u_t^2 dx + c(1+\frac{1}{\varepsilon_1})\int_0^1 \phi_t^2 dx \\ & + c\int_0^1 \int_0^\infty g(p)\varphi_x^2(p)dpdx + c\int_0^1 f^2(\phi_t)dx \\ & + c\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y^2(x,1,\varrho,t)\, d\varrho dx,\end{aligned} \tag{4.3.11}$$

where $\widehat{\mu} = \xi - \frac{b^2}{\mu} > 0$.

Proof. Direct computation using integration by parts and Young's inequality, for $\varepsilon_1 > 0$, yields

$$\begin{aligned}
D_1'(t) &= -l\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + \frac{b\rho}{\mu}\int_0^1 \phi_t \int_0^x u_t(y)\, dydx \\
&\quad + \int_0^1 \phi_x \int_0^\infty g(p)\,\varphi_x(p)\, dpdx + \alpha(t)\int_0^1 \phi f(\phi_t) dx \\
&\quad + J\int_0^1 \phi_t^2 dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t) d\varrho dx \\
&\leq -l\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx \\
&\quad + \varepsilon_1 \int_0^1 \left(\int_0^x u_t(y)\, dy\right)^2 dx + \int_0^1 \phi_x \int_0^\infty g(p)\,\varphi_x(p)\, dpdx \\
&\quad - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t) d\varrho dx + \alpha(t)\int_0^1 \phi f(\phi_t) dx. \qquad (4.3.12)
\end{aligned}$$

By Cauchy-Schwartz inequality, it is clear that

$$\int_0^1 \left(\int_0^x u_t(y)\, dy\right)^2 dx \leq \int_0^1 \left(\int_0^1 u_t dx\right)^2 dx \leq \int_0^1 {u_t}^2 dx.$$

So, estimate (4.3.12) becomes

$$\begin{aligned}
D_1'(t) &\leq -\delta\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx \\
&\quad + \varepsilon_1 \int_0^1 {u_t}^2 dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t) d\varrho dx \\
&\quad + \int_0^1 \phi_x \int_0^\infty g(p)\,\varphi_x(p)\, dpdx + \alpha(t)\int_0^1 \phi f(\phi_t) dx. \qquad (4.3.13)
\end{aligned}$$

The last term in the RHS of (4.3.13) is estimated as follows

$$\int_0^1 \phi_x \int_0^\infty g(p)\,\varphi_x(p)\, dpdx \leq c\delta_1 \int_0^1 \phi_x^2 dx + \frac{c}{4\delta_1}\int_0^1\int_0^\infty g(p)\,\varphi_x^2(p) dpdx, \qquad (4.3.14)$$

where we have used Cauchy-Schwartz, Young and poincare's inequalities, for $\delta_1, \delta_2, \delta_3 > 0$.

By substituting (4.3.14) into(4.3.12), we obtain

$$\begin{aligned}
D_1'(t) &\leq -(l - c\delta_1 - \mu_1 c\delta_2 - c\delta_3)\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx \\
&\quad + \varepsilon_1 \int_0^1 u_t^2 dx + c(1 + \frac{1}{\varepsilon_1})\int_0^1 \phi_t^2 dx + \frac{c}{4\delta_1}\int_0^1\int_0^\infty g(p)\,\varphi_x^2(p) dpdx \\
&\quad + \frac{1}{4\delta_2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx + \frac{1}{4\delta_3}\int_0^1 f^2(\phi_t) dx. \qquad (4.3.15)
\end{aligned}$$

Bearing in mind that $\mu\xi > b^2$ and letting $\delta_1 = \frac{l}{6}$, $\delta_2 = \frac{l}{6c\mu_1}$ and $\delta_3 = \frac{l}{6c}$, we obtain estimate (4.3.11). $\square$

Lemma 12. *Then, for any $\varepsilon_2 > 0$ the functional*

$$D_2(t) := \int_0^1 \phi_x u_t dx + \int_0^1 \phi_t u_x dx - \frac{\rho}{\mu J}\int_0^1 u_t \int_0^\infty g(p)\,\phi_x(t-p)\,dpdx,$$

satisfies

$$\begin{aligned} D_2'(t) \leq & -\frac{b}{2J}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx + c\varepsilon_2\int_0^1 u_t^2 dx + c\int_0^1 \phi_t^2 dx \\ & + c\int_0^1\int_0^\infty g(p)\,\varphi_x^2(p)\,dpdx - \frac{c}{\varepsilon_2}\int_0^1\int_0^\infty g'(p)\,\varphi_x^2(p)\,dpdx \\ & + c\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)d\varrho dx + c\int_0^1 f^2(\phi_t)dx \\ & + \left(\frac{\delta}{J} - \frac{\mu}{\rho}\right)\int_0^1 u_x\phi_{xx}dx. \end{aligned} \tag{4.3.16}$$

Proof. By differentiating D_2, then using (4.1.10), integration by parts and (4.1.11) we obtain

$$\begin{aligned} D_2'(t) = & -\frac{b}{J}\int_0^1 u_x^2 dx + \left(\frac{l+g_0}{J} - \frac{\mu}{\rho}\right)\int_0^1 u_x\phi_{xx}dx + \left(\frac{b}{\rho} - \frac{bg_0}{\mu J}\right)\int_0^1 \phi_x^2 dx \\ & - \frac{\xi}{J}\int_0^1 u_x\phi dx - \frac{b}{\mu J}\int_0^1 \phi_x\int_0^\infty g(p)\varphi_x(p)\,dpdx \\ & - \frac{\rho}{\mu J}\int_0^1 u_t\int_0^\infty g'(p)\,\varphi_x(p)\,dpdx - \frac{\alpha(t)}{\mu J}\int_0^1 u_x f(\phi_t)dx \\ & - \frac{\mu_1}{J}\int_0^1 \phi_t u_x dx - \frac{1}{J}\int_0^1 u_x\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)d\varrho dx. \end{aligned} \tag{4.3.17}$$

In what follows, we estimate the last six terms in the RHS of (4.3.17), using Young, Cauchy-Schwartz and Poincare's inequalities. For $\delta_4, \delta_5, \varepsilon_2 > 0$, we have

$$-\frac{\xi}{J}\int_0^1 u_x\phi dx \leq \frac{\xi}{J}\delta_4\int_0^1 u_x^2 dx + \frac{\xi}{4J\delta_4}\int_0^1 \phi^2 dx.$$

By letting $\delta_4 = \frac{b}{6\xi}$, using Poincar's inequality, we get

$$-\frac{\xi}{J}\int_0^1 u_x\phi dx \leq \frac{b}{6J}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx, \tag{4.3.18}$$

and by Young and Chauchy-Schawrz's inequalities, we get

$$-\frac{b}{\mu J}\int_0^1 \phi_x\int_0^\infty g(p)\,\varphi_x(p)\,dpdx \leq c\delta_5\int_0^1 \phi_x^2 dx + \frac{c}{4\delta_5}\int_0^1\int_0^\infty g(p)\,\varphi_x^2(p)\,dpdx.$$

By letting $\delta_5 = \frac{b}{6cJ}$, we obtain

$$-\frac{b}{\mu J}\int_0^1 \phi_x \int_0^\infty g\left(p\right)\varphi_x\left(p\right)dpdx \leq \frac{b}{6J}\int_0^1 \phi_x^2 dx + c\int_0^1\int_0^\infty g\left(p\right)\varphi_x^2\left(p\right)dpdx. \tag{4.3.19}$$

Similarly, $\forall \varepsilon_2 > 0$ we have

$$\frac{\rho}{\mu J}\int_0^1 u_t \int_0^\infty g'\left(p\right)\varphi_x\left(p\right)dpdx \leq c\varepsilon_2\int_0^1 u_t^2 dx + \frac{c}{\varepsilon_2}\int_0^1\int_0^\infty g'\left(p\right)\varphi_x^2\left(p\right)dpdx, \tag{4.3.20}$$

and

$$-\frac{\mu_1}{J}\int_0^1 \phi_t u_x dx \leq \frac{\mu_1\delta_6}{2J}\int_0^1 u_x^2 dx + \frac{\mu_1}{2J\delta_6}\int_0^1 \phi_t^2 dx, \tag{4.3.21}$$

and

$$\begin{aligned}\frac{1}{J}\int_0^1 u_x \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t)d\varrho dx \quad \leq \quad & \frac{\delta_7\mu_1}{2J}\int_0^1 u_x^2 dx \\ & + \frac{1}{2J\delta_7}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)d\varrho,\end{aligned} \tag{4.3.22}$$

and

$$-\frac{\alpha(t)}{J}\int_0^1 u_x f(\phi_t)dx \leq \frac{\alpha(0)\delta_8}{2J}\int_0^1 u_x^2 dx + \frac{\alpha(0)}{2J\delta_8}\int_0^1 f^2(\phi_t)dx. \tag{4.3.23}$$

Replacing (4.3.18)-(4.3.23) into (4.3.17) and letting $\delta_6 = \delta_7 = \frac{b}{6\mu_1}$ and $\delta_8 = \frac{b}{6\alpha(0)}$, yields (4.3.16).

□

Lemma 13. *The functional*

$$D_3\left(t\right) := -\rho\int_0^1 u_t u dx,$$

satisfies

$$D_3'\left(t\right) \leq -\rho\int_0^1 u_t^2 dx + \frac{3\mu}{2}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx. \tag{4.3.24}$$

Proof. Direct computations give

$$D_3'\left(t\right) = -\rho\int_0^1 u_t^2 dx + \mu\int_0^1 u_x^2 dx + b\int_0^1 u_x \phi dx.$$

The estimat (4.3.24) easily follows by using Young and Poincaré inequalities.

$$\begin{aligned} D_3'\left(t\right) \quad & \leq \quad -\rho\int_0^1 u_t^2 dx + \mu\int_0^1 u_x^2 dx + b\varepsilon\int_0^1 u_x^2 dx + \frac{b}{4\varepsilon}\int_0^1 \phi^2 dx \\ & \leq \quad -\rho\int_0^1 u_t^2 dx + \mu\int_0^1 u_x^2 dx + b\varepsilon\int_0^1 u_x^2 dx + \frac{bc}{4\varepsilon}\int_0^1 \phi_x^2 dx,\end{aligned}$$

by taking $\varepsilon = \frac{\mu}{2b}$, we obtain (4.3.24). □

Lemma 14. *The functional*

$$D_4(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} |\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx,$$

satisfies

$$\begin{aligned} D_4'(t) \le & -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx + \mu_1 \int_0^1 \phi_t^2 dx \\ & -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)\, d\varrho dx, \end{aligned} \tag{4.3.25}$$

where η_1 is a positive constant.

Proof. By differentiating D_4, with respect to t and using the equation $(4.1.10)_3$, we have

$$\begin{aligned} D_4'(t) &= -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\varrho\rho} |\mu_2(\varrho)| y y_\rho(x,\rho,\varrho,t)\, d\varrho d\rho dx \\ &= -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} |\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx \\ &\quad - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| [e^{-\varrho} y^2(x,1,\varrho,t) - y^2(x,0,\varrho,t)] d\varrho dx. \end{aligned}$$

Using the fact that $y(x,0,\varrho,t) = \phi_t(x,t)$ and $e^{-\varrho} \le e^{-\varrho\rho} \le 1$, for all $0 < \rho < 1$, we obtain

$$\begin{aligned} D_4'(t) &= -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx \\ &\quad - \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\varrho} |\mu_2(\varrho)| y^2(x,1,\varrho,t)\, d\varrho dx + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \int_0^1 \phi_t^2 dx. \end{aligned}$$

□

Since $-e^{-\varrho}$ is an increasing function, we have $-e^{-\varrho} \le -e^{-\tau_2}$, for all $\varrho \in [\tau_1, \tau_2]$. Finally, setting $\eta_1 = e^{-\tau_2}$ and recalling (4.1.9), we obtain (4.3.25). We are now ready to prove the main result.

Theorem 16. *Assume (4.1.4)-(4.1.9) hold. Let $h(t) = \alpha(t)$. $\eta(t)$ be a positive non-increasing function. Then, for any $U_0 \in \mathcal{D}(\mathcal{A})$, satisfying for some $c_0 > 0$*

$$max\Big\{ \int_0^1 \phi_{0x}^2(x,s)dx, \int_0^1 \phi_{0sx}^2(x,s)dx \Big\} \le c_0, \quad \forall s > 0, \tag{4.3.26}$$

there exist positive constants β_1, β_2 and β_3 such that the energy functional given by (4.3.1) satisfies

$$\mathcal{E}(t) \leq \beta_1 G_0^{-1}\left(\frac{\beta_2 + \beta_3 \int_0^t h(p)\varpi(p)dp}{\int_0^t h(p)dp}\right), \tag{4.3.27}$$

where

$$G_0(t) = tG'(\varepsilon_0 t), \forall \varepsilon_0 \geq 0, \ \ and \ \ \varpi(s) = \int_s^\infty g(\sigma)d\sigma. \tag{4.3.28}$$

Proof. We define a Lyapunov functional

$$\mathcal{L}(t) := N\mathcal{E}(t) + N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t), \tag{4.3.29}$$

where N, N_1, N_2, and N_4 are positive constants to be chosen later. By differentiating (4.3.29) and using (4.3.2), (4.3.11), (4.3.16), (4.3.24), (4.3.25), we have

$$\begin{aligned}
\mathcal{L}'(t) \ \leq \ & -\left[\frac{lN_1}{2} - cN_2 - c\right]\int_0^1 \phi_x^2 dx - \left[\rho - N_1\varepsilon_1 - N_2 c\varepsilon_2\right]\int_0^1 u_t^2 dx \\
& -\left[\frac{bN_2}{2J} - \frac{3\mu}{2}\right]\int_0^1 u_x^2 dx + c\left[N_1 + N_2\right]\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \\
& -\left[\eta_0 N - cN_1(1 + \frac{1}{\varepsilon_1}) - N_2 c - \mu_1 N_4\right]\int_0^1 \phi_t^2 dx \\
& -\left[N_4\eta_1 - cN_1 - cN_2\right]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)\, d\varrho dx \\
& -N_1\widehat{\mu}\int_0^1 \phi^2 dx + \left[\frac{N}{2} - \frac{cN_2}{\varepsilon_2}\right]\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx \\
& -N_4\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx \\
& +c\left[N_1 + N_2\right]\int_0^1 f^2(\phi_t)dx + N_2\chi\int_0^1 u_{xx}\phi_x dx,
\end{aligned}$$

where $\chi = (\frac{\mu}{\rho} - \frac{\delta}{J})$ and by setting

$$\varepsilon_1 = \frac{\rho}{4N_1}, \varepsilon_2 = \frac{\rho}{4cN_2},$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{lN_1}{2} - cN_2(1+N_2) - c\right]\int_0^1 \phi_x^2 dx - \frac{\rho}{2}\int_0^1 u_t^2 dx \\
& -\left[\frac{bN_2}{2J} - \frac{3\mu}{2}\right]\int_0^1 u_x^2 dx + c\left[N_1 + N_2\right]\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \\
& -\left[\eta_0 N - cN_1(1+N_1) - cN_2 - \mu_1 N_4\right]\int_0^1 \phi_t^2 dx \\
& -\left[N_4\eta_1 - cN_1 - cN_2\right]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2\left(x,1,\varrho,t\right)d\varrho dx \\
& -N_1\widehat{\mu}\int_0^1 \phi^2 dx + \left[\frac{N}{2} - cN_2^2\right]\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx \\
& -N_4\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(s)| y^2\left(x,\rho,\varrho,t\right)d\varrho d\rho dx \\
& +c\left[N_1+N_2\right]\int_0^1 f^2(\phi_t)dx + N_2\chi\int_0^1 u_{xx}\phi_x dx.
\end{aligned}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive. We choose N_2 large enough such that

$$\alpha_1 = \frac{bN_2}{2J} - \frac{3\mu}{2} > 0,$$

then we choose N_1 large enough such that

$$\alpha_2 = \frac{lN_1}{4} - cN_2\left(1+N_2\right) - c > 0,$$

then we choose N_4 large enough such that

$$\alpha_3 = N_4\eta_1 - cN_1 - cN_2 > 0,$$

thus, we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\alpha_2\int_0^1 \phi_x^2 dx - \alpha_0\int_0^1 \phi^2 dx - \frac{\rho}{2}\int_0^1 u_t^2 dx - \alpha_1\int_0^1 u_x^2 dx \\
& -[\eta_0 N - c]\int_0^1 \phi_t^2 dx + \left[\frac{N}{2} - c\right]\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx \\
& +c\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx - \alpha_3\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2\left(x,1,\varrho,t\right)d\varrho dx \\
& -\alpha_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y^2\left(x,\rho,\varrho,t\right)d\varrho d\rho dx \\
& +c\int_0^1 f^2(\phi_t)dx + \alpha_5\int_0^1 u_{xx}\phi_x dx. \qquad (4.3.30)
\end{aligned}$$

where $\alpha_0 = \widehat{\mu} N_1 = \left(\xi - \frac{b^2}{\mu}\right) N_1$, *and* $\alpha_5 = N_2\chi = N_2(\frac{\mu}{\rho} - \frac{\delta}{J})$. On the other hand, if we let

$$\mathfrak{L}(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t),$$

then

$$\begin{aligned} |\mathfrak{L}(t)| &\leq JN_1 \int_0^1 |\phi\phi_t|\, dx + \frac{b\rho N_1}{\mu} \int_0^1 \left|\phi \int_0^x u_t(y)\, dy\right| dx \\ &+N_2 \int_0^1 \left|\phi_x u_t + u_x \phi_t - \frac{\rho}{\mu J} u_t \int_0^\infty g(p)\, \phi_x(t-p)\, dp\right| dx \\ &+\rho \int_0^1 |u_t u|\, dx + N_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} |\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx. \end{aligned}$$

Exploiting Young, Cauchy-Schwartz and Poincaré inequalities, we obtain

$$\begin{aligned} |\mathfrak{L}(t)| &\leq c \int_0^1 \left(u_t^2 + \phi_t^2 + \phi_x^2 + u_x^2 + \phi^2\right) dx + c \int_0^1 \int_0^\infty g(p)\varphi_x^2(p) dp dx \\ &+c \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(s)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho \\ &\leq c\mathcal{E}(t). \end{aligned}$$

Consequently, we obtain

$$|\mathfrak{L}(t)| = |\mathcal{L}(t) - N\mathcal{E}(t)| \leq c\mathcal{E}(t),$$

that is

$$(N-c)\,\mathcal{E}(t) \leq \mathcal{L}(t) \leq (N+c)\,\mathcal{E}(t). \tag{4.3.31}$$

Now, by choosing N large enough such that

$$\frac{N}{2} - c > 0, N - c > 0, N\eta_0 - c > 0,$$

and exploiting (4.3.1), estimates (4.3.30) and (4.3.31), respectively, give

$$c_2\mathcal{E}(t) \leq \mathcal{L}(t) \leq c_3\mathcal{E}(t), \forall\, t \geq 0, \tag{4.3.32}$$

and

$$\begin{aligned} \mathcal{L}'(t) &\leq -k_1\mathcal{E}(t) + k_2 \int_0^1 \int_0^\infty g(p)\varphi_x^2(p) dp dx \\ &\quad +k_3 \int_0^1 (\phi_t^2 + f^2(\phi_t))dx + \alpha_5 \int_0^1 u_{xx}\phi_x dx, \end{aligned} \tag{4.3.33}$$

for some $k_1, k_2, k_3, c_2, c_3 > 0$.

Case 1. *If $\chi = (\frac{\mu}{\rho} - \frac{\delta}{J}) = 0$, in this case, (4.3.33) takes the from*

$$\begin{aligned}\mathcal{L}'(t) &\leq -k_1\mathcal{E}(t) + k_2\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \\ &\quad +k_3\int_0^1(\phi_t^2 + f^2(\phi_t))dx. \end{aligned} \tag{4.3.34}$$

By multiplying (4.3.34) by $h(t) = \alpha(t).\eta(t)$, we obtain

$$\begin{aligned}h(t)\mathcal{L}'(t) &\leq -k_1h(t)\mathcal{E}(t) + k_2h(t)\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \\ &\quad +k_3h(t)\int_0^1(\phi_t^2 + f^2(\phi_t))dx. \end{aligned} \tag{4.3.35}$$

We distinguish two cases

- *G is linear on $[0,\varepsilon]$. In this case, using the assumption $(4.1.7)_1$ and (4.3.2), we can write*

$$k_3h(t)\int_0^1(\phi_t^2 + f^2(\phi_t))dx \leq k_3h(t)\int_0^1\phi_t f(\phi_t))dx \leq -k_3\eta(t)\mathcal{E}'(t), \tag{4.3.36}$$

and by (4.1.5) we have

$$\begin{aligned}h(t)\int_0^1\int_0^t g(p)\varphi_x^2(p)dpdx &= \alpha(t)\int_0^1\int_0^t \eta(s)g(p)\varphi_x^2(p)dpdx \\ &\leq -\alpha(t)\int_0^1\int_0^t g'(p)\varphi_x^2(p)dpdx \\ &\leq -\alpha(t)\int_0^1\int_0^\infty g'(p)\varphi_x^2(p)dpdx \\ &\leq -2\alpha(t)\mathcal{E}'(t), \end{aligned} \tag{4.3.37}$$

and by (4.3.26 we obtain

$$\begin{aligned}\int_0^1\varphi_x^2(s)dx &= 2\int_0^1\phi_x^2(x,t)dx + 2\int_0^1\phi_x^2(x,t-s)dx \\ &\leq 4\sup_{s>0}\int_0^1\phi_x^2(x,s)dx + 2\sup_{\tau>0}\int_0^1\phi_{0x}^2(x,\tau)dx \\ &\leq \frac{8\mathcal{E}(0)}{l} + 2c_0, \end{aligned} \tag{4.3.38}$$

then, we get

$$h(t)\int_0^1\int_t^\infty g(p)\varphi_x^2(p)dpdx \leq (\frac{8\mathcal{E}(0)}{l} + 2c_0)h(t)\int_t^\infty g(p)dp. \tag{4.3.39}$$

Hence

$$h(t)\int_0^1\int_0^\infty g(p)\varphi_x^2(p)dpdx \;\leq\; -2\alpha(t)\mathcal{E}'(t) + (\frac{8\mathcal{E}(0)}{l} + 2c_0)h(t)\varpi(t). \tag{4.3.40}$$

Inserting (4.3.36) and (4.3.40) in (4.3.35). Since $h'(t) \leq 0, \alpha'(t) \leq 0, \eta'(t) \leq 0$. *Then, we have*

$$\mathcal{L}_1'(t) \leq -k_1 h(t)\mathcal{E}(t) + \gamma h(t)\varpi(t), \tag{4.3.41}$$

and

$$m_1\mathcal{E}(t) \leq \mathcal{L}_1(t) \leq m_2\mathcal{E}(t), \tag{4.3.42}$$

with

$$m_1 = \tau_1, \quad m_2 = c_2 h(0) + k_3\eta(0) + 2k_2\alpha(0) + \tau_1,$$

where

$$\begin{aligned} \mathcal{L}_1(t) &= h(t)\mathcal{L}(t) + (k_3\eta(t) + 2k_2\alpha(t) + \tau_1)\mathcal{E}(t) \sim \mathcal{E}(t), \\ \gamma &= (\frac{8\mathcal{E}(0)}{l} + 2c_0), \;\; \tau_1 > 0 \;\; \text{and} \;\; \varpi(t) = \int_t^\infty g(p)dp. \end{aligned} \tag{4.3.43}$$

Since $\mathcal{E}'(t) \leq 0, \forall\, t \geq 0$. *By using (4.3.41), we have*

$$\mathcal{E}(T)\int_0^T h(t)dt \leq \left(\frac{\mathcal{L}_1(0)}{k_1} + \frac{\gamma}{k_1}\int_0^T h(t)\varpi(t)dt\right). \tag{4.3.44}$$

Using the fact that G_0^{-1} *is linear. Then*

$$\mathcal{E}(T) \leq \zeta G_0^{-1}\left(\frac{\frac{\mathcal{L}_1(0)}{k_1} + \frac{\gamma}{k_1}\int_0^T h(t)\varpi(t)dt}{\int_0^T h(t)dt}\right). \tag{4.3.45}$$

with $\beta_1 = \zeta, \;\; \beta_2 = \frac{\mathcal{L}_1(0)}{k_1}, \;\; \beta_3 = \frac{\gamma}{k_1}$. *This completes the proof.*

- G *is nonlinear on* $[0,\varepsilon]$, *we choose* $0 \leq \varepsilon_1 \leq \varepsilon$ *and we consider*

$$I_1(t) = \{x \in (0,1), \;\; |\phi_t| \leq \varepsilon_1\}, \;\; I_2 = \{x \in (0,1), \;\; |\phi_t| > \varepsilon_1\},$$

we define

$$I = \int_{I_1}\phi_t f(\phi_t)dt.$$

Using Jensen's inequality and the assumption $(4.1.7)_1$, *we have*

$$
\begin{aligned}
k_3h(t)\int_0^1(\phi_t^2+f^2(\phi_t))dx &\leq k_3h(t)\int_0^1\phi_t f(\phi_t))dx \\
&\leq k_3'h(t)G^{-1}(I(t))-k_3'\eta(t)\mathcal{E}'(t). \qquad (4.3.46)
\end{aligned}
$$

Inserting (4.3.46) in (4.3.35), since $\alpha'(t)\leq 0,\eta'(t)\leq 0$ *and* $\mathcal{E}'(t)\leq 0$, *we obtain*

$$\mathcal{L}_2'(t)\leq -k_1h(t)\mathcal{E}(t)+\gamma h(t)\varpi(t)+k_3'h(t)G^{-1}(I(t)). \qquad (4.3.47)$$

and

$$m_3\mathcal{E}(t)\leq\mathcal{L}_2(t)\leq m_4\mathcal{E}(t), \qquad (4.3.48)$$

with

$$m_3=\tau_1,\quad m_4=c_2h(0)+k_3'\eta(0)+2k_2\alpha(0)+\tau_1,$$

where

$$\mathcal{L}_2(t) = h(t)\mathcal{L}(t)+(k_3'\eta(t)+2k_2\alpha(t)+\tau_1)\mathcal{E}(t)\sim\mathcal{E}(t).$$

Now, for $\varepsilon_0<\varepsilon_1$ *and by using* $\mathcal{E}'(t)\leq 0, G'>0$ *and* $G''>0$ *on* $(0,\varepsilon]$, *we define the functional* $\mathcal{L}_3(t)$ *by,*

$$\mathcal{L}_3(t) = G'(\varepsilon_0\mathcal{E}(t))\mathcal{L}_2(t)+\tau_2\mathcal{E}(t)\sim\mathcal{E}(t),\ \ \tau_2>0,$$

satisfies

$$
\begin{aligned}
\mathcal{L}_3'(t) &= \mathcal{E}'(t)(\varepsilon_0G'(\varepsilon_0\mathcal{E}(t))\mathcal{L}_2(t)+\tau_2)+\mathcal{L}_2'(t)G'(\varepsilon_0\mathcal{E}(t)) \\
&\leq -k_1h(t)G_0(\mathcal{E}(t))+\gamma G'(\varepsilon_0\mathcal{E}(t))h(t)\varpi(t) \\
&\quad +k_3'h(t)G'(\varepsilon_0\mathcal{E}(t))G^{-1}(I(t)). \qquad (4.3.49)
\end{aligned}
$$

To estimate the last term of (4.3.41), using the general Young's inequality

$$AB \leq G^*(A)+G(B),\quad if\ \ A\in(0,G'(\varepsilon)),\ \ B\in(0,\varepsilon),$$

where

$$G^*(A) = s(G')^{-1}(s)-G((G')^{-1}(s)),\quad if\ \ s\in(0,G'(\varepsilon)),$$

satisfies

$$k_3' h(t) G'(\varepsilon_0 \mathcal{E}(t)) G^{-1}(I(t)) \;\; \leq \;\; k_3' \varepsilon_0 h(t) G_0(\mathcal{E}(t)) - k_3' \eta(t) \mathcal{E}'(t). \qquad (4.3.50)$$

Inserting (4.3.50) in (4.3.41) and letting $\varepsilon_0 = \frac{k_1}{2k_3'}$ *, we get*

$$\mathcal{L}_3'(t) + k_3' \eta(t) \mathcal{E}'(t) \;\; \leq \;\; -k_1 h(t) G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0 \mathcal{E}(t)) h(t) \varpi(t). \qquad (4.3.51)$$

Since $\eta'(t) \leq 0$*, then*

$$\mathcal{L}_4'(t) \;\; \leq \;\; -k_1 h(t) G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0 \mathcal{E}(t)) h(t) \varpi(t),$$

where

$$\mathcal{L}_4(t) \;\; = \;\; \mathcal{L}_3(t) + k_3' \eta(t) \mathcal{E}(t) \sim \mathcal{E}(t).$$

Since $\alpha(t), G_0(\mathcal{E}(t)), G'(\varepsilon_0 \mathcal{E}(t))$ *are non-increasing functions,*

then, for any $T > 0$

$$\begin{aligned} k_1 G_0(\mathcal{E}(T)) \int_0^T h(t) dt \;\; &\leq \;\; k_1 \int_0^T h(t) G_0(\mathcal{E}(t)) dt \\ &\leq \;\; \mathcal{L}_4(0) + \gamma G'(\varepsilon_0 \mathcal{E}(0)) \int_0^T h(t) \varpi(t) dt, \end{aligned}$$

which gives (4.3.27) with $\beta_1 = 1, \;\; \beta_2 = \dfrac{\mathcal{L}_4(0)}{k_1}$ *and* $\beta_3 = \dfrac{\gamma G'(\varepsilon_0 \mathcal{E}(0))}{k_1}$.

The proof is now completed.

Case 2. *If* $\chi = (\frac{\mu}{\rho} - \frac{\delta}{J}) \neq 0$ *and*

$$\begin{cases} |\chi| < \dfrac{k_1 \mu^2 l}{2N_2(l\rho + b\mu)} & if \quad \chi < 0 \\ |\chi| < \dfrac{k_1 \mu^2}{2N_2 \rho} & if \quad \chi > 0. \end{cases}$$

This case is more important from the physical point of view, where waves are not necessarily of equal speeds. Let

$$\mathcal{E}(t) = \mathcal{E}(u, \phi, y, \varphi) = \mathcal{E}_1(t).$$

Denotes the first-order energy defined in (4.3.1) and

$$\mathcal{E}_2(t) = \mathcal{E}(u_t, \phi_t, y_t, \varphi_t).$$

Denotes the second-order energy, Then, we have

$$\begin{aligned}
\mathcal{E}_2'(t) &\leq -\eta_0 \int_0^1 \phi_{tt}^2 dx + \frac{1}{2}\int_0^1 \int_0^\infty g'(p)\varphi_{tx}^2(p)dp \\
&\quad -\alpha'(t)\int_0^1 \phi_{tt} f(\phi_t)dx - \alpha(t)\int_0^1 \phi_{tt}^2 f'(\phi_t)dx \\
&= -\eta_0 \int_0^1 \phi_{tt}^2 dx + \frac{1}{2}\int_0^1 \int_0^\infty g'(p)\varphi_{tx}^2(p)dp \\
&\quad +\alpha(t)\left(\frac{-\alpha'(t)}{\alpha(t)}\int_0^1 \phi_{tt} f(\phi_t)dx - \int_0^1 \phi_{tt}^2 f'(\phi_t)dx\right). \qquad (4.3.52)
\end{aligned}$$

Since f, g are non-decreasing functions, $\alpha(t)$ is a positive function and $\lim_{t\to\infty} \frac{-\alpha'(t)}{\alpha(t)} = 0$, we deduce that

$$\begin{aligned}
\mathcal{E}_2'(t) &\leq -\eta_0 \int_0^1 \phi_{tt}^2 dx + \frac{1}{2}\int_0^1 \int_0^\infty g'(s)\varphi_{tx}^2 \\
&\leq -\eta_0 \int_0^1 \phi_{tt}^2 dx, \qquad (4.3.53)
\end{aligned}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho > 0$.
The last term in (4.3.33), by using $(4.1.10)_1$, Young's inequality and by setting $K = \frac{\chi N_2 \rho}{\mu} = \frac{\alpha_5 \rho}{\mu}$ and $\alpha_5 = \chi N_2$ as follows

$$\begin{aligned}
\alpha_5 \int_0^1 u_{xx}\phi_x dx &= \frac{\alpha_5 \rho}{\mu}\int_0^1 \phi_x u_{tt} dx - \frac{b\alpha_5}{\mu}\int_0^1 \phi_x^2 dx \\
&= K\left(\frac{d}{dt}\left[\int_0^1 \phi_t u_x dx + \int_0^1 \phi_x u_t dx\right]\right) \\
&\quad -K\int_0^1 u_x \phi_{tt}^2 dx - \frac{b\alpha_5}{\mu}\int_0^1 \phi_x^2 dx \\
&\leq K\left(\frac{d}{dt}\left[\int_0^1 \phi_t u_x dx + \int_0^1 \phi_x u_t dx\right]\right) \\
&\quad +\frac{|K|}{4}\int_0^1 \phi_{tt}^2 dx + |K|\int_0^1 u_x^2 dx. \qquad (4.3.54)
\end{aligned}$$

Let

$$\mathcal{N}(t) = \left(\int_0^1 \phi_t u_x dx + \int_0^1 \phi_x u_t dx\right),$$

then (4.3.33)

$$
\begin{aligned}
\mathcal{L}'(t) + K\mathcal{N}'(t) &\leq -k_1\mathcal{E}_1(t) + k_2\int_0^1\int_0^\infty g(p)\varphi_p^2 dpdx + \frac{|K|}{4}\int_0^1 \phi_{tt}^2 dx \\
&\quad +|K|\int_0^1 u_x^2 dx + k_3\int_0^1 (\phi_t^2 + f^2(\phi_t))dx \\
&\leq -k_4\mathcal{E}_1(t) + k_2\int_0^1\int_0^\infty g(p)\varphi_p^2 dpdx \\
&\quad +\frac{|K|}{4}\int_0^1 \phi_{tt}^2 dx + k_3\int_0^1 (\phi_t^2 + f^2(\phi_t))dx, \qquad (4.3.55)
\end{aligned}
$$

where

$$
k_4 = k_1 - 2\frac{|K|}{\mu} > 0.
$$

Let

$$
\mathcal{R}(t) = \mathcal{L}(t) + K\mathcal{N}(t) + N_5(\mathcal{E}_1(t) + \mathcal{E}_2(t)). \qquad (4.3.56)
$$

Indeed, by using Young's inequality, we obtain

$$
\begin{aligned}
|\mathcal{N}(t)| &= |\int_0^1 \phi u_{xt} dx| + |\int_0^1 \phi_t u_x dx| \\
&\leq \frac{1}{2}\int_0^1 u_t^2 dx + \frac{1}{2}\int_0^1 \phi_t^2 dx + \frac{1}{2}\int_0^1 \phi_x^2 dx + \frac{1}{2}\int_0^1 u_x^2 dx \\
&\leq C_0\mathcal{E}_1(t), \qquad (4.3.57)
\end{aligned}
$$

where $C_0 = max\{\frac{1}{J}, \frac{1}{\xi}, \frac{1}{\rho}, \frac{1}{\mu}\}$.

By (4.3.32) and (4.3.57), we get

$$
|\mathcal{R}(t) - N_5(\mathcal{E}_1(t) + \mathcal{E}_2(t))| \leq (c_3 + C_0)\mathcal{E}_1(t) \leq c(\mathcal{E}_1(t) + \mathcal{E}_2(t)), \qquad (4.3.58)
$$

and

$$
(N_5 - c)(\mathcal{E}_1(t) + \mathcal{E}_2(t)) \leq \mathcal{R}(t) \leq (N_5 + c)(\mathcal{E}_1(t) + \mathcal{E}_2(t)), \qquad (4.3.59)
$$

and by using (4.3.53), (4.3.55) and (2), we obtain

$$
\begin{aligned}
\mathcal{R}'(t) &= \mathcal{L}'(t) + K\mathcal{N}'(t) + N_5(\mathcal{E}_1'(t) + \mathcal{E}_2'(t)) \\
&\leq -k_4\mathcal{E}_1(t) + k_2\int_0^1\int_0^\infty g(p)\varphi_p^2 dpdx \\
&\quad +k_3\int_0^1 (\phi_t^2 + f^2(\phi_t))dx - (\eta_0 N_5 - \frac{|K|}{4})\int_0^1 \phi_{tt}^2 dx. \qquad (4.3.60)
\end{aligned}
$$

We choose N_5 *large enough, such that*

$$\eta_0 N_5 - \frac{|K|}{4} > 0, \quad N_5 - c > 0,$$

we obtain

$$\mathcal{R}(t) \sim (\mathcal{E}_1(t) + \mathcal{E}_2(t)), \tag{4.3.61}$$

and

$$\begin{aligned} \mathcal{R}'(t) \quad &\leq \quad -k_4\mathcal{E}_1(t) + k_2 \int_0^1 \int_0^\infty g(p)\varphi_p^2 dpdx \\ &+k_3 \int_0^1 (\phi_t^2 + f^2(\phi_t))dx. \end{aligned} \tag{4.3.62}$$

By multiplying (4.3.62) by $h(t) = \alpha(t).\eta(t)$*, we obtain*

$$\begin{aligned} h(t)\mathcal{R}'(t) \quad &\leq \quad -k_4 h(t)\mathcal{E}(t) + k_2 h(t) \int_0^1 \int_0^\infty g(p)\varphi_x^2(p) dpdx \\ &+k_3 h(t) \int_0^1 (\phi_t^2 + f^2(\phi_t))dx. \end{aligned} \tag{4.3.63}$$

We distinguish two cases

- G *is linear on* $[0, \varepsilon]$*. In the same way that in the previous case, we obtain*

$$\mathcal{R}_1'(t) \leq -k_4 h(t)\mathcal{E}(t) + \gamma h(t)\varpi(t), \tag{4.3.64}$$

and

$$m_1(\mathcal{E}_1(t) + \mathcal{E}_2(t)) \leq \mathcal{R}_1(t) \leq m_2(\mathcal{E}_1(t) + \mathcal{E}_2(t)), \tag{4.3.65}$$

with

$$m_1 = \tau_1, \quad m_2 = c_2 h(0) + k_3\eta(0) + 2k_2\alpha(0) + \tau_1,$$

where

$$\begin{aligned} \mathcal{R}_1(t) \quad &= \quad h(t)\mathcal{R}(t) + (k_3\eta(t) + 2k_2\alpha(t) + \tau_1)\mathcal{E}(t) \sim (\mathcal{E}_1(t) + \mathcal{E}_2(t)) \\ \gamma \quad &= \quad \left(\frac{8\mathcal{E}(0)}{l} + 2c_0\right), \ \ \tau_1 > 0 \ \ \text{and} \ \ \varpi(t) = \int_t^\infty g(p)dp. \end{aligned}$$

Since $\mathcal{E}'(t) \leq 0, \forall t \geq 0$*. By using (4.3.64), we have*

$$\mathcal{E}(T) \int_0^T h(t)dt \leq \left(\frac{\mathcal{R}_1(0)}{k_4} + \frac{\gamma}{k_4} \int_0^T h(t)\varpi(t)dt \right). \tag{4.3.66}$$

Using the fact that G_0^{-1} *is linear. Then*

$$\mathcal{E}(T) \leq \zeta G_0^{-1}\left(\frac{\frac{\mathcal{R}_1(0)}{k_4}+\frac{\gamma}{k_4}\int_0^T h(t)\varpi(t)dt}{\int_0^T h(t)dt}\right), \tag{4.3.67}$$

with $\beta_1 = \zeta, \quad \beta_2 = \frac{\mathcal{R}_1(0)}{k_4}, \quad \beta_3 = \frac{\gamma}{k_4}$. *This completes the proof.*

- G *is nonlinear on* $[0, \varepsilon]$, *we choose* $0 \leq \varepsilon_1 \leq \varepsilon$. *And in a similar way to that in the previous case, we get*

$$\mathcal{R}_2'(t) \leq -k_1 h(t)\mathcal{E}(t) + \gamma h(t)\varpi(t) + k_3' h(t) G^{-1}(I(t)), \tag{4.3.68}$$

and

$$m_3(\mathcal{E}_1(t)+\mathcal{E}_2(t)) \leq \mathcal{R}_2(t) \leq m_4(\mathcal{E}_1(t)+\mathcal{E}_2(t)), \tag{4.3.69}$$

with

$$m_3 = \tau_1, \quad m_4 = c_2 h(0) + k_3'\eta(0) + 2k_2\alpha(0) + \tau_1,$$

where

$$\mathcal{R}_2(t) \quad = \quad h(t)\mathcal{R}(t) + (k_3'\eta(t) + 2k_2\alpha(t) + \tau_1)\mathcal{E}(t) \sim (\mathcal{E}_1(t)+\mathcal{E}_2(t)).$$

Now, for $\varepsilon_0 < \varepsilon_1$ *and by using* $\mathcal{E}'(t) \leq 0, G' > 0$ *and* $G'' > 0$ *on* $(0, \varepsilon]$, *we define the functional* $\mathcal{L}_3(t)$ *by,*

$$\mathcal{R}_3(t) \quad = \quad G'(\varepsilon_0\mathcal{E}(t))\mathcal{R}_2(t) + \tau_2\mathcal{E}(t) \sim (\mathcal{E}_1(t)+\mathcal{E}_2(t)), \;\; \tau_2 > 0,$$

satisfies

$$\begin{aligned}\mathcal{R}_3'(t) &= \mathcal{E}'(t)(\varepsilon_0 G'(\varepsilon_0\mathcal{E}(t))\mathcal{R}_2(t) + \tau_2) + \mathcal{R}_2'(t)G'(\varepsilon_0\mathcal{E}(t)) \\ &\leq -k_4 h(t) G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0\mathcal{E}(t))h(t)\varpi(t) \\ &\quad + k_3' h(t) G'(\varepsilon_0\mathcal{E}(t)) G^{-1}(I(t)).\end{aligned} \tag{4.3.70}$$

To estimate the last term of (4.3.70), using the general Young's inequality

$$AB \;\; \leq \;\; G^*(A) + G(B), \;\;\; if \;\; A \in (0, G'(\varepsilon)), \;\; B \in (0, \varepsilon),$$

where

$$G^*(A) \;=\; s(G')^{-1}(s) - G((G')^{-1}(s)), \quad if \;\; s \in (0, G'(\varepsilon)),$$

satisfies

$$k_3' h(t) G'(\varepsilon_0 \mathcal{E}(t)) G^{-1}(I(t)) \;\leq\; k_3' \varepsilon_0 h(t) G_0(\mathcal{E}(t)) - k_3' \eta(t) \mathcal{E}'(t). \tag{4.3.71}$$

Inserting (4.3.71) in (4.3.70) and letting $\varepsilon_0 = \frac{k_1}{2k_3'}$*, we get*

$$\mathcal{R}_3'(t) + k_3' \eta(t) \mathcal{E}'(t) \;\leq\; -k_4 h(t) G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0 \mathcal{E}(t)) h(t) \varpi(t). \tag{4.3.72}$$

Since $\eta'(t) \leq 0$*, then*

$$\mathcal{R}_4'(t) \;\leq\; -k_4 h(t) G_0(\mathcal{E}(t)) + \gamma G'(\varepsilon_0 \mathcal{E}(t)) h(t) \varpi(t),$$

where

$$\mathcal{R}_4(t) \;=\; \mathcal{R}_3(t) + k_3' \eta(t) \mathcal{E}(t) \sim (\mathcal{E}_1(t) + \mathcal{E}_2(t)).$$

Since $\alpha(t), G_0(\mathcal{E}(t)), G'(\varepsilon_0 \mathcal{E}(t))$ *are non-increasing functions, then, for any* $T > 0$

$$\begin{aligned} k_4 G_0(E(T)) \int_0^T h(t) dt \;&\leq\; k_4 \int_0^T h(t) G_0(\mathcal{E}(t)) dt \\ &\leq\; \mathcal{R}_4(0) + \gamma G'(\varepsilon_0 \mathcal{E}(0)) \int_0^T h(t) \varpi(t) dt, \end{aligned}$$

which gives (4.3.27) with $\beta_1 = 1, \;\; \beta_2 = \frac{\mathcal{R}_4(0)}{k_4}$ *and* $\beta_3 = \frac{\gamma G'(\varepsilon_0 \mathcal{E}(0))}{k_4}$.

The proof is completed.

□

Conclusion and perspective

Damping arises from the removal of energy by dissipation. In the past few years, damped systems has been actively studied both quantitatively and qualitatively, which is associated with the evolutional equations (Systems). The importance of the present research lies with describing the role of three different damping terms in a system of three wave equations with three different variables in the presence of strong external forces that make the issue very important in application point of view, the complete study concerning to existence and uniqueness in addition to the nature of decay for the energy function makes it easy for applications in the sciences. This type of problem is not previously considered, it is new, especially in the presence of the memory functions. In this direction, one can ask the next question. For some similar problem with n equations and n independent variable, can one obtains a results of the existence/nonexistence and asymptotic behavior of solution over time?

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