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By

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Title

# Mean-Field Optimal Control of Diffusion with Regime Switching

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# Abstract

The objective of this thesis is to study a problem of optimal control with regime switching jump-diffusion model of mean-field type. In the first part we recall a result in the stochastic maximum principle whose horizon is finite. In the second part, we devote ourselves to presenting the two main results of this thesis, in the first result we give the necessary and sufficient conditions of optimality whose control system is governed by a stochastic differential equation with regime switching of infinite horizon and by way of illustration, we have given two examples where in both cases the equation of state is linear and the objective function is of utility form. The second contribution on the maximum principle for a control problem of conditional mean field type of finite horizon, we illustrate our result by a model which gives an explicit solution

**Keys words**. Stochastic maximum principle, Optimal control, Partial information, Regime switching, Jump-diffusion model, Mean-field type.

# Résumé

L'objectif de cette thèse est d'étudier un problème de contrôle optimal pour un système de diffusion avec saut à changement de régime de type champs moyen. Dans la première partie nous rappelons un résultat sur le principe de maximum stochastique dont l'horizon est fini. Dans la deuxième partie, on se consacre a présenté les deux résultat principaux de cette thèse, dans le premier résultat on donne les conditions nécessaires et suffisantes d'optimalité dont le système contrôle est gouverné par une équation différentielle stochastique à changement de régime d'horizon infini et à titre d'illustration, nous avons donné deux exemples où dans les deux cas, l'équation d'état est linéaire et la fonction objectif est de forme utilitaire. La deuxième contribution sur le principe de maximum pour un problème de contrôle de type champs moyen conditionnel d'horizon fini, nous illustrons notre résultat par un modèle qui donne une solution explicite.

**Mots Clés.** Principe du maximum stochastique, Contrôle optimal, Information partielle, Changement de régime, Modèle diffusion-saut, Type champ moyen.

# Contents

# Index of notations

Here we give some the different symbols and abbreviations used in this thesis:

| | | |
|---|---|---|
| $(\Omega, \mathcal{F}, P)$ | : | Probability space. |
| $\left(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P\right)$ | : | Filtred probability space. |
| $\{\mathcal{F}\}_{t\geq 0}$ | : | Filtration. |
| $B(t)_{t\in[0,T]}$ | : | Brownian motion. |
| $a.s$ | : | Almost surely. |
| $a.e$ | : | Almost everywhere. |
| $P - a.s$ | : | Almost surely with respect to the probability measure. |
| *càdlàg* | : | Right continuous with left limits. |
| $\mathbb{R}$ | : | Real numbers. |
| $\mathbb{R}^d$ | : | $d-$ dimensional real Euclidean space. |
| $\mathbb{R}^{n\times d}$ | : | The set of all $n \times d$ real matrixes. |
| $\mathbb{N}$ | : | Natural numbers. |
| $L^2([0,T];\mathbb{R}^n)$ | : | The set of continuous $\mathcal{F}_t$-measurable process $\{\varphi_t\}_{t\in[0,T]}$ which satisfy $\mathbf{E}\left[\sup_{0\leq t\leq T}\lvert\varphi_t\rvert^2\right] < \infty$. |
| $\mathcal{S}$ | : | Finite state space. |
| $U$ | : | The set of values taken by control $u$. |
| $\mathcal{U}$ | : | The set of admissible controls. |
| $u^*$ | : | Opimal control. |
| $\Lambda$ | : | Rate matrix. |
| $y^\top$ | : | Transpose of a victor $y$. |
| $\mathcal{B}(\mathbb{R}^+)$ | : | The Borel $\sigma$-field generated by the open subsets of $\mathbb{R}^+$. |
| $\mathcal{N}$ | : | The compensated poisson random measure. |
| $\widetilde{\mathcal{N}}_\alpha$ | : | The compensated Markov regime-switching random measure. |
| $\mathbf{P}_{ij}^{(n)}$ | : | $n$-step transition probability of a Markov chain. |

| | | |
|---|---|---|
| $\mathbf{P}$ | : | Transition probability matrix of the Markov chain. |
| $I_A(s)$ | : | The indicator function of $A$. |
| $\mathbf{E}[X]$ | : | Expectation at $X$. |
| $\mathbf{E}[X \mid \mathcal{F}_t]$ | : | Conditional expectation. |
| $SDEs$ | : | Stochastic differential equations. |
| $SDEJs$ | : | Stochastic differential equation with jumps. |
| $BSDE$ | : | Backward stochastic differential equation. |
| $BSDEJs$ | : | Backward stochastic differential equation with jumps. |
| $HMM$ | : | Hidden Markov Model. |
| $\gcd\{E\}$ | : | Greatest common divisor of $E$. |

# Introduction

In the past years, regime switching models have been widely used in finance and stochastic optimal controls. The basic idea of such models is to modulate the model with a continuous time, finite state Markov chain where each state represents a regime of the system or level of economic indicator. For example, in the stock market, the up-trend volatility of a stock tends to be smaller than its down-trend volatility (see Zhang [56] for further details). Therefore, it is reasonable to describe the market trends by a two-state Markov chain, much work has been done on stability and stochastic control problems for the regime switching system, such as [[31]-[57]]. The regime switching model in economic and finance fields was first introduced by Hamilton in [29] to describe a time series model and then intesively investigated in the past two decades in mathematical finance.

In the deterministic case, the maximum principle was introduced by Pontryagin & al [45] in the 1950's. Since then, a lot of works have been done for systems driven by Brownian motion such as Bismut [10], Kushner [32], Bensoussan [9] and Haussman [30]. Peng [42] derived a general stochastic maximum principle where the control domain is not necessarly convex and the diffusion coefficient can contain the control variable. Mezerdi [11] generalized the principle of Kushner to the case of a SDE with non smooth drift. It was extended to systems with jumps by Tang & al [53], and later by Framstad & al [24]. In infinite horizon, Haadam & al [25], introduced a maximum principle for infinite horizon jump diffusion processes for partial information. They proved necessary and sufficient maximum principles for this problem. The results obtained are applied to several problems which appear in finance. However, Maslowski and Veverka [34] establish a sufficient stochastic maximum principle for infinite horizon discounted control problem. As an application, they study the controlled stochastic logistic equation of population dynamics.

The concept of mean-field theory is widely used for the description of interacting many-body systems in physics and probability theory. The behind idea is not to treat the many-body system by summing up all mutual two-body interactions of the particles but to discribe the interaction of one particle with the remaining ones by an average potential created by the other particles.

Stochastic optimal control problems for the mean-field stochastic differential equations (SDEs) have attracted an increasing attention. The history of the mean field SDEs can trace their roots to the Mckean-Valsov model, which was first introduced by Kac (1956) and Mckean (1966) to study physical systems with a large number of interacting particles. Lasry and Lions (2007) extended applications of the mean-field models to economics and finance. Intuitively speaking, the adjoint equation of a controlled state process driven by the mean field SDE is a mean-field backward stochastic differential equation (BSDE). In 2009, Backdahn et al established the theory of the mean-field BSDEs that the stochastic maximum principle for the optimal control system of mean-field type has become a popular topic. Interested readers may refer to Andersson and Djehich (2011), Backdahn et al (2011), Li (2012). In Shen and Siu [49], the authers proved the existance and uniqueness of solutions to mean-field BSDEs driven by Brownian motions and Poisson jumps. All these works established a solide foundation to cope with stochastic optimal control problems of mean-field models.

The optimal control problem for the Markov regime switching model has seen great interest in recent years. See, for example [36],[51], [59], [20], what characterizes these models is that there are two components, a diffusion part which is continuous and another discert one representes by continuous Markov chain, moreover in an empirical sense these last models are more preferred than the classical one for example, Mean -Variance with regime switching [61], Option pricing [23], American options [17] . The first version on mean field stochastic optimal control with regime switching is due to Zhang et al [60] in their paper they gave the necessary and sufficient conditions of optimality of an optimal control when the coefficients of the system depend on the solution as well as it's expect value, inspired by the paper of Buckdahn et al [16] and another very powerfull model proposed by Nguyen et al [39], a very interesting paper introduced by So et al [38] in that paper the authors treat the convexe case with full information.

In this thesis, we present a mean-field optimal control of diffusion with regime-switching.

Let us briefly describe the contents of this thesis:

In **Chapter 1**, We recall some result about stochastic calculs with jumps in which we define the Lévy processes, brownian motion and state few important properties such as the Markov property, stochastic integral with respect to Lévy process.

In **Chapter 2,** We present some notions about Markov chains in continuous and discret time in which we define the transition function and transition rate matrix. Finnaly, we give some models that illustrate regime switching.

In **Chapter 3,** We recall a result in stochastic optimal control of a Jump-Diffusion with finite horizon, sufficient and necessary maximum principles are presented under partial information.

In **Chapter 4,** We give our first result about optimal control of jump-diffusion with Markov regime switching in infinite horizon. Firstly we prove that our system have unique solution, then sufficient and necessary maximum principles are developed under partial information . An optimal portfolio and consumption in a switching diffusion market are studied.

In **Chapter 5,** We present our second result about stochastic optimal control problem for a Markov regime switching in the conditional mean-field model. Sufficient and necessary maximum principles for optimal control under partial information are obtained. Finally we illustrate our result through a model which gives an explicit solution.

# Relevant Papers

The content of this thesis was the subject of the following papers:

1. Benabdallah, Hani, Lazhar Tamer, and Nassima Chaouchkhouane. "Stochastic maximum principle for a Markov regime switching jump-diffusion in infinite horizon." International Journal of Nonlinear Analysis and Applications (2022).

2. "Partial Information Maximum Principle for Optimal Control Problem with Regime Switching in the Conditional Mean-Field Model"; paper accepted for publication.

# Chapter 1

# Some Elements of Jump Processes

In this chapter, we recall some result about stochastic calculs with jumps in which we define the Lévy processes, brownian motion and state few important properties such as the Markov property, stochastic integral with respect to Lévy process.

## 1.1 Stochastic processes

### 1.1.1 Filtration and processes

**Definition 1.1.1** *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration is an increasing family of $\sigma$-algebras $(\mathcal{F}_t)_{t\in[0,T]}$ : $\forall t \geq s \geq 0$, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$.*

**Definition 1.1.2** *A stochastic process is collection of random variables $(X_t)_{t\in[0,T]}$ such that for each fixed $t \in [0, T]$ , $X_t$ is a random variable from $(\Omega, \mathcal{F}, P)$ to $(E, \vartheta)$ .*

**Definition 1.1.3 (adapt process)** *A process $(X_t)_{[0,T]}$ is adapted (with respect to $(\mathcal{F}_t)_{[0,T]}$) if for all $[0, T]$, $X_t$ is $\mathcal{F}_t$-measurable.*

**Definition 1.1.4 (Progressively measurable, optional and predictable process)**

**(1)** A process $(X_t)_{[0,T]}$ is progressively measurable if for any $[0, T]$, the mapping $(s, w) \to X_s(w)$ is measurable on $[0, t] \times \Omega$ equipped with the product $\sigma$-field $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$.

**(2)** A process $(X_t)_{t\geq 0}$ is optional if the mapping $(s, w) \to X_s(w)$ is measurable on $[0, T] \times \Omega$ equipped with the $\sigma$-field generated by the $(\mathcal{F}_t)_{t[0,T]}$-adapted and càdlàg processes.

**(3)** A process $(X_t)_{[0,T]}$ is predictable if the mapping $(s, w) \to X_s(w)$ is measurable on $[0, T] \times \Omega$ equipped with the $\sigma$-field generated by the $(\mathcal{F}_t)_{[0,T]}$-adapted and continuous processes.

**Proposition 1.1.1** *If the process $X$ is optional, it is progressively measurable. In particular, if it is càdlàg and adapted, it is progressively measurable.*

### 1.1.2 Brownian motion

**Definition 1.1.5** *A standard d-dimensional Brownian motion on $[0, T]$ is continuous process valued in $\mathbb{R}^d$, $(B_t)_{[0,T]} = \left(B_t^1, ..., B_t^d\right)_{[0,T]}$ such that:*

**(i)** $B_0 = 0$.

**(ii)** *For all $0 \leq s \leq t$ in $[0, T]$, the increment $B_t - B_s$ is independent of $\sigma(B_u, u \leq s)$ and follows a centered Gaussian distribution with variance-covariance matrix $(t - s) I_d$.*

**Definition 1.1.6 (Brownian motion with respect to filtration)** *A vectorial (d-dimensional) Brownian motion on $[0, T]$ with respect to a filtration $(\mathcal{F}_t)_{[0,T]}$ is a continuous $(\mathcal{F}_t)_{[0,T]}$-adapted process, valued in $\mathbb{R}^d$, $(B_t)_{t \geq 0} = \left(B_t^1, ..., B_t^d\right)_{[0,T]}$ such that:*

**(i)** $B_0 = 0$.

**(ii)** *For all $0 \leq s \leq t$ in $[0, T]$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$ and follows a centered Gaussian distribution with variance-covariance matrix $(t - s) I_d$.*

**Remark 1.1.1** *A standard Brownian motion is a Brownian motion with respect to its natural filtration.*

**Proposition 1.1.2** *Let $(B_t)_{[0,T]}$ be a Brownian motion with respect to $(\mathcal{F}_t)_{[0,T]}$.*

**(1)** *Symmetry: $(-B_t)_{[0,T]}$ is also a Brownian motion.*

**(2)** *Scaling: for all $\lambda > 0$, the process $((1/\lambda) B_{\lambda^2 t})_{[0,T]}$ is also a Brownian motion.*

**(3)** *Invariance by translation: for all $s > 0$, the process $(B_{t+s} - B_s)_{[0,T]}$ is a standard Brownian motion independent of $\mathcal{F}_s$.*

## 1.2 Lévy process and strong Markov property

**Definition 1.2.1 (Lévy process)** *Let $X = (X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued stochastic process. We say $X$ is a Lévy process if it satisfies the following conditions:*

**(1)** $X_0 = 0$ *a.s.;*

**(2)** $X$ *has càdlàg trajectories a.s.;*

**(3)** $X$ *has independent and stationary increments.*

*The third item in the definition above means that, for all $n \in \mathbb{N}$ and $0 = t_0 < t_1 < ... < t_n$, the random variables $(X_{t_i} - X_{t_i-1})_{1\leq i\leq n}$ and $(X_{t_i+h} - X_{t_i-1+h})_{1\leq i\leq n}$ have the same law.*

**Example 1.2.1** *Brownian motions (with constant drift and standard deviation) and compound Poisson processes are Lévy processes.*

**Definition 1.2.2** *A stochastic process $B = (B_t)_{t\geq 0}$ on $\mathbb{R}^d$ is a Brownian motion if it is a Lévy process and if*

**(1)** *For all $t > 0$, has a Gaussian distribution with mean $0$ and covariance $tTd$.*

**(2)** *There is $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$, for every $w \in \Omega_0$, $B(t, w)$ is continuous in $t$*

**Definition 1.2.3 (Poisson process)** *A poisson process $\pi(t)$ of intensity $\lambda > 0$ is a Lévy process taking values in $\mathbb{N} \cup \{0\}$ and such that*

$$P[\pi(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}; \quad n = 0, 1, 2, ...$$

Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ be the natural filtration associated to $X$. We recall that a $[0, +\infty[$-valued random variable $T$ is a stopping time with respect to $\mathcal{F}$ if for all $t \geq 0$, the event $\{T \leq t\}$ belongs to $\mathcal{F}_t$. We also denote by

$$\mathcal{F}_T := \{A \in A : A \cap \{T \leq t\} \in \mathcal{F}_t, \ \forall t \geq 0\} \tag{1.1}$$

the $\sigma$-algebra of events prior to this stopping time. We may now state the following proposition.

**Proposition 1.2.1 (Strong Markov property)** *Let $X = (X_t)_{t\geq 0}$ be a Lévy process and $T$ a stopping time such that $T < \infty$ a.s. The process $(X_{T+t} - X_T)_{t\geq 0}$ is a Lévy process independent of $\mathcal{F}_T$ and distributed as $X$.*

*Of course, the strong Markov property implies the simple version of it, when $T$ is a deterministic time.*

### 1.2.1 The Itô formula and related results

**Theorem 1.2.1 (The One-Dimensional Itô Formula)** *Suppose $X(t) \in \mathbb{R}$ is an Itô-Lévy process of the form*

$$dX(t) = \alpha(t,w)\,dt + \sigma(t,w)\,dB(t) + \int_{\mathbb{R}} \gamma(t,z,w)\,\widetilde{\mathcal{N}}(dt,dz)$$

*where*

$$\widetilde{\mathcal{N}}(dt,dz) = \begin{cases} \mathcal{N}(dt,dz) - v(dz)\,dt & \text{if} \quad |z| < R \\ \mathcal{N}(dt,dz) & \text{if} \quad |z| \geq R \end{cases}$$

*for some $R \in [0,\infty)$.*

*Let $f \in C^2(\mathbb{R}^2)$ and define $Y(t) = f(t, X(t))$. Then $Y(t)$ is again an Itô-Lévy process and*

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}(t,X(t))\,dt + \frac{\partial f}{\partial x}(t,X(t))\left[\alpha(t,w)\,dt + \sigma(t,w)\,dB(t)\right] \\ &+\frac{1}{2}\sigma^2(t,w)\frac{\partial^2 f}{\partial x^2}(t,X(t))\,dt \\ &+\int_{|z|<R}\left\{f(t,X(t^-)+\gamma(t,z,w)) - f(t,X(t^-)) \right. \\ &\left. -\frac{\partial f}{\partial x}(t,X(t^-))\,\gamma(t,z,w)\right\} v(dz)\,dt \\ &+\int_{\mathbb{R}}\left\{f(t,X(t^-)+\gamma(t,z,w)) - f(t,X(t^-))\right\}\widetilde{\mathcal{N}}(dt,dz). \end{aligned}$$

**Example 1.2.2 (The Geometric Lévy Process)** *Consider the stochastic differential equation*

$$dX(t) = X(t-)\left[a\ dt + b\ dB(t) \quad + \int_{\mathbb{R}} c(t,z)\,\widetilde{\mathcal{N}}(dt,dz)\right],$$

*where $a$, $b$ are constants and $c(t,z) \geq -1$. To find the solution $X(t)$ of this equation we rewrite* *it as follows:*

$$\frac{dX(t)}{X(t-)} = a\ dt + b\ dB(t) \quad + \int_{\mathbb{R}} c(t,z)\widetilde{\mathcal{N}}(dt,dz).$$

*Now define*

$$Y(t) = \ln\ X(t).$$

*Then by Itô formula,*

$$\begin{aligned} dY(t) &= \tfrac{X(t)}{X(t)}\left[a\ dt\ + b\ dB(t)\right] - \tfrac{1}{2}b^2 X^{-2}(t)X^2(t)\,dt \\ &+ \int_{|z|<R} \{\ln(X(t^-) + c(t,z)X(t^-)) - \ln(X(t^-)) \\ &- X^{-1}(t^-)c(t,z)X(t^-)\}\,v(dz)\,dt \\ &+ \int_{\mathbb{R}} \{\ln(X(t^-) + c(t,z)X(t^-)) - \ln(X(t^-))\}\,\widetilde{\mathcal{N}}(dt,dz) \\ &= \left(a - \tfrac{1}{2}b^2\right)dt + b\ dB(t) + \int_{|z|<R} \{\ln(1 + c(t,z)) - c(t,z)\}\,v(dz)\,dt \\ &+ \int_{\mathbb{R}} \ln(1 + c(t,z))\,\widetilde{\mathcal{N}}(dt,dz). \end{aligned}$$

*Hence*

$$\begin{aligned} Y(t) &= Y(0) + \left(a - \frac{1}{2}b^2\right)t + b\ dB(t) + \int_0^t \int_{|z|<R} \{\ln(1 + c(s,z)) \\ &- c(s,z)\}\,v(dz)\,ds + \int_0^t \int_{\mathbb{R}} \ln(1 + c(s,z))\,\widetilde{\mathcal{N}}(ds,dz) \end{aligned}$$

*and this gives the solution*

$$\begin{aligned} X(t) \quad & X(0)\exp\left\{\left(a - \frac{1}{2}b^2\right)t + b\ dB(t)\right. \\ & + \int_0^t \int_{|z|<R} \{\ln(1 + c(s,z)) - c(s,z)\}\,v(dz)\,ds \\ & \left. + \int_0^t \int_{\mathbb{R}} \ln(1 + c(s,z))\,\widetilde{\mathcal{N}}(ds,dz)\right\}. \end{aligned} \tag{1.2}$$

*In analogy with the diffusion case* ($\mathcal{N} = 0$) *we call this process* $X(t)$ *a geometric Lévy process.*

## 1.3 Stochastic integral with respect to Lévy process

Let $(\Omega, \mathcal{F}, P)$ be a given probability space with the $\sigma$-algebra $(\mathcal{F}_t)_{t\geq 0}$ generated by the underline driven processes; Brownian motion $B(t)$ and independent compensated Poisson random measure

$\widetilde{\mathcal{N}}$, such that

$$\widetilde{\mathcal{N}}(dt,dz) := \mathcal{N}(dt,dz) - \nu(dz)\,dt. \tag{1.3}$$

For any $t$, let $\widetilde{\mathcal{N}}(dt,dz)$, $z \in \mathbb{R}_0$, $s \leq t$, augmented for all the sets of $P$-zero probability. For any $\mathcal{F}_t$-adapted stochastic process $\theta = \theta(t,z)$, $t \geq 0$, $z \in \mathbb{R}_0$ such that

$$\mathbf{E}\left[\int_0^T \int_{\mathbb{R}_0} \theta^2(t,z)\, v(dz)\, dt\right] < \infty, \text{ for some } T > 0, \tag{1.4}$$

we can see that the process

$$R_n(t) := \int_0^t \int_{|z|\geq \frac{1}{n}} \theta(s,z)\, \widetilde{\mathcal{N}}(ds,dz),\ 0 \leq t \leq T \tag{1.5}$$

is a martingale in $L^2(P)$ and its limit

$$R(t) \lim_{n\to\infty} R_n(t) := \int_0^T \int_{\mathbb{R}_0} \theta(s,z)\, \widetilde{\mathcal{N}}(ds,dz),\ 0 \leq t \leq T \tag{1.6}$$

in $L^2(P)$ is also martingale. Moreover, we have the Itô isometry

$$\mathbf{E}\left[\left(\int_0^T \int_{\mathbb{R}_0} \theta(s,z)\, \widetilde{\mathcal{N}}(ds,dz)\right)^2\right] = \mathbf{E}\left[\left(\int_0^T \int_{\mathbb{R}_0} \theta^2(t,z)\, \nu(dz)\, dt\right)\right]. \tag{1.7}$$

The Itô- Lévy decomposition.is a sum of two independent parts, a continuous part and a part expressible as a compensated sum of independent jumps.

**Theorem 1.3.1 (Itô-Lévy decomposion)** *The Itô-Lévy decomposion for a Lévy process $X$ is given by*

$$X(t) = b_0 t + \sigma_0 B(t) + \int_{|z|<1} z\widetilde{\mathcal{N}}(dt,dz) + \int_{|z|\geq 1} z\mathcal{N}(dt,dz), \tag{1.8}$$

*where $b_0, \sigma_0 \in \mathbb{R}$, $\widetilde{\mathcal{N}}(dt,dz)$ is the compensated Poisson measure of $X(.)$ and $B(t)$ is an independent Brownian motion with the jump measure $\mathcal{N}(dt,dz)$.*

We assume that

$$\mathbf{E}\left[X^2(t)\right] < \infty, \quad t \geq 0, \tag{1.9}$$

then

$$\int_{|z|\geq 1} |z|^2 \nu (dz) < \infty.$$

We can represent (1.8) as

$$X(t) = b_0' t + \sigma_0 B(t) + \int_0^t \int_{\mathbb{R}_0} z \widetilde{\mathcal{N}}(ds, dz), \tag{1.10}$$

where $b_0' t = b_0 + \int_{|z|\geq 1} z\nu (dz)$ . if $\sigma_0 = 0$, then a Lévy process is called a pure jump Lévy process. Let us consider that the process $X(t)$ admits the stochastic integral representation as follows

$$X(t) = x + \int_0^t b(s)\, ds + \int_0^t \sigma(s)\, dB(s) + \int_0^t \int_{\mathbb{R}_0} \theta(s,z) \widetilde{\mathcal{N}}(ds, dz), \tag{1.11}$$

where $b(t)$ , $\sigma(t)$ , and $\theta(t,.)$ are predictable processes such that, for all $t > 0$, $z \in \mathbb{R}_0$,

$$\int_0^t \left[ |b(s)| + \sigma^2(s) + \int_{\mathbb{R}_0} \theta^2(s,z)\, \nu(dz) \right] ds < \infty \quad P - a.s. \tag{1.12}$$

Under this assumption, the stochastic integrals are well-defined and local martingales. If we strengthened the condition

$$\mathbf{E}\left[ \int_0^t \left[ |b(s)| + \sigma^2(s) + \int_{\mathbb{R}_0} \theta^2(s,z)\, \nu(dz) \right] ds \right] < \infty, \tag{1.13}$$

for all $t > 0$, then the corresponding stochastic integrals are martingales.

**Theorem 1.3.2** *We call such a process an Itô-Lévy process. In analogy with the Brownian motion case, we use the short-hand differential notation*

$$\begin{cases} dX(t) &= b(t)\, dt + \sigma(t)\, dB(t) + \int_{\mathbb{R}_0} \theta(t,z) \widetilde{\mathcal{N}}(dt, dz), \\ X(0) &= x \in \mathbb{R}. \end{cases} \tag{1.14}$$

The conditions satisfied by the coefficients to obtain existence and uniqueness of the solution of a SDEs with jumps, are given in the following theorem.

### 1.3.1 Stochastic differential equations driven by Lévy processes

By the Itô-Lévy decomposition, we can introduce the SDE for Lévy process.

For simplicity, we only consider the one dimentional case. The extension to several dimensions is straightforward.

**Theorem 1.3.3 (Existence and uniqueness)** *Consider the following Lévy SDE in* $\mathbb{R}$ :

$$\begin{cases} dX(t) &= b(t, X(t))\,dt + \sigma(t, X(t))\,dB(t) + \int_{\mathbb{R}_0} \theta(t, X(t^-, z))\,\widetilde{N}(dt, dz) \\ X(0) &= x \in \mathbb{R}, \end{cases} \tag{1.15}$$

*where*

$$b : [0, T] \times \mathbb{R} \to \mathbb{R},$$

$$\sigma : [0, T] \times \mathbb{R} \to \mathbb{R},$$

$$\theta : [0, T] \times \mathbb{R} \times \mathbb{R}_0 \to \mathbb{R}.$$

*We assume that the coefficients satisfy the following assumptions*

**1.** *(At most linear growth) There exists a constant* $C_1 < \infty$ *such that*

$$\|\sigma(t, x)\|^2 + |b(t, x)|^2 + \int_{\mathbb{R}_0} |\theta(t, x, z)|^2 \nu(dz) \leq C_1 \left(1 + |x|^2\right), \ x \in \mathbb{R}. \tag{1.16}$$

**2.** *(Lipschitz continuity) There exists a constant* $C_2 < \infty$ *such that*

$$\begin{aligned} \|\sigma(t, x) - \sigma(t, y)\|^2 & + |b(t, x) - b(t, y)|^2 + \int_{\mathbb{R}_0} |\theta(t, x, z) - \theta(t, y, z)|^2 \nu(dz) \\ & \leq C_2 \left(1 + |x - y|^2\right), \end{aligned}$$

*for all* $x, y \in \mathbb{R}$.

*Then there exists a unique càdlàg adapted solution* $X(t)$ *such that* (1.9) *is satisfied.*

# Chapter 2

# Markov Regime Switching Model

In this chapter, we present some notions about Markov chains in continuous and discret time in which we define the transition function and transition rate matrix. Finnaly, we give some models that illustrate regime switching.

## 2.1 Introduction

The Markov regime switching model, first described by G. Lindgren, 1978, is a type of specification in which the main point is handling processes driven by different states, or regimes of the world. The behaviour of the time series is characterized by multiple equations, decided by the different states of the model.

The difference between the Markov regime switching model and othor switching models is that the switching mechanism is controlled by an unobservable variable which follows the hidden Markov chain. By means of Markov properties, the present value depends only on its previous value. This means that a structure in the chain may prevail for a random period of time, before being replaced by another structure when the switch occurs. Through this method, the Markov system switching model is able to capture more complex dynamic patterns.

The idea of the financial market finding itself in different countries and times is attractive. On the other hand, it has been found that financial time series display some facts that can be usefully reproduced by hidden Markov model. This has made the Markov system shift model one of the most popular nonlinear time series models in the literature (Cont, 2001, Hamilton, 1989, 2005, Lindgren, 1978).

## 2.2 Markov chains

**Definition 2.2.1** *Markov chain* $\{X_n\}_{n\geq 0}$ *is a stochastic process that satisfies the following relationship (Markov property)*
*For all natural numbers* $n$ *and all states* $x_n$,

$$\begin{aligned} & P\left(X_{n+1}=x_{n+1} \mid X_n=x_n, X_{n-1}=x_{n-1}, \ldots, X_0=x_0\right) \\ & \quad = P\left(X_{n+1}=x_{n+1} \mid X_n=x_n\right). \end{aligned} \tag{2.1}$$

**Definition 2.2.2** *The conditional probabilities* (2.1)*, now written as* $P\left(X_{n+1}=j \mid X_n=i\right)$ *are called the transition probabilities of the Markov chain. They are denoted by:*

$$\mathbf{P}_{i,j}=P\left(X_{n+1}=j \mid X_n=i\right),$$

*and we define the transition probability matrix* $\mathbf{P}$ *of the Markov chain as :*

$$\mathbf{P}=\left(\mathbf{P}_{i,j}\right)_{i,j\in S}.$$

**Definition 2.2.3** *A Markov chain* $\{X_n\}_{n\geq 0}$ *on a state space* $S$ *is said to be homogeneous if, for all* $n,k\in\mathbb{N}$ *and* $i,j\in S$*, we have :*

$$P\left(X_{n+k}=j \mid X_k=i\right)=P\left(X_n=j \mid X_0=i\right)$$

### 2.2.1 The nth-step transition matrix

In this subsection, we are going to investigate the $n$-step transition probability $\mathbf{P}_{ij}^{(n)}$ of a Markov chain process.

**Definition 2.2.4** *Define* $\mathbf{P}_{ij}^{(n)}$ *to be the probability that a process in state* $i$ *will be in state* $j$ *after* $n$ *additional transitions. In particular, we have* $\mathbf{P}_{ij}^{(0)}=\mathbf{P}_{ij}$.

**Proposition 2.2.1** *We have* $\mathbf{P}^{(n)} = \mathbf{P}^n$ *where* $\mathbf{P}^{(n)}$ *is the* $n$*-step transition probability matrix and* $\mathbf{P}$ *is the one-step transition matrix.*

***Proof.*** *Clearly the proposition is true when* $n = 1$*. We then assume that the proposition is true for* $n$*.*

*We note that*

$$\mathbf{P}^n = \underbrace{\mathbf{P} \cdot \mathbf{P} \cdot .... \cdot \mathbf{P}}_{n\ times}.$$

*Then we have*

$$\mathbf{P}_{ij}^{(n+1)} = \sum_{k \in M}^{(n)} \mathbf{P}_{ik}\ \mathbf{P}_{kj}^{(1)} = \sum_{k \in M}^{(n)} \mathbf{P}_{ik}\ \mathbf{P}_{kj} = \left[\mathbf{P}^{n+1}\right]_{ij}.$$

*By the principle of mathematical induction the proposition is true for all non-negative integer* $n$*.*

■

### 2.2.2 Irreducible Markov chain and classifications of states

**Definition 2.2.5** *State* $i$ *is said to be reachable from state* $j$ *if* $\mathbf{P}_{ij}^{(n)} > 0$ *for some* $n \geq 0$*. This means that starting from state* $j$*, it is possible to enter state* $i$ *in a finite number of transitions.*

**Definition 2.2.6** *State* $i$ *and state* $j$ *are said to communicate if state* $i$ *and state* $j$ *are reachable from each other.*

**Remark 2.2.1** *The definition of communication defines an equivalent relation.*

**(1)** *state* $i$ *communicates with state* $i$ *in* $0$ *step because*

$$\mathbf{P}_{ii}^{(0)} = P\left(X^{(0)} = i \;\middle|\; X^{(0)} = i\right) = 1 > 0$$

**(2)** *If state* $i$ *communicates with state* $j$*.* $\Rightarrow$ *State* $j$ *communicates with state* $i$*.*

**(3)** *If state* $i$ *communicates with state* $j$ *and state* $j$ *communicates with state* $k$*.* $\Rightarrow$ *State* $i$ *communicates with state* $k$*. Since* $\mathbf{P}_{ji}^{(m)}, \mathbf{P}_{kj}^{(n)} > 0$ *for some* $m$ *and* $n$*, we have*

$$\mathbf{P}_{ki}^{(m+n)} = \sum_{h \in M}^{(m)} \mathbf{P}_{hi}^{(n)} \mathbf{P}_{kh} \geq \mathbf{P}_{ji}^{(m)} \mathbf{P}_{kj}^{(n)} > 0.$$

**Definition 2.2.7** *A Markov chain is said to be irreducible, if all states belong to the same class (The states that communicate), i.e. they communicate with each other.*

**Definition 2.2.8** *For any state $i$ in a Markov chain, let $f_i$ be the probability that starting in state $i$, the process will ever re-enter state $i$. State $i$ is said to be recurrent if $f_i = 1$ and transient if $f_i < 1$.*

**Proposition 2.2.2** *In a finite Markove chain, if state $i$ is recurrent (transient) and state $i$ communicates with state $j$ then state $j$ is also recurrent (transient).*

### 2.2.3 Aperiodic Markov chains

**Definition 2.2.9** *The period $d(i)$ of a state $i \in S$ is defined by:*

$$d(i) = \gcd\{n \geq 1; \quad (\mathbf{P}^n)_{ii} > 0\},$$

*using the convention $d(i) = 0$ if $(\mathbf{P}^n)_{ii} = 0$, for all $n \geq 1$. If $d(i) = 1$ then the state $i$ is said to be aperiodic.*

**Remark 2.2.2** $\gcd\{E\}$ *the greatest common divisor of $E$, that is the largest integer that divides all integers of $E$.*

**Theorem 2.2.1** *if $i \Leftrightarrow j$ then $d(i) = d(j)$.*

**Proof.** See Theorem 1.20 [48]. ■

**Definition 2.2.10** *A Markov chain is said to be aperiodic if all its states have the same period equal to* 1.

## 2.3 Continuous-Time Markov chains

**Definition 2.3.1** *A stochastic process $X = \{X_t\}_{t \geq 0}$ with values in a countable state space $S$ is a continuous-time Markov chain if for all $n \geq 0$, for all instants $0 \leq s_0 < ... < s_n < s < t$ and for all states $i_0, ..., i_n, i, j \in S$, we have:*

$$P(X_t = j \mid X_s = i, X_{s_n} = i_n, ..., X_{s_0} = i_0) = P(X_t = j \mid X_s = i).$$

**Definition 2.3.2** *A continuous time Markov chain $X = \{X_t\}_{t\geq 0}$ is homogeneous if $t, s \geq 0$ and $i, j \in S$, we have:*

$$P\left(X_{t+s} = j \mid X_s = i\right) = P\left(X_t = j \mid X_0 = i\right).$$

### 2.3.1 Transition function and Q-Matrix

**Definition 2.3.3** *Let $X = \{X_t\}_{t\geq 0}$ be a continuous-time Markov chain on a countable state space $S$. For all $i, j \in S$ and $t \geq 0$, we set $\mathbf{P}_{i,j}(t) = P\left(X_t = j \mid X_0 = i\right)$ and we define the matrix $\mathbf{P}(t)$ by $\mathbf{P}(t) = \left(\mathbf{P}_{i,j}(t)\right)_{i,j\in S}$. The functions $\mathbf{P}_{i,j}(t)$ are called the transition functions.*

**Lemma 2.3.1** *If $X = \{X_t\}_{t\geq 0}$ is a continuous-time Markov chain then, for all $n \geq 1$, for all instants $0 \leq t_1 < ... < t_n$ and for all states $i_0, i_1, ..., i_n \in S$, we have:*

$$\begin{aligned} &P\left(X_{t_n} = i_n,\ X_{t_{n-1}} = i_{n-1}, ..., X_{t_1} = i_1 \middle|\ X_0 = i_0\right) \\ &= \mathbf{P}_{i_0,i_1}(t_1)\,\mathbf{P}_{i_1,i_2}(t_2 - t_1)\cdots\mathbf{P}_{i_{n-1},i_n}(t_n - t_{n-1}). \end{aligned}$$

**Proof.** The result is true for $n = 1$ from definition of the transition functions $\mathbf{P}_{i,j}(t)$. Let us assume that the result is true at step $n-1$. By conditioning and then using the Markov property as well as the homogeneity of $X$, we have:

$$\begin{aligned} &P\left(X_{t_n} = i_n,\ X_{t_{n-1}} = i_{n-1}, ..., X_{t_1} = i_1 \middle|\ X_0 = i_0\right) \\ &\quad = P\left(X_{t_n} = i_n, \mid X_{t_{n-1}} = i_{n-1}\right) P\left(X_{t_{n-1}} = i_{n-1}, ..., X_{t_1} = i_1 \middle|\ X_0 = i_0\right) \\ &= \mathbf{P}_{i_0,i_1}(t_1)\,\mathbf{P}_{i_1,i_2}(t_2 - t_1)\cdots\mathbf{P}_{i_{n-1},i_n}(t_n - t_{n-1}), \end{aligned}$$

which completes the proof. ■

**Remark 2.3.1** *At time $t = 0$, we have, by definition $\mathbf{P}(0) = I$, where $I$ denotes the intensity matrix whose dimension is defined by the contex.*

**Lemma 2.3.2** *the transition functions $\mathbf{P}_{i,j}(t)$ are right-continuous at 0, that is for all $i, j \in S$, we have:*

$$\lim_{t\to 0} \mathbf{P}_{i,j}(t) = \mathbf{P}_{i,j}(0) = 1_{\{i=j\}}.$$

**Proof.** See Lemma 2.2 [48]. ■

**Lemma 2.3.3** *For all $s,t \geq 0$, we have $\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(s)$, that is for all $i,j \in S$,*

$$\mathbf{P}_{i,j}(t+s) = \sum_{k \in S} \mathbf{P}_{i,k}(t)\,\mathbf{P}_{k,j}(s).$$

**Proof.** See Lemma 2.3 [48]. ■

**Definition 2.3.4** *The Q-matrix ( transition rate matrix or infinitesimal generator) of a continuous-time Markov chain allows us to encode all properties the chain $(X_t)_{t\geq 0}$ in a single matrix. By differentiating the semigroup relation with respect to t we get, by componentwise differentiation :*

$$\begin{aligned}\mathbf{P}'(t) & \lim_{h\to 0} \frac{\mathbf{P}(t+h) - \mathbf{P}(t)}{h} \\ &= \lim_{h\to 0} \frac{\mathbf{P}(t)\mathbf{P}(h) - \mathbf{P}(t)}{h} \\ &= \mathbf{P}(t)\,Q,\end{aligned}$$

*where*

$$Q := \mathbf{P}'(0) = \lim_{h\to 0} \frac{\mathbf{P}(h) - \mathbf{P}(0)}{h}$$

*is called the Q-matrix of $(X_t)_{t\geq 0}$.*

When $S = \{0, 1, ..., N\}$ we will denote by $\lambda_{i,j}$, $i,j \in S$ the entries of the transition rate matrix $Q = (\lambda_{i,j})_{i,j\in S}$, i.e.

$$\begin{aligned} Q &= \left.\frac{d\mathbf{P}(t)}{dt}\right|_{t=0} \\ &= [\lambda_{i,j}]_{0\leq i,j\leq N}\end{aligned}$$

Denoting $Q = [\lambda_{i,j}]_{i,j\in S}$, for all $i \in S$ we have

**(i)** $0 \leq -\lambda_{i,i} \leq \infty$ for all $i$;

**(ii)** $\lambda_{i,j} \geq 0$ for all $i \neq j$;

**(iii)** $\sum_{j\in S} \lambda_{i,j} = 0$ for all $i$.

**Example 2.3.1** *Assume the matrix $Q$ is defined as*

$$Q = \begin{pmatrix} -0.5 & 0.5 & 0 & 0 \\ 0.5 & -1.5 & 1 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 3 & -3 \end{pmatrix}$$

## 2.4 The hidden Markov model

### 2.4.1 Introduction

The term Hidden Markov Model (HMM) has become quite familiar in the speech signal processing community and is gaining acceptance for communication systems. It can be less difficult, but more obscure than the term partially observed dynamic stochastic system model, which is a translation familiar to people in systems.

**Definition 2.4.1** *A hidden Markov model (HMM) is a bivariate discrete time process $\{S_t, Y_t\}_{t \geq 0}$, where $\{S_t\}$ is an underlying Markov chain and $\{Y_t\}$ is a sequence of independent random variable, of which follows that the conditional distribution of $Y_t$ only depends on $S_t$. Since the Markov chain $S_t$ is hidden, only the stochastic process $\{Y_t\}$.*

A HMM has an interesting dependence structure, which comes handy when dealing with e.g. financial time series. For an intuitive hint on how this dependency works, it is represented like this: :model

$$\begin{array}{ccccccc} .... & \rightarrow & S_t & \rightarrow & S_{t+1} & \rightarrow & .... \\ & & \downarrow & & \downarrow & & \\ & & Y_t & & Y_{t+1} & & \end{array}$$

As form implies, the distribution of a variable $S_{t+1}$ conditional on the history of the process $S_0, .., S_t$, is determined only by the value of the preceding variable, $S_t$. Future events are completely independent of the past, depending only on the present state. In addition, the distribution of $Y_t$ is conditionally determined on the previous observations $Y_0, ..., Y_{t-1}$ and the previous value of the case, $S_0, .., S_t$, by $S_t$ only (Rydén et al, 2005). state

$$P\left(S_{t+1} \middle| S_t, ..., S_1\right) = P\left(S_{t+1} \middle| S_t\right) \tag{2.2}$$

$$P\left(Y_t \middle| S_{t-1}, ..., S_1, Y_{t-1}, ..., Y_1\right) = P\left(Y_t \middle| S_t\right) \tag{2.3}$$

### 2.4.2 Assumptions of the hidden Markov model

Some assumptions about the HMM used here must be made in order to benefit from the model. First, the hidden Markov chain is supposed to be time independent. This means that the chain transmission probabilities;

$$\mathbf{P}_{i,j} = P\left(S_{t+1} = j \middle| S_t = i\right) = P\left(S_{t+1} = j \middle| S_t = i, S_{t-1} = k, ..., S_1 = l\right) \tag{2.4}$$

between two states $i$ and $j$ in a finite state space $\Omega = \{1, ..., N\}$ needs to be constant over time. This is convenient, since said transition probabilities and the Markov chain's initial probabilities;

$$\pi_i = P\left(S_1 = i\right), \qquad 1 \leq i \leq N,$$

are all that is needed to define the dynamic of the HMM.
Secondly, the Markov chain is assumed to be ergodic (aperiodic and positive recurrent) This is necessary in order to ensure consistency of the estimates of the model (Campigotto, 2009).

## 2.5 Method : specification of chosen Markov regime switching

The model applied is based on a mixture of normal distributions, mainly based on Campigotto, 2009, Hamilton, 2005 and Perlin, 2015.
The model is assumed with a process of the following :

$$Y_t = \mu_{S_t} + \varepsilon_t. \tag{2.5}$$

Where;

$Y_t$ is the observed return of the time series at time $t$

$\mu_{S_t}$ is the intercept, or expected return, while in state $S_t$.

$\varepsilon_t$ is a normal random stochastic variable, $\varepsilon_t \sim N\left(0, \sigma^2_{S_t}\right)$

This is a simple case of a model with a switching dynamic. The model in equation (2.5) is switching states with respect to an indicator value $S_t$, meaning that with $N$ states there will be $N$ values for $\mu_{S_t}$ and $\sigma^2_{S_t}$. Here, the residuals $\varepsilon_t$ are assumed to be normal distributed.

### 2.5.1 Markov regime switching model with N regims

Now, assume that the number of states (or regimes) in $N$, i.e. $S_t \in \Omega = \{1, ..., N\}$ . This implies that e.g. the log returns of a financial time series are drawn from $N$ distinct normal distributions, depending on what state the HMM is currently in. This would give us the following model to work with:

$$Y_t = \mu_1 + \varepsilon_t \quad \text{for state 1} \tag{2.6}$$

$$Y_t = \mu_2 + \varepsilon_t \quad \text{for state 2} \tag{2.7}$$

$$\vdots$$

$$Y_t = \mu_N + \varepsilon_t \quad \text{for state } N. \tag{2.8}$$

Where;

$$\varepsilon_t \sim N\left(0, \sigma_1^2\right) \quad \text{for state 1} \tag{2.9}$$

$$\varepsilon_t \sim N\left(0, \sigma_2^2\right) \quad \text{for state 2} \tag{2.10}$$

$$\vdots$$

$$\varepsilon_t \sim N\left(0, \sigma_N^2\right) \quad \text{for state } N. \tag{2.11}$$

This means that the HMM state for time $t$ is 1, the expectation of the dependent variable is $\mu_1$

and the variance of innovations is $\sigma_1^2$, etc.

Since the underlying Markov chain is hidden, one cannot directly observe which state the HMM is in, but only infer that it is operating from the observed behavior of $Y_t$. In order to arrive at the probability law that governs the observed data $Y_t$ a probabilistic model of what causes the change from state $S_t = i$ to state $S_t = j$. This can be determined using the transition probabilities of the N state HMM (Hamilton, 2005);

$$\mathbf{P}_{i,j} = P\left(S_{t+1} = j \middle| S_t = i\right) \;\; i, j \in \Omega = \{1, 2, .., N\}\,. \tag{2.12}$$

The transition probability (2.12) is by the Markov property described in (2.4) dependent of the past only through the value of the most recent state. This in one of the central points of the structure of a stochastic process itself.

# Chapter 3

# Stochastic Maximum Principle with Partial Information

In this chapter, We recall a result in stochastic optimal control of a Jump-Diffusion with finite horizon, sufficient and necessary maximum principles are presented under partial information.

## 3.1 Finite horizon

### 3.1.1 Formulation of the problem

Let $B(t) = (B_1(t), .., B_k(t))^\top$ (where $()^\top$ denotes transposed) and $\eta(t) = (\eta_1(t), ..., \eta_n(t))^\top$ be $n$-dimensional Brownian motion and $n$ independent pure jump Lévy martingales, respectively, on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$.

If $\mathcal{N}_i(dt, dz)$ denote the jump measure of $\eta_i(.)$ and $v_i(dz)$ denotes the Lévy measure of $n_i(.)$, then we can write

$$n_i(t) = \int_0^t \int_{\mathbb{R}_0} z\widetilde{\mathcal{N}}_i(ds, dz)$$

where

$$\widetilde{\mathcal{N}}_i(ds, dz) = \mathcal{N}_i(ds, dz) - v_i(dz)\, ds$$

is the compensated jump measure of $n_i(.)$, $1 \leq i \leq n$ ; $\mathbb{R}_0 = \mathbb{R} - \{0\}$.

For simplicity we assume that

$$\int_{\mathbb{R}_0} z^2 v_i\left(dz\right) < \infty \qquad \text{for} \quad i = 1, ..., n.$$

**Definition 3.1.1** *An admissible control is a measurable, adapted processes* $u : [0, T] \times \Omega \to \mathcal{U}$, *such that* $\mathbf{E}\left[\int_0^T u\left(s\right) ds\right] < \infty$.

Suppose the state process $X\left(t\right) = X^{(u)}\left(t\right) \in \mathbb{R}^n$ is given by a controlled stochastic differential equation of the form

$$\begin{cases} dX\left(t\right) & = b\left(t, X\left(t\right), u\left(t\right)\right) dt + \sigma\left(t, X\left(t\right), u\left(t\right)\right) dB\left(t\right) \\ & \quad + \int_{\mathbb{R}_0^n} \theta\left(t, X\left(t\right), u\left(t\right), z\right) \widetilde{N}\left(dt, dz\right); \quad 0 \leq t \leq T \\ X\left(0\right) & = x \in \mathbb{R}^n. \end{cases}$$

Here $b : [0, T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^{n \times n}$ and $\theta : [0, T] \times \mathbb{R}^n \times \mathcal{U} \times \mathbb{R}_0 \to \mathbb{R}^{n \times n}$ are given functions, $C^1$ with respect to $x$ and $u$, and $T > 0$ is a given constant. The process $u\left(t\right)$ is our control process, required to have values in a given set $\mathcal{U} \subset \mathbb{R}^k$ and required to be adapted to a given filtration $\left\{\varepsilon_t\right\}_{t \geq 0}$, where

$$\varepsilon_t \subseteq \mathcal{F}_t, \qquad \text{for all } t$$

For example, $\varepsilon_t$ could be the $\delta$-delayed information defined by

$$\varepsilon_t = \mathcal{F}_{(t-\delta)^+} \quad ; \quad t \geq 0$$

where $\delta > 0$ is a given constant delay.

We let $\mathcal{A} = \mathcal{A}_{\varepsilon}$ denotes a given family of $\varepsilon_t$-adapted control process

$$u\left(t\right) = u\left(t, w\right) : [0, T] \times \Omega \to U.$$

Suppose we are given a performance functionnal (cost fuctionnal)

$$J(u) = \mathbf{E}\left[\int_0^T f(t, X(t), u(t))\, dt + g(X(T))\right], \quad u \in \mathcal{A}$$

where $f : [0,T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given $C^1$ functions satisfying the condition

$$\mathbf{E}\left[\int_0^T |f(t, X(t), u(t))|\, dt + |g(X(T))|\right] < \infty \quad ; \quad u \in \mathcal{A}. \tag{3.1}$$

The partial information control problem is to find $\Phi_\varepsilon$ and $u^* \in \mathcal{A}$ such that

$$\Phi_\varepsilon = \sup_{u \in \mathcal{A}} J(u) = J(u^*),$$

where $u^*$ is an optimal control which maximized the cost fuctionnal.

### 3.1.2 A partial information sufficient maximum principle

In this subsection we state and prove a sufficient maximum principle for the partial information control problem $(3.1)$.

Let $R$denote the set of functions $r : [0,T] \times \mathbb{R}_0 \to \mathbb{R}^{n\times n}$ such that

$$\int_{\mathbb{R}_0} |\theta_{i,j}(t, x, u, z)\, r_{ij}(t, z)|\, v_j(dz) < \infty \text{ for all } i, j, t, x$$

We define the Hamiltonian $H : [0,T] \times \mathbb{R}^n \times \mathcal{U} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \times R \times \Omega \to \mathbb{R}$; by

$$H(t, x, u, p, q, r(.)) = f(t, x, u) + b^\top(t, x, u)\, p + tr\left(\sigma^\top(t, x, u)\, q\right) + \sum_{i,j=1}^{n} \int_{\mathbb{R}_0} \theta_{i,j}(t, x, u, z)\, r_{ij}(t, z)\, v_j(dz) \tag{3.2}$$

The adjoint equation in the unknown $\mathcal{F}$-predictable processes $p(t)$, $q(t)$, $r(t,z)$ is the following backward stochastic differential equation (BSDE) :

$$dp(t) = -\nabla_x H(t, x(t), u(t), p(t), q(t), r(t,.))\, dt + q(t)\, B(t) \qquad (3.3)$$
$$+ \int_{\mathbb{R}_0} r(t,z)\, \widetilde{\mathcal{N}}(dt, dz);\ 0 \leq t \leq T,$$
$$p(T) = \nabla g(X(T)). \qquad (3.4)$$

where $\bigtriangledown_y \varphi(.) = \left(\frac{\partial \varphi}{\partial y_1}, ..., \frac{\partial \varphi}{\partial y_n}\right)^T$ is the gradient of $\varphi : \mathbb{R}^n \to \mathbb{R}$ with respect to $y = (y_1, ..., y_n)$.

**Theorem 3.1.1 (Partial information sufficient maximum principle)** *Let $u^* \in \mathcal{A}_\varepsilon$ with corresponding state process $X^*(t) = X^{(u^*)}(t)$ and suppose there exists a solution $(p^*(t), q^*(t), r^*(t,.))$ of the corresponding adjoint equation $(3.3) - (3.4)$ satisfying*

$$\mathbf{E}\left[\int_0^T \left(X^*(t) - X^{(u)}(t)\right)^\top [q^*(q^*)^\top + \int_{\mathbb{R}_0^n} r^*(r^*)^\top (t,z)\, v(dz)] \left(X^*(t) - X^{(u)}(t)\right) dt\right] < \infty, \qquad (3.5)$$

$$\mathbf{E}\left[\int_0^T p^*(t)^\top [\sigma\sigma^\top(t, X(t), u(t)) + \int_{\mathbb{R}_0^n} \theta\theta^\top(t, X(t), u(t))\, v(dz)] p(t)\, dt\right] < \infty, \qquad (3.6)$$

*for all $u \in \mathcal{A}$*

*and*

$$\mathbf{E}\left[\int_0^T |\nabla_u H(t, X^*(t), u^*(t), p^*(t), q^*(t), r^*(t,.))|^2\, dt\right] < \infty, \qquad (3.7)$$

*assume that $H(t, x, u, p^*(t), q^*(t), r^*(t,.))$ and $g$ are concave with respect to $x$, $u$ respectively. (the partial information maximization condition)*

$$\mathbf{E}[H(t, X^*(t), u^*(t), p^*(t), q^*(t), r^*(t,.))|\, \varepsilon_t]$$
$$= \max_{v \in \mathcal{U}} \mathbf{E}[H(t, X^*(t), v, p^*(t), q^*(t), r^*(t,.))|\, \varepsilon_t]. \qquad (3.8)$$

*Then $u^*$ is a partial information optimal control.*

**Proof.** Choose $u \in \mathcal{A}$. and consider

$J(u) - J(u^*) = I_1 + I_2$, where

$$I_1 = \mathbf{E}\left[\int_0^T \{f(t, X(t), u(t)) - f(t, X^*(t), u^*(t))\}\, dt\right], \qquad (3.9)$$

and

$$I_2 = \mathbf{E}\left[g\left(X\left(T\right)\right) - g\left(X^*\left(T\right)\right)\right],$$

by definition of $H$ $(3.2)$,

$$I_1 = I_{1,1} - I_{1,2} - I_{1,3} - I_{1,4},$$

with

$$I_{1,1} = \mathbf{E}\left[\int_0^T \left\{H\left(t, X\left(t\right), u\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right)\right) - H\left(t, X^*\left(t\right), u^*\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right)\right)\right\} dt\right] \tag{3.10}$$

$$I_{1,2} = \mathbf{E}\left[\int_0^T \left\{b\left(t, X\left(t\right), u\left(t\right)\right) - b\left(t, X^*\left(t\right), u^*\left(t\right)\right)\right\}^\top p^*\left(t\right) dt\right] \tag{3.11}$$

$$I_{1,3} = \mathbf{E}\left[\int_0^T tr\left[\left(\sigma\left(t, X\left(t\right), u\left(t\right)\right) - \sigma^*\left(t, X^*\left(t\right), u^*\left(t\right)\right)\right)^\top q^*\left(t\right)\right] dt\right], \tag{3.12}$$

$$\begin{aligned} I_{1,4} = \mathbf{E}\Bigg[\sum_{i,j=1}^{n} \int_0^T \int_{\mathbb{R}_0} &\left(\theta_{i,j}\left(t, X\left(t\right), u\left(t\right), z\right)\right. \\ &\left.-\theta_{i,j}\left(t, X^*\left(t\right), u^*\left(t\right), z\right)\right) r^*_{i,j}\left(t,z\right) v_j\left(dz\right) dt\Bigg]. \end{aligned}$$

$H$ is concave, we have

$$\begin{aligned} &H\left(t, X\left(t\right), u\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right)\right) - H\left(t, X^*\left(t\right), u^*\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right)\right) \\ &\leq \nabla_x H\left(t, X^*\left(t\right), u^*\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right)\right)^\top \left(X\left(t\right) - X^*\left(t\right)\right) \\ &+ \nabla_u H\left(t, X^*\left(t\right), u^*\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right)\right)^\top \left(u\left(t\right) - u^*\left(t\right)\right). \end{aligned} \tag{3.13}$$

Since $u \to \mathbf{E}\left[H\left(t, X^*\left(t\right), u, p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right)\right) \middle| \varepsilon_t\right]$; $u \in U$ is maximal for $u = u^*\left(t\right)$ and $u\left(t\right), u^*\left(t\right)$ are $\varepsilon_t$-measurable, we get by $(3.7)$

$$\begin{aligned} 0 &\geq \nabla_u \mathbf{E}\left[\left(t, X^*\left(t\right), u, p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right)\right) \middle| \varepsilon_t\right]^\top_{u=u^*(t)} \left(u\left(t\right) - u^*\left(t\right)\right) \\ &= \mathbf{E}\left[\nabla_u \left(t, X^*\left(t\right), u^*\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right)\right)^\top \left(u\left(t\right) - u^*\left(t\right)\right) \middle| \varepsilon_t\right]. \end{aligned} \tag{3.14}$$

Combining (**??**) , (3.5) , (3.10) , (3.13) and (3.14) , we obtain

$$\begin{aligned} I_{1,1} &\leq \mathbf{E}\left[\int_0^T \nabla_x \left(t, X^*\left(t\right), u^*\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right)\right)^\top \left(X\left(t\right) - X^*\left(t\right)\right) dt\right] \\ &= -\mathbf{E}\left[\int_0^T \left(X\left(t\right) - X^*\left(t\right)\right)^\top dp^*\left(t\right)\right] = -J_1. \end{aligned}$$

Using (3.4) and $g$ is concave together and by the Itô formula,

$$\begin{aligned} I_2 &= \mathbf{E}\left[g\left(X\left(T\right)\right) - g\left(X^*\left(T\right)\right)\right] \leq \mathbf{E}\left[\nabla g\left(X^*\left(T\right)\right)\left(X\left(t\right) - X^*\left(t\right)\right)\right] \\ &= \mathbf{E}\left[\left(X\left(t\right) - X^*\left(t\right)\right)^\top p^*\left(T\right)\right] \\ &= \mathbf{E}\left[\int_0^T \left(X\left(t\right) - X^*\left(t\right)\right)\left(-\nabla_x H\left(t, X^*\left(t\right), u^*\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right)\right)\right) dt\right. \\ &+ \int_0^T p^*\left(t\right)^\top \left\{b\left(t, X\left(t\right), u\left(t\right)\right) - b\left(t, X^*\left(t\right), u^*\left(t\right)\right)\right\} dt \\ &+ \int_0^T tr\left[\left\{\sigma\left(t, X\left(t\right), u\left(t\right)\right) - \sigma\left(t, X^*\left(t\right), u^*\left(t\right)\right)\right\}^\top q^*\left(t\right)\right] dt \\ &\left. + \int_0^T \sum_{i,j=1}^n \int_{\mathbb{R}_0} \left\{\theta_{i,j}\left(t, X\left(t\right), u\left(t\right), z_j\right) - \theta_{i,j}\left(t, X^*\left(t\right), u^*\left(t\right), z_j\right)\right\} r^*\left(t, z_j\right) v\left(dz_j\right) dt\right] \\ &= J_1 + I_{1,2} + I_{1,3} + I_{1,4}. \end{aligned}$$

So we have,

$$\begin{aligned} J\left(u\right) - J\left(u^*\right) &= I_1 + I_2 = I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4} + I_2 \\ &\leq -J_1 - I_{1,2} - I_{1,3} - I_{1,4} + J_1 + I_{1,2} + I_{1,3} + I_{1,4} \\ &= 0. \end{aligned}$$

Then $u^*$ is a partial information optimal control. ■

### 3.1.3 A partial information necessary maximum principle

We assume the folowing:

**(A1)** For all $t, h$ such that $0 \leq t \leq t + h \leq T$, all $i = 1, .., k$ and all bounded $\varepsilon_t$-measurable $\alpha = \alpha\left(w\right)$, the control $\beta\left(s\right) := \left(0, ..., \beta_i\left(s\right), 0, ..., 0\right) \in U \subset \mathbb{R}^k$ defined by

$$\beta_i\left(s\right) = \alpha_i \chi_{[t,t+h]}\left(s\right); \qquad s \in [0, T] \tag{3.15}$$

belonge to $\mathcal{A}_\varepsilon$.

**(A2)** For all $u, \beta \in \mathcal{A}_\varepsilon$ with $\beta$ bounded, there exists $\delta > 0$ such that $u + y\beta \in \mathcal{A}_\varepsilon$ for all $y \in (-\delta, \delta)$.

We define the derivative process $\xi(t) = \xi^{(u,\beta)}(t)$ by

$$\xi(t) = \frac{d}{dy} X^{(u+y\beta)}(t)\Big|_{s=0} = (\xi_1(t), ..., \xi_n(t))^\top. \tag{3.16}$$

Note that

$$\xi(0) = 0.$$

$$d\xi_i(t) = \lambda_i(t)\,dt + \sum_{j=1}^{n} \varphi_{ij}(t)\,dB_j(t) + \sum_{j=1}^{n} \int_{\mathbb{R}_0^n} \zeta_{ij}(t,z)\,\widetilde{N}_j(dz,dt),$$

where

$$\begin{aligned}
\lambda_i(t) &= \nabla_x b_i(t, X(t), u(t))^\top \xi(t) + \nabla_u b_i(t, X(t), u(t))^\top \beta(t), \\
\varphi_{ij}(t) &= \nabla_x \sigma_{i,j}(t, X(t), u(t))^\top \xi(t) + \nabla_u \sigma_{ij}(t, X(t), u(t))^\top \beta(t), \\
\zeta_{ij}(t,z) &= \nabla_x \theta_{i,j}(t, X(t), u(t), z)^\top \xi(t) + \nabla_u \theta_{ij}(t, X(t), u(t), z)^\top \beta(t),
\end{aligned}$$

**Theorem 3.1.2 (Partail Information Necessary Maximum Principle)** *Suppose $u^* \in \mathcal{A}_\varepsilon$ is a local maximum for $J(u)$, meaning that for all bounded $\beta \in \mathcal{A}_\varepsilon$ there exists a $\delta > 0$ such that $u^* + y\beta \in \mathcal{A}_\varepsilon$ for all $y \in (-\delta, \delta)$ and*

$$h(y) := J(u^* + y\beta), \ \ y \in (-\delta, \delta) \tag{3.17}$$

*is maximal at $y = 0$. Suppose there exists a solution $(p^*(t), q^*(t), r^*(t,.))$ to the adjoint equation*

$$\begin{aligned}
dp^*(t) \quad & = -\nabla_x H(t, X^*(t), u^*(t), p^*(t), q^*(t), r^*(t,.))\,dt + q^*(t)\,dB(t) \\
& \quad + \int_{\mathbb{R}_0^n} r^*(t,z)\,\widetilde{N}(dt,dz); \quad 0 \le t \le T \\
p^*(T) \quad & \nabla g(X^*(T)), \quad \textit{where} \quad X^* = X^{(u^*)}.
\end{aligned}$$

*Moreover assume that if $\xi^*(t) = \xi^{(u^*,\beta)}(t)$, with corresponding coefficients $\lambda_i^*(t)$, $\varphi_{ij}^*(t)$, $\zeta_{ij}^*(t,z)$,* *we have*

$$\mathbf{E}\left[\xi^*(t)^\top\left[q^*q^{*\top}(t)+\int_{\mathbb{R}_0^n} r^*r^{*\top}(t,z)\,v(dz)\right]\xi^*(t)\,dt\right]<\infty, \tag{3.18}$$

*and*

$$\mathbf{E}\left[\int_0^T p^*(t)^\top\left[\varphi\varphi^\top(t,X^*(t),u^*(t))+\int_{\mathbb{R}_0^n}\theta\theta^\top(t,X^*(t),u^*(t),z)\,v(dz)\right]p^*(t)\,dt\right]<\infty. \tag{3.19}$$

*Then $u^*$ is a stationary point for $\mathbf{E}[H|\,\varepsilon_t]$ in the sense that for all $t\in[0,T]$,*

$$\mathbf{E}\left[\nabla_u H(t,X^*(t),u^*(t),p^*(t),q^*(t),r^*(t,.))|\,\varepsilon_t\right]=0.$$

**Proof.** Put $X^*(t)=X^{(u^*)}(t)$. Then with $h$ as in (3.17) we have

$$\begin{aligned}
0 &= h'(0)\\
&= \mathbf{E}\left[\int_0^T\left\{\nabla_x f(t,X^*(t),u^*(t))^\top\frac{d}{dy}X^{u^*+y\beta}(t)|_{y=0}+\nabla_u f(t,X^*(t),u^*(t))^\top\beta(t)\right\}dt\right.\\
&+\mathbf{E}\left[\nabla g(X^*(T))^\top\frac{d}{dy}X^{u^*+y\beta}(T)|_{y=0}\right]\\
&= \mathbf{E}\left[\int_0^T\nabla_x f(t,X^*(t),u^*(t))^\top\xi^*(t)\,dt\right.\\
&\left.\int_0^T\nabla_u f(t,X^*(t),u^*(t))^\top\beta(t)\,dt+\mathbf{E}\left[\nabla g(X^*(T))^{\top\top}\xi^*(T)\right]\right].
\end{aligned} \tag{3.20}$$

By (3.18), (3.19), and Itô's formula we get

$$\begin{aligned}
&\mathbf{E}\left[\nabla g(X^*(T))^\top\xi^*(T)\right]=\mathbf{E}\left[p^*(T)^\top\xi^*(T)\right]\\
&=\mathbf{E}\Big[\sum_{i=1}^n\int_0^T\Big\{p_i^*(t)\Big(\nabla_x b_i(t,X^*(t),u^*(t))^\top\xi^*(t)+\nabla_u b_i(t,X^*(t),u^*(t))^\top\beta(t)\\
&+\xi_i^*(t)\left(-\nabla_x H(t,X^*(t),u^*(t),p^*(t),q^*(t),r^*(t,.))\right)_i\\
&+\sum_{j=1}^n q_{ij}^*(t)\Big(\nabla_x\sigma_{ij}(t,X^*(t),u^*(t))^\top\xi^*(t)+\nabla_u\sigma_{ij}(t,X^*(t),u^*(t))^\top\beta(t)\Big)\\
&+\sum_{j=1}^n\int_{\mathbb{R}}r_{ij}^*(t,z)\Big(\nabla_x\theta_{ij}(t,X^*(t),u^*(t),z)^\top\xi^*(t)+\nabla_u\theta_{ij}(t,X^*(t),u^*(t),z)^\top\beta(t)\Big)\}dt].
\end{aligned}$$

Now

$$\begin{aligned}\nabla_u H(t,x,u,p,q,r) \quad & \nabla_u f(t,x,u) + \sum_{j=1}^{n} \nabla_u b_j(t,x,u)\, p_j + \sum_{k,j=1}^{n} \nabla_u \sigma_{kj}(t,x,u)\, q_{kj} \\ & + \sum_{k,j=1}^{n} \int_{\mathbb{R}_0} \nabla_u \theta_{kj}(t,x,u,z)\, r_{kj}(t,z)\, v_j(dz),\end{aligned}$$

and

$$\begin{aligned}\nabla_x H(t,x,u,p,q,r) \quad & \nabla_x f(t,x,u) + \sum_{j=1}^{n} \nabla_x b_j(t,x,u)\, p_j + \sum_{k,j=1}^{n} \nabla_x \sigma_{kj}(t,x,u)\, q_{kj} \\ & + \sum_{k,j=1}^{n} \int_{\mathbb{R}_0} \nabla_x \theta_{kj}(t,x,u,z)\, r_{kj}(t,z)\, v_j(dz).\end{aligned}$$

Combined with (3.19) and (3.20) this gives

$$\begin{aligned}0 \quad & = \mathbf{E}\Bigg[\int_0^T \sum_{i=1}^{n} \Big\{ \tfrac{\partial f}{\partial u_i}(t, X^*(t), u^*(t)) \\ & + \sum_{i=1}^{n} \Big( p_j^*(t) \tfrac{\partial b_j}{\partial u_i}(t, X^*(t), u^*(t)) + \sum_{k=1}^{n} \Big[ q_{kj}^*(t) \tfrac{\partial \sigma_{kj}}{\partial u_i}(t, X^*(t), u^*(t)) \Big) \\ & + \int_{\mathbb{R}_0} r\,{}_{kj}^{*}(t,z) \tfrac{\partial \theta_{kj}}{\partial u_i}(t, X^*(t), u^*(t), z)\, v_j(dz) \Big] \Big\} \beta_i(t)\, dt \Bigg] \\ & = \mathbf{E}\left[\int_0^T \nabla_u H(t, X^*(t), u^*(t), p^*(t), q^*(t), r^*(t,.))^\top \beta(t)\, dt\right].\end{aligned}$$

Fix $t \in [0,T]$ apply the above to $\beta = (0, ..., ..., \beta_i, ..., 0)$ where

$$\beta_i(s) = \alpha_i \chi_{[t,t+h]}(s), \quad s \in [0,T]$$

where $t+h \leq T$ and $\alpha_i = \alpha_i(w)$ is bounded, $\varepsilon_t$-measurable

$$\mathbf{E}\left[\int_t^{t+h} \frac{\partial}{\partial u_i} H(s, X^*(s), u^*(s), p^*(s), q^*(s), r^*(s,.))\, \alpha_i ds\right] = 0$$

Differentiating with respect to $h$ at $h = 0$ gives

$$\mathbf{E}\left[\frac{\partial}{\partial u_i} H(t, X^*(t), u^*(t), p^*(t), q^*(t), r^*(t,.))\, \alpha_i\right] = 0.$$

Since this holds for all $\varepsilon$ measurable $\alpha$, using (3.7), we have that

$$\mathbf{E}\left[\frac{\partial}{\partial u_i} H\left(t, X^*\left(t\right), u^*\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t, .\right)\right) \mid \varepsilon_t\right] = 0.$$

which proves the theorem. ■

## Chapter 4

# Stochastic Maximum Principle for a Markov Regime Switching Jump-Diffusion in Infinite Horizon

In this chapter we have been studied an optimal control problem with regime switching and infinite horizon. In section 1 , we present the optimal control problem for our Markov regime switching jump-diffusion model and the main assumptions . In Sect. 2 , we prove the existence–uniqueness theorem for BSDE with jumps and regimes. In Sects. 3 and 4 sufficient and necessary maximum principles are developed under partial information. An optimal portfolio and consumption in a switching diffusion market is studied in Sect 5 .

### 4.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}\}_{t\geq 0}, P)$ be complete filtered probability space. The filtration $\{\mathcal{F}\}_{t\geq 0}$ is right-continuous, $P$-completed and all of the processes defined below including the Markov chain, the Brownian motions and the Poisson random measures are adapted to it. We consider a continuous-time, finie-state Markov chain $\{\alpha(t)/t \geq 0\}$ with a finite state space $\mathcal{S} = \{e_1, ..., e_D\}$ ,where $D \in \mathbb{N}$, $e_i \in \mathbb{R}^D$, and jthe component of $e_i$ is the Kronecker delta $\delta_{ij}$ for each $i, j = 1, 2, ..., D$. the state space $\mathcal{S}$ is called a canonical state space and its use faciliates the mathematics. We suppose that the chain is homogeneous and irreducible. To specify statistical or probabilistic properties of the chain $\alpha$. we define the generator $\Lambda = \{\lambda_{ij}\ 1 \leq i \leq j \leq D\}$ of the chain under $P$. this is also called the rate matrix, or the $Q$-matrix. Here, for each $i, j = 1, 2, .., D$, $\lambda_{ij}$ is the constant transition intensity of the chain from state $e_i$ to state $e_j$ at time $t$. Note that $\lambda_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{D} \lambda_{ij} = 0$, so $\lambda_{ii} \leq 0$. In what follows for each $i, j = 1, 2, .., D$ which $i \neq j$, we suppose that $\lambda_{ij} > 0$, so $\lambda_{ii} < 0$.

Elliott et al. [22] obtained the following semimartingale dynamics for the chain $\alpha$ :

$$\alpha(t) = \alpha(0) + \int_0^t \Lambda^\top \alpha(u)\, du + \mathcal{M}(t)$$

where $\{\mathcal{M}(t) \setminus t \geq 0\}$ is an $\mathbb{R}^D$-valued, $\left(\{\mathcal{F}\}_{t\geq 0}, P\right)$-martingale and $y^\top$ denotes the transpose of a matrixe (or, in particular, a victor).

To model the controlled state process, we first need to introduce a set of Markov jump martingales associated with the chain $\alpha$. Here we follow the results of Elliott et al. [22].

For each $i, j = 1, 2, .., D$, wich $i \neq j$, and $t \in [0, \infty[$ let $J^{ij}(t)$ be the number of jumps from state $e_i$ to state $e_j$ up to time $t$. Then

$$\begin{aligned}
J^{ij}(t) &= \sum_{0 \leq s \leq t} \langle \alpha(s-), e_i \rangle \langle \alpha(s), e_j \rangle \\
&= \sum_{0 \leq s \leq t} \langle \alpha(s-), e_i \rangle \langle \alpha(s) - \alpha(s-), e_j \rangle \\
&= \int_0^t \langle \alpha(s-), e_i \rangle \langle d\alpha(s), e_j \rangle \\
&= \int_0^t \langle \alpha(s-), e_i \rangle \langle \Lambda^\top \alpha(s), e_j \rangle ds + \int_0^t \langle \alpha(s-), e_i \rangle \langle d\mathcal{M}(s), e_j \rangle ds \\
&= \lambda_{ij} \int_0^t \langle \alpha(s-), e_i \rangle ds + m_{ij}(t),
\end{aligned}$$

where $m_{ij} = \{m_{ij}(t) \setminus t \in \tau\}$ with $m_{ij}(t) = \int_0^t \langle \alpha(s-), e_i \rangle \langle d\mathcal{M}(s), e_j \rangle$ is an $\left(\{\mathcal{F}\}_{t\geq 0}, P\right)$-martingale, the $m_{ij}$'s are called the basie martingales associated with the chain $\alpha$.

Now, for each fined $j = 1, 2, .., D$, let $\Phi_j(t)$ be the number of jumps into state $e_j$ up to time $t$.

Then

$$\begin{aligned}\Phi_j(t) &= \sum_{i=1,\ i\neq j}^{D} J^{ij}(t) \\ &= \sum_{i=1,\ i\neq j}\lambda_{ij}\int_0^t \langle\alpha(s), e_i\rangle ds + \widetilde{\Phi}_j(t),\end{aligned}$$

where $\widetilde{\Phi}_j(t) = \sum_{i=1,\ i\neq j}^{D} m_{ij}(t)$ and, for each $j = 1, 2, .., D$, $\widetilde{\Phi}_j(t) = \left\{\widetilde{\Phi}_j(t) \backslash t \in \tau\right\}$ is a an $\left(\{\mathcal{F}_t\}_{t\geq 0}, P\right)$-martingale.

Write for each $j = 1, 2, ..., D$

$$\lambda_j(t) = \sum_{i=1,\ i\neq j}^{D}\lambda_{ij}\int_0^t \langle\alpha(s), e_i\rangle ds. \tag{4.1}$$

Then for each $j = 1, 2, .., D$,

$$\widetilde{\Phi}_j(t) = \Phi_j(t) - \lambda_j(t), \tag{4.2}$$

is an $\left(\{\mathcal{F}\}_{t\geq 0}, P\right)$-martingale.

We now introduce a Markov regime-switching Poisson random measures. Let $\mathbb{R}^+ = [0, +\infty[$ be the time index set and $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ be a measurable space. Where $\mathcal{B}(\mathbb{R}^+)$ is the Borel $\sigma$-field generated by the open subsets of $\mathbb{R}^+$.

Let $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$ and $\mathcal{B}_0$ the Borel $\sigma$-field generated by open subset $O$ of $\mathbb{R}_0$ whose closure $\overline{O}$ does not contain the point 0. In what follows, suppose that $\mathcal{N}^i(dz, dt)$, $\int = 1, ..., M$, are independent Poisson random measure on $(\mathbb{R}^+ \times \mathbb{R}_0, \mathcal{B}(\mathbb{R}^+) \times \mathcal{B}_0)$ where $M \in \mathbb{N}$. Assume that the Poisson random measures $\mathcal{N}^i(dz, dt)$ has the following compensator :

$$\eta^i_\alpha(dt, dz) = \nu^i_{\alpha(t-)}(dz)\,dt = \langle\alpha(t-), \nu^i(dz)\rangle dt, \tag{4.3}$$

where

$$\nu^i(dz) = \left(\nu^i_{e_1}(dz), \nu^i_{e_2}(dz), .., \nu^i_{e_D}(dz)\right)^\top \in \mathbb{R}^D$$

For each $i = 1, 2, ..., M$, $j = 1, 2, .., D$, $\nu^i_{e_j}$ is assumed to be $\sigma-$finite measure on $\mathbb{R}_0$ satisfying $\nu^i_{e_j}(O) < \infty$ , $\forall O \in \mathcal{B}_0$ and $\int_{\mathbb{R}_0}\min\left(1,\ z^2\right)\nu^i_{e_j}(dz) < \infty$. Here we use the subscript $\alpha$ in $\eta^i_\alpha$ to indicate the dependence of the probability law of the Poisson random measures on the Markov chain. Indeed, $\nu_{e_j}^i(dz)$ is the conditional Lévy density of jump sizes of the random measure $\mathcal{N}^i(dz, dt)$ when $\alpha(t-) = e_j$. Moreover, denote the compensated Poisson random measuren $\widetilde{\mathcal{N}}_\alpha(dz, dt)$ by

$$\widetilde{\mathcal{N}}_\alpha(dz, dt) := \left(\mathcal{N}_\alpha^1(dz, dt) - v_\alpha^1(dz)\,dt, ..., \mathcal{N}_\alpha^M(dz, dt) - v_\alpha^M(dz)\,dt\right)^\top. \tag{4.4}$$

We now introduce the state process $X = \{X(t) \backslash t \in [0, \infty[\}$. Suppose that we are given a set $U \subset \mathbb{R}^K$ and a control process $u(t) = u(t, w) : [0, \infty[ \times \Omega \to U$. We also require that $\{u(t, w) \backslash t \in [0, \infty[\}$ is $\mathcal{F}_t$-predictable and has right limits. Let $X(t) = X^{(u)}(t)$ be a controlled Markov regime-switching jumps-diffusion in $\mathbb{R}^L$ described by the stochastic differential equation

$$\begin{cases} dX(t) = & b(t, X(t), u(t), \alpha(t))\,dt + \sigma(t, X(t), u(t), \alpha(t))\,dB(t) \\ & +\int_{\mathbb{R}_0} \eta(t, X(t), u(t), \alpha(t), z)\,\widetilde{\mathcal{N}}_\alpha(dz, dt) \\ & +\gamma(t, X(t), u(t), \alpha(t))\,d\widetilde{\Phi}(t) \qquad 0 \le t \le \infty, \\ X(0) = & x_0. \end{cases} \tag{4.5}$$

Here $b : [0, \infty[ \times \mathbb{R}^L \times U \times \mathcal{S} \to \mathbb{R}^L$, $\sigma : [0, \infty[ \times \mathbb{R}^L \times U \times \mathcal{S} \to \mathbb{R}^{L\times N}$, $\eta : [0, \infty[ \times \mathbb{R}^L \times U \times \mathcal{S} \times \mathbb{R}_0 \to \mathbb{R}^{L\times M}$ and $\gamma : [0, \infty[ \times \mathbb{R}^L \times U \times \mathcal{S} \to \mathbb{R}^{L\times D}$, are given continuous functions , $B(t) := (B_1(t), ..., B_N(t))^T$ is an $N-$dimensional standard Brownian motion, $\widetilde{\mathcal{N}}_\alpha(dz, dt)$ is M-dimentional Markov regime-switching random measures definied by (4.4) $\widetilde{\Phi}(t) = \left(\widetilde{\Phi}_1, .., \widetilde{\Phi}_D\right)$ whith $\widetilde{\Phi}_j(t)$, $j = 1, 2, .., D$, defined by $(5.2)$.

Let $\varepsilon_t \subset \mathcal{F}_t$ be a given subfiltration, representing the information avialable to the controller at time $t$, $t \ge 0$. The control process $u(t)$ assumed to be $\{\varepsilon_t\}_{t\ge 0}$ predictable and with value in a convexe set $U \subset \mathbb{R}^K$. Let $\mathcal{A}_\varepsilon$ be our family of $\varepsilon_t$-predictable controls.

Consider a performance criterion defined for each $x \in \mathbb{R}^L, e_i \in \mathcal{S}$ as

$$J(x, e_i, u) = \mathbf{E}_{x,e_i}\left[\int_0^\infty f(t, X(t), u(t), \alpha(t))\,dt\right].$$

Here $\mathbf{E}_{x,e_i}$ is the conditional expectation given $X(0) = 0$ and $\alpha(0) = e_i$ under $P$, and

$$\mathbf{E}\left[\int_0^\infty \left\{|f(t, X(t), u(t), \alpha(t))| + \left|\frac{\partial f}{\partial x}((t, X(t), u(t), \alpha(t)))\right|^2\right\} dt\right] < \infty,$$

for all $u \in \mathcal{A}_\varepsilon$ , we study the problem to find $u^* \in \mathcal{A}_\varepsilon$ such that

$$J(x^*, e_i, u^*) = \sup_{u \in \mathcal{A}_\varepsilon} J(x, e_i, u). \tag{4.6}$$

Denote by $\mathcal{R}$ the set of functions $r : [0, \infty[ \times \mathbb{R}_0^L \to \mathbb{R}^{L \times M}$ such that

$$\int_{\mathbb{R}_0} |\eta_{nm}(t, x, u, e_i, z) r_{nm}(t, z)| \nu_{e_i}^m(dz) < \infty, \text{ forl all } n, m, x, t,$$

and $\mathcal{M}^2$ the set of functions $s(\cdot) : [0, \infty[ \to \mathbb{R}^{L \times D}$ such that

$$\sum_{m=1}^{D}\sum_{n=1}^{L} \gamma_{nm}(t, x, u, e_i) s_{nm}(t) \lambda_{im}(t) < \infty, \text{ forl all } n, m, x, t,$$

and define the Hamiltonian $H : [0, \infty[ \times \mathbb{R}^L \times U \times \mathcal{S} \times \mathbb{R}^L \times \mathbb{R}^{L \times N} \times \mathcal{R} \times \mathbb{R}^{L \times D} \to \mathbb{R}$ by

$$\begin{aligned} H(t, x, u, e_i, p, q, r, s) &= f(t, x, u, e_i) + b^T(t, x, u, e_i) p + tr\left(\sigma^T(t, x, u, e_i) q\right) \\ &\quad + \int_{\mathbb{R}_0} \sum_{n=1}^{L}\sum_{m=1}^{M} \eta_{nm}(t, x, u, e_i, z) r_{nm}(t, z) \nu_{e_i}^m(dz) \\ &\quad + \sum_{m=1}^{D}\sum_{n=1}^{L} \gamma_{nm}(t, x, u, e_i) s_{nm}(t) \lambda_{im}. \end{aligned} \tag{4.7}$$

The adjoint equation in the unknown $\mathcal{F}_t$-predictable processes $(p(t), q(t), r(t, z), s(t))$ where $p(t) \in \mathbb{R}^L, q(t) \in \mathbb{R}^{L \times N}, r(t, z) \in \mathbb{R}^{L \times M}, s(t) \in \mathbb{R}^{L \times D}$ is the following backward stochastic differential equation (BSDE)

$$\begin{aligned} dp(t) = &\ -\frac{\partial H}{\partial x}(t, X(t), u(t), \alpha(t), p(t), q(t), r(t, .), s(t)) dt \\ &+ q(t) dB(t) + \int_{\mathbb{R}_0} r(t, z) \widetilde{\mathcal{N}}_\alpha(dz, dt) + s(t) d\widetilde{\Phi}(t), \quad t \geq 0. \end{aligned} \tag{4.8}$$

## 4.2 Existence and uniqueness

In this section, we prove the existence and uniqueness of the solution $(Y(t), Z(t), K(t,\varsigma), V(t))$ of infinite horizon BSDEs of the form:

$$\begin{cases} dY(t) = & -g(t,\alpha(t), Y(t), Z(t), K(t,.), V(t))\,dt + Z(t)\,dB(t) \\ & +\int_{\mathbb{R}_0} - K(t,\varsigma)\widetilde{\mathcal{N}}_\alpha(d\varsigma, dt) + V(t)\,d\widetilde{\Phi}(t), \quad 0 \le t \le \tau, \\ \lim_{t\to\tau} Y(t) = & \xi(\tau)\,1_{[0,\infty[}(\tau), \end{cases} \tag{4.9}$$

where $\tau \le \infty$ is a given $\mathcal{F}_t$-stopping time, possibly infinite. We assume the following.

**(H1)** The function $g : \Omega \times \mathbb{R}_+ \times \mathcal{S} \times \mathbb{R}^L \times \mathbb{R}^{L\times N} \times \mathcal{R} \times \mathbb{R}^{L\times D} \to \mathbb{R}^L$, is such that there exist real numbers $\mu, \lambda, K_1, K_2$ and $K_3$ such that $K_1, K_2$ and $K_3 > 0$, and $\lambda > 2\mu + K_1^2 + K_2^2 + K_3^2$.

We assume that the function $g$ satisfies the following requirement:

$(a)$ $g(.e_i, y, z, k, v)$ is progressively measurable for all $y$, $z$, $k$, $v$ and

$$\left|g(t,e_i,y,z,k,v) - g\left(t,e_i,y,z',k',v'\right)\right| \le K_1 \left\|z-z'\right\| + K_2\left\|k-k'\right\|_{\mathcal{R}} + K_3 \left\|v-v'\right\|_{\mathcal{M}^2},$$

where

$$\begin{aligned} \|z\|^2 &= trace(zz^*), \\ \|k(.)\|^2_{\mathcal{R}} &= \textstyle\sum_{l=1}^L \sum_{m=1}^M \int |k_{lm}(z)|^2 v^m_{e_i}(dz), \\ \|v\|^2_{\mathcal{M}^2} &= \textstyle\sum_{l=1}^L \displaystyle\sum_{j=1}^D |\nu_{lj}(t)|^2 \lambda_j(t). \end{aligned}$$

$(b)$

$$\left\langle y-y', g(t,e_i,y,z,k,v) - g\left(t,e_i,y',z,k,v\right)\right\rangle \le \mu\left|y-y'\right|, \text{ for all } y, y', z, k, v \ P-a.s.$$

$(c)$

$$\mathbf{E}\int_0^\tau e^{\lambda t}\left|g(t,e_i,0,0,0,0)\right|^2 dt < \infty,$$

$(d)$ $y \mapsto g(t,e_i,y,z,k,v)$ is continuous for all $t$, $e_i$, $z, k$, $v$. $P-a.s.$

**(H2)** A final condition $\xi$ which is a an $\mathcal{F}_\tau-$mesurable and m-dimensional random variablen such that

$$\mathbf{E}\left[e^{\lambda\tau}\left|\xi\right|^2\right]<\infty,$$
$$\mathbf{E}\int_0^\tau e^{\lambda t}\left|g\left(t,e_i,\xi_t,\eta_t,\psi_t,\varphi_t\right)\right|^2dt<\infty,$$

where $\tau$ is an $\mathcal{F}_t$-stopping time , $\xi_t=\mathbf{E}\left(\xi/\mathcal{F}_t\right),\eta\in L^2_{\mathcal{F},p},\psi\in F^2_p$ and $\varphi\in M^2_p$ such that:

$$\xi=\mathbf{E}\left(\xi\right)+\int_0^\infty\eta\left(s\right)dB_s+\int_0^\infty\int_{\mathbb{R}_0}\psi\left(s,\varsigma\right)\widetilde{\mathcal{N}}_\alpha\left(d\varsigma,ds\right)+\int_0^\infty\varphi\left(s\right)d\widetilde{\Phi}\left(s\right),$$

where

$$\begin{aligned}
L^2_{\mathcal{F},p}&=\left\{f:\mathbb{R}^{L\times N}\text{-valued }\mathcal{F}_t-\text{predictable process, s,t. }\mathbf{E}\left[\int_0^\infty\left|f\left(t\right)\right|^2dt\right]<\infty\right\}.\\
F^2_p&=\left\{f:\mathbb{R}^{L\times M}\text{-valued }\mathcal{F}_t-\text{predictable process, s,t. }\mathbf{E}\left[\int_0^\infty\left\|f\left(t,.\right)\right\|^2_{\mathcal{R}}dt\right]<\infty\right\}.\\
M^2_p&=\left\{f:\mathbb{R}^{L\times D}\text{-valued }\mathcal{F}_t-\text{predictable process, s,t. }\mathbf{E}\left[\int_0^\infty\left\|f\left(t\right)\right\|^2_{\mathcal{M}^2}dt\right]<\infty\right\}.
\end{aligned}$$

A solution of the BSDE (4.9),is a quadreplet $(Y,Z,K,V)$ of progressively measurable processes with values in $\mathbb{R}^L\times\mathbb{R}^{L\times N}\times\mathbb{R}^{L\times M}\times\mathbb{R}^{L\times D}$ s.t $Z_t,K_t,V_t=0$, when $t>\tau$, and

$$\begin{cases}
\mathbf{E}\left(\sup_{t\geq0}e^{\lambda t}\left|Y\left(t\right)\right|^2+\int_0^\tau e^{\lambda t}\left\|Z\left(t\right)\right\|^2dt+\int_0^\tau e^{\lambda t}\left\|K\left(t\right)\right\|^2_{\mathcal{R}}dt+\int_0^\tau e^{\lambda t}\left\|V\left(t\right)\right\|^2_{\mathcal{M}^2}dt\right)<\infty,\\
Y\left(t\right)=Y\left(T\right)+\int_{t\wedge\tau}^{T\wedge\tau}g\left(s,\alpha\left(s\right),Y\left(s\right),Z\left(s\right),K\left(t,.\right),V\left(s\right)\right)ds-\int_{t\wedge\tau}^{T\wedge\tau}Z\left(s\right)dB\left(s\right)\\
\qquad-\int_{t\wedge\tau}^{T\wedge\tau}\int_{\mathbb{R}_0}K\left(s,\varsigma\right)\widetilde{\mathcal{N}}_\alpha\left(d\varsigma,ds\right)-\int_{t\wedge\tau}^{T\wedge\tau}V\left(s\right)d\widetilde{\Phi}\left(s\right);\text{ for all deterministic }T<\infty.\\
Y_t=\xi\text{ on the set }\left\{t\geq\tau\right\}.
\end{cases}$$

**Theorem 4.2.1** ***(Existence and Uniqueness)*** *Under the above conditions there exists a unique solution* $(Y_t,Z_t,K_t,V_t)$ *of the BSDE* (4.9)*, which satisfies moreover, for any* $\lambda>2\mu+K_1^2+K_2^2+K_3^2$,

$$\begin{aligned}
&\mathbf{E}\left(\sup_{0\leq t\leq\tau}e^{\lambda t}\left|Y\left(t\right)\right|^2+\int_0^\tau e^{\lambda t}\left\|Z\left(t\right)\right\|^2dt+\int_0^\tau e^{\lambda t}\left\|K\left(t\right)\right\|^2_{\mathcal{R}}dt+\int_0^\tau e^{\lambda t}\left\|V\left(t\right)\right\|^2_{\mathcal{M}^2}dt\right)\\
&<c\mathbf{E}\left(e^{\lambda\tau}\left|\xi\right|^2+\int_0^\tau e^{\lambda t}\left|g\left(t,e_i,0,0,0,0\right)\right|^2dt\right).
\end{aligned}\tag{4.10}$$

**Proof of uniqueness.** Let $(Y,Z,K,V)$ and $(Y',Z',K',V')$ be two solutions, which satisfy (4.9) and let $\left(\bar{Y},\bar{Z},\bar{K},\bar{V}\right)=(Y-Y',Z-Z',K-K',V-V')$. It follows from Itô's formula, and the above assumption that

$$
\begin{aligned}
&e^{\lambda(T\wedge\tau)}\left|\bar{Y}(T)\right|^2-e^{\lambda(t\wedge\tau)}\left|\bar{Y}(t)\right|^2\\
&=-\int_{(t\wedge\tau)}^{(T\wedge\tau)}e^{\lambda s}\left\langle g\left(s,\alpha(s),Y(s),Z(s),K(s,.),V(s)\right)\right.\\
&\quad\left.-\,g\left(s,\alpha(s),Y'(s),Z'(s),K'(s,.),V'(s)\right),Y(s)-Y'(s)\right\rangle ds\\
&\quad+\int_{(t\wedge\tau)}^{(T\wedge\tau)}(e^{\lambda s}\left\|\bar{Z}(s)\right\|^2+\lambda e^{\lambda s}\left|\bar{Y}(s)\right|^2)ds\\
&\quad+\int_{(t\wedge\tau)}^{(T\wedge\tau)}e^{\lambda s}\left\|\bar{K}(s,\zeta)\right\|_{\mathcal{R}}^2ds+\textstyle\int_{(t\wedge\tau)}^{(T\wedge\tau)}e^{\lambda s}\left\|V(s)\right\|_{\mathcal{M}^2}^2ds\\
&\quad+2\int_{(t\wedge\tau)}^{(T\wedge\tau)}e^{\lambda s}\left\langle\bar{Y}(s),\bar{Z}(s)dB(s)\right\rangle+\textstyle\int_{(t\wedge\tau)}^{(T\wedge\tau)}\displaystyle\int_{\mathbb{R}_0}e^{\lambda s}\left(\bar{K}^2(s,\zeta)-2\left\langle\bar{Y}(s),\bar{K}(s,\zeta)\right\rangle\right)\widetilde{\mathcal{N}}_\alpha(d\varsigma,ds)\\
&\quad+\int_{(t\wedge\tau)}^{(T\wedge\tau)}e^{\lambda s}\left(\bar{V}^2(s)-2\left\langle\bar{Y}(s),\bar{V}(s)\right\rangle\right)d\widetilde{\Phi}(s).
\end{aligned}
$$

so

$$
\begin{aligned}
&e^{\lambda(t\wedge\tau)}\left|\bar{Y}(t)\right|^2+\int_{(t\wedge\tau)}^{(T\wedge\tau)}e^{\lambda s}\left(\lambda\left|\bar{Y}(s)\right|^2+\left\|\bar{Z}(s)\right\|^2\right)ds+\int_{(t\wedge\tau)}^{(T\wedge\tau)}e^{\lambda s}\left(\left\|\bar{K}(s,\zeta)\right\|_{\mathcal{R}}^2+\left\|\bar{V}(s)\right\|_{\mathcal{M}^2}^2\right)ds\\
&\le e^{\lambda(T\wedge\tau)}\left|\bar{Y}(T)\right|^2\\
&\quad+2\int_{(t\wedge\tau)}^{(T\wedge\tau)}e^{\lambda s}\left(\mu\left|\bar{Y}(s)\right|^2+K_1\left|\bar{Y}(s)\right|\left\|\bar{Z}(s)\right\|+K_2\left|\bar{Y}(s)\right|\left\|\bar{K}(s,\zeta)\right\|_{\mathcal{R}}+K_3\left|\bar{Y}(s)\right|\left\|\bar{V}(s)\right\|_{\mathcal{M}^2}\right)ds\\
&\quad-2\int_{(t\wedge\tau)}^{(T\wedge\tau)}e^{\lambda s}\left\langle\bar{Y}(s),\bar{Z}(s)dB(s)\right\rangle-\int_{(t\wedge\tau)}^{(T\wedge\tau)}\int_{\mathbb{R}_0}e^{\lambda s}\left(\bar{K}^2(s,\zeta)-2\left\langle\bar{Y}(s),\bar{K}(s,\zeta)\right\rangle\right)\widetilde{\mathcal{N}}_\alpha(d\varsigma,ds)\\
&\quad-\int_{(t\wedge\tau)}^{(T\wedge\tau)}e^{\lambda s}\left(\bar{V}^2(s)-2\left\langle\bar{Y}(s),\bar{V}(s)\right\rangle\right)d\widetilde{\Phi}(s).
\end{aligned}
$$

By the fact that

$$
\begin{aligned}
2K_1\left|\bar{Y}(s)\right|\left\|\bar{Z}(s)\right\|\le&\quad\left\|\bar{Z}(s)\right\|^2+K_1^2\left|\bar{Y}(s)\right|^2,\\
2K_2\left|\bar{Y}(s)\right|\left\|\bar{K}(s,\zeta)\right\|_{\mathcal{R}}\le&\quad\left\|\bar{K}(s,\zeta)\right\|_{\mathcal{R}}^2+K_2^2\left|\bar{Y}(s)\right|^2,\\
2K_3\left|\bar{Y}(s)\right|\left\|\bar{V}(s)\right\|_{\mathcal{M}^2}\le&\quad\left\|\bar{V}(s)\right\|_{\mathcal{M}^2}^2+K_3^2\left|\bar{Y}(s)\right|^2,
\end{aligned}
$$

and since $\lambda > 2\mu + K_1^2 + K_2^2 + K_3^2$,we deduce that for $t < T$,

$$\mathbf{E}\left(e^{\lambda(t\wedge\tau)}\left|\bar{Y}(t)\right|^2\right) \leq \mathbf{E}\left(e^{\lambda(T\wedge\tau)}\left|\bar{Y}(T)\right|^2\right).$$

The same result holds with $\lambda$ remplaced by $\lambda'$, with

$$2\mu + K_1^2 + K_2^2 + K_3^2 < \lambda' < \lambda.$$

Hence

$$\mathbf{E}\left(e^{\lambda'(t\wedge\tau)}\left|\bar{Y}(t)\right|^2\right) \leq e^{(\lambda-\lambda')T}\mathbf{E}\left(e^{\lambda(T\wedge\tau)}\left|\bar{Y}(T)\right|^2 \mathbf{I}_{\{T<\tau\}}\right).$$

With our conditions the second factor of the right hand side remains bounded as $T \to \infty$, while the first factor tend to 0 as $T \to \infty$. Uniquenessis is proved.

**Proof of existence.** For each $n$, we construct a solution $\{(Y^n(t), Z^n(t), K^n(t), V^n(t))\ ; t \geq 0\}$ of the BSDE

$$\begin{cases} Y^n(t) &= \xi + \int_{t\wedge\tau}^{n\wedge\tau} g(s,\alpha(s), Y^n(s), Z^n(s), K^n(t,.), V^n(s))\, ds - \int_{t\wedge\tau}^{n\wedge\tau} Z^n(s)\, dB(s) \\ & -\int_{t\wedge\tau}^{n\wedge\tau}\int_{\mathbb{R}_0} K^n(s,\varsigma)\widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) - \int_{t\wedge\tau}^{n\wedge\tau} V^n(s)\, d\widetilde{\Phi}(s),\ t \geq 0, \end{cases}$$

as follows. $\{(Y^n(t), Z^n(t), K^n(t), V^n(t)); 0 \leq t \leq n\}$ is defined as the solution of the following BSDE on the fixed intervall $[0,n]$ :

$$\begin{cases} Y^n(t) &= \mathbf{E}(\xi/\mathcal{F}_n) + \int_t^n \mathbf{I}_{[0,\tau]} g(s,\alpha(s), Y^n(s), Z^n(s), K^n(t,.), V^n(s))\, ds \\ & -\int_t^n Z^n(s)\, dB(s) - \int_t^n \int_{\mathbb{R}_0} K^n(s,\varsigma)\widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) \\ & -\int_t^n V^n(s)\, d\widetilde{\Phi}(s), \quad 0 \leq t \leq n, \end{cases}$$

$\{(Y^n(t), Z^n(t), K^n(t), V^n(t))\ ; t \geq n\}$ is defined by

$$Y^n(t) = \xi_t, Z^n(t) = \eta(s), K^n(t) = \psi(s,\varsigma), V^n(t) = \varphi(s).$$

For any $\varepsilon > 0,\ 0 < \rho < 1, 0 < \alpha < 1\ , 0 < \beta < 1$ , we have for all $t \geq 0, y \in \mathbb{R}^L, e_i \in D, z \in \mathbb{R}^{L\times N}, k \in \mathbb{R}^{L\times M}, v \in \mathbb{R}^{L\times D}$ if $c = \frac{1}{\epsilon}$,

$$\begin{aligned}
2\left\langle y, g\left(t, e_i, y, z, k, v\right)\right\rangle &= 2\left\langle y, g\left(t, e_i, y, z, k, v\right) - g\left(t, e_i, 0, z, k, v\right)\right\rangle \\
&\quad +2\left\langle y, g\left(t, e_i, 0, z, k, v\right) - g\left(t, e_i, 0, 0, k, v\right)\right\rangle \\
&\quad +2\left\langle y, g\left(t, e_i, 0, 0, k, v\right) - g\left(t, e_i, 0, 0, 0, v\right)\right\rangle \\
&\quad +2\left\langle y, g\left(t, e_i, 0, 0, 0, v\right) - g\left(t, e_i, 0, 0, 0, 0\right)\right\rangle \\
&\quad +2\left\langle y, g\left(t, e_i, 0, 0, 0, 0\right)\right\rangle \\
&\leq \left(2\mu + \tfrac{1}{\rho}K_1^2 + \tfrac{1}{\alpha}K_2^2 + \tfrac{1}{\beta}K_3^2 + \varepsilon\right)|y|^2 \\
&\quad +\rho\left\|z\right\| + \alpha\left\|k\left(.\right)\right\|_{\mathcal{R}}^2 + \beta\left\|v\right\|_{\mathcal{M}^2}^2 \\
&\quad +c\left|g\left(t, e_i, 0, 0, 0, 0\right)\right|^2 .
\end{aligned}$$

From these and Itô′s formula, we deduce that

$$\begin{aligned}
& e^{\lambda(t\wedge\tau)}\left|Y^n\left(t\wedge\tau\right)\right|^2 + \int_{(t\wedge\tau)}^{\tau} e^{\lambda s}\left(\bar{\lambda}\left|Y^n\left(s\right)\right|^2 + \bar{\rho}\left\|Z^n\left(s\right)\right\|^2\right)ds \\
& + \int_{(t\wedge\tau)}^{\tau} \bar{\alpha} e^{\lambda s}\left\|K^n\left(s,\zeta\right)\right\|_{\mathcal{R}}^2 ds + \int_{(t\wedge\tau)}^{\tau} \bar{\beta} e^{\lambda s}\left\|V^n\left(s\right)\right\|_{\mathcal{M}^2}^2 ds \\
& \leq e^{\lambda\tau}\left|\xi\right|^2 + c\int_{(t\wedge\tau)}^{\tau} e^{\lambda s}\left|g\left(s, e_i, 0, 0, 0, 0\right)\right|^2 ds \\
& -2\int_{(t\wedge\tau)}^{\tau} e^{\lambda s}\left\langle Y^n\left(s\right), Z^n\left(s\right)dB\left(s\right)\right\rangle \\
& -\int_{(t\wedge\tau)}^{\tau}\int_{\mathbb{R}_0} e^{\lambda s}\left(\left(K^n\right)^2\left(s,\zeta\right) + 2\left\langle Y^n\left(s\right), K^n\left(s,\zeta\right)\right\rangle\right)\widetilde{\mathcal{N}}_\alpha\left(d\varsigma, ds\right) \\
& -\int_{(t\wedge\tau)}^{\tau} e^{\lambda s}\left(\left(V^n\right)^2\left(s\right) + 2\left\langle Y^n\left(s\right), V^n\left(s\right)\right\rangle\right)d\widetilde{\Phi}\left(s\right),
\end{aligned}$$

with $\bar{\lambda} = \lambda - 2\mu - \frac{1}{\rho}K_1^2 - \frac{1}{\alpha}K_2^2 - \frac{1}{\beta}K_3^2 - \varepsilon > 0$ , $\bar{\rho} = 1 - \rho > 0, \bar{\alpha} = 1 - \alpha$ and $\bar{\beta} = 1 - \beta$. It then follows from Burkholder's inequality

$$\begin{aligned}
&\mathbf{E}\left[\sup_{t\geq s} e^{\lambda(t\wedge\tau)} \left|Y^n\left(t\wedge\tau\right)\right|^2 + \int_{(t\wedge\tau)}^{\tau} e^{\lambda r}\left(\left|Y^n\left(r\right)\right|^2 + \left\|Z^n\left(r\right)\right\|^2\right) dr\right.\\
&\left.+\int_{(t\wedge\tau)}^{\tau} e^{\lambda r}\left(\left\|K^n\left(r,\zeta\right)\right\|_{\mathcal{R}}^2 + \left\|V^n\left(r\right)\right\|_{\mathcal{M}^2}^2\right) dr\right]\\
&\leq C\mathbf{E}\left[e^{\lambda\tau}\left|\xi\right|^2 + \int_{(t\wedge\tau)}^{\tau} e^{\lambda r}\left|g\left(r,e_i,0,0,0,0\right)\right|^2 dr\right].
\end{aligned}$$

Let now $m > n$, and define

$$\triangle Y\left(t\right) = Y^m\left(t\right) - Y^n\left(t\right),\ \triangle Z\left(t\right) = Z^m\left(t\right) - Z^n\left(t\right),$$
$$\triangle K\left(t\right) = K^m\left(t\right) - K^n\left(t\right),\ \triangle V\left(t\right) = V^m\left(t\right) - V^n\left(t\right).$$

We first have that for $n \leq t \leq m$,

$$\begin{aligned}
\triangle Y\left(t\right) &= \int_{t\wedge\tau}^{m\wedge\tau} g\left(s,\alpha\left(s\right),Y^m\left(s\right),Z^m\left(s\right),K^m\left(t,.\right),V^m\left(s\right)\right) ds\\
&\quad -\int_{t\wedge\tau}^{m\wedge\tau} \triangle\, Z^m\left(s\right) dB\left(s\right) - \int_{t\wedge\tau}^{m\wedge\tau}\int_{\mathbb{R}_0} \triangle\, K^m\left(s,\varsigma\right)\widetilde{\mathcal{N}}_\alpha\left(d\varsigma,ds\right)\\
&\quad -\int_{t\wedge\tau}^{m\wedge\tau} \triangle\, V^m\left(s\right) d\widetilde{\Phi}\left(s\right).
\end{aligned}$$

Consequently, again for $n \leq t \leq m$,

$$\begin{aligned}
& e^{\lambda(t\wedge\tau)} \left|\triangle Y(t)\right|^2 + \int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left(\lambda \left|\triangle Y(s)\right|^2 + \left\|\triangle Z(s)\right\|^2\right) ds \\
& + \int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left(\left\|\triangle K(s,\zeta)\right\|_{\mathcal{R}}^2 + \left\|\triangle V(s)\right\|_{\mathcal{M}^2}^2\right) ds \\
& = 2\int_{(t\wedge\tau)}^{(m\wedge\tau)} \left(e^{\lambda s} \left\langle g\left(s,\alpha(s), Y^m(s), Z^m(s), K^m(s,.), V^m(s)\right), \triangle Y(s)\right\rangle\right) ds \\
& -2\int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left\langle \triangle Y(s), \triangle Z(s)\, dB(s)\right\rangle - \int_{(t\wedge\tau)}^{(m\wedge\tau)} \int_{\mathbb{R}_0} e^{\lambda s} \left(\left(\triangle K\right)^2(s,\zeta) + 2\left\langle \triangle Y(s), \triangle K(s,\zeta)\right\rangle\right) \widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) \\
& - \int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left(\left(\triangle V\right)^2(s) + 2\left\langle \triangle Y(s), \triangle V(s)\right\rangle\right) d\widetilde{\Phi}(s). \\
& \leq 2\int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \Big\{\mu \left|\triangle Y(s)\right|^2 + K_1 \left|\triangle Y(s)\right| \left\|\triangle Z(s)\right\| + K_2 \left|\triangle Y(s)\right| \left\|\triangle K(s,\zeta)\right\|_{\mathcal{R}} \\
& + K_3 \left|\triangle Y(s)\right| \left\|\triangle V(s)\right\|_{\mathcal{M}^2}\} ds \\
& -2\int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left|\triangle Y(s)\right| \left|g\left(s, e_i, \xi_s, \eta_s, \psi_s, \varphi_s\right)\right|^2 ds - 2\int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left\langle \triangle Y(s), \triangle Z(s)\, dB(s)\right\rangle \\
& - \int_{(t\wedge\tau)}^{(m\wedge\tau)} \int_{\mathbb{R}_0} e^{\lambda s} \left(\left(\triangle K\right)^2(s,\zeta) + 2\left\langle \triangle Y(s), \triangle K(s,\zeta)\right\rangle\right) \widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) \\
& - \int_{(t\wedge\tau)}^{(m\wedge\tau)} e^{\lambda s} \left(\left(\triangle V\right)^2(s) + 2\left\langle \triangle Y(s), \triangle V(s)\right\rangle\right) d\widetilde{\Phi}(s).
\end{aligned}$$

We then deduce, by an argument that already used , that

$$\begin{aligned}
& \mathbf{E}\Big[\sup\nolimits_{n\leq t\leq m} e^{\lambda(t\wedge\tau)} \left|Y(t\wedge\tau)\right|^2 + \int_{n\wedge\tau}^{m\wedge\tau} e^{\lambda s} \Big(\left|\triangle Y(s)\right|^2 + \left\|\triangle Z(s)\right\|^2 \\
& + \left\|\triangle K(s,\zeta)\right\|_{\mathcal{R}}^2 + \left\|\triangle V(s)\right\|_{\mathcal{M}^2}^2\Big) ds\Big] \\
& \leq C\int_{(n\wedge\tau)}^{\tau} e^{\lambda s} \left|g\left(s, e_i, \xi_s, \eta_s, \psi_s, \varphi_s\right)\right|^2 ds,
\end{aligned}$$

and this last term tends to zero, as $n \to \infty$. Next , for $t \leq n$,

$$\begin{aligned}
\triangle Y(t) &= \triangle Y(n) + \int_{(t\wedge\tau)}^{(n\wedge\tau)} \{g\left(s,\alpha(s), Y^m(s), Z^m(s), K^m(s,.), V^m(s)\right) \\
&\quad - g\left(s,\alpha(s), Y^n(s), Z^n(s), K^n(s,.), V^n(s)\right)\} ds^2 \\
&\quad - \int_{t\wedge\tau}^{n\wedge\tau} \triangle Z(s)\, dB(s) - \int_{t\wedge\tau}^{n\wedge\tau} \int_{\mathbb{R}_0} \triangle K(s,\varsigma)\, \widetilde{\mathcal{N}}_\alpha(d\varsigma, ds) - \int_{t\wedge\tau}^{n\wedge\tau} \triangle V(s)\, d\widetilde{\Phi}(s).
\end{aligned}$$

It follows from the same argument as in the proof of uniqueness that

$$\begin{aligned}\mathbf{E}\left(e^{\lambda(t\wedge\tau)}\left|\triangle Y(t)\right|^{2}\right) &\leq \mathbf{E}\left(e^{\lambda(n\wedge\tau)}\left|\triangle Y(n)\right|^{2}\right)\\ &\leq C\int_{(n\wedge\tau)}^{\tau} e^{\lambda s}\left|g\left(s,e_i,\xi_s,\eta_s,\psi_s,\varphi_s\right)\right|^{2} ds.\end{aligned}$$

It now follows that the sequence $(Y^n, Z^n, K^n, V^n)$ is Cauchy with the norm

$$\left\|(Y^n, Z^n, K^n, V^n)\right\|^2 = \mathbf{E}\left[\sup_{0\leq t\leq\tau} e^{\lambda t}\left|Y(t)\right|^2 + \int_0^{\tau} e^{\lambda t}\left(\left|Y(t)\right|^2 + \left\|Z(t)\right\|^2 + \left\|K(t)\right\|_{\mathcal{R}}^2 + \left\|V(t)\right\|_{\mathcal{M}^2}^2\right) dt\right],$$

and that the limit $(Y, Z; K, V)$ is a solution of the BSDE (4.9) The proof is complete. ■

## 4.3 Optimal control with partial information and infinite horizon

In the following we assume that $L = M = N = 1$.

Now, let us get back to the problem of maximizing the performance functional

$$J(x, e_i, u) = \mathbf{E}_{x,e_i}\left[\int_0^{\infty} f\left(t, X(t), u(t), \alpha(t)\right) dt\right],$$

where $X(t)$ is of the from (4.5). Our goal is to find a $u^* \in \mathcal{A}_\varepsilon$ such that

$$J(x^*, e_i, u^*) = \sup_{u\in\mathcal{A}_\varepsilon} J(x, e_i, u),$$

where $u(t)$ is a control which adapted to subfiltration $\varepsilon_t \subset \mathcal{F}_t$, with value in a set $U \subset \mathbb{R}$.

Let $H$ be the Hamiltonian defined by (4.7) and $(p, q, r, s)$ the solution to the adjoint equation (4.8). Then we have the following maximum principle.

**Theorem 4.3.1** ***(Sufficient Infinite Horizon Maximum Principle)*** *Let $u^* \in \mathcal{A}_\varepsilon$ and let $(p^*(t), q^*(t), r^*(t,z), s^*(t))$ be an associated solution to Eq (4.8). Assume that for all $u \in \mathcal{A}_\varepsilon$*

the following terminal condition holds :

$$0 \leq \mathbf{E}\left[\overline{\lim_{t\rightarrow\infty}}\left[p^{*}\left(t\right)\left(X\left(t\right)-X^{*}\left(t\right)\right)\right]\right] < \infty. \tag{4.11}$$

Moreover, assume that $H\left(t,x,u,e_i,p^{*}\left(t\right),q^{*}\left(t\right),r^{*}\left(t,\cdot\right),s^{*}\left(t\right)\right)$ is concave in $x$ and $u$ and

$$\begin{aligned} &\mathbf{E}\left[H\left(t,X^{*}\left(t\right),u^{*}\left(t\right),\alpha\left(t\right),p^{*}\left(t\right),q^{*}\left(t\right),r^{*}\left(t,.\right),s^{*}\left(t\right)\right)/\varepsilon_t\right] \\ &= \max_{u\in U}\mathbf{E}\left[H\left(t,X^{*}\left(t\right),u,\alpha\left(t\right),p^{*}\left(t\right),q^{*}\left(t\right),r^{*}\left(t,.\right),s^{*}\left(t\right)\right)/\varepsilon_t\right]. \end{aligned} \tag{4.12}$$

In addition we assume that for all $T<\infty$,

$$\mathbf{E}\left[\int_0^T\left(X^{*}\left(t\right)-X^{u}\left(t\right)\right)^2\left\{\left(q^{*}\right)^2\left(t\right)+\int_{\mathbb{R}_0}\left(r^{*}\right)^2\left(t,z\right)\nu_{\alpha}\left(dz\right)+\sum_{j=1}^{D}\left(s_j^{*}\right)^2\left(t\right)\lambda_j\left(t\right)\right\}dt\right]<\infty, \tag{4.13}$$

and

$$\mathbf{E}\left[\int_0^T\left(p^{*}\right)^2\left(t\right)\left\{\left(\sigma\left(t\right)\right)^2+\int_{\mathbb{R}_0}\left(\eta\left(t,z\right)\right)^2\nu_{\alpha}\left(dz\right)+\sum_{j=1}^{D}\left(\gamma^{j}\right)^2\lambda_j\left(t\right)\right\}dt\right]<\infty \tag{4.14}$$

$$\mathbf{E}\left[\left|\frac{\partial}{\partial u}H\left(t,X^{*}\left(t\right),u^{*}\left(t\right),\alpha\left(t\right),p^{*}\left(t\right),q^{*}\left(t\right),r^{*}\left(t,.\right),s^{*}\left(t\right)\right)\right|^2\right]<\infty, \tag{4.15}$$

and that

$$\mathbf{E}\left[\int_0^{\infty}\left|H\left(t,X\left(t\right),u\left(t\right),\alpha\left(t\right),p^{*}\left(t\right),q^{*}\left(t\right),r^{*}\left(t,.\right),s^{*}\left(t\right)\right)\right|\right]<\infty, \tag{4.16}$$

for all $u$. Then we have that $u^{*}\left(t\right)$ is optimal.

**Proof.** Let

$$\begin{aligned} I^{\infty} &:= \mathbf{E}\left[\int_0^{\infty}\left\{f\left(t,X\left(t\right),u\left(t\right),\alpha\left(t\right)\right)-f\left(t,X^{*}\left(t\right),u^{*}\left(t\right),\alpha\left(t\right)\right)\right\}dt\right] \\ &= J\left(x,e_i,u\right)-J\left(x^{*},e_i,u^{*}\right). \end{aligned}$$

Then $I^{\infty} = I_1^{\infty} - I_2^{\infty} - I_3^{\infty} - I_4^{\infty} - I_5^{\infty}$, where

$$\begin{aligned} I_1^{\infty} := \mathbf{E}\Big[\int_0^{\infty} & \left(H\left(s, X\left(s\right), u\left(s\right), \alpha\left(s\right), p^*\left(s\right), q^*\left(s\right), r^*\left(s,.\right), s^*\left(s\right)\right)\right. \\ & - \left. H\left(s, X^*\left(s\right), u^*\left(s\right), \alpha\left(s\right), p^*\left(s\right), q^*\left(s\right), r^*\left(s,.\right), s^*\left(s\right)\right)\right) ds\Big], \end{aligned}$$

$$I_2^{\infty} := \mathbf{E}\left[\int_0^{\infty} p^*\left(s\right)\left(b\left(s, X\left(s\right), u\left(s\right), \alpha\left(s\right)\right) - b^*\left(s, X^*\left(s\right), u^*\left(s\right), \alpha\left(s\right)\right)\right) ds\right],$$

$$I_3^{\infty} := \mathbf{E}\left[\int_0^{\infty} q^*\left(s\right)\left(\sigma\left(s, X\left(s\right), u\left(s\right), \alpha\left(s\right)\right) - \sigma^*\left(s, X^*\left(s\right), u^*\left(s\right), \alpha\left(s\right)\right)\right) ds\right],$$

$$I_4^{\infty} := \mathbf{E}\left[\int_0^{\infty}\int_{\mathbb{R}_0}\left(\eta\left(s, X\left(s\right), u\left(s\right), \alpha\left(s\right), z\right) - \eta^*\left(s, X^*\left(s\right), u^*\left(s\right), \alpha\left(s\right), z\right)\right) r^*\left(s, z\right)\nu_{\alpha(s)}\left(dz\right) ds\right],$$

$$I_5^{\infty} := \mathbf{E}\left[\int_0^{\infty}\sum_{j=1}^{D}\left(\gamma^j\left(s, X\left(s\right), u\left(s\right), \alpha\left(s\right)\right) - \gamma^{*j}\left(s, X^*\left(s\right), u^*\left(s\right), \alpha\left(s\right)\right)\right) s_j^*\left(s\right)\lambda_j\left(s\right) ds\right].$$

For the simplification we put

$$H_{t,x,u,\alpha,p^*,q^*,r^*,s^*} := H\left(t, x, u, \alpha\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t,\cdot\right), s^*\left(t\right)\right),$$

and the same for the other expressions. We have from concavity that

$$\begin{aligned} & H_{t,X,u,\alpha,p^*,q^*,r^*,s^*} - H_{t,X^*,u^*,\alpha,p^*,q^*,r^*,s^*} \\ & \leq \frac{\partial}{\partial x}H\left(t, X^*\left(t\right), u^*\left(t\right), \alpha\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right), s^*\left(t\right)\right)\left(X\left(t\right) - X^*\left(t\right)\right) \\ & + \frac{\partial}{\partial u}H\left(t, X^*\left(t\right), u^*\left(t\right), \alpha\left(t\right), p^*\left(t\right), q^*\left(t\right), r^*\left(t,.\right), s^*\left(t\right)\right)\left(u\left(t\right) - u^*\left(t\right)\right) \end{aligned} \tag{4.17}$$

Then we have from (4.12),(4.15) and that $u\left(t\right)$ is adapted to $\varepsilon_t$,

$$\begin{aligned} 0 & \geq \frac{\partial}{\partial u}\mathbf{E}\left[H_{t,X^*,u,\alpha,p^*,q^*,r^*,s^*}/\varepsilon_t\right]_{u=u^*(t)}\left(u\left(t\right) - u^*\left(t\right)\right) \\ & = \frac{\partial}{\partial u}\mathbf{E}\left[H_{t,X^*,u^*,\alpha,p^*,q^*,r^*,s^*}\left(u\left(t\right) - u^*\left(t\right)\right)/\varepsilon_t\right]. \end{aligned} \tag{4.18}$$

Combining (4.8), (4.13), (4.17) and (4.18), we get

$$\begin{aligned} I_1^{\infty} & \leq \mathbf{E}\left[\int_0^{\infty}\frac{\partial}{\partial x}H_{t,X^*,u^*,\alpha,p^*,q^*,r^*,s^*}\left(X\left(s\right) - X^*\left(s\right)\right) ds\right] = \mathbf{E}\left[\int_0^{\infty} dp^*\left(s\right)\left(X\left(s\right) - X^*\left(s\right)\right)\right] \\ & := -J_1. \end{aligned}$$

From (4.13) , (4.14), and Ito's formula , we have that

$$
\begin{aligned}
0 &\leq \mathbf{E}\left[\overline{\lim_{t\to\infty}}\left[p^*(t)\left(X(t)-X^*(t)\right)\right]\right] \\
&= \mathbf{E}\left[\overline{\lim_{t\to\infty}}\int_0^t p^*(s)\left(b\left(s,X(s),u(s),\alpha(s)\right)-b\left(s,X^*(s),u^*(s),\alpha(s)\right)\right)ds\right. \\
&\quad+\int_0^t p^*(s)\left(\sigma\left(s,X(s),u(s),\alpha(s)\right)-\sigma^*\left(s,X^*(s),u^*(s),\alpha(s)\right)\right)dB(s) \\
&\quad+\int_0^t\int_{\mathbb{R}_0} p^*(s)\left(\eta\left(s,X(s),u(s),\alpha(s),z\right)-\eta^*\left(s,X^*(s),u^*(s),\alpha(s),z\right)\right)\widetilde{\mathcal{N}}_\alpha(ds,dz) \\
&\quad+\int_0^t p^*(s)\left(\gamma\left(s,X(s),u(s),\alpha(s)\right)-\gamma^*\left(s,X^*(s),u^*(s),\alpha(s)\right)\right)d\widetilde{\Phi}(t)+\int_0^\infty\left(X(s)-X^*(s)\right) \\
&\quad\times\left(-\frac{\partial}{\partial x}H^*\left(s,X^*(s),u^*(s),\alpha(s),p^*(s),q^*(s),r^*(s,.),s^*(s)\right)\right)ds \\
&\quad+\int_0^t q^*(s)\left(X(s)-X^*(s)\right)dB(s)+\int_0^\infty\int_{\mathbb{R}_0} r^*(s,z)\left(X(s)-X^*(s)\right)\widetilde{\mathcal{N}}_\alpha(ds,dz) \\
&\quad+\int_0^t s^*(s)\left(X(s)-X^*(s)\right)d\widetilde{\Phi}(t) \\
&\quad+\int_0^t q^*(s)\left(\sigma\left(s,X(s),u(s),\alpha(s)\right)-\sigma^*\left(s,X^*(s),u^*(s),\alpha(s)\right)\right)ds \\
&\quad+\int_0^t\int_{\mathbb{R}_0} r^*(s,z)\left(\eta\left(s,X(s),u(s),\alpha(s),z\right)-\eta^*\left(s,X^*(s),u^*(s),\alpha(s),z\right)\right)v_{\alpha(s)}(dz)\,ds \\
&\quad+\int_0^t\int_{\mathbb{R}_0} r^*(s,z)\left(\eta\left(s,X(s),u(s),\alpha(s),z\right)-\eta^*\left(s,X^*(s),u^*(s),\alpha(s),z\right)\right)\widetilde{\mathcal{N}}_\alpha(ds,dz) \\
&\quad+\int_0^t s^*(s)\left(\gamma\left(s,X(s),u(s),\alpha(s)\right)-\gamma^*\left(s,X^*(s),u^*(s),\alpha(s)\right)\right)d\widetilde{\Phi}(t) \\
&\quad\left.+\int_0^t\sum_{j=1}^D s_j^*(s)\left(\gamma^j\left(s,X(s),u(s),\alpha(s)\right)-\gamma^{*j}\left(s,X^*(s),u^*(s),\alpha(s)\right)\right)\lambda_j(s)\,ds\right].
\end{aligned}
$$

From (4.13) and (4.14) , we have that

$$
\begin{aligned}
& E\left[\int_0^\infty p^*(s)\left(b(s,X(s),u(s),\alpha(s))-b(s,X^*(s),u^*(s),\alpha(s))\right)ds+\int_0^\infty (X(s)-X^*(s))\right. \\
& \times\left(-\frac{\partial}{\partial x}H^*(s,X^*(s),u^*(s),\alpha(s),p^*(s),q^*(s),r^*(s,.),s^*(s))\right)ds \\
& +\int_0^\infty q^*(s)\left(\sigma(s,X(s),u(s),\alpha(s))-\sigma^*(s,X^*(s),u^*(s),\alpha(s))\right)ds \\
& +\int_0^\infty\int_{\mathbb{R}_0} r^*(s,z)\left(\eta(s,X(s),u(s),\alpha(s),z)-\eta^*(s,X^*(s),u^*(s),\alpha(s),z)\right)v_{\alpha(s)}(dz)\,ds \\
& \left.+\int_0^\infty\sum_{j=1}^{D}s_j^*(s)\left(\gamma^j(s,X(s),u(s),\alpha(s))-\gamma^{*j}(s,X^*(s),u^*(s),\alpha(s))\right)\lambda_j(s)\,ds\right] \\
& = I_2^\infty+J_1^\infty+I_3^\infty+I_4^\infty+I_5^\infty.
\end{aligned}
$$

Finally, combining the above we get

$$
\begin{aligned}
J(x,e_i,u)-J(x^*,e_i,u^*) &\leq I_1^\infty-I_2^\infty-I_3^\infty-I_4^\infty-I_5^\infty \\
&\leq -J_1^\infty-I_2^\infty-I_3^\infty-I_4^\infty-I_5^\infty \\
&\leq 0.
\end{aligned}
$$

This holds for all $u\in\mathcal{A}_\varepsilon$, so the proof is complete. ■

## 4.4 Necessary maximum principle

In this section, we establish optimality necessary conditions for our control problem. We will to prove : if $u^*$ is optimal does it satisfy

$$
\begin{aligned}
& \mathbf{E}\left[H\left(t,\ X^*(t),\ u^*(t),\ \alpha(t),\ p^*(t),\ q^*(t),\ r^*(t,.),\ s^*(t)\right)/\varepsilon_t\right] \\
& =\max_{u\in U}\mathbf{E}\left[H\left(t,\ X^*(t),\ u,\ \alpha(t),\ p^*(t),\ q^*(t),\ r^*(t,.),\ s^*(t)\right)/\varepsilon_t\right].
\end{aligned}
\tag{4.19}
$$

We assume the following:

**(A1)** For all $t$, $h$ such that $0\leq t\leq t+h\leq\infty$ and for all bounded $\varepsilon_t$-measurable random variables $\theta=\theta(\omega)$, the control process $\beta(s)$ defined by

$$\beta(s) = \theta 1_{[t,t+h]}(s),$$

belongs to $\mathcal{A}_\varepsilon$. Here

$$1_{[t,t+h]}(s) = \begin{cases} 1 & \text{if } t \in [t,\ t+h], \\ 0 & \text{otherwise.} \end{cases}$$

**(A2)** For all $u \in \mathcal{A}_\varepsilon$ and all $\beta \in \mathcal{A}_\varepsilon$ bounded, there exists $\epsilon > 0$ such that

$$u + \epsilon\beta \in \mathcal{A}_\varepsilon \text{ for all } \epsilon \in [-\delta, \delta].$$

**(A3)** The derivative process

$$\xi(t) := \frac{d}{d\epsilon} X^{u+\epsilon\beta}(t)\bigg|_{\epsilon=0},$$

exists and belongs to $L^2(m \times P)$, where $m$ denotes the Lebesgue measure on $\mathbb{R}$.

$$\begin{aligned} d\xi(t) &= \left\{\frac{\partial b}{\partial x}(t)\xi(t) + \frac{\partial b}{\partial u}(t)\beta(t)\right\} dt + \left\{\frac{\partial \sigma}{\partial x}(t)\xi(t) + \frac{\partial \sigma}{\partial u}(t)\beta(t)\right\} dB(t) \\ &+ \int_{\mathbb{R}_0} \left\{\frac{\partial \eta}{\partial x}(t,z)\xi(t) + \frac{\partial \eta}{\partial u}(t,z)\beta(t)\right\} \widetilde{\mathcal{N}}_\alpha(dt, dz) \\ &+ \left\{\frac{\partial \gamma}{\partial x}(t)\xi(t) + \frac{\partial \gamma}{\partial u}(t)\beta(t)\right\} d\widetilde{\Phi}(t), \end{aligned}$$

where, for simplicity of notation, we define

$$\frac{\partial b}{\partial x}(t) := \frac{\partial b}{\partial x}(t,\ X(t),\ \alpha(t),\ u(t)).$$

Note that

$$\xi(0) = 0.$$

**(A4)** Assume that $f$ satisfies a Lipschitz condition of the form

$$|f(x_1, u_1, e_j) - f(x_2, u_2, e_j)| \le C(t)(|x_1 - x_2| + |u_1 - u_2|),$$

for any $t, x_i, u_i,\ i = 1, 2, e_j \in \mathcal{S}$.

We have the following theorem.

**Theorem 4.4.1** ***(Partial Information Necessary Maximum Principle)*** *Suppose $u^* \in \mathcal{A}_\varepsilon$ is a local maximum for $J(u)$ meaning that for all bounded $\beta \in \mathcal{A}_\varepsilon$ there exists a $\delta > 0$ such that $u^* + \epsilon\beta \in \mathcal{A}_\varepsilon$ for all $\epsilon \in (-\delta, \delta)$ and $h(\epsilon) := J(u^* + \epsilon\beta)\,,\ \epsilon \in (-\delta, \delta)$ is maximal at $\epsilon = 0$. Let $(p^*(t)\,,\ q^*(t)\,,\ r^*(t,z)\,,\ s^*(t))$ be the solution to the adjoint equation*

$$\begin{aligned} dp^*(t) = &-\frac{\partial H}{\partial x}(t,\ X^*(t)\,,\ u^*(t)\,,\ \alpha(t)\,,\ p^*(t)\,,\ q^*(t)\,,\ r^*(t,.)\,,\ s^*(t))\,dt \\ &+ q^*(t)\,dB(t) + \int_{\mathbb{R}_0} r^*(z,t)\,\widetilde{\mathcal{N}}_\alpha(dz,dt) + s^*(t)\,d\widetilde{\Phi}(t)\,. \end{aligned}$$

Moreover assume that if $\xi^*(t) = \xi^{(u^*,\beta)}(t)$, with corresponding coefficients $\pi_t^*$, $\tau_t^*$, $\varsigma_{t,z}^*$, $\varphi_t^*$, where

$$\begin{aligned} \pi_t &= \left(\frac{\partial b_{t,X,u,\alpha}}{\partial x}\right)\xi(t) + \left(\frac{\partial b_{t,X,u,\alpha}}{\partial u}\right)\beta(t)\,, \\ \tau_t &= \left(\frac{\partial \sigma_{t,X,u,\alpha}}{\partial x}\right)\xi(t) + \left(\frac{\partial \sigma_{t,X,u,\alpha}}{\partial u}\right)\beta(t)\,, \\ \varsigma_{t,z} &= \left(\frac{\partial \eta_{t,X,u,z,\alpha}}{\partial x}\right)\xi(t) + \left(\frac{\partial \eta_{t,X,u,z,\alpha}}{\partial u}\right)\beta(t)\,, \\ \varphi_t &= \left(\frac{\partial \gamma_{t,X,u,\alpha}}{\partial x}\right)\xi(t) + \left(\frac{\partial \gamma_{t,X,u,\alpha}}{\partial u}\right)\beta(t)\,, \end{aligned}$$

we have

$$\lim_{T\to\infty} \mathbf{E}\left[p^*(T)\,\xi^*(T)\right] = 0, \tag{4.20}$$

$$\mathbf{E}\left[\int_0^\infty C(t)\,(1 + |\xi^*(t)|)\,dt\right] < \infty, \tag{4.21}$$

$$\mathbf{E}\left[\int_0^T (\xi^*(t))^2 \left\{(q^*)^2(t) + \int_{\mathbb{R}_0} (r^*(t,z))^2\,v_\alpha(dz) + \sum_{j=1}^D \left(\gamma^j\right)^2(t)\,\lambda_j(t)\right\} dt\right] < \infty \tag{4.22}$$

where $\lambda(t) = (\lambda_1(t)\,, .., \lambda_D(t))^T$ ,and

$$\begin{aligned} &\left[\int_0^T (p^*(t))^2 \left[(\tau^*)^2(t,\ X^*(t)\,,\ \alpha(t)\,,\ u^*(t)) + \int_{\mathbb{R}_0} (\varsigma^*)^2(t,\ X^*(t)\,,\ \alpha(t)\,,\ u^*(t)\,,\ z)\,v_\alpha(dz)\right.\right. \\ &\qquad \left.\left. + \sum_{j=1}^D \left(\varphi^{j*}\right)^2(t,\ X^*(t)\,,\ \alpha(t)\,,\ u^*(t))\,\lambda_j(t)\right] dt\right] < \infty, \end{aligned} \tag{4.23}$$

for all $T < \infty$. Then $u^*$ is a stationary point for $E[H \ / \ \varepsilon_t]$ in the sense that for all $t \geq 0$,

$$\mathbf{E}\left[\frac{\partial}{\partial u} H\left(t,\ X^*(t),\ e_i,\ u^*,\ p^*(t),\ q^*(t),\ r^*(t,.),\ s^*(t)\right) \ /\varepsilon_t\right] = 0. \tag{4.24}$$

**Proof.** First note that by $(A3)$, $(A4)$ and (4.21) we have that

$$\begin{aligned} 0 &= \frac{\partial}{\partial \epsilon} J\left(u^* + \epsilon\beta\right)\Big|_{\epsilon=0} \\ &= \frac{\partial}{\partial \epsilon}\mathbf{E}\left[\int_0^\infty f\left(t,\ X^{u^*+\epsilon\beta}(t),\ u^*(t) + \epsilon\beta,\ \alpha(t)\right) dt\right]\Big|_{\epsilon=0} \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbf{E}\left[\int_0^\infty \left\{f\left(t,\ X^{u^*+\epsilon\beta}(t),\ u^*(t)+\epsilon\beta,\ \alpha(t)\right) - f\left(t,\ X^{u^*}(t),\ u^*(t)\ ,\alpha(t)\right)\right\} dt\right] \\ &= \mathbf{E}\left[\int_0^\infty \left\{\frac{\partial f}{\partial x}\left(t,\ X^{u^*}(t),\ u^*(t)\ ,\alpha(t)\right)\xi^*(t) + \frac{\partial f}{\partial u}\left(t,\ X^{u^*}(t),\ u^*(t)\ ,\alpha(t)\right)\beta(t)\right\} dt\right]. \end{aligned} \tag{4.25}$$

We Know by the definition of $H$ that

$$\frac{\partial f}{\partial x}(t) = \frac{\partial H}{\partial x}(t) - \frac{\partial b}{\partial x}(t)\,p(t) - \frac{\partial \sigma}{\partial x}(t)\,q(t) - \int_{\mathbb{R}_0}\frac{\partial \eta}{\partial x}(t,z)\,r(t,z)\,v_\alpha(dz) - \sum_{j=1}^{D}\frac{\partial \gamma^j}{\partial x}(t)\,s_j(t)\,\lambda_j(t) \tag{4.26}$$

and the same for $\frac{\partial f}{\partial u}(t)$.

Applying the Itô formula to

$$p^*(t)\,\xi^*(t),$$

we obtain by (4.20), (A2), (4.22) and (4.23)

$$
\begin{aligned}
0 &= \lim_{T\to\infty} \mathbf{E}\left[p^*(T)\,\xi(T)\right] \\
&= \lim_{T\to\infty} \mathbf{E}\Bigg[\int_0^T p^*(t)\left\{\tfrac{\partial b}{\partial x}(t)\,\xi^*(t) + \tfrac{\partial b}{\partial u}(t)\,\beta(t)\right\}dt + \int_0^T \xi^*(t)\left(-\tfrac{\partial H^*(t)}{\partial x}\right)dt \\
&+ \int_0^T q^*(t)\left\{\tfrac{\partial \sigma}{\partial x}(t)\,\xi^*(t) + \tfrac{\partial \sigma}{\partial u}(t)\,\beta(t)\right\}dt \\
&+ \int_0^T \int_{\mathbb{R}_0} r^*(t,z)\left\{\tfrac{\partial \eta}{\partial x}(t,z)\,\xi^*(t) + \tfrac{\partial \eta}{\partial u}(t,z)\,\beta(t)\right\}v_\alpha(dz)\,dt \\
&+ \int_0^T \sum_{j=1}^{D} s_j^*(t)\left\{\tfrac{\partial \gamma^j}{\partial x}(t)\,\xi^*(t) + + \tfrac{\partial \gamma^j}{\partial u}(t)\,\beta(t)\,\lambda_j(t)\right\}dt\Bigg] \\
&= \lim_{T\to\infty} \mathbf{E}\Bigg[\int_0^T \xi^*(t)\Bigg\{\tfrac{\partial b}{\partial x}(t)\,p^*(t) + \tfrac{\partial \sigma}{\partial x} q^*(t) + \int_{\mathbb{R}_0} \tfrac{\partial \eta}{\partial x}(t,z)\,r^*(t,z)\,v_\alpha(dz) \\
&+ \sum_{j=1}^{D} \tfrac{\partial \gamma^j}{\partial x}(t)\,s_j^*(t) - \tfrac{\partial H^*(t)}{\partial x}\Bigg\}dt \\
&+ \int_0^T \beta(t)\left\{\tfrac{\partial b}{\partial u}(t)\,p^*(t) + \tfrac{\partial \sigma}{\partial u} q^*(t) + \int_{\mathbb{R}_0} \tfrac{\partial \eta}{\partial u}(t,z)\,r^*(t,z)\,v_\alpha(dz) + \sum_{j=1}^{D} \tfrac{\partial \gamma^j}{\partial u}(t)\,s_j^*(t)\right\}dt \\
&= \lim_{T\to\infty} \mathbf{E}\left[\int_0^T \xi^*(t)\left\{-\tfrac{\partial f}{\partial x}(t)\right\}dt + \int_0^T \beta(t)\left\{\tfrac{\partial H^*(t)}{\partial u} - \tfrac{\partial f}{\partial u}(t)\right\}dt\right] \\
&= -\lim_{T\to\infty} \mathbf{E}\left[\int_0^T \left\{\tfrac{\partial f}{\partial x}(t)\,\xi^*(t) + \tfrac{\partial f}{\partial u}(t)\,\beta(t)\right\}dt\right] + \lim_{T\to\infty} \mathbf{E}\left[\int_0^T \beta(t)\,\tfrac{\partial H^*(t)}{\partial u}\right]
\end{aligned}
$$

Hence

$$
\frac{d}{d\epsilon} J\left(u^* + \epsilon\beta\right)\big|_{\epsilon=0} = \lim_{T\to\infty} \mathbf{E}\left[\int_0^T \frac{\partial H^*}{\partial u}(t)\,\beta(t)\,dt\right]
$$

If

$$
\beta(s) = \theta 1_{[t,t+h]}(s),
$$

then

$$
\mathbf{E}\left[\int_t^{t+h} \frac{\partial}{\partial u} H^*\left(s,\ X_s^*,\ e_i,\ u_s^*,\ p_s^*,\ q_s^*,\ r^*(s,.),\ s_s^*\right)\theta ds\right] = 0.
$$

Differentiating with respect to $h$ at $h = 0$, we have

$$
\mathbf{E}\left[\frac{\partial}{\partial u} H^*\left(t,\ X_t^*,\ e_i,\ u_t^*,\ p_t^*,\ q_t^*,\ r^*(t,.),\ s_t^*\right)\theta\right] = 0.
$$

This holds for all $\varepsilon_t$-measurable $\theta$ and hence we obtain that

$$\mathbf{E}\left[\frac{\partial}{\partial u}H^*\left(t,\ X_t^*,\ e_i,\ u_t^*,\ p_t^*,\ q_t^*,\ r^*\left(t,.\right),\ s_t^*\right)\ /\varepsilon_t\right]=0.$$

Which proves the theorem. ■

## 4.5 Applications

### 4.5.1 Example 01(Optimal portfolio and consumption with regime switching)

We consider a continuous-time, finite-state, hidden Markov chain $\alpha=\{\alpha(t), t\in[0,\infty[\}$ taking values in a finite-state space $S=\{1,2,...,n\}$.

The financial market consists of two assets with $S_0$ the prices of the risk-free asset and $S_1$ of the stock are given

$$dS_0(t)=\rho S_0(t)\,dt \text{ for all } t\in[0,\infty[\,, S_0(0)>0, \tag{4.27}$$

and

$$dS_k(t)=S_k(t)\left\{b(t,\alpha(t))\,dt+\sigma(t,\alpha(t))\,dB(t)\right\}, \tag{4.28}$$

respectively, where the interest rate $\rho$ is a constant, the appreciation rate $b(t,i)$ and the volatility $\sigma(t,i)\neq 0$ are assumed to be deterministic and bounded.

The wealth of an agent $x(t)$ defined as

$$\begin{cases} dx(t) &= x(t)\left[(\pi(t)(b(t,\alpha(t))-\rho)+\rho-c(t))\,dt+\pi(t)\,\sigma(t,\alpha(t))\right]dB(t), \\ x(0) &= x_0>0, \end{cases} \tag{4.29}$$

where $\pi(.)$ is the fraction of the agent's wealth that is invested in the risky asset and $c(.)$ is the consumption of the agent and the control process $u(t)=(\pi(t),c(t))$, we have that

$$x(t)=x_0\exp\left[\int_0^t\left\{\rho+\pi(s)(b(s,\alpha(s))-\rho)-c(s)-\frac{1}{2}\pi^2(s)\,\sigma^2(s,\alpha(s))\right\}ds+\int_0^t\pi(s)\,\sigma(s,\alpha(s))\,dB(s)\right], \tag{4.30}$$

and the associated cost functional is

$$J(u) = \mathbf{E}\left[\int_0^\infty \mathrm{e}^{-\delta t} \ln\left(c(t)\, x(t)\right) dt\right], \tag{4.31}$$

where $\delta > 0$.The objective is to find an optimal control $u^*(.) = (c^*(.), \pi^*(.))$ that maximizes (4.31).

Now the Hamiltonian is

$$H(t,x,c,\pi,i,p,q) = \mathrm{e}^{-\delta t} \ln(cx) + (\pi(b(t,i)-\rho)+\rho-c)\, xp + \pi\sigma(t,i)\, xq, \tag{4.32}$$

then

$$\nabla_x H(t,x,c,\pi,i,p,q) \quad = e^{-\delta t}\tfrac{1}{x} + (\pi(b(t,i)-\rho)+\rho-c)\, p + \pi\sigma(t,i)\, q,$$

on the other hand we have

$$\begin{aligned} dp(t) \quad &= -\left(e^{-\delta t}\tfrac{1}{x(t)} + (\pi(b(t,\alpha(t))-\rho)+\rho-c(t))\, p(t) + \pi\sigma(t,\alpha(t))\, q(t)\right) dt \\ &+q(t)\, dB(t)\, s(t) + s(t)\, d\widetilde{\Phi}(t), \end{aligned} \tag{4.33}$$

$$\nabla_\pi H(t,x,c,\pi,i,p,q) \quad = (b(t,i)-\rho)\, px + \sigma(t,i)\, qx \quad , \tag{4.34}$$

$$\nabla_c H(t,x,c,\pi,i,p,q) \quad = e^{-\delta t}\tfrac{1}{c} - px \tag{4.35}$$

so that

$$q(t) \quad = -\tfrac{(b(t,i)-\rho)}{\sigma(t,i)} p(t), \tag{4.36}$$

and

$$c^*(t) \quad = e^{-\delta t}\tfrac{1}{p(t)x(t)} \tag{4.37}$$

then

$$\begin{aligned} dp(t) \quad &= -\left[\left(e^{-\delta t}\tfrac{1}{x(t)} + \pi(b(t,\alpha(t))-\rho) + \rho - e^{-\delta t}\tfrac{1}{p(t)x(t)}\right) p(t) - \pi(b(t,\alpha(t))-\rho)\, p(t)\right] dt \\ &-\tfrac{(b(t,\alpha(t))-\rho)}{\sigma(t,\alpha(t))} p(t)\, dB(t) + s(t)\, d\widetilde{\Phi}(t) \\ &= -\rho p(t)\, dt - \tfrac{(b(t,\alpha(t))-\rho)}{\sigma(t,\alpha(t))} p(t)\, dB(t) + s(t)\, d\widetilde{\Phi}(t) \\ &= -p(t)\left(\rho dt + \tfrac{(b(t,\alpha(t))-\rho)}{\sigma(t,\alpha(t))} dB(t)\right) + s(t)\, d\widetilde{\Phi}(t), \end{aligned}$$

Let us try to choose $s(t) = 0$. Then we have that

$$p(t) = p(0)\exp\left[\int_0^t \left\{-\rho - \frac{1}{2}\frac{(b(s,\alpha(s))-\rho)^2}{\sigma^2(s,\alpha(s))}\right\} ds - \int_0^t \frac{(b(s,\alpha(s))-\rho)}{\sigma(s,\alpha(s))} dB(s)\right]. \tag{4.38}$$

So to ensure that the requirement

$$\mathbf{E}\left[\overline{\lim_{t\to\infty}}\left[p(t)(x(t)-x^*(t))\right]\right] \geq 0,$$

is satisfied it suffices that

$$\mathbf{E}\left[\overline{\lim_{t\to\infty}}\left[p^*(t)(x^*(t))\right]\right] \leq 0. \tag{4.39}$$

Let us try to choose $c^*(t,\omega) = c^*$ and $\pi^*(t,\omega) = \pi^*$.

Then from (4.37) we get

$$\begin{aligned} p(t) &= e^{-\delta t}\tfrac{1}{c^* x(t)} \\ &= \tfrac{1}{c^* x_0}\exp\left[\int_0^t -\left\{\rho + \pi^*(b(s,\alpha(s))-\rho) - c^* - \tfrac{1}{2}\pi^{*2}\sigma^2(s,\alpha(s)) + \delta\right\} ds - \int_0^t \pi^*\sigma(s,\alpha(s))\, dB(s)\right] \end{aligned} \tag{4.40}$$

comparing (4.38) with (4.40) we get

$$\begin{aligned} &\rho + \pi^*(b(t,i)-\rho) - c^* - \tfrac{1}{2}\pi^{*2}\sigma^2(t,i) + \delta = \rho + \tfrac{1}{2}\tfrac{(b(t,i)-\rho)^2}{\sigma^2(t,i)} \\ &\pi^*\sigma(t,i) = \tfrac{(b(t,i)-\rho)}{\sigma(t,i)} \end{aligned}$$

then

$$c^* = \pi^*(b(t,i)-\rho) - \frac{1}{2}\left(\pi^{*2}\sigma^2(t,i) + \frac{(b(t,i)-\rho)^2}{\sigma^2(t,i)}\right) + \delta \tag{4.41}$$

$$\pi^* = \frac{(b(t,i)-\rho)}{\sigma^2(t,i)} \tag{4.42}$$

Substituting into (4.41) this gives

$$c^* = \frac{(b(t,i)-\rho)^2}{\sigma^2(t,i)} - \frac{1}{2}\left(\frac{(b(t,i)-\rho)^2}{\sigma^2(t,i)} + \frac{(b(t,i)-\rho)^2}{\sigma^2(t,i)}\right) + \delta = \delta \tag{4.43}$$

By (4.30) and (4.40) we have

$$\begin{aligned} p(t)\,x^*(t) &= p(0)\exp\left[\int_0^t \left\{-\rho(s,\alpha(s)) - \frac{1}{2}\frac{(b(t,i)-\rho)^2}{\sigma^2(t,i)}\right\}ds - \int_0^t \frac{(b(t,i)-\rho)}{\sigma(t,i)}dB(s)\right] \\ &\cdot x_0 \exp\left[\int_0^t \left\{\rho(s,\alpha(s)) + \frac{(b(t,i)-\rho)^2}{\sigma^2(t,i)} - \hat{c} - \frac{1}{2}\frac{(b(t,i)-\rho)^2}{\sigma^2(t,i)}\right\}ds + \int_0^t \frac{(b(t,i)-\rho)}{\sigma(t,i)}dB(s)\right] \\ &= p(0)\,x_0\exp\left[-c^*t\right] \end{aligned}$$

Therefore (4.39) holds.

We have proved the following theorem.

**Theorem 6.1** The optimal control of (4.29) – (4.31) are given by (4.42) and (4.43).

### 4.5.2 Exemple 02

We consider the following optimization problem which is to maximize the performance functional:

$$J(u) = \mathbf{E}\left[2\int_0^\infty e^{-\beta t}\sqrt{u(t)}dt\right], \tag{4.44}$$

where $x(t)$ is subject to

$$\begin{cases} dx(t) &= (A(t,\alpha(t))\,x(t) - u(t))\,dt - C(t,\alpha(t))\,x(t)\,dB(t), \\ x(t) &= x_0, \end{cases} \tag{4.45}$$

where $\beta, x_0 > 0$, $A(t,i), C(t,i) > 0$, for all $i \in \mathcal{S} = \{1, 2, ..., n\}$.

In this case the Hamiltonian function takes the form

$$H(t,x,u,i,p,q) = 2\sqrt{u}e^{-\beta t} + (A(t,i)\,x - u)\,p - C(t,i)\,xq,$$

then

$$\begin{aligned} H_u(t,x,u,i,p,q) &= e^{-\beta t}\frac{1}{\sqrt{u}} - p \\ H_x(t,x,u,i,p,q) &= (A(t,i))\,p - C(t,i)\,q. \end{aligned}$$

Therefore, if $H_u = 0$ we get

$$e^{-\beta t}\frac{1}{\sqrt{u}} - p = 0 \tag{4.46}$$

The adjoint equation is given by

$$\begin{aligned} dp(t) &= -\left[A(t,\alpha(t))\,p(t) - C(t,\alpha(t))\,q(t)\right]dt \\ &+ q(t)\,dB(t) + s(t)\,d\widetilde{\Phi}(t). \end{aligned}$$

Let us try to choose $q(t) = s(t) = 0$. So

$$dp(t) = -A(t,\alpha(t))\,p(t)\,dt,$$

this leads to

$$p(t) = p(0)\,e^{-\int_0^t A(s,\alpha(s))ds}, \tag{4.47}$$

for some constant $p(0)$ and by $(4.46)$,

$$u^*(t) = \frac{e^{-2\beta t}}{\left(p(0)\,e^{-\int_0^t A(s,\alpha(s))ds}\right)^2} \tag{4.48}$$

Inserting $u^*(t)$ into $(4.45)$, we get

$$\begin{cases} dx^*(t) &= x^*(t)\,A(t,\alpha(t)) - p(0)^{-2}\,e^{2\int_0^t (A(s,\alpha(s))-\beta)ds}dt - x^*(t)\,C(t,\alpha(t))\,dB(t), \\ x(t) &= x_0, \end{cases}$$

Let us consider the process $\Gamma(.)$ defined by

$$\Gamma(t) = \exp\left(\int_0^t -C(s,\alpha(s))\,dB(s) + A(s,\alpha(s))\,ds - \frac{1}{2}\int_0^t C^2(s,\alpha(s))\,ds\right),$$

Using integration by part we get

$$x^*(t) = x^*(0)\,\Gamma(t) - p(0)^{-2}\int_0^t \frac{e^{2\int_0^s (A(r,\alpha(r))-\beta)dr}}{\Gamma(s)}\Gamma(t)\,ds.$$

Hence

$$\mathbf{E}\left[x^*(t)\,e^{-\int_0^t A(s,\alpha(s))ds}\right] = x^*(0) - p(0)^{-2}\int_0^t E\left(e^{\int_0^s (A(r,\alpha(r))-2\beta)dr}\right)ds,$$

Therefore to ensure the positivity condition , we get the optimal $p(0)$ as

$$p^*(0) = \left[ \frac{x^*(0)}{\int_0^\infty \mathbf{E}\left( e^{\int_0^s (A(r,\alpha(r))-2\beta)dr} \right) ds} \right]^{-\frac{1}{2}}, \tag{4.49}$$

and we can verify that

$$\lim_{T\to\infty} \mathbf{E}\left[x^*(T)\, p^*(T)\right] = 0.$$

Therefore the transversality condition is verified, then with $p(0) = p^*(0)$ given by (4.49), the control $u^*$ given by (4.48) is optimal.

# Chapter 5

# Partial Information Maximum Principle for Optimal Control Problem with Regime Switching in the Conditional Mean-Field Model

In this chapter, we present our second main result. In Sects. 1 and 2 sufficient and necessary maximum principles are developed under partial information. An example of switching optimal control problem in conditional mean field setting is studied in Sect 3 .

We consider the following controlled regime-switching diffusion equation:

$$\begin{cases} dX(t) & = & b\left(t, X(t), \mathbf{E}\left(\phi(X(t)) / \mathcal{F}_{t-}^{\alpha}\right), u(t), \alpha(t-)\right) dt \\ & + & \sigma\left(t, X(t), \mathbf{E}\left(\varphi(X(t)) / \mathcal{F}_{t-}^{\alpha}\right), u(t), \alpha(t-)\right) dB(t) \\ X(0) & = & x_0, \end{cases} \tag{5.1}$$

where $x_0$ is a real number. This mean-field SDE is obtained as the mean-square limit as $n \to \infty$ of a system of interacting particles of the form

$$\begin{aligned} dX^{i,n}(t) &= b\left(t, X^{i,n}(t), \frac{1}{n}\sum_{i=1}^{n}\phi\left(X^{i,n}(t)\right), u(t), \alpha(t-)\right) dt \\ &+\sigma\left(t, X^{i,n}(t), \frac{1}{n}\sum_{i=1}^{n}\varphi\left(X^{i,n}(t)\right), u(t), \alpha(t-)\right) dB^i(t) \end{aligned}$$

where $\left(B^{i}(.), i \geq 1\right)$ is a collection of independent standard Brownian motions. Note that for more generality we consider the mean-field term as nonlinear functions of the state with the use of $\phi(.)$ and $\varphi(.)$, respectively. Moreover, in (5.1), the conditional expectations $\mathbf{E}\left(\phi(X(t))/\mathcal{F}_{t-}^{\alpha}\right)$ and $\mathbf{E}\left(\varphi(X(t))/\mathcal{F}_{t-}^{\alpha}\right)$ appear instead of the expectations $\mathbf{E}(\phi(X(t)))$ and $\mathbf{E}(\varphi(X(t)))$ because of the effect of the common switching process $\alpha(t)\ \ t \geq 0$. Because all the particles depend on the history of this process, their average (mean-field term) must depend on the history of $\alpha(t)$; see [39].

Here :

$$\begin{aligned} b\ &:[0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathcal{S} \rightarrow \mathbb{R} \\ \sigma\ &:[0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathcal{S} \rightarrow \mathbb{R} \\ f\ &:[0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathcal{S} \rightarrow \mathbb{R} \\ \phi, \varphi, \psi, \varrho\ &\ \mathbb{R} \rightarrow \mathbb{R}, \end{aligned}$$

are given continuous functions. $B(t)$ is one dimensional standard Brownian motion and the control process $u(t, \omega):[0,T] \times \Omega \rightarrow \mathcal{U}\ \ (\mathcal{U} \subset \mathbb{R})$ required to be $\mathcal{E}_{t}-$predictable and

$$\mathbf{E}\left[\int_{0}^{T}|u(t)|^{2} dt\right]<\infty.$$

Where $\mathcal{E}_{t} \subset \mathcal{F}_{t}$ be a given subfiltration, representing the information avialable to the controller at time $t$. We denote by $\mathcal{A}_{\varepsilon}$ the set of all admissible controls.

For each $e_{i} \in \mathcal{S}$ we introduce the following assumptions:

**H1)** The functions $\phi(.)$,$\varphi(.)$ ,$\psi(.)$ and $\varrho(.)$ are continuously differentiable; $g(.,., e_{i})$ is continuously differentiable with respect to $(x, y)$; $b(.,.,.,., e_{i}); \sigma(.,.,.,., e_{i})$, and $f(.,.,.,., e_{i})$ are continuous in $t$ and continuously differentiable with respect to $(x, y, u)$.

**H2)** For each $t$ and $e_{i} \in \mathcal{S}$ , all derivatives of $\phi(.)$, $\varphi(.)$ , $\psi(.)$ , $g(.,., e_{i})$, $b(t,.,.,., e_{i})$; $\sigma(t,.,.,., e_{i})$, and $f(t,.,.,., e_{i})$ with respect to $x, y$, and $u$ are Lipschitz continuous and bounded.

The existence and uniqueness of (5.1) is given in [38].

The cost functional is defined as follows:

$$\begin{aligned} J\left(x_{0}, e_{i}, u\right) &= \mathbf{E}\left[\int_{0}^{T} f\left(t, X(t), \mathbf{E}\left(\psi(X(t))/\mathcal{F}_{t-}^{\alpha}\right), u(t), \alpha(t-)\right) dt\right. \\ &\left.+ g\left(X(T), \mathbf{E}\left(\varrho(X(T))/\mathcal{F}_{T-}^{\alpha}\right), \alpha(T)\right)\right], \end{aligned} \tag{5.2}$$

where

$$f: \quad [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathcal{S} \to \mathbb{R}$$
$$g: \quad \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathcal{S} \to \mathbb{R}$$

are functions such that for all $x_i = x, y, u$

$$\begin{aligned}&\mathbf{E}\Big[\int_0^T \Big(\left|f\left(t, X(t), \mathbf{E}\left(\psi(X(t))/\mathcal{F}_{t-}^{\alpha}\right), u(t), \alpha(t-)\right)\right| dt \\ &+\left|\frac{\partial f}{\partial x_i}\left(t, X(t), \mathbf{E}\left(\psi(X(t))/\mathcal{F}_{t-}^{\alpha}\right), u(t), \alpha(t-)\right)\right|^2\Big) dt \\ &+\left|g\left(X(T), \mathbf{E}\left(\varrho(X(T))/\mathcal{F}_{T-}^{\alpha}\right), \alpha(T)\right)\right| + \left|\frac{\partial g}{\partial x_i}\left(X(T), \mathbf{E}\left(\varrho(X(T))/\mathcal{F}_{T-}^{\alpha}\right), \alpha(T)\right)\right|^2\Big] < \infty.\end{aligned}$$

Our control problem is to find $u^* \in \mathcal{A}_{\mathcal{E}}$ such that

$$J(x_0, e_i, u^*) = \sup_{u \in \mathcal{A}_{\mathcal{E}}} J(x_0, e_i, u) \tag{5.3}$$

Now let us define the Hamiltonian as follows:

$H : \mathbb{R}^4 \times \mathbb{R} \times \mathbb{R} \times \mathbb{S} \to \mathbb{R}$,

$$\begin{aligned}H(t, \bar{x}, u, p, q, e_i) &= f(t, x, y_1, u, e_i) + b(t, x, y_2, u, e_i)\, p \\ &+\sigma(t, x, y_3, u, e_i)\, q\end{aligned}$$

where $\bar{x} = (x, y_1, y_2, y_3)$. For simplicity, for a random variable $x$, $H(t, x, u, p, q, e_i)$ will be used instead of $H(t, x, \mathbf{E}\left(\phi(X(t))/\mathcal{F}_{t-}^{\alpha}\right), \mathbf{E}\left(\varphi(X(t))/\mathcal{F}_{t-}^{\alpha}\right), \mathbf{E}\left(\psi(X(t))/\mathcal{F}_{t-}^{\alpha}\right), u, p, q, e_i)$ with luttle abuse of notation. That is,

$$\begin{aligned}H(t, x, u, p, q, e_i) &= f\left(t, x, \mathbf{E}\left(\psi(x)/\mathcal{F}_{t-}^{\alpha}\right), u, e_i\right) + b\left(t, x, \mathbf{E}\left(\phi(x)/\mathcal{F}_{t-}^{\alpha}\right), u, e_i\right) p \\ &+\sigma\left(t, x, \mathbf{E}\left(\varphi(x)/\mathcal{F}_{t-}^{\alpha}\right), u, e_i\right) q\end{aligned} \tag{5.4}$$

The adjoint equation corresponding to $u^*$ and $X^{u^*}(.)$ in the unknown, adapted processes $(p(t), q(t), s(t))$

is the backward stochastic differential equation

$$\begin{cases} dp(t) &= -\left[b_x^*(t)\,p(t) + \sigma_x^*(t)\,q(t) + f_x^*(t)\right]dt \\ &\quad + \left[\mathbf{E}\left(b_y^*(t)\,p(t)/\mathcal{F}_{t-}^{\alpha}\right)\phi_x^*(t) + \mathbf{E}\left(\sigma_y^*(t)\,q(t)/\mathcal{F}_{t-}^{\alpha}\right)\varphi_x^*(t) + \mathbf{E}\left(f_y^*(t)/\mathcal{F}_{t-}^{\alpha}\right)\psi_x^*(t)\right]dt \\ &\quad + q(t)\,dB(t) + s(t)\,d\widetilde{\Phi}(t) \\ p(T) &= g_x^*(T) + \mathbf{E}\left(g_y^*(T)/\mathcal{F}_{T-}^{\alpha}\right)\varrho_x^*(T) \end{cases} \tag{5.5}$$

In view of [38] this backward equation has a unique solution $(p(t), q(t), s(t)) \in S_{\mathcal{F}}^2([0,T], \mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^2([0,T], \mathbb{R}) \times \mathcal{M}_{\mathcal{F}}^2([0,T], \mathbb{R}^D)$. Where

$$\begin{aligned} S_{\mathcal{F}}^2([0,T],\mathbb{R}) &= \left\{ f : \mathbb{R}\text{−valued } \mathcal{F}_t - \text{adapted càdlàg processes, s.t.:}\mathbf{E}\left[\sup_{0\le t\le T} |f(t)|^2\right] < \infty \right\}, \\ \mathcal{L}_{\mathcal{F}}^2([0,T],\mathbb{R}) &= \left\{ f : \mathbb{R}\text{−valued } \mathcal{F}\text{−progressively measurable process} : \|f\|_2^2 = \mathbf{E}\left[\int_0^T |f(t)|^2\,dt\right] < \infty \right\}, \\ \mathcal{M}_{\mathcal{F}}^2([0,T],\mathbb{R}^D) &= \left\{ f : \mathbb{R}^D - \text{valued } \mathcal{F}_t\text{-predictable processes, s.t.}\mathbf{E}\left[\int_0^T \sum_{j=1}^D |f_j(t)|^2 \lambda_j(t)\,dt\right] < \infty \right\}. \end{aligned}$$

For an admissible control $u(.)$, denote the corresponding trajectory of (5.1) by $X^u(.)$. In particular, if $u^*(\cdot)$ is an optimal control, then $X^{u^*}(.)$ is the associated optimal trajectory. In the sequel we use the following abbreviation:

$$\begin{aligned} b^*(t) &= b\left(t, X^{u^*}(t), \mathbf{E}\left(\phi\left(X^{u^*}(t)\right)/\mathcal{F}_{t-}^{\alpha}\right), u^*(t), \alpha(t-)\right) \\ \sigma^*(t) &= \sigma\left(t, X^{u^*}(t), \mathbf{E}\left(\varphi\left(X^{u^*}(t)\right)/\mathcal{F}_{t-}^{\alpha}\right), u^*(t), \alpha(t-)\right) \\ f^*(t) &= f\left(t, X^{u^*}(t), \mathbf{E}\left(\psi\left(X^{u^*}(t)\right)/\mathcal{F}_{t-}^{\alpha}\right), u^*(t), \alpha(t-)\right) \\ g^*(t) &= g\left(t, X^{u^*}(t), \mathbf{E}\left(\varrho\left(X^{u^*}(t)\right)/\mathcal{F}_{t-}^{\alpha}\right), \alpha(t)\right) \\ \phi^*(t) &= \phi\left(X^{u^*}(t)\right), \varphi^*(t) = \varphi\left(X^{u^*}(t)\right) \\ \psi^*(t) &= \psi\left(X^{u^*}(t)\right), \varrho^*(t) = \varrho\left(X^{u^*}(t)\right), \end{aligned}$$

also we use $h_a = \frac{\partial h}{\partial a}$ for all $a = x, y, u$, and $h = b, \sigma, f, g, b^*, \sigma^*, .....$

## 5.1 Partial information sufficient maximum principle

In this section we state and prove a sufficient maximum principle for the partial information control problem $(5.1) - (5.3)$.

**Theorem 5.1.1** *(Partial Information Sufficient Maximum Principle). Let $u^* \in \mathcal{A}_{\mathcal{E}}$ with corresponding state process $X^*(t) = X^{u^*}(t)$ and suppose there exists a solution $(p^*(t), q^*(t), s^*(t))$ of the corresponding adjoint equations* (5.5) *satisfying*

$$\begin{aligned}&\mathbf{E}\Big[\textstyle\int_0^T (p^*(t)(\sigma(t) - \sigma^*(t)))^2 + (q^*(t)((t) - X(t)))^2 \\ &+ \textstyle\sum_j \int_0^T (X^*(t) - X(t))^2 |s_j(t)|^2 \lambda_j(t)\Big] dt < \infty,\end{aligned} \tag{5.6}$$

*and*

$$\mathbf{E}\left[\int_0^T \left|\frac{\partial}{\partial u} H(t, X^*(t), u^*(t), p^*(t), q^*(t), e_i)\right|^2 dt\right] < \infty \tag{5.7}$$

*for all admissible controls $u \in \mathcal{A}_{\mathcal{E}}$. Further suppose that for all $t \in [0,T]$.*

1. The functions $\phi(.), \varphi(.), \psi(.)$, and $\varrho(.)$ are concave, the function $g(.,.,.)$ is concave in $(x, y)$, and the Hamiltonian $H(.,.,.,.,.,.)$ is concave in $(x^*, u)$.
2. The functions $b_y(.,.,.,.,.)$, $\sigma_y(.,.,.,.,.)$, $f_y(.,.,.,.,.)$, and $g_y(.,.,.)$ are nonnegative.
3. $$\mathbf{E}[H(t, X^*(t), u^*(t), p^*(t), q^*(t), e_i) / \mathcal{E}_t] = \max_{u \in U} \mathbf{E}[H(t, X^*(t), u, p^*(t), q^*(t), e_i) / \mathcal{E}_t]. \tag{5.8}$$

Then $u^*$ is a partial information optimal control.

**Proof.** Choose $u \in \mathcal{A}_{\mathcal{E}}$ and $X^u(t)$ the corresponding state trajectory , we set

$$\begin{aligned}
&\phi(t) = \phi(X^u(t)), \varphi(t) = \varphi(X^u(t)), \\
&\psi(t) = \psi(X^u(t)), \varrho(t) = \varrho(X^u(t)) \\
&b(t) = b\left(t, X^u(t), \mathbf{E}\left(\phi(X^u(t)) / \mathcal{F}^\alpha_{t-}\right), u(t), \alpha(t-)\right) \\
&\sigma(t) = \sigma\left(t, X^u(t), \mathbf{E}\left(\varphi(X^u(t)) / \mathcal{F}^\alpha_{t-}\right), u(t), \alpha(t-)\right) \\
&f(t) = f\left(t, X^u(t), \mathbf{E}\left(\psi(X^u(t)) / \mathcal{F}^\alpha_{t-}\right), u(t), \alpha(t-)\right) \\
&g(T) = g\left(T, X^u(T), \mathbf{E}\left(\varrho(X^u(T)) / \mathcal{F}^\alpha_{T-}\right), \alpha(T)\right) \\
&H(t) = H(t, X^u(t), u(t), p^*(t), q^*(t), \alpha(t-)) \\
&\hat{H}(t) = H\left(t, X^{u^*}(t), \hat{u}(t), p^*(t), q^*(t), \alpha(t-)\right),
\end{aligned}$$

and consider

$$J(u) - J(u^*) = J_1 + J_2,$$

where

$$\begin{aligned} J_1 &= \mathbf{E}\left[\int_0^T f\left(t, X^u(t), \mathbf{E}\left(\psi(X^u(t))/\mathcal{F}_{t-}^{\alpha}\right), u(t), \alpha(t-)\right)\right. \\ &\left. - f\left(t, X^{u^*}(t), \mathbf{E}\left(\psi\left(X^{u^*}(t)\right)/\mathcal{F}_{t-}^{\alpha}\right), u^*(t), \alpha(t-)\right)\right] \\ J_2 &= \mathbf{E}\left[g\left(\left(T, X^u(T), \mathbf{E}\left(\varrho(X^u(T))/\mathcal{F}_{T-}^{\alpha}\right), \alpha(T)\right)\right)\right. \\ &\left. - g\left(T, X^{u^*}(T), \mathbf{E}\left(\varrho\left(X^{u^*}(T)\right)/\mathcal{F}_{T-}^{\alpha}\right), \alpha(T)\right)\right]. \end{aligned}$$

Note that

$$J_1 = J_{1.1} - J_{1.2} - J_{1.3},$$

where

$$\begin{aligned} J_{1.1} &= \mathbf{E}\left[\int_0^T (H(t) - H^*(t))\, dt\right] \\ J_{1.2} &= \mathbf{E}\left[\int_0^T (b(t) - b^*(t))\, p^*(t)\, dt\right] \\ J_{1.3} &= \mathbf{E}\left[\int_0^T (\sigma(t) - \sigma^*(t))\, q^*(t)\, dt\right] \end{aligned}$$

By concavity we have

$$\begin{aligned} & H(t) - H^*(t) \\ & \leq H_x^*(t)(X^u(t) - X^*(t)) + b_y^*(t)\,\mathbf{E}\left(\phi(t) - \phi^*(t)/\mathcal{F}_{t-}^{\alpha}\right) p^*(t) \\ & + \sigma_y^*(t)\,\mathbf{E}\left(\varphi(t) - \varphi^*(t)/\mathcal{F}_{t-}^{\alpha}\right) q^*(t) + f_y^*(t)\,\mathbf{E}\left(\psi(t) - \psi^*(t)/\mathcal{F}_{t-}^{\alpha}\right) \\ & + H_u^*(t)(u(t) - u^*(t)) \\ & \leq H_x^*(t)(X^u(t) - X^*(t)) + b_y^*(t)\,\mathbf{E}\left(\phi_x(t)(X^u(t) - X^*(t))/\mathcal{F}_{t-}^{\alpha}\right) \hat{p}(t) \\ & + \sigma_y^*(t)\,\mathbf{E}\left(\varphi_x(t)(X^u(t) - X^*(t))/\mathcal{F}_{t-}^{\alpha}\right) q^*(t) + \hat{f}_y(t)\,\mathbf{E}\left(\psi_x(t)(X^u(t) - X^*(t))/\mathcal{F}_{t-}^{\alpha}\right) \\ & + H_u^*(t)(u(t) - u^*(t)) \end{aligned}$$

since $u \to \mathbf{E}[H(t, X^*(t), u, p^*(t), q^*(t), i)/\mathcal{E}_t]$ is maximal for $u = u^*(t)$ and $u(t), u^*(t)$ are $\mathcal{E}_t-$mesurable, we get :

$$\begin{aligned} 0 &\geq \frac{\partial}{\partial u}\mathbf{E}\left[H(t, X^*(t), u, p^*(t), q^*(t), e_i)/\mathcal{E}_t\right]_{u=u^*(t)}(u(t) - u^*(t)) \\ &= \mathbf{E}\left[H_u(t, X^*(t), u, p^*(t), q^*(t), e_i)(u(t) - u^*(t))/\mathcal{E}_t\right]_{u=u^*(t)} \end{aligned} \tag{5.9}$$

Combining (5.6) – (5.7) – (5.8) and (5.9) we obtain

$$
\begin{aligned}
J_{1.1} \quad & \leq \mathbf{E}\Big[\int_0^T \big\{H_x^*(t)\left(X^u(t)-X^*(t)\right)+b_y^*(t)\,\mathbf{E}\left(\phi_x^*(t)\left(X^u(t)-X^*(t)\right)/\mathcal{F}_{t-}^{\alpha}\right)p^*(t) \\
& +\sigma_y^*(t)\,\mathbf{E}\left(\varphi_x^*(t)\left(X^u(t)-X^*(t)\right)/\mathcal{F}_{t-}^{\alpha}\right)q^*(t)+f_y^*(t)\,\mathbf{E}\left(\psi_x^*(t)\left(X^u(t)-X^*(t)\right)/\mathcal{F}_{t-}^{\alpha}\right)\big\}\,dt\Big]
\end{aligned}
$$

Similarly, since $g$ is concave we get,

$$
\begin{aligned}
J_2 \quad & =\mathbf{E}\left[g\left(\left(T,X^u(T),\mathbf{E}\left(\varrho\left(X^u(T)\right)/\mathcal{F}_{T-}^{\alpha}\right),\alpha(T)\right)\right)-g\left(T,X^{u^*}(T),\mathbf{E}\left(\varrho\left(X^{u^*}(T)\right)/\mathcal{F}_{T-}^{\alpha}\right),\alpha(T)\right)\right] \\
& \leq \mathbf{E}\left\{g_x^*(T)\left(X^u(T)-X^*(T)\right)+g_y^*(T)\,\mathbf{E}\left(\varrho\left(X^u(T)\right)-\varrho\left(X^*(T)\right)/\mathcal{F}_{T-}^{\alpha}\right)\right\} \\
& \leq \mathbf{E}\left\{g_x^*(T)\left(X^u(T)-X^*(T)\right)+g_y^*(T)\,\mathbf{E}\left(\varrho_x^*(T)\left(X^u(T)-X^*(T)\right)/\mathcal{F}_{T-}^{\alpha}\right)\right\} \\
& \leq \mathbf{E}\left[p^*(T)\left(X^u(T)-X^*(T)\right)\right].
\end{aligned}
$$

By the Itô formula

$$
\begin{aligned}
& \mathbf{E}\left[p^*(T)\left(X^u(T)-X^*(T)\right)\right] \\
& =\mathbf{E}\left[\int_0^T\left(X^u(t)-X^*(t)\right)dp^*(t)+p^*(t)\,d\left(X^u(t)-X^*(t)\right)+q^*(t)\left(\sigma(t)-\sigma^*(t)\right)dt\right] \\
& =-\mathbf{E}\Big\{\int_0^T\left(X^u(t)-X^*(t)\right)\Big[b_x^*(t)\,p^*(t)+\sigma_x^*(t)\,q^*(t)+f_x^*(t)+\mathbf{E}\left(b_y^*(t)\,p(t)/\mathcal{F}_{t-}^{\alpha}\right)\phi_x^*(t) \\
& +\ \mathbf{E}\left(\sigma_y^*(t)\,q(t)/\mathcal{F}_{t-}^{\alpha}\right)\varphi_x^*(t)+\mathbf{E}\left(f_y^*(t)/\mathcal{F}_{t-}^{\alpha}\right)\hat{\psi}_x(t)\Big]\,dt\Big\} \\
& +\mathbf{E}\left[\int_0^T\left[p^*(t)\left(b(t)-b^*(t)\right)+q^*(t)\left(\sigma(t)-\sigma^*(t)\right)\right]dt\right] \\
& =-\mathbf{E}\Big\{\int_0^T\left(X^u(t)-X^*(t)\right)\Big[H_x^*(t)+\mathbf{E}\left(b_y^*(t)\,p(t)/\mathcal{F}_{t-}^{\alpha}\right)\hat{\phi}_x(t) \\
& +\ \mathbf{E}\left(\sigma_y^*(t)\,q(t)/\mathcal{F}_{t-}^{\alpha}\right)\varphi_x^*(t)+\mathbf{E}\left(f_y^*(t)/\mathcal{F}_{t-}^{\alpha}\right)\psi_x^*(t)\Big]\,dt\Big\} \\
& +\mathbf{E}\left[\int_0^T\left[p^*(t)\left(b(t)-b^*(t)\right)+q^*(t)\left(\sigma(t)-\sigma^*(t)\right)\right]dt\right]
\end{aligned}
$$

Then

$$
\begin{aligned}
&J(u) - J(u^*) \\
&\leq \mathbf{E}\Big[\int_0^T \big\{H_x^*(t)(X^u(t) - X^*(t)) + b_y^*(t)\,\mathbf{E}\big(\phi_x^*(t)(X^u(t) - X^*(t))/\mathcal{F}_{t-}^{\alpha}\big)p^*(t) \\
&+\hat{\sigma}_y(t)\,\mathbf{E}\big(\hat{\varphi}_x(t)(X^u(t) - X^*(t))/\mathcal{F}_{t-}^{\alpha}\big)q^*(t) + f_y^*(t)\,\mathbf{E}\big(\psi_x^*(t)(X^u(t) - X^*(t))/\mathcal{F}_{t-}^{\alpha}\big)\big\}\,dt\Big] \\
&-\mathbf{E}\Big[\int_0^T (b(t) - b^*(t))p^*(t)\,dt\Big] - \mathbf{E}\Big[\int_0^T (\sigma(t) - \sigma^*(t))q^*(t)\,dt\Big] \\
&-\mathbf{E}\Big\{\int_0^T (X^u(t) - X^*(t))\big[H_x^*(t) + \mathbf{E}\big(b_y^*(t)p(t)/\mathcal{F}_{t-}^{\alpha}\big)\phi_x^*(t) \\
&+\ \mathbf{E}\big(\sigma_y^*(t)q(t)/\mathcal{F}_{t-}^{\alpha}\big)\varphi_x^*(t) + \mathbf{E}\big(f_y^*(t)/\mathcal{F}_{t-}^{\alpha}\big)\psi_x^*(t)\big]\,dt\Big\} \\
&+\mathbf{E}\Big[\int_0^T [p^*(t)(b(t) - b^*(t)) + q^*(t)(\sigma(t) - \sigma^*(t))]\,dt\Big] \\
&= \mathbf{E}\Big[\int_0^T \big\{H_x^*(t)(X^u(t) - X^*(t)) + b_y^*(t)\,\mathbf{E}\big(\phi_x^*(t)(X^u(t) - X^*(t))/\mathcal{F}_{t-}^{\alpha}\big)p^*(t) \\
&+\sigma_y^*(t)\,\mathbf{E}\big(\varphi_x^*(t)(X^u(t) - X^*(t))/\mathcal{F}_{t-}^{\alpha}\big)q^*(t) + f_y^*(t)\,\mathbf{E}\big(\psi_x^*(t)(X^u(t) - X^*(t))/\mathcal{F}_{t-}^{\alpha}\big)\big\}\,dt\Big] \\
&-\mathbf{E}\Big\{\int_0^T (X^u(t) - X^*(t))\big[H_x^*(t) + b_y^*(t)\,\mathbf{E}\big(\phi_x^*(t)/\mathcal{F}_{t-}^{\alpha}\big)p^*(t) \\
&+\ \sigma_y^*(t)\,\mathbf{E}\big(\varphi_x^*(t)/\mathcal{F}_{t-}^{\alpha}\big)q^*(t) + f_y^*(t)\,\mathbf{E}\big(\psi_x^*(t)/\mathcal{F}_{t-}^{\alpha}\big)\big]\,dt\Big\} \\
&= 0.
\end{aligned}
$$

Since this holds for all $u \in \mathcal{A}_{\mathcal{E}}$ , the result follows. ■

## 5.2 A partial information necessary maximum principle

In the previous section we proved that (under some conditions) an admissible control $u^*$ satisfying the partial information maximum condition (5.8) is indeed optimal. We now turn to the converse question: If $u^*$ is optimal, does it satisfy (5.8)

In addition to the assumptions in Section 2 we now assume the following:

$(A1)$ For all $t$, $h$ such that $0 \leq t < t+h \leq T$ and all bounded $\mathcal{E}_t$-measurable random variables $\alpha$, the control process $\beta(t)$ defined by

$$\beta(s) = \alpha 1_{[t,t+h]}(s)\ ,\ s \in [0,T]$$

belongs to $\mathcal{A}_{\mathcal{E}}$

$(A2)$ For all $u, \beta \in \mathcal{A}_{\mathcal{E}}$ with $\beta$ bounded, there exists $\epsilon > 0$ such that

$$u + y\beta \in \mathcal{A}_{\mathcal{E}} \text{ for all } y \in (-\epsilon, \epsilon).$$

$(A3)$ For given $u, \beta \in \mathcal{A}_{\mathcal{E}}$ with $\beta$ bounded we define the derivative process $Y(t) = Y^{(u,\beta)}(t)$ by

$$Y(t) := \frac{d}{dy} X^{u+y\beta}(t)\Big|_{y=0}$$

note $Y(0) = 0$ and

$$dY(t) = K(t)\,dt + L(t)\,dB(t),$$

where

$$\begin{aligned} K(t) &= b_x(t)\,Y(t) + b_y(t)\,\mathbf{E}\left(Y(t)\,\phi_x(X(t))\,/\mathcal{F}^{\alpha}_{t-}\right) + b_u(t)\,\beta(t) \\ L(t) &= \sigma_x(t)\,Y(t) + \sigma_y(t)\,\mathbf{E}\left(Y(t)\,\varphi_x(X(t))\,/\mathcal{F}^{\alpha}_{t-}\right) + \sigma_u(t)\,\beta(t) \end{aligned} \tag{5.10}$$

**Theorem 5.2.1** *(Partial Information Necessary Maximum Principle). Suppose that $u^* \in \mathcal{A}_{\mathcal{E}}$ is a local maximum for $J(u)$, in the sense that for all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ there exists $\varepsilon > 0$ such that $u + y\beta \in \mathcal{A}_{\mathcal{E}}$ for all $y \in (-\epsilon, \epsilon)$ and*

$$k(y) := J(u^* + y\beta) \tag{5.11}$$

*is maximal at $(y = 0)$. Suppose there exists a solution $(p^*(t), q^*(t), s^*(t))$ of the associated adjoint Equations (5.5), that is,*

$$\begin{cases} dp^*(t) &= -\left[b^*_x(t)\,p^*(t) + \sigma^*_x(t)\,q^*(t) + f^*_x(t)\right]dt \\ &\quad + \left[\mathbf{E}\left(b^*_y(t)\,p^*(t)\,/\mathcal{F}^{\alpha}_{t-}\right)\phi^*_x(t) + \mathbf{E}\left(\sigma^*_y(t)\,q^*(t)\,/\mathcal{F}^{\alpha}_{t-}\right)\varphi^*_x(t) + \mathbf{E}\left(f^*_y(t)\,/\mathcal{F}^{\alpha}_{t-}\right)\psi^*_x(t)\right]dt \\ &\quad + q^*(t)\,dB(t) + s^*(t)\,d\widetilde{\Phi}(t) \\ p^*(T) &= g^*_x(T) + \mathbf{E}\left(g^*_y(T)\,/\mathcal{F}^{\alpha}_{T-}\right)\varrho^*_x(T). \end{cases}$$

*Moreover, suppose that, if $Y^*(t) = Y^{(u^*,\beta)}(t)$ and $K^*(t)$ and $L^*(t)$ are the corresponding coef-*

*ficients* (5.10). *Moreover, let us assume that,*

$$
\begin{array}{ll}
\mathbf{E}\left[\int_0^T \left(p^*(t)^2 \left\{\left(\frac{\partial\sigma}{\partial x}\right)^2(t)\, Y^*(t)^2 + \left(\frac{\partial\sigma}{\partial u}\right)^2(t)\,\beta(t)^2\right\} + Y^*(t)^2\, q^*(t)^2\right) dt\right] & < \infty \\
\mathbf{E}\left[(Y^*(t))^2 \sum_j \int_0^T |s_j(t)|^2\, \lambda_j(t)\, dt\right] & < \infty
\end{array}
\tag{5.12}
$$

*Then* $u^*$ *is a stationary point for* $\mathbf{E}[H(t)/\mathcal{E}_t]$ *in the sense that for a. a.* $t \in [0,T]$ *we have*

$$
\mathbf{E}\left[H_u\left(t, X^*(t), u^*, p^*(t), q^*(t), e_i\right)/\mathcal{E}_t\right] = 0
$$

**Proof.** Put $X^*(t) = X^{u^*}(t)$. Then with $k$ as in (5.11) we have

$$
\begin{aligned}
0 \;&= k'(0) = \frac{d}{dy} J(u^* + y\beta)\Big|_{y=0} \\
&= \mathbf{E}\Bigg[\int_0^T \Big\{ f_x\left(t, X^{u^*}(t), \mathbf{E}\left(\psi\left(X^{u^*}(t)\right)/\mathcal{F}^{\alpha}_{t-}\right), u^*, \alpha(t-)\right) \frac{d}{dy} X^{u^*+y\beta}(t)\Big|_{y=0} \\
&+ f_y\left(t, X^{u^*}(t), \mathbf{E}\left(\psi\left(X^{u^*}(t)\right)/\mathcal{F}^{\alpha}_{t-}\right), u^*, \alpha(t-)\right) \mathbf{E}\left(\frac{d}{dy}\psi\left(X^{u^*+y\beta}(t)\right)\Big|_{y=0}/\mathcal{F}^{\alpha}_{t-}\right) \\
&+ f_u\left(t, X^{u^*}(t), \mathbf{E}\left(\psi\left(X^{u^*}(t)\right)/\mathcal{F}^{\alpha}_{t-}\right), u^*(t), \alpha(t-)\right)\beta(t)\Big\}\, dt \\
&+ g_x\left(X^{u^*}(T), \mathbf{E}\left(\varrho\left(X^{u^*}(T)\right)/\mathcal{F}^{\alpha}_{T-}\right), \alpha(T)\right) \frac{d}{dy} X^{u^*+y\beta}(t)\Big|_{y=0} \\
&+ g_y\left(X^{u^*}(T), \mathbf{E}\left(\varrho\left(X^{u^*}(T)\right)/\mathcal{F}^{\alpha}_{T-}\right), \alpha(T)\right) \mathbf{E}\left(\frac{d}{dy}\varrho\left(X^{u^*+y\beta}(t)\right)\Big|_{y=0}/\mathcal{F}^{\alpha}_{t-}\right)\Bigg] \\
&= \mathbf{E}\Big[\int_0^T \Big\{ f_x\left(t, X^{u^*}(t), \mathbf{E}\left(\psi\left(X^{u^*}(t)\right)/\mathcal{F}^{\alpha}_{t-}\right), u^*, \alpha(t-)\right) Y^*(t) \\
&+ f_y\left(t, X^{u^*}(t), \mathbf{E}\left(\psi\left(X^{u^*}(t)\right)/\mathcal{F}^{\alpha}_{t-}\right), u^*, \alpha(t-)\right) \mathbf{E}\left(Y^*(t)\,\psi_x\left(X^{u^*}(t)\right)/\mathcal{F}^{\alpha}_{t-}\right) \\
&+ f_u\left(t, X^{u^*}(t), \mathbf{E}\left(\psi\left(X^{u^*}(t)\right)/\mathcal{F}^{\alpha}_{t-}\right), u^*(t), \alpha(t-)\right)\beta(t)\Big\}\, dt\Big] \\
&+ \mathbf{E}\Big[g_x\left(X^{u^*}(T), \mathbf{E}\left(\varrho\left(X^{u^*}(T)\right)/\mathcal{F}^{\alpha}_{T-}\right), \alpha(T)\right) Y^*(T) \\
&+ g_y\left(X^{u^*}(T), \mathbf{E}\left(\varrho\left(X^{u^*}(T)\right)/\mathcal{F}^{\alpha}_{T-}\right), \alpha(T)\right) \mathbf{E}\left(Y^*(T)\,\varrho_x\left(X^{u^*}(T)\right)/\mathcal{F}^{\alpha}_{t-}\right)\Big].
\end{aligned}
\tag{5.13}
$$

By (5.12), and the Itô formula,

$$\begin{aligned}
&\mathbf{E}\left[g_x^*(T)\hat{Y}(T)+g_y^*(T)\mathbf{E}\left(Y^*(t)\varrho_x\left(X^{u^*}(T)\right)/\mathcal{F}_{t-}^{\alpha}\right)\right] \\
&=\mathbf{E}\left[p^*(T)Y^*(T)\right] \\
&=\mathbf{E}\left[\int_0^T p^*(t)\left\{b_x^*(t)Y^*(t)+b_y^*(t)\mathbf{E}\left(Y^*(t)\phi_x(X^*(t))/\mathcal{F}_{t-}^{\alpha}\right)+b_u^*(t)\beta(t)\right.\right. \\
&+Y^*(t)\{b_x^*(t)p^*(t)+\sigma_x^*(t)q^*(t)+f_x^*(t) \\
&+\mathbf{E}\left(b_y^*(t)p^*(t)/\mathcal{F}_{t-}^{\alpha}\right)\phi_x^*(t)+\mathbf{E}\left(\sigma_y^*(t)q^*(t)/\mathcal{F}_{t-}^{\alpha}\right)\varphi_x^*(t)+\mathbf{E}\left(f_y^*(t)/\mathcal{F}_{t-}^{\alpha}\right)\psi_x^*(t)\} \\
&+\left.\left.q^*(t)\left(\sigma_x^*(t)Y^*(t)+\sigma_y^*(t)\mathbf{E}\left(Y^*(t)\varphi_x(X^*(t))/\mathcal{F}_{t-}^{\alpha}\right)+\sigma_u^*(t)\beta(t)\right)\right\}\right]
\end{aligned} \tag{5.14}$$

Now

$$\begin{aligned}
H_x(t) &= f_x(t)+f_y(t)\mathbf{E}\left(\psi_x(t)/\mathcal{F}_{t-}^{\alpha}\right)+\left(b_x(t)+b_y(t)\mathbf{E}\left(\phi_x(t)/\mathcal{F}_{t-}^{\alpha}\right)\right)p(t) \\
&+\left(\sigma_x(t)+\sigma_y(t)\mathbf{E}\left(\varphi_x(t)/\mathcal{F}_{t-}^{\alpha}\right)\right)q(t), \\
H_u(t) &= f_u(t)+b_u(t)p(t)+\sigma_u(t)q(t).
\end{aligned}$$

Combined with (5.13) and (5.14) this gives

$$\begin{aligned}
0 &= \mathbf{E}\int_0^T\left[f_u\left(t,X^{u^*}(t),\mathbf{E}\left(\psi\left(X^{u^*}(t)\right)/\mathcal{F}_{t-}^{\alpha}\right),u^*(t),\alpha(t-)\right)\right. \\
&+\left.p^*(t)b_u^*(t)\beta(t)+q^*(t)\sigma_u^*(t)\right]\beta(t)\,dt \\
&=\mathbf{E}\left[\int_0^T H_u(t,X^*(t),u^*(t),p^*(t),q^*(t),e_i)\beta(t)\,dt\right]
\end{aligned}$$

Fix $t\in[0,T]$ and apply the above to $\beta$ where

$$\beta(s)=\alpha 1_{[t,t+h]}(s)\ ,\ s\in[0,T]$$

where $t+h\leq T$ and $\alpha$ is bounded $\mathcal{E}_t$-measurable random variables. Then

$$\mathbf{E}\left[\int_t^{t+h}H_u(s,X^*(s),u^*(s),p^*(s),q^*(s),e_i)\alpha ds\right]=0$$

Differentiating with respect to $h$ at $h=0$ gives

$$\mathbf{E}\left[H_u(s,X^*(s),u^*(s),p^*(s),q^*(s),e_i)\alpha\right]=0$$

Since this holds for all bounded $\mathcal{E}_t$-measurable random variables $\alpha$ we have that

$$\mathbf{E}\left[H_u\left(s, X^*\left(s\right), u^*\left(s\right), p^*\left(s\right), q^*\left(s\right), e_i\right)/\mathcal{E}_t\right] = 0,$$

which proves the theorem. ■

## 5.3 Application

As an example, consider the following optimization problem which is to maximize the performance functional:

$$J\left(u\right) = \frac{-1}{2}\mathbf{E}\left[S\left(\alpha\left(T\right)\right)\left(X\left(T\right)\right)^2\right] \tag{5.15}$$

where $X(t)$ is subject to

$$\begin{cases} dX\left(t\right) & = \left[A\left(\alpha\left(t-\right)\right)X\left(t\right) + A^*\left(\alpha\left(t-\right)\right)\mathbf{E}\left(X\left(t\right)/\mathcal{F}_{t-}^{\alpha}\right) + B\left(\alpha\left(t-\right)\right)u\left(t\right)\right]dt \\ & + \left[C\left(\alpha\left(t-\right)\right)u\left(t\right)\right]dB\left(t\right) \\ X\left(0\right) & = x_0 \end{cases} \tag{5.16}$$

Here,$A(i), A^*(i), B(i), S(i), x_0 \in \mathbb{R}$ and $C(i) > 0$ for each $i \in \mathcal{S}$. ($\mathcal{S} = \{1, 2, 3, ..., D\}$)

We associate to this problem the Hamiltonian

$$\begin{aligned} H\left(t, x, u, p, q, i\right) & = \left[A\left(i\right)x + A^*\left(i\right)\mathbf{E}\left(x/\mathcal{F}_{t-}^{\alpha}\right) + B\left(i\right)u\right]p \\ & + \left[C\left(i\right)u\right]q \end{aligned} \tag{5.17}$$

and the adjoint equation

$$\begin{cases} dp\left(t\right) & = -\left[A\left(\alpha\left(t-\right)\right)p\left(t\right) + A^*\left(\alpha\left(t-\right)\right)\mathbf{E}\left(p\left(t\right)/\mathcal{F}_{t-}^{\alpha}\right)\right]dt \\ & + q\left(t\right)dB\left(t\right) + s\left(t\right)\widetilde{d\Phi}\left(t\right) \\ p\left(T\right) & = -S\left(\alpha\left(T\right)\right)X\left(T\right). \end{cases} \tag{5.18}$$

For simplicity, put $X^*\left(t\right) = \mathbf{E}\left(X\left(t\right)/\mathcal{F}_{t-}^{\alpha}\right), p^*\left(t\right) = \mathbf{E}\left(p\left(t\right)/\mathcal{F}_{t-}^{\alpha}\right), q^*\left(t\right) = \mathbf{E}\left(q\left(t\right)/\mathcal{F}_{t-}^{\alpha}\right)$ and $L\left(t\right) = L\left(\alpha\left(t-\right)\right)$,for all $L = A, A^*, B, C, S$.

Via the conjecture of Peng to solve this system, we put

$$p(t) = \upsilon(t, \alpha(t-)) X(t) + \gamma(t, \alpha(t-)) \mathbf{E}\left(X(t) / \mathcal{F}_{t-}^{\alpha}\right)$$

for some functions $\upsilon(.,.), \gamma(.,.) : [0;T] \times \mathcal{S} \rightarrow \mathbb{R}$ differentiable in $t$ to be determined. For each $i \in \mathcal{S}$ and $t \geq 0$, denote $\upsilon'(t,i) = \frac{d}{dt}\upsilon(t,i)$ and $\gamma'(t,i) = \frac{d}{dt}\gamma(t,i)$ .We have

$$d\upsilon(t,i) \quad = \left[\upsilon'(t,i) + \sum_j (\upsilon(t,j) - \upsilon(t,i)) \lambda_j(t)\right] dt + \sum_j (\upsilon(t,j) - \upsilon(t,i)) d\widetilde{\Phi}(t),$$

A similar equation holds for $\gamma(t, \alpha(t-))$. Denote $\upsilon(t) = \upsilon(t, \alpha(t-)), \upsilon'(t) = \upsilon'(t, \alpha(t-)), \gamma(t) = \gamma(t, \alpha(t-)), \gamma'(t) = \gamma'(t, \alpha(t-))$. Then by the Itô formula

$$\begin{aligned}
dp(t) \quad &= d\left(\upsilon(t, \alpha(t-)) X(t) + \gamma(t, \alpha(t-)) \mathbf{E}\left(X(t) / \mathcal{F}_{t-}^{\alpha}\right)\right) \\
&= X(t)\, d\upsilon(t) + \upsilon(t)\, dX(t) + \mathbf{E}\left(X(t) / \mathcal{F}_{t-}^{\alpha}\right) d\gamma(t) + \gamma(t)\, d\mathbf{E}\left(X(t) / \mathcal{F}_{t-}^{\alpha}\right) \\
&= X(t)\left[\upsilon'(t,i) + \sum_j (\upsilon(t,j) - \upsilon(t,i)) \lambda_j(t)\right] dt + X(t) \sum_j (\upsilon(t,j) - \upsilon(t,i))\, \widetilde{d\Phi}(t) \\
&+ \upsilon(t)\left[A(t) X(t) + A^*(t) \mathbf{E}\left(X(t) / \mathcal{F}_{t-}^{\alpha}\right) + B(t) u(t)\right] dt \\
&+ \upsilon(t) [C(t) u(t)]\, dB(t) + \mathbf{E}\left(X(t) / \mathcal{F}_{t-}^{\alpha}\right)\left[\gamma'(t,i) + \sum_j (\gamma(t,j) - \gamma(t,i)) \lambda_j(t)\right] dt \\
&+ \mathbf{E}\left(X(t) / \mathcal{F}_{t-}^{\alpha}\right) \sum_j (\gamma(t,j) - \gamma(t,i))\, \widetilde{d\Phi}(t) \\
&+ \gamma(t)\left((A(t) + A^*(t)) \mathbf{E}\left(X(t) / \mathcal{F}_{t-}^{\alpha}\right) + B(t) \mathbf{E}\left(u(t) / \mathcal{F}_{t-}^{\alpha}\right)\right) dt,
\end{aligned} \tag{5.19}$$

and by (5.18) we get

$$
\begin{aligned}
&-\left[A(t)\,p(t)+A^{*}(t)\,p^{*}(t)\right]\\
&=X(t)\left[\upsilon'(t,i)+\sum_{j}\left(\upsilon(t,j)-\upsilon(t,i)\right)\lambda_{j}(t)\right]+\upsilon(t)\left[A(t)\,X(t)+A^{*}(t)\,X^{*}(t)+B(t)\,u(t)\right]\\
&+X^{*}(t)\left[\gamma'(t,i)+\sum_{j}\left(\gamma(t,j)-\gamma(t,i)\right)\lambda_{j}(t)\right]\\
&+\gamma(t)\left(\left(A(t)+A^{*}(t)\right)X^{*}(t)+B(t)\,\mathbf{E}\left(u(t)/\mathcal{F}_{t-}^{\alpha}\right)\right)\\
&q(t)=\upsilon(t)\left[C(t)\,u(t)\right]\\
&s(t)=X(t)\sum_{j}\left(\upsilon(t,j)-\upsilon(t,i)\right)+\mathbf{E}\left(X(t)/\mathcal{F}_{t-}^{\alpha}\right)\sum_{j}\left(\gamma(t,j)-\gamma(t,i)\right)
\end{aligned}
\tag{5.20}
$$

Let $\bar{u}(t)\in\mathcal{A}_{\mathcal{E}}$ be a candidate for an optimal control and let $\bar{X}(t),(\bar{p}(t),\bar{q}(t),\bar{s}(t))$ be the corresponding solutions of (5.16), (5.18). If $\mathcal{E}_t=\sigma\{\alpha(s),s\leq t\}$, then

$$
\begin{aligned}
\mathbf{E}\left[H\left(t,\bar{X}(t),u,\bar{p}(t),\bar{q}(t),i\right)/\mathcal{E}_t\right] &= A(t)\,\mathbf{E}\left[\bar{X}(t)\,\bar{p}(t)/\mathcal{E}_t\right]\\
&+\hat{A}(t)\,\mathbf{E}\left[\bar{X}(t)/\mathcal{E}_t\right]+B(t)\,\mathbf{E}\left[\bar{p}(t)/\mathcal{E}_t\right]u\\
&+C(t)\,\mathbf{E}\left[\bar{q}(t)/\mathcal{E}_t\right]u.
\end{aligned}
$$

Since this is a linear expression in $u$, we get

$$
B(t)\,\mathbf{E}\left[\bar{p}(t)/\mathcal{E}_t\right]+C(t)\,\mathbf{E}\left[\bar{q}(t)/\mathcal{E}_t\right]=0, \tag{5.21}
$$

and by $(5.20)$,

$$
C(t)\,q(t)=\upsilon(t)\,C^{2}(t)\,u(t)
$$

$$
C(t)\,\hat{q}(t)=\upsilon(t)\,C^{2}(t)\,u(t)
$$

and

$$
B(t)\,p^{*}(t)=B(t)\left(\upsilon(t)+\gamma(t)\right)X^{*}(t)
$$

then by (5.21)

$$
B(t)\left(\upsilon(t)+\gamma(t)\right)X^{*}(t)+\upsilon(t)\,C^{2}(t)\,u(t)=0
$$

then

$$u(t) = -\frac{B(t)(\upsilon(t) + \gamma(t)) X^*(t)}{\upsilon(t) C^2(t)}, \tag{5.22}$$

since

$$dX^*(t) = \left( A(t) + A^*(t) - B^2(t) \frac{(\upsilon(t) + \gamma(t))}{\upsilon(t) C^2(t)} \right) X^*(t)\, dt \tag{5.23}$$

by $(5.20) - (5.22) - (5.23)$

$$\begin{aligned}
& -[A(t) p(t) + A^*(t) p^*(t)] \\
& = -[A(t)(\upsilon(t) X(t) + \gamma(t) X^*(t)) + A^*(t)(\upsilon(t) + \gamma(t)) X^*(t)] \\
& = X(t) \left[ \upsilon'(t,i) + \sum_j (\upsilon(t,j) - \upsilon(t,i)) \lambda_j(t) \right] \\
& + \upsilon(t) \left[ A(t) X(t) + A^*(t) X^*(t) - B^2(t) \tfrac{(\upsilon(t)+\gamma(t))}{\upsilon(t) C^2(t)} X^*(t) \right] \\
& + X^*(t) \left[ \gamma'(t,i) + \sum_j (\gamma(t,j) - \gamma(t,i)) \lambda_j(t) \right] \\
& + \gamma(t) \left( A(t) + A^*(t) - B^2(t) \tfrac{(\upsilon(t)+\gamma(t))}{\upsilon(t) C^2(t)} \right) X^*(t)
\end{aligned}$$

then

$$\begin{aligned}
& \upsilon'(t,i) + \sum_j (\upsilon(t,j) - \upsilon(t,i)) \lambda_j(t) + 2A(t) \upsilon(t) && = 0 \\
& \upsilon(T) = -S(T)
\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}
& \gamma'(t) + 2\left( A(t) + A^*(t) - \tfrac{B^2(t)}{C^2(t)} \right) \gamma(t) - \tfrac{B^2(t)}{\upsilon(t) C^2(t)} \gamma^2(t) + \left( 2A^*(t) - \tfrac{B^2(t)}{C^2(t)} \right) \upsilon(t) + \sum_j (\gamma(t,j) - \gamma(t,i)) \lambda_j(t) \quad = \\
& \gamma(T) = 0
\end{aligned} \tag{5.25}$$

**Theorem 5.3.1** *The solution* $u^*$ *of the optimal control* (5.15) (5.16) *is given by* (5.22) *with* $\upsilon(t), \gamma(t)$ *given by* $(5.24) - (5.25)$.

# Conclusion

This thesis contains two main results. The first one is the necessary and sufficient conditions of optimality where the control systeme is governed by stochastic differential equation (SDE) with regime switching in infinite horizon, which is mentioned in [7]. The second main result is the maximum principle of optimal control for conditional mean field type in finite horizon, cited in [1], where we motivate our study by two examples in finance.

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