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Probabilities and Stochastic Differential Equations

Title

On Maximum Principle of Non Linear Stochastic Mckean-Vlasov System with Applications

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Aknowledgemnt

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ملخص

مسئلة التحكم الأمثل التي يتم مراقبتها جزئيا لها مجموعة متنوعة من التطبيقات الهامة في العديد من المجالات وتوفر طرق عملية لمعالجة تحديات التحكم في العالم الحقيقي ومشاكل اتخاذ القرارات ، مثل الهندسة والاقتصاد والتمويل.

الهدف من هذه الأطروحة هو دراسة هذا النوع من مشكلة التحكم الأمثل التي تمت ملاحظتها جزئياً للمعادلات التفاضلية العشوائية التقدمية والتراجعية من نوع ماكين-فلاسوف. تعتمد معاملات النظام ووظيفة التكلفة على حالة عملية الحل بالإضافة إلى قانون الاحتمالية ومتغير التحكم.

نبدأ بتعريف الأداة الرئيسية (الانحياز الجزئي نسبة إلى قياس الاحتمال في فضاء واسرشتاين) المستخدمة لتوضيح نتيجتنا الرئيسية. ثم نثبت الشروط الضرورية والكافية للأمثلية لمعادلات التفاضلية الجزئية ذات الصيغة مكين-فلاسوف مع افتراض أن مجال التحكم من المفترض أن يكون محدب. تستند هذه النتيجة إلى نظرية جيرسافوف.

وأخيرًا ، نثبت مبدأ الحد الأقصى العشوائي الجديد لهذا النوع من مشكلات التحكم الأمثل الجزئي المراقب من نوع مكين-فلاسوف والتي تقودها قياس عشوائي للبواسون وحركة براونية مستقلة. على سبيل المثال، تمت دراسة مشكلة التحكم الخطي التربيعي التي تمت ملاحظتها جزئيًا من حيث التذبذب العشوائي.

الكلمات الرئيسية: المبدأ الأقصى العشوائي, المعادلات التفاضلية الجزئية الأمامية والخلفية مع عمليات القفزات العشوائية, التحكم الأمثل المراقب جزئيًا, معادلات مكين-فلاسوف التفاضلية, المشتقة بالنسبة لقياسات الاحتمال.

Résumé

Le problème de contrôle optimal partiellement observé a une variété d'applications importantes dans de nombreux domaines et offre des solutions pratiques pour enlever des défis de contrôle du monde réel et des problèmes de prise de décision, tels que l'ingénierie, l'économie et la finance.

Le but de cette thèse est d'étudier ce type du problème de contrôle optimal partiellement observé pour les équations différentielles stochastiques progressives rétrogrades (EDSPRs) de type McKean-Vlasov. Les coefficients du système et la fonction de coût dépendent de l'état du processus de solution ainsi que de sa loi de probabilité et du contrôle.

Nous commençons par définir l'outil principal (la dérivée partielle par rapport à une mesure de probabilité dans l'espace de Wasserstein) utilisé pour illustrer notre résultat principal. Ensuite, nous prouvons les conditions nécessaires et suffisantes d'optimalité pour les EDSPRs de type McKean-Vlasov en supposant que le domaine de contrôle est convexe. Ce résultat est basé sur le théorème de Girsanov.

Enfin, nous prouvons un nouveau principe du maximum stochastique pour ce type de problème de contrôle optimal partiellement observé de type McKean-Vlasov gouverné par une mesure aléatoire de Poisson et d'un mouvement brownien indépendant. À titre d'exemple, un problème de contrôle linéaire quadratique partiellement observé a été étudié en termes de filtrage stochastique.

Mots-clés : Principe du maximum stochastique, Équations différentielles stochastiques progressives rétrogrades, Processus de saut, Contrôle optimal partiellement observé, Dérivées par rapport aux mesures de probabilité.

Abstract

Partially observed optimal control problem has a variety of important applications in many fields and offers practical avenues for addressing real-world control challenges and decision-making problems, such as engineering, economics, and finance.

The aim of this thesis is to study this kind of partially observed optimal control problem for forward-backward stochastic differential equations of the McKean– Vlasov type. The coefficients of the system and the cost functional depend on the state of the solution process as well as of its probability law and the control variable.

We start by defining the primary tool (the partial derivative with respect to the probability measure in Wasserstein space) used to illustrate our main result. Then, we prove the necessary and sufficient conditions of optimality for FBSDEs of the McKean– Vlasov type under the assumption that the control domain is supposed to be convex. This result is based on Girsavov's theorem.

Finally, we prove a stochastic maximum principle for this kind of partially observed optimal control problems of McKean– Vlasov type driven by a Poisson random measure and an independent Brownian motion. As an illustration, a partially observed linear–quadratic control problem is studied in terms of stochastic filtering.

Key words. Stochastic maximum principle, Forward-backward stochastic differential equations with jump processes Partially observed optimal control, McKean–Vlasov differential equations, Derivatives with respect to probability measures.

Contents

Symbols

- $(\Omega, \mathcal{F}, P)$: probability space.
- $\{\mathcal{F}_t\}_{t\geq 0}$: filtration.
- $\Big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P\Big)$: filtered probability space.
- $\mathbb{R}$: Real numbers.
- $\mathbb{N}$: Natural numbers.
- $L^2(r, s; \mathbb{R}^n)$: the space of $\mathbb{R}^n$-valued deterministic function $\eta(t)$, such that

$$\int_r^s |\eta(t)|^2 \, dt < +\infty.$$

- $L^2(\mathcal{F}_t; \mathbb{R}^n)$: the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variable φ, such that

$$\mathbb{E}\,|\varphi|^2 < +\infty.$$

- $L^2_{\mathcal{F}}(r, s; \mathbb{R}^n)$: the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\psi(\cdot)$, such that

$$\mathbb{E}\int_r^s |\psi(t)|^2 \, dt < +\infty.$$

- $\mathbb{M}^2([0, T]; \mathbb{R})$: the space of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted measurable process $c(\cdot)$, such that

$$\mathbb{E}\int_0^T \int_\Theta |c(t, \theta)|^2 \, \pi(de)dt < +\infty.$$

- $\mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$: is the Hilbert space.
- $Q_2\left(\mathbb{R}^d\right)$: the space of all probability measures μ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$.

- P_X: the law of the random variable $X(\cdot)$.
- $\mathbb{E}(\cdot)$: Expectation.
- $\mathbb{E}(\cdot|\, F_t)$: Conditional expectation.
- $\sigma(A)$: $\sigma-$algebra generated by A.
- 1_A : Indicator function of the set A.
- $\mathbb{E}^v$: denotes expectation on $(\Omega, \mathcal{F}, \mathbb{F}, P^v)$.
- $k(\cdot)$: be a stationary $\mathcal{F}_t$-Poisson point process with the characteristic measure $\pi(de)$.
- $N(de, dt)$: the counting measure or Poisson measure induced by $k(\cdot)$.
- Θ : is a fixed nonempty subset of $\mathbb{R}$.
- $\mathcal{F}^X$: The filtration generated by the process X.
- $W(\cdot)$: Brownian motions.
- $\mathcal{F}_t^W$: the natural filtration generated by the brownian motion $W(\cdot)$.
- $\mathcal{F}_1 \vee \mathcal{F}_2$: denotes the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$.
- $\partial_\mu f$: the derivatives with respect to measure μ.
- $\mathcal{D}_\zeta f(\mu_0)$: the *Fréchet-derivative* of f at μ_0 in the direction ξ.

Acronyms

- *a.e.,:* almost everywhere.
- *a.s.,:* almost surely.
- *e.g.*: for example (abbreviation of Latin exempli gratia).
- *i.e,.* that is (abbreviation of Latin id est).
- *SDE*: Stochastic differential equations.
- *BSDE:* Backward stochastic differential equation.
- *FBSDEs:* Forward-Backward stochastic differential equations.
- *PDE*: Partial differential equation.
- *ODE:* Ordinary differential equation.
- $\frac{\partial f}{\partial x}, f_x$: The derivatives with respect to x.
- $\mathbb{P}\otimes dt$: The product measure of P with the Lebesgue measure dt on $[0,T]$.
- $W(\cdot)$: Brownian motions.
- $\mathcal{F}_t^W$: the natural filtration generated by the brownian motion $W(\cdot)$.
- $\mathcal{F}_1 \vee \mathcal{F}_2$ denotes the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$.
- $\partial_\mu f$: the derivatives with respect to measure μ.
- $\mathcal{D}_\zeta f(\mu_0)$: the *Fréchet-derivative* of f at μ_0 in the direction ξ.

Introduction

This doctoral thesis is part of the framework of stochastic analysis and stochastic optimization problems. Two major tools for studying optimal control are Bellman's dynamic programming method and Pontryagin's maximum principle. The stochastic maximum principle gives some necessary conditions for optimality for a stochastic optimal control problem. See the pioneering works on the stochastic maximum principle were written by Kushner [27, 28]. Since then there has been a lot of work on this subject, among them, in particular, those by Bensoussan [5], Peng [44].

The mean-field stochastic system was introduced by Kac [38] as a stochastic model for the Vlasov-Kinetic equation of plasma and the study of which was initiated by the McKean model [25]. Since then, the mean-field theory has found important applications and has become a powerful tool in many fields, such as mathematical finance, economics, optimal control and stochastic mean-field games; see for instance [38, 2, 9, 10, 45, 3, 17, 18, 19, 20, 21, 22, 23, 24, 31, 56, 47]

In this thesis, the central theme is to establish a set of necessary and sufficient conditions in the form of Pontryagin's stochastic maximum of the mean field type of optimal control and these applications. More precisely, our objective in this work is to study partially observed optimal control the problem of forward-backward stochastic differential equations (FBSDEs for short) systems of McKean–Vlasov type, which are governed by Poisson random measure and an independent Brownian motion. This kind of partially observed optimal control problems have a variety of important applications in many fields such as engineering, economics, and finance.

It is assumed so far that the controller completely observes the state system. In many real applications, she is only able to observe partially the state via other variables and there is noise in the observation system. Then it is natural to study this kind of optimal control problems under partial observation. There is rich literature on partially observed optimal control problems, see for example [4, 13, 15] and references therein. The stochastic maximum principle for partially observed optimal control problems of general McKean–Vlasov equations has been proved by Lakhdari et al. [29]. Ma and Liu [33] studied the maximum principle for partially observed risk-sensitive optimal control problems of mean-field type. Miloudi et al. [40] established the necessary conditions of partially observed optimal control of general McKean–Vlasov stochastic differential equations with jumps.

Partially observed stochastic optimal control of forward-backward stochastic differential equations has been studied by many authors, see for example, Wu [52] proved the maximum principle for partially observed optimal control of forward-backward stochastic control systems. The maximum principle for partially observed optimal control of FBSDEs driven by Teugels martingales and independent Brownian motion has been proved by Bougherara and Khelfallah [48]. Shi and Wu [46] established the maximum principle for partially observed optimal control of fully coupled forward-backward stochastic systems. Li and Fu [30] established a general maximum principle for partially observed optimal control problems of mean-field FBSDEs under general control domains, with the help of Ekeland's variational principle and reduction method. Nie and Yan [42] studied an extended mean-field control problem with partial observation, where the state and the observation all depend on the joint distribution of the state and the control process. Wang et al. [53] studied three versions of stochastic maximum principle for partially observed optimal control problem for FBSDEs in the sense of weak solution by utilizing a direct method, an approximation method, and a Malliavin derivative method. Xiao [55] proved the maximum principle for partially observed optimal control of forward-backward stochastic systems with random jumps. Partially observed optimal control problem of the forward-backward stochastic jump-diffusion differential system has been discussed by Wang et al. [54]. Partially observed maximum principle by using Malliavin calculus has been studied by Zhou et al. [59]. Partially observed optimal control problem for FBSDEs driven by Lévy processes with Markov regime-switching has been investigated by Zhang et al. [60].

In this thesis, we aim to establish a stochastic maximum principle for a class of partially observed optimal control problems involving stochastic differential forward-backward equations of McKean-Vlasov type.

The dynamics of the controlled system in the first part of our study are governed by the following stochastic differential equation:

$$\begin{cases} dx_t^v = b\left(t, x_t^v, P_{x_t^v}, v_t\right) dt + g\left(t, x_t^v, P_{x_t^v}, v_t\right) dW_t + \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) d\widetilde{W}_t^v \\ -dy_t^v = f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, v_t\right) dt - z_t^v dW_t - \bar{z}_t^v dY_t \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v, P_{x_T^v}\right), \end{cases}$$

$W(\cdot)$ represents a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. $\widetilde{W}(\cdot)$ denotes a stochastic process that depends on the control variable $v(\cdot)$. P_X represents the probability distribution of the random variable X.
The coefficients in the problem are characterized by: $b : [0, T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$, $g, \sigma : [0, T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$. The space $Q_2\left(\mathbb{R}^d\right)$ corresponds to the set of probability measures μ defined on $\mathbb{R}^d$ and is equipped with the 2-Wasserstein metric. The associated cost function is also of the McKean-Vlasov type and is described as

$$\begin{aligned} J(v) &= \mathbb{E}^v\left[\int_0^T l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, v_t\right) dt\right] \\ &\quad + \mathbb{E}^v\left[M\left(x_T^v, P_{x_T^v}\right) + h\left(y_0^v, P_{y_0^v}\right)\right], \end{aligned}$$

where $\mathbb{E}^v$ represents the expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P^v)$ and $l : [0, T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$, $M, h : \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}$.
For the partially observable control problem of general Mckean-Vlasov Forward-backward stochastic differential equations, where the coefficients depend nonlinearly on both the state process and its law, the aim of this thesis is to establish a stochastic maximum principle. It is assumed that the control domain is convex.
In another section of this thesis, we provide a stochastic maximum principle for a class of Mckean-Vlasov type partially observed optimal control problems with jumps. The stochastic system under discussion is controlled by a stochastic differential forward-backward

equation with independent Brownian motion and Poisson random measure.

It is defined in the following way:

$$\begin{cases} dx_t^v = b\left(t, x_t^v, P_{x_t^v}, v_t\right)dt + g\left(t, x_t^v, P_{x_t^v}, v_t\right)dW_t + \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right)d\widetilde{W}_t^v \\ \qquad + \int_\Theta c\left(t, x_{t\_}^v, P_{x_{t\_}^v}, v_t, e\right)\widetilde{N}(de, dt), \\ -dy_t^v = f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, r_t^v, P_{r_t^v}, v_t\right)dt - z_t^v dW_t - \bar{z}_t^v dY_t \\ \qquad - \int_\Theta r_t^v(e)\widetilde{N}(de, dt), \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v, P_{x_T^v}\right), \end{cases}$$

where $P_{x_t}, P_{y_t}, P_{z_t}, P_{\bar{z}_t}$ and P_{r_t} denotes the law of the random variable $x_t, y_t, z_t, \bar{z}_t$ and r_t respectively. The maps $b : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$, $g, \sigma : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$, $c : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \times \Theta \to \mathbb{R}$, $\varphi : \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}$, $f : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$ are given deterministic functions.

The cost functional to be minimized over the class of admissible controls is also of McKean-Vlasov type, which has the following form

$$J(v(\cdot)) = \mathbb{E}^v\left[\int_0^T l\left(t, x^v(t), \mathbb{P}_{x^v(t)}, v(t)\right)dt + \psi(x^v(T), \mathbb{P}_{x^v(T)})\right],$$

where, $l : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}) \times U \to \mathbb{R}$, $\psi : \mathbb{R}^n \times Q_2(\mathbb{R}) \to \mathbb{R}$ and $\mathbb{E}^v$ stands for the mathematical expectation on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^v)$.

The derivatives with respect to probability measure and the associated Itô-formula is used in this study to prove our main results. It is worth noting that our generic McKean-Vlasov partially observed control the problem occurs naturally in probabilistic analyses of financial optimization problems. Our class of partially observed control problems was motivated by the recent research of McKean-Vlasov games, which have lately played an important role in multiple fields of economics and finance. As an example, using our maximum approach, we consider a McKean-Vlasov-type linear quadratic control problem with the jump, where the partially observed optimum control is achieved directly in the feedback form.

This thesis is divided into three chapters:

The first chapter is essentially a reminder, we present some concepts and results that allow us to prove our results, such as stochastic processes, natural filtration, Lévy Processus, admissible control, feedback controls, relaxed controls...etc.

In the second chapter, we define the primary tool used to illustrate our main result. Then, we prove the necessary and sufficient conditions of optimality for FBSDEs of the McKean– Vlasov type under the assumption that the control domain is supposed to be convex. This result is based on Girsavov's theorem and fundamental variational techniques.

Finally, we prove a stochastic maximum principle for this kind of partially observed optimal control problems of McKean– Vlasov type driven by a Poisson random measure and an independent Brownian motion. As an illustration, a partially observed linear–quadratic control problem is studied in terms of stochastic fltering.

Scientific Contributions

Publications based on this thesis

- Abba, K., Lakhdari, I. E. (2023). A Stochastic Maximum Principle for Partially Observed Optimal Control Problem of Mckean–Vlasov FBSDEs with Random Jumps. Bulletin of the Iranian Mathematical Society, 49(5), 56.

Conference papers

- Abba, K. Lakhdari, I. E. **Stochastic Maximum Principle for Partially Observed Optimal Control Problems of McKean-Vlasov FBSDEs with Jumps**. *The first National Conference on Pure and Applied Mathematics, NC-PAM'2021*. Laghouat, Algeria. December 11-12, 2021.

- Abba, K. Lakhdari, I. E. **Forward-backward stochastic differential equation for partially observed control problem in Wesserstein space** . *The First International Workshop on Applied Mathematics,1st-IWAM*. Constantine, Algeria. December 06-08, 2022.

- Abba, K. Lakhdari, I. E. **A Risk Sensetive Optimal Control Problems**. *The First National Applied Mathematics,1st-NAMS'23*. Biskra, Algeria. May 14–15, 2023.

CHAPTER 1

Stochastic calculus

1.1 Stochastic processes and Brownian motion

Definition 1.1 (***Stochastic processes***)

Consider a set of indices denoted by T. Let $(\Omega, \mathcal{F}, P)$ represent a probability space. A stochastic process is a collection of variables $\{X(t); t \in T\}$, where for each t in the set T, X_t is a variable that maps from $(\Omega, \mathcal{F}, P)$ to $\mathbb{R}^n$. The function that associates each t with $X(t, w)$ is known as a sample path for any given w, in the set Ω.

Definition 1.2 (***Natural filtration***)

Lets consider the stochastic process $X = (X_t, t \geq 0)$, on the probability space $(\Omega, \mathcal{F}, P)$ which we denote as $\mathcal{F}_t^X$ for the filtration of X. This filtration is defined as $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$. We also refer to it as the filtration generated by X.

Definition 1.3 (***Brownian motion***)

A stochastic process $(W(t), t \geq 0)$ is called a standard Brownian motion if :

- $P[W(0) = 0] = 1$.
- $t \to W(t, w)$ *is continuous.* $\mathbb{P}-p.s.$
- $\forall s \leq t$, $W(t) - W(s)$ *is normally distributed; center with variation* $(t - s)$ *i.e* $W(t) - W(s) \sim \mathcal{N}(0, t - s)$.
- $\forall n$, $\forall\ 0 \leq t_0 \leq t_1 \leq ... \leq t_n$, *the variables* $\left(W_{t_n} - W_{t_{n-1}}, ..., W_{t_1 -} W_{t_0}, W_{t_0}\right)$ *are independents.*

Definition 1.4 (*Stochastically equivalent*)

Two processes X_t and Y_t are said to be stochastically equivalent *if*

$$X_t = Y_t,\ P - a.s,\ \forall t \in [0, T].$$

In this case, one is called a modification of the other.
If X_t and Y_t are stochastically equivalent, then for any $t \in [0, T]$ there exists a P -null set $N_t \in \mathcal{F}$ such that

$$X_t = Y_t, \quad \forall w \in \Omega \mid N_t.$$

Example: Let $\Omega = [0, 1], T \geq 1$, P the Lebesgue measure, $X(w, t) = 0$, and

$$Y_t(w) = \begin{cases} 0, & w \neq t, \\ 1, & w = t. \end{cases}$$

Then X_t and Y_t are said to be *stochastically equivalent*. But each sample path $X(., t)$ is continuous, and none of the sample paths $Y(., t)$ is continuous. In the present case, we have

$$\bigcup_{t \in [0,T]} N_t = [0, 1] = \Omega.$$

Definition 1.5

The process at $s \in [0, T]$ if for any $\varepsilon > 0$

$$\lim_{t \to s} P(w \in \Omega, |X_t(w) - X_s(w)| > \varepsilon) = 0.$$

Moreover, X_t is said to be continuous if there exists a P -null set $N \in \mathcal{F}$ such that for any $w \in \Omega \mid N$, the sample path $X(\cdot, t)$ is continuous. Then X_t and Y_t are said to be stochastically equivalent. But each sample path $X(\cdot, t)$ is continuous, and none of the sample paths $Y_t(\cdot, w)$ is continuous. In the present case, we have

$$\bigcup_{t \in [0,T]} N_t = [0, 1] = \Omega.$$

Definition 1.6

The process at $s \in [0, T]$ if for any $\varepsilon > 0$

$$\lim_{t \to s} P\left(w \in \Omega, \left|X_t\left(w\right) - X_s\left(w\right)\right| > \varepsilon\right) = 0.$$

Moreover, X_t is said to be continuous if there exists a P -null set $N \in \mathcal{F}$ such that for any $w \in \Omega \mid N$, the sample path $X(\cdot, t)$ is continuous.

1.2 Stochastic integral with respect to Lévy process

1.2.1. Lévy process

To capture the fluctuations, in the finance field it is only logical to incorporate jumps, in the model as it adds a touch of realism. This type of modeling can be described using Lévy processes, which have been widely employed in this study. The term "Lévy process" pays tribute to the contributions made by Paul Lévy, a mathematician, from France.

Definition 1.7

A stochastic process $X = (X(t))_{t \geq 0}$ which takes values in the set of numbers $\mathbb{R}$ is considered a Lévy process if it satisfies the following conditions:

1. $P[X(0) = 0] = 1$.
2. *The paths of X are P-almost surely right continuous with left limits.*
3. *Stationary increments, i.e., for $0 \leqslant s \leqslant t$, $X(t) - X(s)$ has the same distribution as $X(t - s)$.*
4. *Independent increments, i.e., for $0 \leqslant s \leqslant t$, $X(t) - X(s)$ is independent of $X(u)$, $u \leqslant s$.*

Example. The known examples are the standard Brownian motion and the Poisson process.

Definition 1.8

A stochastic process denoted as $W = (W(t))_{t\geq 0}$ in the space $\mathbb{R}^n$ is referred to as a Brownian motion when it satisfies the conditions of being both a Lévy process and meeting the flowing criteria :

1. *For all $t > 0$, has a Gaussian distribution with mean 0 and covariance matrix tI_d.*
2. *There exists $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ such that, for every $w \in \Omega_0$, $W(t, w)$ is continuous in t.*

Definition 1.9

A stochastic process $N = (N(t))_{t\geq 0}$ on $\mathbb{R}$ such that

$$\mathbb{P}\left[N(t) = n\right] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}; \quad n = 0, 1,$$

is a Poisson process with parameter $\lambda > 0$ if it is a Lévy process and for $t > 0$, $N(t)$ has a Poissson distribution with mean λt.

Remark 1.1

- *It is worth noting that when we talk about the characteristics of stationarity and independent increments we can conclude that a Lévy process needs a Markov process.*
- *Thanks to the continuity of paths it is possible to demonstrate that Lévy processes are also considered strong Markov processes.*
- *Each random variable can be characterized by its characteristic function. In the case of a Lévy process X, this characterization for all time t gives the* Lévy-Khintchine formula *and it is also called Lévy-Khintchine representation.*

Consider a probability space $(\Omega, \mathcal{F}, P)$, with a σ algebra $(\mathcal{F}_t)_{t\geq 0}$ generated by stochastic processes. These processes include motion denoted as $W(t)$ and an independent compensated Poisson random measure called $\widetilde{N}$, where:

$$\widetilde{N}(dt, de) := N(dt, de) - \pi(de)dt.$$

For any t, let $\widetilde{N}(ds, de)$, $e \in \mathbb{R}$, $s \leqslant t$, augmented for all the sets of P-zero probability. For any $\mathcal{F}_t-$adapted stochastic process $\theta = \theta(t, e)$, $t \geqslant 0$, such that

$$E\left[\int_0^T \int_{\mathbb{R}} \theta^2(t, e)\pi(de)dt\right] < \infty, \text{ for some } T > 0,$$

we can see that the process

$$M_n(t) = \int_0^t \int_{|e|\geq \frac{1}{n}} \theta(s, e)\widetilde{N}(ds, de), \quad 0 \leqslant t \leqslant T,$$

is a martingale in $L^2(\Omega, \mathcal{F}, P)$ and its limit

$$M(t) = \lim_{n\to\infty} M_n(t) := \int_0^T \int_{|e|\geq \frac{1}{n}} \theta(s, e)\widetilde{N}(ds, de), \quad 0 \leqslant t \leqslant T,$$

in $L^2(\Omega, \mathcal{F}, P)$ is also a martingale. Moreover, we have the Itô isometry

$$E\left[\left(\int_0^T \int_{\mathbb{R}_0} \theta(s, e)\widetilde{N}\,(ds, de)\right)^2\right] = E\left[\left(\int_0^T \int_{\mathbb{U}} \theta^2(t, e)\pi(de)dt\right)\right].$$

Such processes can be expressed by the sum of two independent parts, a continuous part and a part expressible as a compensated sum of independent jumps. That is the *Itô-Lévy* decomposition.

Theorem 1.2 (*Itô-Lévy decomposition*)

The Itô-Lévy decomposition for a Lévy process X is given by

$$X(t) = \alpha t + \beta W(t) + \int_{|e|<1} e\widetilde{N}(dt, de) + \int_{|e|\geqslant 1} eN(dt, de), \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}$, and $\widetilde{N}(dt, de)$ is the compensated Poisson random measure of $X(.)$ and $B(t)$ is an independent Brownian motion with the jump measure $N(dt, de)$. We assume that

$$E\left[X^2(t)\right] < \infty, \quad t \geqslant 0,$$

then

$$\int_{|e|\geq 1} |e|^2 \pi(de) < \infty.$$

We can represent (1.1) as

$$X(t) = \alpha t + \beta W(t) + \int_{\mathbb{R}} e\widetilde{N}(dt, de),$$

where $X(t) = \alpha + \int_{|e|\geq 1} e\pi(de)$. *If* $\beta = 0$, *then a Lévy process is called a pure jump Lévy process.*

Let us consider that the process $X(t)$ *admits the stochastic integral representation as follows*

$$X(t) = x + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW(s) + \int_0^t \int_{\mathbb{R}} \theta(s, e)\widetilde{N}(ds, de),$$

where $\alpha(t)$, $\beta(t)$, *and* $\theta(t, \cdot)$ *are predictable processes such that, for all* $t > 0, e \in \mathbb{R}$,

$$\int_0^t \left[|b(s)| + \sigma^2(s) + \int_{\mathbb{R}} \theta^2(s, e)\pi(de)\right] ds < \infty \quad P - a.s.$$

Under this assumption, the stochastic integrals are well-defined and local martingales. If we strengthened the conditions to

$$E\left[\int_0^t \left[|b(s)| + \sigma^2(s) + \int_{\mathbb{R}} \theta^2(s, e)\pi(de)\right] ds\right] < \infty,$$

for all $t > 0$, *then the corresponding stochastic integrals are martingales.*

We call such a process an Itô–Lévy process. In analogy with the Brownian motion case, we use the short-hand differential notation

$$\begin{cases} dX(t) = b(t)dt + \sigma(t)dW(t) + \int_{\mathbb{R}} \theta(t, e)\widetilde{N}(dt, de), \\ X(0) = x \in \mathbb{R}. \end{cases}$$

1.2.2 The Itô's formula and related results

We now come to the important Itô formula for Itô-Lévy processes. Let $X(t)$ be a process given by theorem 1.2

$$X(t) = \alpha(t) + \beta(t) W(t) + \int_{\mathbb{R}} \gamma(t, e) \widetilde{N}(dt, de), \tag{1.2}$$

where $f:\mathbb{R}^2\to\mathbb{R}$ is a C^2 function is the process $Y(t):=f(t,X(t))$ again an Itô-Lévy process and if so, how do we represent it in the form (1.2).
Let $X^c(t)$ be the continuous part of $X(t)$, i.e $X^c(t)$ is obtained by removing the jumps from $X(t)$.

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}(t,X(t))\,dt + \frac{\partial f}{\partial x}(t,X(t))\,dX^c(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X(t))\,\beta^2(t)\,dt \\ &+ \int_R \left\{ f\left(t,X(t^-)+\gamma(t,e)\right) - f\left(t,X(t^-)\right)\right\} \widetilde{N}(dt,de). \end{aligned}$$

It can be proved that our guess is correct. Since

$$dX^c(t) = \left(\alpha(t)dt - \int_{|F|<r}\gamma(t,e)\pi(de)\right) + \beta(t)dW(t),$$

this gives the following result;

Theorem 1.3

Let $X(t)\in\mathbb{R}$ is an Itô-Lévy process of the form

$$dX(t) = \alpha(t) + \beta(t)W(t) + \int_{\mathbb{R}}\gamma(t,e)\widetilde{N}(dt,de), \tag{1.3}$$

where

$$\widetilde{N}(dt,de) = \begin{cases} N(dt,de) - \pi(de)\,dt, & \text{if } |F|<r. \\ N(dt,de) & \text{if } |F|\geq r, \end{cases}$$

for some $r\in[0,\infty]$. Let $f\in C^2\left(R^2\right)$ and define $Y(t)=f(t,X(t))$. Then $Y(t)$ is again an Itô -Lévy process

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}(t,X(t))\,dt + \frac{\partial f}{\partial x}(t,X(t))\,(\alpha(t)dt+\beta(t)dW(t)) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X(t))\,\beta^2(t)\,dt \\ &+ \int_{|F|<r}\left\{ f\left(t,X(t^-)+\gamma(t,e)\right) - f\left(t,X(t^-)\right) - \frac{\partial f}{\partial x}(t,X(t))\,\gamma(t,e)\right\}\pi(de) \\ &\int_{\mathbb{R}}\left\{ f\left(t,X(t^-)+\gamma(t,e)\right) - f\left(t,X(t^-)\right)\right\}\widetilde{N}(dt,de), \end{aligned}$$

Remark 1.4

if $r=0$ then $\widetilde{N}=N$ every where. If $r=\infty$ then $\widetilde{N}=N$ every where.

Theorem 1.5 (*The multi-dimensional Itô formula*)

Let $X(t) \in \mathbb{R}^n$ be an Itô-Lévy process of the form

$$dX(t) = \alpha(t,w)\,dt + \sigma(t, X(t,w))\,dW(t) + \int_{\mathbb{R}^n} \gamma(t,e,w)\,\widetilde{N}(dt,de),$$

where $\alpha : [0,T]\times\Omega \to \mathbb{R}^n, \sigma : [0,T]\times\Omega \to \mathbb{R}^{n+m}$ and $\gamma : [0,T]\times\mathbb{R}^n\times\Omega\times \to \mathbb{R}^{n\times l}$ are adapted processes such that the integrals exist. Here $W(t)$ is an multidimensional Brownian motion and

$$\begin{aligned}\widetilde{N}(dt,de)^T &= \left(\widetilde{N}_1(dt,de),...,\widetilde{N}_l(dt,de)\right)\\ &= \left(\widetilde{N}_1(dt,de) - I_{|e_1|<r}\pi_1(de_1)\,dt,...,\widetilde{N}_l(dt,de) - I_{|e_l|<r_l}\pi_l(de_l)\,dt\right),\end{aligned}$$

where $(N_j(\cdot,\cdot))$ are independent Poisson random measures with Lévy processes $(\eta_1,...,\eta_l)$. Note that each column $\gamma^{(k)}$ of the $n\times l$ matrix $\gamma = (\gamma_{ij})$ depends on e only through the k^{th}coordinate e_k, i.e.,

$$\gamma^{(k)}(t,e,w) = \gamma^{(k)}((t,e_k,w));\ e = (e_1,...,e_l)\in\mathbb{R}^l.$$

Thus the integral on the right of (1.3) is just a short-hand matrix notation. When written out in detail component number i of $X(t)$ in (1.3), $X_i(t)$, gets the form

$$dX_i(t) = \alpha_i(t;w)\,dt + \sum_{j=1}^{m}\sigma_{ij}(t,w)\,dW_j(t) + \sum_{j=1}^{l}\int_{\mathbb{R}^n}\gamma_{ij}(t,e_j,w)\,\widetilde{N}_j(dt,de_j),$$

$$1\le i\le n.$$

Theorem 1.6 (The Itô-Lévy isometry)

Let $X(t)\in\mathbb{R}^n$ is be as in (1.3) but with $X(0)$ and $\alpha = 0$. Then

$$\begin{aligned}E\left[X^2(t)\right] &= E\left[\int_0^T\left\{\sum_{j=1}^{m}\sigma_{ij}^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{l}\int_{\mathbb{R}^n}\gamma_{ij}^2(t,e_j)\,\pi_j(de_j)\right\}dt\right],\\ &= \sum_{i=1}^{n}E\left[\int_0^T\left\{\sum_{j=1}^{m}\sigma_{ij}^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{l}\int_{\mathbb{R}^n}\gamma_{ij}^2(t,e_j)\,\pi_j(de_j)\right\}dt\right].\end{aligned}$$

Chapter 2

Stochastic control problem with partial observations in Wasserstein space

2.1 Partial derivative with respect to the probability measure

We recall briefly the main tool used to prove our main result. We apply the differentiability with respect to probability measures, which was introduced by Lions. The fundamental concept is to find a distribution $\mu \in Q_2(\mathbb{R}^n)$ with a random variable $\vartheta \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ so that $\mu \in P_\vartheta$. We suppose that the probability space $(\Omega, \mathcal{F}, P)$ is rich enough that, for every $\mu \in Q_2(\mathbb{R}^n)$, there is a random variable $\vartheta \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ such that $\mu = P_\vartheta$. (For example, the probability space $([0,1], \mathbb{B}[0,1], dx)$, where dx is the Borel measure, satisfy this property). We presume there is a sub-σ field $\mathcal{F}_0 \subset \mathcal{F}$ such that the Brownian motion $W(\cdot)$ is independent of $\mathcal{F}_0$, and $\mathcal{F}_0$ is rich enough, ie

$$Q_2(\mathbb{R}^n) \triangleq P_\vartheta : \vartheta \in \mathbb{L}^2(\mathcal{F}_0, \mathbb{R}^n). \tag{2.1}$$

$\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ denotes the filter produced by $W(\cdot)$, that has been finished and complemented by $\mathcal{F}_0$. Next, we construct a function $\widetilde{f} : Q_2(\mathbb{R}^n) \to \mathbb{R}$ such that for every function $f : Q_2(\mathbb{R}^n) \to \mathbb{R}$, such that

$$\widetilde{f} \triangleq f(P_\vartheta), \vartheta \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n). \tag{2.2}$$

It is obvious that the function $\mathcal{F}$, also known as the lift of f, depends simply on the law of $\vartheta \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ and is independent to the selection of the representative ϑ.

Definition 2.1 (***Differentiable function in*** $Q_2(\mathbb{R}^d)$)

A function $f : Q_2\left(\mathbb{R}^d\right) \to \mathbb{R}$ *is said to be differentiable at* $\mu_0 \in Q_2\left(\mathbb{R}^d\right)$ *if there exists* $\vartheta_0 \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ *with* $\mu_0 = P_{\vartheta_0}$ *such that its lift* $\widetilde{f}$ *is Fréchet differentiable at* ϑ_0*. More precisely, there exists a continuous linear functional* $D\widetilde{f}(\vartheta_0) : \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right) \to \mathbb{R}$ *such that*

$$\widetilde{f}(\vartheta_0+\alpha) - \widetilde{f}(\vartheta_0) = \left\langle D\widetilde{f}(\vartheta_0), \alpha\right\rangle + O\left(\|\alpha\|_2\right) = D_\alpha f(\mu_0) + O\left(\|\alpha\|_2\right), \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ *is the dual product on* $\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$*, and we will refer to* $D_\alpha f(\mu_0)$ *as the Fréchet derivative of* f *at* μ_0 *in the direction* α*. In this case, we have*

$$D_\alpha f(\mu_0) = \left\langle D\widetilde{f}(\vartheta_0), \alpha\right\rangle = \frac{d}{dt}\widetilde{f}(\vartheta_0+t\alpha)\Big|_{t=0}, \text{ with } \mu_0 = P_{\vartheta_0}.$$

Note that by Riesz's representation theorem, there is a unique random variable $\Lambda_0 \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ *such that* $\left\langle D\widetilde{f}(\vartheta_0), \alpha\right\rangle = (\Lambda_0,\alpha)_2 = \mathbb{E}\left[(\Lambda_0,\alpha)_2\right]$*, where* $\alpha \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$. *It was shown (see [12]). Then there exists a Borel function* $h[\mu_0] : \mathbb{R}^d \to \mathbb{R}^d$*, depending only on the law* $\mu_0 = P_{\vartheta_0}$ *but not on the particular choice of the representative* ϑ_0 *such that* $\Lambda_0 = h[\mu_0](\vartheta_0)$*. So, we can write equation (2.3) as*

$$f(P_\vartheta) - f(P_{\vartheta_0}) = \left(h[\mu_0](\vartheta_0), \vartheta-\vartheta_0\right)_2 + O\left(\|\vartheta-\vartheta_0\|_2\right), \quad \forall\vartheta \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right).$$

We shall denote $\partial_\mu f(P_{\vartheta_0}, x) = h[\mu_0](x)$, $x \in \mathbb{R}^d$*. Moreover, we have the following identities:*

$$\begin{aligned} D\widetilde{f}(\vartheta_0) &= \Lambda_0 = h[\mu_0](\vartheta_0) = \partial_\mu f(P_{\vartheta_0}, \vartheta_0), \\ D_\alpha f(P_{\vartheta_0}) &= \langle \partial_\mu f(P_{\vartheta_0}, \vartheta_0), \alpha\rangle, \end{aligned}$$

where $\alpha = \vartheta - \vartheta_0$*, and for each* $\mu \in Q_2\left(\mathbb{R}^d\right)$, $\partial_\mu f(P_\vartheta,\cdot) = h[P_\vartheta](\cdot)$ *is only defined in a* $P_\vartheta(dx)-a.e$ *sense, where* $\mu = P_\vartheta$*.*

Definition 2.2

We say that the function $f \in \mathbb{C}_b^{1,1}\left(Q_2\left(\mathbb{R}^d\right)\right)$ *if for all* $\vartheta \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$, *there exists a* P_ϑ*-modification of* $\partial_\mu f\left(P_\vartheta,\cdot\right)$ *such that* $\partial_\mu f : Q_2\left(\mathbb{R}^d\right)\times\mathbb{R}^d \to \mathbb{R}^d$ *is bounded and Lipchitz continuous. That is for some* $C>0$, *it holds that*

1. $|\partial_\mu f(\mu,x)| \leq C, \forall \mu \in Q_2\left(\mathbb{R}^d\right), \forall x \in \mathbb{R}^d;$
2. $|\partial_\mu f(\mu,x) - \partial_\mu f(\acute{\mu},\acute{x})| \leq C\left(\mathbb{D}_2(\mu,\acute{\mu}) + |x-\acute{x}|\right), \forall \mu,\acute{\mu} \in Q_2\left(\mathbb{R}^d\right), \forall x,\acute{x} \in \mathbb{R}^d.$

Remark. If $f \in \mathbb{C}_b^{1,1}\left(Q_2\left(\mathbb{R}^d\right)\right)$, the derivative $\partial_\mu f\left(P_\vartheta,\cdot\right), \vartheta \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ indicated in definition (2.2) is unique.

2.1.1 Hypotheses

The following established assumptions relating the coefficients will be used.

Condition(A1) :

1. For all $t \in [0,T]$, the function $\rho(\cdot,0,0,0) \in \mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R})$ for $\rho = b,g,\sigma$ and $\xi(\cdot,0,0) \in \mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R}), f(\cdot,0,0,0,0,0,0,0,0,0) \in \mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R})$ and $\varphi(0,0) \in \mathcal{L}^2_{\mathcal{F}}(\Omega,\mathbb{R})$.

2. For any $t \in [0,T]$, the functions b,g,σ are continuously differentiable in (x,v) and they are bounded by $C(1+|x|+|v|)$. The function ξ is continuously differentiable in x.

3. The functions f and l are continuously differentiable in $(x,y,z,\bar{z},v)$, and they are bounded by $C(1+|x|+|y|+|z|+|\bar{z}|+|v|)$ and $C\left(1+|x|^2+|y|^2+|z^{|2}+|\bar{z}|^2+|v|^2\right)$ respectively. The derivatives of f and l with respect to $(x,y,z,\bar{z},v)$ are uniformly bounded.

4. The functions φ and M are continuously differentiable in x, and the function h is continuously differentiable in y. The derivatives M_x, h_y are bounded by $C(1+|x|)$ and $C(1+|y|)$respectively.

5. The derivatives $b_x, b_v, g_x, g_v, \sigma_x, \sigma_v, \xi_x$ are continuous and uniformly bounded.

Condition(A2) :

1. The functions $b, g, \sigma, f, l, \xi, M, h, \varphi \in \mathbb{C}_b^{1,1}\left(Q_2\left(\mathbb{R}\right)\right)$.

2. The derivatives $\partial_\mu^{P_x} b, \partial_\mu^{P_x} g, \partial_\mu^{P_x} \sigma, \partial_\mu^{P_x} \xi, \left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}\right)(f, l)$ are bounded and Lipchitz continuous, such that, for some $C > 0$, it holds that

   i. For $\rho = b, g, \sigma, \xi$, and $\forall \mu, \mu' \in Q_2\left(\mathbb{R}\right), \forall x, x' \in \mathbb{R}$,

$$\begin{aligned}
&\left|\partial_\mu^{P_x} \rho\left(t, x, \mu\right)\right| \leq C,\\
&\left|\partial_\mu^{P_x} \rho\left(t, x, \mu\right) - \partial_\mu^{P_x} \rho\left(t, x', \mu'\right)\right| \leq C\left(\mathbb{D}_2\left(\mu, \mu'\right) + \left|x - x'\right|\right),
\end{aligned}$$

   ii. For $\rho = M, \varphi$, and $\forall \mu, \mu' \in Q_2\left(\mathbb{R}\right), \forall x, x' \in \mathbb{R}$,

$$\begin{aligned}
&\left|\partial_\mu^{P_x} \rho\left(x, \mu\right)\right| \leq C,\\
&\left|\partial_\mu^{P_x} \rho\left(x, \mu\right) - \partial_\mu^{P_x} \rho\left(x', \mu'\right)\right| \leq C\left(\mathbb{D}_2\left(\mu, \mu'\right) + \left|x - x'\right|\right);
\end{aligned}$$

   iii. For $\rho = f, l$, and $\forall \mu_1, \mu_1', \mu_2, \mu_2', \mu_3, \mu_3', \mu_4, \mu_4' \in Q_2\left(\mathbb{R}\right)$ and $\forall x, x', y, y', z, z'$ $, \bar{z}, \bar{z}' \in \mathbb{R}$,

$$\begin{aligned}
&\left|\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}\right) \rho\left(t, x, \mu_1, y, \mu_2, z, \mu_3, \bar{z}, \mu_4\right)\right| \leq C,\\
&\left|\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}\right) \rho\left(t, x, \mu_1, y, \mu_2, z, \mu_3, \bar{z}, \mu_4\right)\right.\\
&\left.- \left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}\right) \rho\left(t, x', \mu_1', y', \mu_2', z', \mu_3', \bar{z}', \mu_4'\right)\right|\\
&\leq C\left(\left|x - x'\right| + \left|y - y'\right| + \left|z - z'\right| + \left|\bar{z} - \bar{z}'\right| + \mathbb{D}_2\left(\mu_1, \mu_1'\right)\right.\\
&\left.+ \mathbb{D}_2\left(\mu_2, \mu_2'\right) + \mathbb{D}_2\left(\mu_3, \mu_3'\right) + \mathbb{D}_2\left(\mu_4, \mu_4'\right)\right).
\end{aligned}$$

2.2 Notation and problem formulation

Let T be a fixed strictly positive real number and $\left(\Omega, \mathcal{F}, \mathbb{F}, P\right)$ be a complete filtered probability space equipped with two independent standard one-dimensional Brownian motions W and Y . Also assume that $\mathbb{F} = \left\{\mathcal{F}_t\right\}_{t \geq 0}$ and $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^Y \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of P-null set and $\mathcal{F}_t^W, \mathcal{F}_t^Y$ denotes the P-completed natural filtration generated by W, Y respectively. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, and by $(\cdot,\cdot)$ (resp. $|\cdot|$) the inner product (resp. norm). The set of the admissible control variables is denoted by $\mathcal{U}$.

Throughout what follows, we will use the following notations.

- $\mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued square-integrable $\mathcal{F}_t$-adapted processes.
- $\mathcal{L}^2_{\mathcal{F}}(\Omega,\mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued square-integrable $\mathcal{F}_T$-measurable random variables.
- $\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ is the Hilbert space with inner product $(x,y)_2=\mathbb{E}\left[x\cdot y\right], x,y\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ and the norm $\|x\|_2=\sqrt{(x,x)_2}$.
- $Q_2\left(\mathbb{R}^d\right)$ the space of all probability measures μ on $\left(\mathbb{R}^d,\mathcal{B}\left(\mathbb{R}^d\right)\right)$ with finite second moment, i.e, $\int_{\mathbb{R}^d}|x|^2\mu(dx)<\infty$, endowed with the following 2-Wasserstein metric: for $\mu,\nu\in Q_2\left(\mathbb{R}^d\right)$,

$$\mathbb{D}_2(\mu_1,\mu_2)=\inf\left\{\left[\int_{\mathbb{R}^d}|x-y|^2\rho(dx,dy)\right]^{\frac{1}{2}}:\rho\in Q_2\left(\mathbb{R}^{2d}\right),\rho\left(\cdot,\mathbb{R}^d\right)=\mu_1,\rho\left(\mathbb{R}^d,\cdot\right)=\mu_2\right\}.$$

- $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{F}},\widehat{P}\right)$ is a copy of the probability space $(\Omega,\mathcal{F},\mathbb{F},P)$.
- $\left(\widehat{\vartheta},\widehat{\alpha}\right)$ is an independent copy of the random variable (ϑ,α) defined on $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{F}},\widehat{P}\right)$, such that

$$\left(\widehat{\vartheta},\widehat{\alpha}\right)\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)\times\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right).$$

- $\left(\Omega\times\widehat{\Omega},\mathcal{F}\otimes\widehat{\mathcal{F}},\mathbb{F}\otimes\widehat{\mathbb{F}},P\otimes\widehat{P}\right)$ is the product probability space, such that

$$\left(\widehat{\vartheta},\widehat{\alpha}\right)(w,\widehat{w})=\left(\vartheta\left(\widehat{w}\right),\alpha\left(\widehat{w}\right)\right)\text{ for any }(w,\widehat{w})\in\Omega\times\widehat{\Omega}.$$

Let $\left(\widehat{u}_t,\widehat{x}_t,\widehat{y}_t,\widehat{z}_t,\widehat{\bar{z}}_t\right)$ represent an independent replica of $(u_t,x_t,y_t,z_t,\bar{z}_t)$ to ensure that $P_{x_t}=\widehat{P}_{\widehat{x_t}}$, $P_{y_t}=\widehat{P}_{\widehat{y_t}}$, $P_{z_t}=\widehat{P}_{\widehat{z_t}}$, and $P_{\bar{z}_t}=\widehat{P}_{\widehat{\bar{z}}_t}$. We denote the expectation under the probability measure $\widehat{P}$ as $\widehat{\mathbb{E}}[\cdot]$, and $P_X=P\circ X^{-1}$ represents the law of the random variable denoted by X.

Now, let's consider a nonempty convex subset U of $\mathbb{R}^k$. A control $v:\Omega\times[0,T]\to U$

is deemed admissible if it is $\mathcal{F}_t^Y$-adapted and satisfies $\sup_{0\leq t\leq T} \mathbb{E}\left|v_t\right|^2 < \infty$. We proceed to examine the stochastic control system characterized by general McKean–Vlasov FBSDEs.

$$\begin{cases} dx_t^v = b\left(t, x_t^v, P_{x_t^v}, v_t\right) dt + g\left(t, x_t^v, P_{x_t^v}, v_t\right) dW_t + \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) d\widetilde{W}_t^v \\ -dy_t^v = f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, v_t\right) dt - z_t^v dW_t - \bar{z}_t^v dY_t \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v, P_{x_T^v}\right), \end{cases} \tag{2.4}$$

where $P_{x_t}, P_{y_t}, P_{z_t}, P_{\bar{z}_t}$ denotes the law of the random variable $x_t, y_t, z_t, \bar{z}_t$ respectively. The coefficients of the controlled system (2.4) are defined as follows

$$\begin{aligned} b &: [0,T]\times\mathbb{R}\times Q_2(\mathbb{R})\times U \to \mathbb{R}, \quad g,\sigma : [0,T]\times\mathbb{R}\times Q_2(\mathbb{R})\times U \to \mathbb{R}, \\ \varphi &: \mathbb{R}\times Q_2(\mathbb{R}) \to \mathbb{R}, \\ f &: [0,T]\times\mathbb{R}\times Q_2(\mathbb{R})\times\mathbb{R}\times Q_2(\mathbb{R})\times\mathbb{R}\times Q_2(\mathbb{R})\times\mathbb{R}\times Q_2(\mathbb{R})\times U \to \mathbb{R}. \end{aligned}$$

It is worth noting that the above forward-backward stochastic differential equation (2.4) of type McKean–Vlasov is very general, in that the dependence of the coefficients on the probability law of the solution $P_{x_t^v}, P_{y_t^v}, P_{z_t^v}, P_{\bar{z}_t^v}$ could be genuinely nonlinear as an element of the space of probability measures.

We assume that the state processes $(x^v, y^v, z^v, \bar{z}^v)$ cannot be observed directly, but the controllers can observe a related noisy process Y, which is the solution of the following equation

$$\begin{cases} dY_t = \xi\left(t, x_t^v, P_{x_t^v}\right) dt + d\widetilde{W}_t^v, \\ Y_0 = 0, \end{cases} \tag{2.5}$$

where $\xi : [0,T]\times\mathbb{R}\times Q_2(\mathbb{R}) \to \mathbb{R}$ and $\widetilde{W}_t^v$ is stochastic processes depending on the control v.

Inserting (2.5) into (2.4), we have

$$\begin{cases} dx_t^v = \left[b\left(t, x_t^v, P_{x_t^v}, v_t\right) dt - \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) \xi\left(t, x_t^v, P_{x_t^v}\right)\right] dt + g\left(t, x_t^v, P_{x_t^v}, v_t\right) dW_t \\ \qquad +\sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) dY_t \\ -dy_t^v = f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, v_t\right) dt - z_t^v dW_t - \bar{z}_t^v dY_t \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v, P_{x_T^v}\right). \end{cases} \tag{2.6}$$

From Girsanov's theorem, it follows that if

$$Z_t^v = \exp\left\{\int_0^t \xi\left(s, x_s^v, P_{x_s^v}\right) dY_s - \frac{1}{2}\int_0^t \left|\xi\left(s, x_s^v, P_{x_s^v}\right)\right|^2 ds\right\},$$

where Z^v is the unique $\mathcal{F}_t^Y$-adapted solution of the SDE of McKean–Vlasov type

$$\begin{cases} dZ_t^v = Z_t^v \xi\left(t, x_t^v, P_{x_t^v}\right) dY_t, \\ Z_0^v = 1, \end{cases} \tag{2.7}$$

and if $dP^v = Z_t^v dP$, then P^v is a new probability and $\left(W, \widetilde{W}^v\right)$ is a two-dimensional standard Brownian motion under this probability.

The associated cost functional is also of McKean–Vlasov type, defined as

$$\begin{aligned} J(v) &= \mathbb{E}^v\left[\int_0^T l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, v_t\right) dt\right] \\ &\quad + \mathbb{E}^v\left[M\left(x_T^v, P_{x_T^v}\right) + h\left(y_0^v, P_{y_0^v}\right)\right], \end{aligned} \tag{2.8}$$

where $\mathbb{E}^v$ denotes the expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P^v)$ and

$$\begin{aligned} M &: \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}, \quad h : \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}, \\ l &: [0, T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}. \end{aligned}$$

Our partially observed optimal control problem of general McKean–Vlasov FBSDE is to minimize the cost functional (2.8) over $v \in \mathcal{U}$ subject to (2.4) and (2.5), *i.e.*,

$$\min_{v \in \mathcal{U}} J(v).$$

If an admissible control u attains the minimum, we call u an optimal control and $(x, y, z, \bar{z})$ an optimal state, respectively. Obviously, cost functional (2.8) can be rewritten as

$$\begin{aligned} J(v) &= \mathbb{E}\left[\int_0^T Z_t^v l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, v_t\right) dt\right] \\ &\quad + \mathbb{E}\left[Z_T^v M\left(x_T^v, P_{x_T^v}\right) + h\left(y_0^v, P_{y_0^v}\right)\right]. \end{aligned} \tag{2.9}$$

Then the original problem (2.8) is equivalent to minimize (2.9) over $v \in \mathcal{U}$ subject to (2.4) and (2.7). Clearly, under assumptions (**A1**) and (**A2**), with the help of Theorem

(2.3) in Buckdahn et al. [6], and Lemma 2 in the work of Wang et al. [54], for each $v \in \mathcal{U}$, there is a unique solution $(x, y, z, \bar{z}) \in \mathcal{L}_{\mathcal{F}}^2(0, T, \mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^2(0, T, \mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^2(0, T, \mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^2(0, T, \mathbb{R})$ which solves

$$\begin{cases} x_t^v = x_0 + \int_0^t \left[b\left(s, x_s^v, P_{x_s^v}, v_s\right) - \sigma\left(s, x_s^v, P_{x_s^v}, v_s\right) \xi\left(s, x_s^v, P_{x_s^v}\right) \right] ds \\ \qquad + \int_0^t g\left(s, x_s^v, P_{x_s^v}, v_s\right) dW_s + \int_0^t \sigma\left(s, x_s^v, P_{x_s^v}, v_s\right) dY_s, \\ y_t^v = y_T^v - \int_t^T f\left(s, x_s^v, P_{x_s^v}, y_s^v, P_{y_s^v}, z_s^v, P_{z_s^v}, v_s\right) dt + \int_t^T z_s^v dW_s + \int_t^T \bar{z}_s^v dY_s, \end{cases}$$

The main result of this thesis is stated in the following section.

2.3 Necessary conditions for optimal control problem of Mckean–Vlasov FBSDEs

For our partially observed optimal control problem using general Mckean-Vlasov FBSDEs, we establish the necessary conditions of optimality in this section. The Girsanov theorem, derivatives with respect to probability measure, and the introduction of variational equations with approximations of their solutions provide the foundation for the proof.

Let's define the Hamiltonian

$$\begin{aligned} &H(t, x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, v, p, q, k, \bar{k}, Q) \\ &= p\left(b\left(t, x, P_x, v\right) - \sigma\left(t, x, P_x, v\right) \xi\left(t, x, P_x\right)\right) - qf\left(t, x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, v\right) \\ &\quad + kg\left(t, x, P_x, v\right) + \bar{k}\sigma\left(t, x, P_x, v\right) + Q\xi\left(t, x, P_x\right) + l\left(t, x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, v\right). \end{aligned} \tag{2.10}$$

Suppose that u is an optimal control with the optimal trajectory $(x, y, z, \bar{z})$ of FBSDE (2.4). For any $0 \leq \theta \leq 1$ and $v + u \in \mathcal{U}$, we define a perturbed control $u_t^\theta = u_t + \theta v_t$. To simplify our notations, we denote for ξ, and $\psi = b, g, \sigma$

$$\begin{aligned} \xi(t) &= \xi\left(t, x_t, P_{x_t}\right), \qquad \psi(t) = \psi\left(t, x_t, P_{x_t}, u_t\right), \\ \xi_x(t) &= \xi_x\left(t, x_t, P_{x_t}\right), \quad \psi_\rho(t) = \psi_\rho\left(t, x_t, P_{x_t}, u_t\right), \text{for } \rho = x, v, \end{aligned}$$

and the derivative processes

$$\begin{aligned} \partial_\mu^{P_x} \xi\left(t\right) &= \partial_\mu^{P_x} \xi\left(t, \widehat{x}_t, P_{x_t}; x_t\right), \qquad \partial_\mu^{P_x} \psi\left(t\right) = \partial_\mu^{P_x} \psi\left(t, \widehat{x}_t, P_{x_t}, \widehat{u}_t; x_t\right), \\ \partial_\mu^{P_x} \xi\left(t, \widehat{x}_t\right) &= \partial_\mu^{P_x} \xi\left(t, x_t, P_{x_t}; \widehat{x}_t\right), \quad \partial_\mu^{P_x} \psi\left(t, \widehat{x}_t\right) = \partial_\mu^{P_x} \psi\left(t, x_t, P_{x_t}, u_t; \widehat{x}_t\right). \end{aligned}$$

Similarly, we denote for $\Psi = f, l$ and $\rho = x, y, z, \bar{z}, u$

$$\Psi(t) = \Psi\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, u_t\right),$$
$$\Psi_\rho(t) = \Psi_\rho\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, u_t\right).$$

Finally, we denote for $\zeta = x, y, z, \bar{z}$

$$\partial_\mu^{P_\zeta}\Psi\left(t\right) = \partial_\mu^{P_\zeta}\Psi\left(t, \widehat{x}_t, P_{x_t}, \widehat{y}_t, P_{y_t}, \widehat{z}_t, P_{z_t}, \widehat{\bar{z}}_t, P_{\widehat{\bar{z}}_t}, \widehat{u}_t; \zeta\right),$$
$$\partial_\mu^{P_\zeta}\Psi\left(t, \widehat{\zeta}_t\right) = \partial_\mu^{P_\zeta}\Psi\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, u_t; \widehat{\zeta}_t\right).$$

Now, we introduce the following variational equations which is a linear FBSDEs

$$\begin{cases} dx_t^1 = \Big[\left(b_x\left(t\right) - \sigma_x\left(t\right)\xi\left(t\right) - \sigma\left(t\right)\xi_x\left(t\right)\right)x_t^1 + \left[b_v\left(t\right) - \sigma_v\left(t\right)\xi\left(t\right)\right]v_t \\ \qquad + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right] - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right]\xi\left(t\right) - \sigma\left(t\right)\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right]\Big] dt \\ \qquad + \left[g_x\left(t\right)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} g\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right] + g_v\left(t\right)v_t\right] dW_t \\ \qquad + \left[\sigma_x\left(t\right)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right] + \sigma_v\left(t\right)v_t\right] dY_t, \\ -dy_t^1 = \Big[f_x\left(t\right)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right] + f_y\left(t\right)y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} f\left(t, \widehat{y}_t\right)\widehat{y}_t^1\right] \\ \qquad + f_z\left(t\right)z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} f\left(t, \widehat{z}_t\right)\widehat{z}_t^1\right] + f_{\bar{z}}\left(t\right)\bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} f\left(t, \widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right] + f_v\left(t\right)v_t\Big] dt \\ \qquad - z_t^1 dW_t - \bar{z}_t^1 dY_t, \\ x_0^1 = 0, \quad y_T^1 = \varphi_x\left(x_T, P_{x_T}\right)x_T^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(x_T, P_{x_T}, \widehat{x}_t\right)\widehat{x}_T^1\right], \end{cases} \tag{2.11}$$

and a linear SDE

$$\begin{cases} dZ_t^1 = \left[Z_t^1\xi\left(t\right) + Z_t\xi_x\left(t\right)x_t^1 + Z_t\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right]\right] dY_t, \\ Z_0^1 = 0. \end{cases} \tag{2.12}$$

Next, we introduce the following adjoint equations of McKean–Vlasov type

$$\begin{cases} -dp_t = \Big[b_x(t)p_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b\,(t)\,\widehat{p}_t\right] - \sigma\,(t)\,\xi_x\,(t)\,p_t - \sigma\,(t)\,\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\,(t)\,\widehat{p}_t\right] \\ \qquad -\sigma_x\,(t)\,\xi\,(t)\,p_t - \xi\,(t)\,\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma\,(t)\,\widehat{p}_t\right] + g_x(t)k_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} g\,(t)\,\widehat{k}_t\right] \\ \qquad +\sigma_x(t)\bar{k}_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma\,(t)\,\widehat{\bar{k}}_t\right] + \xi_x\,(t)\,Q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\,(t)\,\widehat{Q}_t\right] \\ \qquad -f_x(t)q_t - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f\,(t)\,\widehat{q}_t\right] + l_x(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\,(t)\right]\Big]\,dt \\ \qquad -k_t dW_t - \bar{k}_t d\widetilde{W}_t, \\ dq_t = \Big[f_y(t)q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} f\,(t)\,\widehat{q}_t\right] - l_y\,(t) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\,(t)\right]\Big]\,dt \\ \qquad + \Big[f_z\,(t)\,q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} f\,(t)\,\widehat{q}_t\right] - l_z\,(t) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\,(t)\right]\Big]\,dW_t \\ \qquad + \Big[f_{\bar{z}}\,(t)\,q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} f\,(t)\,\widehat{q}_t\right] - \xi\,(t)\,q_t - l_{\bar{z}}\,(t) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\,(t)\right]\Big]\,d\widetilde{W}_t, \\ p_T = M_x\left(x_T, P_{x_T}\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(\widehat{x}_T, P_{x_T}, x_T\right)\right] \\ \qquad -\varphi_x\left(x_T, P_{x_T}\right) q_t - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{q}_t\right], \\ q_0 = -h_y(y_0, P_{y_0}) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right]. \end{cases} \tag{2.13}$$

It is clear that, under assumptions (**A1**) and (**A2**), there exists a unique $\left(p, k, \bar{k}\,(\cdot)\,, q\right) \in \mathcal{L}_{\mathcal{F}}^2\,(0, T, \mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^2\,(0, T, \mathbb{R}) \times \mathbb{M}^2\left([0, T]\,; \mathbb{R}\right) \times \mathcal{L}_{\mathcal{F}}^2\,(0, T, \mathbb{R})$ satisfying the FBSDE (2.13) of McKean–Vlasov type.

Remark Note that the mean-field nature of FBSDE (2.13) comes from the terms involving Fréchet derivatives $\partial_\mu^{P_x} b\,(t)\,, \partial_\mu^{P_x} g\,(t)\,, \partial_\mu^{P_x}\sigma\,(t)\,, \partial_\mu^{P_x}\xi\,(t)$ and $\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}\right)(f, l)$, which will reduce to a standard BSDE if the coefficients do not explicitly depend on the law of the solution.

Now, we introduce the following BSDE involved in the stochastic maximum principle

$$\begin{cases} -dP_t = l\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, u_t\right) dt \\ \qquad -\bar{Q}_t dW_t - Q_t d\widetilde{W}_t, \\ P_T = M(x_T, P_{x_T}). \end{cases} \tag{2.14}$$

Under assumptions (**A1**) and (**A2**), it is easy to prove that BSDE (2.14) admits a unique strong solution, given by

$$\begin{aligned} P_t &= M(x_T, P_{x_T}) - \int_t^T l\left(s, x_s, P_{x_s}, y_s, P_{y_s}, z_s, P_{z_s}, \bar{z}_s, P_{\bar{z}_s}, v_s\right) ds \\ &\quad + \int_t^T \bar{Q}_s dW_s + \int_t^T Q_s d\widetilde{W}_s. \end{aligned}$$

The following theorem presents the fundamental result of this section.

Theorem 2.1 (*Partial necessary conditions of optimality*)

*Let assumptions (**A1**) and (**A2**) hold. Let $(x, y, z, \bar{z}, u)$ be an optimal solution of our partially observed optimal control problem. Then, there are $(p, q, k, \bar{k})$ and $(P, \bar{Q}, Q)$ of $\mathbb{F}-$adapted processes that satisfy (2.13), (2.14) respectively, and that for all $v \in \mathcal{U}$, we have*

$$\mathbb{E}^u \left[H_v\left(t\right)\left(v_t - u_t\right)/\mathcal{F}_t^Y \right] \geq 0, a.e, a.s, \tag{2.15}$$

where the Hamiltonian function

$$H\left(t\right) = H\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, u_t, p_t, q_t, k_t, \bar{k}_t\left(\cdot\right), Q_t\right),$$

is defined by (2.10).

We give some auxiliary results to present our fundamental result in Theorem 2.1.

Lemma 2.2

*Let assumptions (**A1**) and (**A2**) hold. Then, we have*

$$\lim_{\theta\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T}\left|x_t^\theta - x_t\right|^2\right] = 0, \tag{2.16}$$

$$\lim_{\theta\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T}\left|y_t^\theta - y_t\right|^2 + \int_0^T\left(\left|z_t^\theta - z_t\right|^2 + \left|\bar{z}_t^\theta - \bar{z}_t\right|^2\right)ds\right] = 0, \tag{2.17}$$

$$\lim_{\theta\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T}\left|Z_t^\theta - Z_t\right|^2\right] = 0. \tag{2.18}$$

Proof: The proof was obtained using Ito's Formula and Gronwall's Theorem, moreover for $c, r = 0$ the proof of lemma 3.2.1 in chapter 3 can be derived using the same method. ■

Lemma 2.3

*Under the assumptions (**A1**) and (**A2**), the following estimations holds*

$$\lim_{\theta\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T}\left|\tilde{x}_t^\theta\right|^2\right] = 0, \tag{2.19}$$

$$\lim_{\theta\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T}\left|\tilde{y}_t^\theta\right|^2 + \int_0^T\left(\left|\tilde{z}_t^\theta\right|^2 + \left|\tilde{\bar{z}}_t^\theta\right|^2\right)dt\right], \tag{2.20}$$

$$\mathbb{E}\int_0^T\left|\tilde{Z}_t^\theta\right|^2 dt = 0. \tag{2.21}$$

Proof: The proof was established using Ito's Formula, Taylor's Expansion, Young's Inequality, and Gronwall's Lemma. For $c, r = 0$, the proof of lemma 3.2.1 in chapter 3 may be generated using the same method. ■

Since u is an optimal control, then, we have the following lemma.

Lemma 2.4

*Let assumptions (**A1**) and (**A2**) hold. Then, we have the following variational inequality*

$$\begin{aligned} 0 \leq\ & \mathbb{E}\left[Z_T M_x\left(x_T, P_{x_T}\right) x_T^1 + Z_T \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}, \widehat{x}_T\right)\widehat{x}_T^1\right]\right] \\ & + \mathbb{E}\left[Z_T^1 M\left(x_T, P_{x_T}\right) + h_y\left(y_0, P_{y_0}\right) y_0^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(y_0, P_{y_0}, \widehat{y}_0)\widehat{y}_0^1\right]\right] \\ & + \mathbb{E}\int_0^T \left[Z_t^1 l(t) + Z_t\left(l_x(t) x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right]\right) + Z_t\left(l_y(t) y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t, \widehat{y}_t\right)\widehat{y}_t^1\right]\right)\right. \\ & \left. + Z_t\left(l_z(t) z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{z}_t\right)\widehat{z}_t^1\right]\right) + Z_t\left(l_{\bar{z}}(t)\bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t, \widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right]\right) + Z_t l_v(t) v_t\right] dt. \end{aligned} \tag{2.22}$$

Proof: Using Lemmas 2.3 and Taylor expansion, we have

$$\begin{aligned} 0 &\leq \frac{1}{\theta}\left[J\left(u_t^\theta\right) - J\left(u_t\right)\right] \\ &= \frac{1}{\theta}\mathbb{E}\left[Z_T^\theta M\left(x_T^\theta, P_{x_T^\theta}\right) - Z_T M\left(x_T, P_{x_T}\right)\right] \\ &\quad + \frac{1}{\theta}\mathbb{E}\left[h\left(y_0^\theta\right) - h\left(y_0\right)\right] \\ &\quad + \frac{1}{\theta}\mathbb{E}\int_0^T \left[Z_t^\theta l^\theta(t) - Z_t l(t)\right] dt \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where $l^\theta(t) = l\left(t, x_t^\theta, P_{x_t^\theta}, y_t^\theta, P_{y_t^\theta}, z_t^\theta, P_{z_t^\theta}, \bar{z}_t^\theta, P_{\bar{z}_t^\theta}, u_t^\theta\right)$.

Then, from the results of (2.19), (2.20) and (2.21), we derive

$$\begin{aligned} I_1 &= \frac{1}{\theta}\mathbb{E}\left[Z_T^\theta M\left(x_T^\theta, P_{x_T^\theta}\right) - Z_T M\left(x_T, P_{x_T}\right)\right] \\ &= \frac{1}{\theta}\mathbb{E}\left[\left(Z_T^\theta - Z_T\right) M\left(x_T^\theta, P_{x_T^\theta}\right)\right] \\ &\quad + \frac{1}{\theta}\mathbb{E}\left[Z_T \int_0^1 M_x\left(x_T + \lambda\left(x_T^\theta - x_T\right), P_{x_T + \lambda\left(\widehat{x}_T^\theta - \widehat{x}_T\right)}\right)\left(x_T^\theta - x_T\right) d\lambda\right] \\ &\quad + \frac{1}{\theta}\mathbb{E}\left[Z_T \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T + \lambda\left(\widehat{x}_T^\theta - \widehat{x}_T\right), P_{x_T + \lambda\left(\widehat{x}_T^\theta - \widehat{x}_T\right)}, \widehat{x}_T\right)\left(\widehat{x}_T^\theta - \widehat{x}_T\right)\right] d\lambda\right] \\ &\longrightarrow \mathbb{E}^u\left[\vartheta_T M\left(x_T, P_{x_T}\right)\right] + \mathbb{E}^u\left[\left(M_x\left(x_T, P_{x_T}\right)\right) x_T^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}, \widehat{x}_T\right)\widehat{x}_T^1\right]\right]. \end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_2 &= \frac{1}{\theta}\mathbb{E}\left[h\left(y_0^\theta, P_{y_0^\theta}\right) - h\left(y_0, P_{y_0}\right)\right] \\
&= \frac{1}{\theta}\mathbb{E}\left[\int_0^1 h_y\left(y_0 + \lambda\left(y_0^\theta - y_0\right), P_{y_0+\lambda\left(\widehat{y}_0^\theta - \widehat{y}_0\right)}\right)\left(y_0^\theta - y_0\right)d\lambda\right] \\
&\quad + \frac{1}{\theta}\mathbb{E}\left[\int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h\left(y_0 + \lambda\left(\widehat{y}_0^\theta - \widehat{y}_0\right), P_{y_0+\lambda\left(\widehat{y}_0^\theta - \widehat{y}_0\right)}, \widehat{y}_0\right)\left(\widehat{y}_0^\theta - \widehat{y}_0\right)\right]d\lambda\right] \\
&\longrightarrow \mathbb{E}^u\left[\left(h_y\left(y_0, P_{y_0}\right)\right)y_0^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(y_0, P_{y_0}, \widehat{y}_0)\widehat{y}_0^1\right]\right],
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \frac{1}{\theta}\mathbb{E}\left[\int_0^T \left(Z_t^\theta l^\theta(t) - Z_t l(t)\right)dt\right] \\
&\longrightarrow \mathbb{E}^u\int_0^T \left[\vartheta_t l(t) + l_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right] + l_y(t)y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t, \widehat{y}_t\right)\widehat{y}_t^1\right]\right. \\
&\quad \left. + l_z(t)z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{z}_t\right)\widehat{z}_t^1\right] + l_{\bar{z}}(t)\bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right] + l_v(t)v_t\right]dt.
\end{aligned}$$

Then, the variational inequality (2.22) can be rewritten as

$$\begin{aligned}
0 \leq\ & \mathbb{E}^u\left[M_x\left(x_T, P_{x_T}\right)x^1(T) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}, \widehat{x}_T\right)\widehat{x}_T^1\right]\right] \\
& + \mathbb{E}^u\left[\vartheta_T M\left(x_T, P_{x_T}\right) + h_y\left(y_0, P_{y_0}\right)y^1(0) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(y_0, P_{y_0}, \widehat{y}_0)\widehat{y}_0^1\right]\right] \\
& + \mathbb{E}^u\int_0^T \left[\vartheta_t l(t) + l_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right] + l_y(t)y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t, \widehat{y}_t\right)\widehat{y}_t^1\right]\right. \\
& \left. + l_z(t)z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{z}_t\right)\widehat{z}_t^1\right] + l_{\bar{z}}(t)\bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t, \widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right] + l_v v(t)\right]
\end{aligned} \tag{2.23}$$

Set $\vartheta = Z^{-1}Z^1$, using Itô's formula, we have

$$\begin{cases}
d\vartheta_t = \left[\xi_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right]\right]d\widetilde{W}_t, \\
\vartheta_0 = 0.
\end{cases} \tag{2.24}$$

Applying Itô's formula to $p_t x_t^1$ and $q_t y_t^1$ such that,

$$\begin{aligned}
q_0 &= -h_y(y_0, P_{y_0}) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right], \\
p_T &= M_x\left(x_T, P_{x_T}\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(\widehat{x}_T, P_{x_T}, x_T\right)\right] \\
&\quad - \varphi_x\left(x_T, P_{x_T}\right)q_T - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{q}_T\right],
\end{aligned}$$

and using Fubini's theorem, we get

$$\begin{aligned}
\mathbb{E}^u\left[p_T x_T^1\right] &= \mathbb{E}^u\int_0^T \left[p_t\left(b_v(t) - \sigma_v(t)\xi(t)\right)v_t + \bar{k}_t\sigma_v(t)v_t + k_t g_v(t)v_t\right]dt \\
&\quad + \mathbb{E}^u\int_0^T x_t^1\left[f_x(t)q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f(t)\widehat{q}_t\right] - l_x(t) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l(t)\right]\right]dt \\
&\quad - \mathbb{E}^u\int_0^T x_t^1\left[\xi_x(t)Q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t)\widehat{Q}_t\right]\right]dt,
\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}
&\mathbb{E}^u\left[q_T y_T^1\right]+\mathbb{E}^u\left[h_y(y_0,P_{y_0})+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}h(\widehat{y}_0,P_{y_0},y_0)\right]\right]\\
=\;&-\mathbb{E}^u\int_0^T q_t\left[f_v\left(t\right)v_t+f_x\left(t\right)x_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}f\left(t,\widehat{x}_t\right)\widehat{x}_t^1\right]\right]dt\\
&-\mathbb{E}^u\int_0^T y_t^1\left[l_y\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}l\left(t\right)\right]\right]dt-\mathbb{E}^u\int_0^T z_t^1\left[l_z\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z}l\left(t\right)\right]\right]dt \qquad (2.26)\\
&-\mathbb{E}^u\int_0^T \bar{z}_t^1\left[l_{\bar{z}}\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}}l\left(t\right)\right]\right]dt.
\end{aligned}$$

Now, applying Itô's formula to $\vartheta_t P_t$ and using also Fubini's theorem, we have

$$\begin{aligned}
\mathbb{E}^u\left[\vartheta_T M(x_T)\right]\;=\;&-\mathbb{E}^u\int_0^T\vartheta_t l\left(t\right)dt\\
&+\mathbb{E}^u\int_0^T Q_t\left[\xi_x\left(t\right)x_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t,\widehat{x}_t\right)\widehat{x}_t^1\right]\right]dt. \qquad (2.27)
\end{aligned}$$

From Eqs. (2.25), (2.26), and (2.27), we obtain

$$\begin{aligned}
&\mathbb{E}^u\left[M_x\left(x_T,P_{x_T}\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}M\left(x_T,P_{x_T}\right)\right]\right]\\
&+\mathbb{E}^u\left[h_y(y_0,P_{y_0})+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}h(\widehat{y}_0,P_{y_0},y_0)\right]+\vartheta_T M(x_T)\right]\\
=\;&\mathbb{E}^u\int_0^T\left[p_t\left[b_v\left(t\right)-\sigma_v\xi(t)\right]v_t+\bar{k}_t\sigma_v\left(t\right)v_t+k_t g_v\left(t\right)v_t\right.\\
&-\mathbb{E}^u\int_0^T\vartheta_t l\left(t\right)dt-\mathbb{E}^u\int_0^T x_t^1\left[l_x(t)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}l\left(t\right)\right]\right]dt \qquad (2.28)\\
&-\mathbb{E}^u\int_0^T y_t^1\left[l_y\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}l\left(t\right)\right]\right]dt-\mathbb{E}^u\int_0^T z_t^1\left[l_z\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z}l\left(t\right)\right]\right]dt\\
&-\mathbb{E}^u\int_0^T \bar{z}_t^1\left[l_{\bar{z}}\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}}l\left(t\right)\right]\right]dt,
\end{aligned}$$

thus

$$\begin{aligned}
&\mathbb{E}^u\left[M_x\left(x_T,P_{x_T}\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}M\left(x_T,P_{x_T}\right)\right]\right]\\
&+\mathbb{E}^u\left[h_y(y_0,P_{y_0})+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}h(\widehat{y}_0,P_{y_0},y_0)\right]+\vartheta_T M(x_T)\right]\\
=\;&\mathbb{E}^u\int_0^T H_v\left(t\right)v_t-\mathbb{E}^u\int_0^T l_v(t)v_t dt-\mathbb{E}^u\int_0^T\vartheta_t l\left(t\right)dt-\mathbb{E}^u\int_0^T x_t^1\left[l_x(t)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}l\left(t\right)\right]\right]dt\\
&-\mathbb{E}^u\int_0^T y_t^1\left[l_y\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}l\left(t\right)\right]\right]dt-\mathbb{E}^u\int_0^T z_t^1\left[l_z\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z}l\left(t\right)\right]\right]dt\\
&-\mathbb{E}^u\int_0^T \bar{z}_t^1\left[l_{\bar{z}}\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}}l\left(t\right)\right]\right]dt.
\end{aligned}$$

This together with the variational inequality (2.23) imply (2.15), the proof is then completed. ■

2.4 Sufficient conditions for optimal control problem of Mckean–Vlasov FBSDEs

Following, we will demonstrate that the necessary condition of partially observed optimal control in Theorem 2.1 is also sufficient under certain additional convexity conditions. A function $\phi : \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}$ is convex if, for every $(x^u, P_x^u), (x^v, P_x^v) \in \mathbb{R} \times Q_2(\mathbb{R})$,

$$\phi(x^v, P_x^v) - \phi(x^u, P_x^u) \geq \phi_x(x^u, P_x^u)(x^v - x^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\phi(x^u, P_x^u)(x^v - x^u)\right].$$

For this, we need an additional assumption condition (**A3**) as follows:

Assumption (A3)

1. The functions M, h are convex in (x, P_x) and (y, P_y) respectively.
2. $H\left(t, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, p_t^u, q_t^u, k_t^u, \bar{k}_t^u(\cdot), Q_t^u\right)$ is convex in $(x^u, P_x^u, y^u, P_{y^u}, z^u, P_{z^u}, \bar{z}^u, P_{\bar{z}^u}, u)$ for $a.e.\ t \in [0, T]$, $P - a.s.$

$$\begin{aligned} H^v(t) - H^u(t) \geq\ & H_x^u(t)(x^v - x^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} H^u(t)(\widehat{x}^v - \widehat{x}^u)\right] \\ & + H_y^u(t)(y^v - y^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} H^u(t)(\widehat{y}^v - \widehat{y}^u)\right] \\ & + H_z^u(t)(z^v - z^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} H^u(t)(\widehat{z}^v - \widehat{z}^u)\right] \\ & + H_{\bar{z}}^u(t)(\bar{z}^v - \bar{z}^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} H^u(t)\left(\widehat{\bar{z}}^v - \widehat{\bar{z}}^u\right)\right], \end{aligned}$$

where

$$\begin{aligned} H^v(t) &= H\left(t, x^v, P_x^v, y^v, P_{y^v}, z^v, P_{z^v}, \bar{z}^v, P_{\bar{z}^v}, v, p^u, q^u, k^u, \bar{k}^u(\cdot), Q^u\right), \\ H^u(t) &= H\left(t, x^u, P_x^u, y^u, P_{y^u}, z^u, P_{z^u}, \bar{z}^u, P_{\bar{z}^u}, u, p^u, q^u, k^u, \bar{k}^u(\cdot), Q^u\right). \end{aligned}$$

Now, we can prove the sufficient conditions of optimality for our control problem of McKean–Vlasov FBSDEs, which is the third main result of this Thesis.

Theorem 2.5 (*Partial sufficient conditions of optimality*)

*Suppose (**A1**),(**A2**) and (**A3**) hold. Let Z^v be $\mathcal{F}_t^Y$-adapted, $u \in \mathcal{U}$ be an admissible control, and $(x, y, z, \bar{z})$ be the corresponding trajectories. Let $\left(p, k, \bar{k}(\cdot), q\right)$ and $\left(P, Q, \bar{Q}\right)$ satisfy (2.13) and (2.14), respectively. Moreover, the Hamiltonian H is* *convex in* $(x, P_x, y, P_y, , z, P_z, \bar{z}, P_{\bar{z}}, v)$, *and*

$$\mathbb{E}^u\left[H_v(t)(v_t - u_t)/\mathcal{F}_t^Y\right] \geq 0, a.\mathbb{E}, a.s, .$$

Then u *is a partial observed optimal control for the problem* (2.4) − (2.9) *subject to* (2.7).

Proof: For any $v \in \mathcal{U}$, we have

$$\begin{aligned} J(v) - J(u) &= \mathbb{E}\left[Z_T^v M\left(x_T^v, P_{x_T^v}\right) - Z_T^u M\left(x_T^u, P_{x_T^u}\right)\right] \\ &\quad + \mathbb{E}\left[h\left(y_0^v, P_{y_0^v}\right) - h\left(y_0^u, P_{y_0^u}\right)\right] \\ &\quad + \mathbb{E}\int_0^T \left(Z_t^v l^v(t) - Z_t^u l^u(t)\right) dt, \end{aligned}$$

where

$$\begin{aligned} l^v(t) &= l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}^v, P_{\bar{z}^v}, v_t\right), \\ l^u(t) &= l\left(t, x_t^u, P_{x_t^u}, y_t^u, P_{y_t^u}, z_t^u, P_{z_t^u}, \bar{z}^u, P_{\bar{z}^u}, u_t\right). \end{aligned}$$

By the convexity property of M and h, we get

$$\begin{aligned} \mathbb{E}\left[Z_T^v M\left(x_T^v, P_{x_T^v}\right) - Z_T^u M\left(x_T^u, P_{x_T^u}\right)\right] &\geq \mathbb{E}[(Z_T^v - Z_T^u) M\left(x_T^u, P_{x_T^u}\right)] \\ &\quad + \mathbb{E}^u[M_x\left(x_T^u, P_{x_T^u}\right)(x_T^v - x_T^u)] \\ &\quad + \mathbb{E}^u[\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T^u, P_{x_T^u}\right)\right](x_T^v - x_T^u)]. \end{aligned} \tag{2.29}$$

Similarly,

$$\begin{aligned} \mathbb{E}\left[h\left(y_0^v, P_{y_0^v}\right) - h\left(y_0^u, P_{y_0^u}\right)\right] &\geq \mathbb{E}\left[h_y\left(y_0^u, P_{y_0^u}\right)(y_0^v - y_0^u))\right] \\ &\quad + \mathbb{E}\left[\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h\left(y_0^u, P_{y_0^u}\right)\right](y_0^v - y_0^u)\right], \end{aligned} \tag{2.30}$$

and

$$\mathbb{E}\int_0^T \left(Z_t^v l^v(t) - Z_t^u l^u(t)\right) dt = \mathbb{E}\int_0^T Z_t^v \left(l^v(t) - l^u(t)\right) dt + \mathbb{E}\int_0^T \left(Z_t^v - Z_t^u\right) l^u(t)\, dt. \tag{2.31}$$

From (2.29), (2.30) and (2.31), we can write

$$\begin{aligned} J(v)-J(u) \geq\ & \mathbb{E}^u[M_x\left(x_T^u, P_{x_T^u}\right)(x_T^v - x_T^u)] + \mathbb{E}^u[\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T^u, P_{x_T^u}, \widehat{x}_T^u\right)\right](x_T^v - x_T^u)] \\ & +\mathbb{E}\left[h_y\left(y_0^u, P_{y_0^u}\right)(y_0^v - y_0^u))\right] + \mathbb{E}\left[\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h\left(y_0^u, P_{y_0^u}, \widehat{y}_0^u\right)\right](y_0^v - y_0^u))\right] \\ & +\mathbb{E}\int_0^T Z_t^v\left(l^v(t) - l^u(t)\right)dt + \mathbb{E}\int_0^T \left(Z_t^v - Z_t^u\right) l^u(t)\, dt \\ & +\mathbb{E}\left[\left(Z_T^v - Z_T^u\right)\left(\int_0^T l^u(t)\, dt + M\left(x_T^u, P_{x_T^u}\right)\right)\right]. \end{aligned}$$

Noting that

$$\begin{aligned} q_0 &= -h_y(y_0, P_{y_0}) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right], \\ p_T &= M_x\left(x_T, P_{x_T}\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(\widehat{x}_T, P_{x_T}, x_T\right)\right] \\ &\quad -\varphi_x\left(x_T, P_{x_T}\right) q_T - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{q}_T\right], \end{aligned}$$

we have

$$\begin{aligned} J(v)-J(u) \geq\ & \mathbb{E}^u[p_T^u\left(x_T^v - x_T^u\right)] + \mathbb{E}^u\left[\varphi_x\left(x_T, P_{x_T}\right) q_T\left(x_T^v - x_T^u\right)\right] \\ & +\mathbb{E}^u\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{q}_T\left(x_T^v - x_T^u\right)\right] - \mathbb{E}\left[q_0^u\left(y_0^v - y_0^u\right)\right] \\ & +\mathbb{E}^u\int_0^T \left(l^v(t) - l^u(t)\right)dt + \mathbb{E}\left[\left(Z_T^v - Z_T^u\right)\left(\int_0^T l^u(t)\, dt + M\left(x_T^u, P_{x_T^u}\right)\right)\right]. \end{aligned}$$

Then, we can write

$$\begin{aligned} J(v)-J(u) \geq\ & \mathbb{E}^u[p_T^u\left(x_T^v - x_T^u\right)] + \mathbb{E}^u[q_T^u\left(y_T^v - y_T^u\right)] \\ & -\mathbb{E}\left[q_0^u\left(y_0^v - y_0^u\right)\right] + \mathbb{E}^u\int_0^T \left[l^v(t) - l^u(t)\right]dt \\ & +\mathbb{E}\left[\left(Z_T^v - Z_T^u\right)\left(\int_0^T l^u(t)\, dt + M\left(x_T^u, P_{x_T^u}\right)\right)\right]. \end{aligned}$$

Now, applying Ito's formula respectively to $p_t^u\left(x_t^v - x_t^u\right), q_t^u\left(y_t^v - y_t^u\right)$ and $P_t^u\left(Z_t^v - Z_t^u\right)$, and by taking expectations, we get

$$\begin{aligned} J(v)-J(u) \geq\ & \mathbb{E}^u\int_0^T \left(H^v(t) - H^u(t)\right)dt \\ & -\mathbb{E}^u\int_0^T H_x^u(t)\left(x_t^v - x_t^u\right)dt - \mathbb{E}^u\int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} H^u(t)\right]\left(x_t^v - x_t^u\right)dt \\ & -\mathbb{E}^u\int_0^T H_y^u(t)\left(y_t^v - y_t^u\right)dt - \mathbb{E}^u\int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} H^u(t)\right]\left(y_t^v - y_t^u\right)dt \\ & -\mathbb{E}^u\int_0^T H_z^u(t)\left(z_t^v - z_t^u\right)dt - \mathbb{E}^u\int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} H^u(t)\right]\left(z_t^v - z_t^u\right)dt \\ & -\mathbb{E}^u\int_0^T H_{\bar{z}}^u(t)\left(\bar{z}_t^v - \bar{z}_t^u\right)dt - \mathbb{E}^u\int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} H^u(t)\right]\left(\bar{z}_t^v - \bar{z}_t^u\right)dt. \end{aligned}$$

By the convexity of the functional H in $(x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, v)$, we have

$$\begin{aligned} J(v) - J(u) &\geq \mathbb{E}^u \int_0^T H_v(t)(v_t - u_t)\, dt \\ &= \mathbb{E} \int_0^T Z_t^u \mathbb{E}\left[H_v(t)(v_t - u_t) / \mathcal{F}_t^Y \right] dt. \end{aligned}$$

Since $Z_t^u \geq 0$, and using condition (2.15), we have

$$J(v) - J(u) \geq 0,$$

i.e., u is a partially observed optimal control. ■

Chapter 3

A Stochastic maximum principle for partially observed optimal control problem of McKean–Vlasov FBSDEs with jump

In this chapter, we develop the necessary and sufficient conditions for partially observed Mckean-Vlasov optimal control problems. A controlled forward-backward stochastic differential equations driven by Poisson's random measure and an independent Brownian motion describes the system. The McKean-Vlasov system's coefficients are dependent by the state of the solution process, as well as its probability law and control variable. In general, this may be stated as follows:

$$\begin{cases} dx_t^v = b\left(t, x_t^v, P_{x_t^v}, v_t\right) dt + g\left(t, x_t^v, P_{x_t^v}, v_t\right) dW_t + \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) d\widetilde{W}_t^v \\ \qquad + \int_\Theta c\left(t, x_{t_-}^v, P_{x_{t_-}^v}, v_t, e\right) \widetilde{N}(de, dt), \\ -dy_t^v = f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, r_t^v, P_{r_t^v}, v_t\right) dt - z_t^v dW_t - \bar{z}_t^v dY_t \\ \qquad - \int_\Theta r_t^v(e) \widetilde{N}(de, dt), \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v, P_{x_T^v}\right), \end{cases}$$

The cost function that must be minimized over the class of admissible controls is also of the McKean Vlasov type, with the form:

$$\begin{aligned} J(v) &= \mathbb{E}^v\left[\int_0^T l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, r_t^v, P_{r_t^v}, v_t\right) dt\right] \\ &\quad + \mathbb{E}^v\left[M\left(x_T^v, P_{x_T^v}\right) + h\left(y_0^v, P_{y_0^v}\right)\right], \end{aligned}$$

We utilize Girsanov's theorem, variational equations, and derivatives with respect to probability measure under convexity assumption to demonstrate our result.

This chapter will be organized as follows: Firstly, we will begin with a formulation of the partially observed control problem of general Mckean-Vlasov FBSDs with jump processes. Then, as our key conclusions, we prove the necessary and sufficient conditions of optimality. As an application, a linear quadratic control issue of this type of partially observed control problem is provided.

3.1 Preliminaries

Let T be a fixed strictly positive real number and $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space equipped with two independent standard one-dimensional Brownian motions W and Y. Let η be a stationary $\mathcal{F}_t$-Poisson point process with the characteristic measure $\pi(de)$. We denote by $N(de, dt)$ the counting measure or Poisson measure induced by η defined on $\Theta \times \mathbb{R}_+$, where Θ is a fixed nonempty subset of $\mathbb{R}$ with its Borel σ-field $\mathcal{B}(\Theta)$ and set $\widetilde{N}(de, dt) = N(de, dt) - \pi(de)\,dt$ satisfying $\int_\Theta (1 \wedge |e|^2)\pi(de) < \infty$ and $\pi(\Theta) < +\infty$. Also assume that $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ and $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^Y \vee \mathcal{F}_t^N \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of P-null set and $\mathcal{F}_t^W, \mathcal{F}_t^Y$ and $\mathcal{F}_t^N$ denotes the P-completed natural filtration generated by W, Y and N respectively. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, and by $(\cdot, \cdot)$ (resp. $|\cdot|$) the inner product (resp. norm). The set of the admissible control variables is denoted by $\mathcal{U}$. Let $\left(\widehat{u}_t, \widehat{x}_t, \widehat{y}_t, \widehat{z}_t, \widehat{\bar{z}}_t, \widehat{r}_t\right)$ be an independent copy of $(u_t, x_t, y_t, z_t, \bar{z}_t, r_t)$ so that $P_{x_t} = \widehat{P}_{\widehat{x}_t}$, $P_{y_t} = \widehat{P}_{\widehat{y}_t}$, $P_{z_t} = \widehat{P}_{\widehat{z}_t}$, $P_{\bar{z}_t} = \widehat{P}_{\widehat{\bar{z}}_t}$ and $P_{r_t} = \widehat{P}_{\widehat{r}_t}$. Throughout what follows, we will use the following notations.

- $\mathbb{M}^2([0, T]; \mathbb{R})$ the space of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted measurable process $c(\cdot)$, such that

$$\mathbb{E} \int_0^T \int_\Theta |c(t, e)|^2 \pi(de)\, dt < +\infty.$$

Let U be a nonempty convex subset of $\mathbb{R}^k$. A control $v : \Omega \times [0, T] \to U$ is called admissible if it is $\mathcal{F}_t^Y$-adapted and satisfies $\sup_{0 \leq t \leq T} \mathbb{E} |v_t|^2 < \infty$.

We consider the following stochastic control system with general McKean–Vlasov FBSDEs

$$\begin{cases} dx_t^v = b\left(t, x_t^v, P_{x_t^v}, v_t\right) dt + g\left(t, x_t^v, P_{x_t^v}, v_t\right) dW_t + \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) d\widetilde{W}_t^v \\ \qquad + \int_\Theta c\left(t, x_{t_-}^v, P_{x_{t_-}^v}, v_t, e\right) \widetilde{N}(de, dt), \\ -dy_t^v = f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, r_t^v, P_{r_t^v}, v_t\right) dt - z_t^v dW_t - \bar{z}_t^v dY_t \\ \qquad - \int_\Theta r_t^v(e) \widetilde{N}(de, dt), \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v, P_{x_T^v}\right), \end{cases} \tag{3.1}$$

where $P_{x_t}, P_{y_t}, P_{z_t}, P_{\bar{z}_t}$ and P_{r_t} denotes the law of the random variable $x_t, y_t, z_t, \bar{z}_t$ and r_t respectively. The coefficients of the controlled system (3.1) are defined as follows

$$\begin{aligned} b &: [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \rightarrow \mathbb{R}, \quad g, \sigma : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \rightarrow \mathbb{R}, \\ c &: [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \times \Theta \rightarrow \mathbb{R}, \quad \varphi : \mathbb{R} \times Q_2(\mathbb{R}) \rightarrow \mathbb{R}, \\ f &: [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \rightarrow \mathbb{R}. \end{aligned}$$

It is worth noting that the above forward-backward stochastic differential equation (3.1) of type McKean–Vlasov is very general, in that the dependence of the coefficients on the probability law of the solution $P_{x_t^v}, P_{y_t^v}, P_{z_t^v}, P_{\bar{z}_t^v}$ and $P_{r_t^v}$ could be genuinely nonlinear as an element of the space of probability measures.

We assume that the state processes $(x^v, y^v, z^v, \bar{z}^v, r^v)$ cannot be observed directly, but the controllers can observe a related noisy process Y, which is the solution of the following equation

$$\begin{cases} dY_t = \xi\left(t, x_t^v, P_{x_t^v}\right) dt + d\widetilde{W}_t^v, \\ Y_0 = 0, \end{cases} \tag{3.2}$$

where $\xi : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \rightarrow \mathbb{R}$ and $\widetilde{W}_t^v$ is stochastic processes depending on the control v.

Inserting (3.2) into (3.1), we have

$$\begin{cases} dx_t^v = \left[b\left(t, x_t^v, P_{x_t^v}, v_t\right) dt - \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) \xi\left(t, x_t^v, P_{x_t^v}\right)\right] dt + g\left(t, x_t^v, P_{x_t^v}, v_t\right) dW_t \\ \qquad + \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) dY_t + \int_\Theta c\left(t, x_{t_-}^v, P_{x_{t_-}^v}, v_t, e\right) \widetilde{N}(de, dt), \\ -dy_t^v = f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, r_t^v, P_{r_t^v}, v_t\right) dt - z_t^v dW_t - \bar{z}_t^v dY_t \\ \qquad - \int_\Theta r_t^v(e) \widetilde{N}(de, dt), \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v, P_{x_T^v}\right). \end{cases} \tag{3.3}$$

From Girsanov's theorem, it follows that if

$$Z_t^v = \exp\left\{\int_0^t \xi\left(s, x_s^v, P_{x_s^v}\right) dY_s - \frac{1}{2}\int_0^t \left|\xi\left(s, x_s^v, P_{x_s^v}\right)\right|^2 ds\right\},$$

where Z^v is the unique $\mathcal{F}_t^Y$-adapted solution of the SDE of McKean–Vlasov type

$$\begin{cases} dZ_t^v = Z_t^v \xi\left(t, x_t^v, P_{x_t^v}\right) dY_t, \\ Z_0^v = 1, \end{cases} \tag{3.4}$$

and if $dP^v = Z_t^v dP$, then P^v is a new probability and $\left(W, \widetilde{W}^v\right)$ is a two-dimensional standard Brownian motion under this probability.

The associated cost functional is also of McKean–Vlasov type, defined as

$$\begin{aligned} J(v) &= \mathbb{E}^v\left[\int_0^T l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, r_t^v, P_{r_t^v}, v_t\right) dt\right] \\ &\quad + \mathbb{E}^v\left[M\left(x_T^v, P_{x_T^v}\right) + h\left(y_0^v, P_{y_0^v}\right)\right], \end{aligned} \tag{3.5}$$

where $\mathbb{E}^v$ denotes the expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P^v)$ and

$$\begin{aligned} M &: \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}, \quad h : \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}, \\ l &: [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}. \end{aligned}$$

Our partially observed optimal control problem of general McKean–Vlasov FBSDE is to minimize the cost functional (3.5) over $v \in \mathcal{U}$ subject to (3.1) and (3.2), *i.e.*,

$$\min_{v \in \mathcal{U}} J(v).$$

If an admissible control u attains the minimum, we call u an optimal control and $(x, y, z, \bar{z}, r)$ an optimal state, respectively. Obviously, cost functional (3.5) can be rewritten as

$$\begin{aligned} J(v) &= \mathbb{E}\left[\int_0^T Z_t^v l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, r_t^v, P_{r_t^v}, v_t\right) dt\right] \\ &\quad + \mathbb{E}\left[Z_T^v M\left(x_T^v, P_{x_T^v}\right) + h\left(y_0^v, P_{y_0^v}\right)\right]. \end{aligned} \tag{3.6}$$

Then the original problem (3.5) is equivalent to minimize (3.6) over $v \in \mathcal{U}$ subject to (3.1) and (3.4).

Let us impose some assumptions on the coefficients of the state and the performance cost functional.

Assumption (A1)

1. For all $t \in [0,T]$, the function $\rho(\cdot,0,0,0) \in \mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R})$ for $\rho = b, g, \sigma$ and $c(\cdot,0,0,0,0) \in \mathbb{M}^2([0,T];\mathbb{R})$, $\xi(\cdot,0,0) \in \mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R})$, $f(\cdot,0,0,0,0,0,0,0,0,0,0,0) \in \mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R})$ and $\varphi(0,0) \in \mathcal{L}^2_{\mathcal{F}}(\Omega,\mathbb{R})$.

2. For any $t \in [0,T]$, the functions b, g, σ and c are continuously differentiable in (x,v) and they are bounded by $C(1+|x|+|v|)$. The function ξ is continuously differentiable in x.

3. The functions f and l are continuously differentiable in $(x,y,z,\bar{z},r,v)$, and they are bounded by $C(1+|x|+|y|+|z|+|\bar{z}|+|r|+|v|)$ and $C\left(1+|x|^2+|y|^2+|z^{|2}+|\bar{z}|^2+|r^{|2}+|v|^2\right)$ respectively. The derivatives of f and l with respect to $(x,y,z,\bar{z},r,v)$ are uniformly bounded.

4. The functions φ and M are continuously differentiable in x, and the function h is continuously differentiable in y. The derivatives M_x, h_y are bounded by $C(1+|x|)$ and $C(1+|y|)$ respectively.

5. The derivatives $b_x, b_v, g_x, g_v, \sigma_x, \sigma_v, \xi_x$ are continuous and uniformly bounded. Moreover, there exists a constants $C = C(T, \pi(\Theta)) > 0$ independent to v and Θ such that $\sup_{e\in\Theta} |c_v(t,x,\mu,v,e)| \le C$, and $\sup_{e\in\Theta} |c_x(t,x,\mu,v,e)| \le C$.

Assumption (A2)

1. The functions $b, g, \sigma, c, f, l, \xi, M, h, \varphi \in \mathbb{C}^{1,1}_b(Q_2(\mathbb{R}))$.

2. The derivatives $\partial_\mu^{P_x} b, \partial_\mu^{P_x} g, \partial_\mu^{P_x} \sigma, \partial_\mu^{P_x} c, \partial_\mu^{P_x} \xi, \left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}, \partial_\mu^{P_r}\right)(f,l)$ are bounded and Lipchitz continuous, such that, for some $C > 0$, it holds that

   i. For $\rho = b, g, \sigma, \xi$, and $\forall \mu, \mu' \in Q_2(\mathbb{R})$, $\forall x, x' \in \mathbb{R}$,

$$\left|\partial_\mu^{P_x} \rho(t,x,\mu)\right| \le C,$$
$$\left|\partial_\mu^{P_x} \rho(t,x,\mu) - \partial_\mu^{P_x} \rho(t,x',\mu')\right| \le C\left(\mathbb{D}_2(\mu,\mu') + |x-x'|\right),$$

and there exists a constants $C = C(T, \pi(\Theta)) > 0$ independent to v and Θ such that,

$$\sup_{e\in\Theta}\left|\partial_\mu^{P_x} c(t,x,\mu,e)\right| \leq C,$$
$$\sup_{e\in\Theta}\left|\partial_\mu^{P_x} c(t,x,\mu,e) - \partial_\mu^{P_x} c(t,x',\mu',e)\right| \leq C\left(|x-x'| + \mathbb{D}_2(\mu,\mu')\right);$$

ii. For $\rho = M, \varphi$, and $\forall \mu, \mu' \in Q_2(\mathbb{R}), \forall x, x' \in \mathbb{R}$,

$$\left|\partial_\mu^{P_x} \rho(x,\mu)\right| \leq C,$$
$$\left|\partial_\mu^{P_x} \rho(x,\mu) - \partial_\mu^{P_x} \rho(x',\mu')\right| \leq C\left(\mathbb{D}_2(\mu,\mu') + |x-x'|\right);$$

iii. For $\rho = f, l$, and $\forall \mu_1, \mu_1', \mu_2, \mu_2', \mu_3, \mu_3', \mu_4, \mu_4', \mu_5, \mu_5' \in Q_2(\mathbb{R})$ and $\forall x, x', y, y', z, z', \bar{z}, \bar{z}', r, r' \in \mathbb{R}$,

$$\left|\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}, \partial_\mu^{P_r}\right)\rho(t,x,\mu_1,y,\mu_2,z,\mu_3,\bar{z},\mu_4,r,\mu_5)\right| \leq C,$$
$$\Big|\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}, \partial_\mu^{P_r}\right)\rho(t,x,\mu_1,y,\mu_2,z,\mu_3,\bar{z},\mu_4,r,\mu_5)$$
$$-\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}, \partial_\mu^{P_r}\right)\rho(t,x',\mu_1',y',\mu_2',z',\mu_3',\bar{z}',\mu_4',r',\mu_5')\Big|$$
$$\leq C\left(|x-x'| + |y-y'| + |z-z'| + |\bar{z}-\bar{z}'| + |r-r'| + \mathbb{D}_2(\mu_1,\mu_1')\right.$$
$$\left. + \mathbb{D}_2(\mu_2,\mu_2') + \mathbb{D}_2(\mu_3,\mu_3') + \mathbb{D}_2(\mu_4,\mu_4') + +\mathbb{D}_2(\mu_5,\mu_5')\right).$$

Clearly, under assumptions (**A1**) and (**A2**), with the help of 2.3 in Buckdahn et al. [6], and Lemma 2 in the work of Wang et al. [54], for each $v \in \mathcal{U}$, there is a unique solution $(x,y,z,\bar{z},r) \in \mathcal{L}_{\mathcal{F}}^2(0,T,\mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^2(0,T,\mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^2(0,T,\mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^2(0,T,\mathbb{R}) \times \mathcal{L}_{\mathcal{F}}^2(0,T,\mathbb{R})$ which solves

$$\begin{cases} x_t^v = x_0 + \int_0^t \left[b\left(s,x_s^v,P_{x_s^v},v_s\right) - \sigma\left(s,x_s^v,P_{x_s^v},v_s\right)\xi\left(s,x_s^v,P_{x_s^v}\right)\right]ds + \int_0^t g\left(s,x_s^v,P_{x_s^v},v_s\right)dW_s \\ \qquad + \int_0^t \sigma\left(s,x_s^v,P_{x_s^v},v_s\right)dY_s + \int_0^t\int_\Theta c\left(s,x_{s_-}^v,P_{x_{s_-}^v},v_s,e\right)\widetilde{N}(de,ds), \\ y_t^v = y_T^v - \int_t^T f\left(s,x_s^v,P_{x_s^v},y_s^v,P_{y_s^v},z_s^v,P_{z_s^v},r_s^v,P_{r_s^v},v_s\right)dt + \int_t^T z_s^v dW_s \\ \qquad + \int_t^T \bar{z}_s^v dY_s + \int_t^T\int_\Theta r_s^v(e)\widetilde{N}(de,ds), \end{cases}$$

To simplify our notations, we denote for ξ, c and $\psi = b, g, \sigma$

$$\xi(t) = \xi(t,x_t,P_{x_t}), \qquad \psi(t) = \psi(t,x_t,P_{x_t},u_t),$$
$$\xi_x(t) = \xi_x(t,x_t,P_{x_t}), \qquad \psi_\rho(t) = \psi_\rho(t,x_t,P_{x_t},u_t),$$
$$c(t,e) = c\left(t,x_{t_-}^v,P_{x_{t_-}^v},u_t,e\right), \quad c_\rho(t,e) = c_\rho\left(t,x_{t_-}^v,P_{x_{t_-}^v},u_t,e\right), \text{for } \rho = x,u,$$

and the derivative processes

$$\partial_\mu^{P_x}\xi(t)=\partial_\mu^{P_x}\xi(t,\widehat{x}_t,P_{x_t};x_t), \qquad \partial_\mu^{P_x}\psi(t)=\partial_\mu^{P_x}\psi(t,\widehat{x}_t,P_{x_t},\widehat{u}_t;x_t),$$
$$\partial_\mu^{P_x}\xi(t,\widehat{x}_t)=\partial_\mu^{P_x}\xi(t,x_t,P_{x_t};\widehat{x}_t), \qquad \partial_\mu^{P_x}\psi(t,\widehat{x}_t)=\partial_\mu^{P_x}\psi(t,x_t,P_{x_t},u_t;\widehat{x}_t),$$
$$\partial_\mu^{P_x}c(t,e)=\partial_\mu^{P_x}c\left(t,\widehat{x}_{t_-},P_{x_{t_-}},\widehat{u}_t,e;x_t\right), \quad \partial_\mu^{P_x}c(t,e,\widehat{x}_t)=\partial_\mu^{P_x}c\left(t,x_{t_-},P_{x_{t_-}},u_t,e;\widehat{x}_t\right).$$

Similarly, we denote for $\Psi=f,l$ and $\rho=x,y,z,\bar{z},r,u$

$$\begin{aligned}\Psi(t)&=\Psi\left(t,x_t,P_{x_t},y_t,P_{y_t},z_t,P_{z_t},\bar{z}_t,P_{\bar{z}_t},r_t,P_{r_t},u_t\right),\\ \Psi_\rho(t)&=\Psi_\rho\left(t,x_t,P_{x_t},y_t,P_{y_t},z_t,P_{z_t},\bar{z}_t,P_{\bar{z}_t},r_t,P_{r_t},u_t\right).\end{aligned}$$

Finally, we denote for $\zeta=x,y,z,\bar{z},r$

$$\begin{aligned}\partial_\mu^{P_\zeta}\Psi(t)&=\partial_\mu^{P_\zeta}\Psi\left(t,\widehat{x}_t,P_{x_t},\widehat{y}_t,P_{y_t},\widehat{z}_t,P_{z_t},\widehat{\bar{z}}_t,P_{\widehat{\bar{z}}_t},\widehat{r}_t,P_{r_t},\widehat{u}_t;\zeta\right),\\ \partial_\mu^{P_\zeta}\Psi\left(t,\widehat{\zeta}_t\right)&=\partial_\mu^{P_\zeta}\Psi\left(t,x_t,P_{x_t},y_t,P_{y_t},z_t,P_{z_t},\bar{z}_t,P_{\bar{z}_t},r_t,P_{r_t},u_t;\widehat{\zeta}_t\right).\end{aligned}$$

Now, we introduce the following variational equations which is a linear FBSDEs

$$\begin{cases} dx_t^1=\Big[\left(b_x(t)-\sigma_x(t)\xi(t)-\sigma(t)\xi_x(t)\right)x_t^1+\left[b_v(t)-\sigma_v(t)\xi(t)\right]v_t\\ \quad +\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}b(t,\widehat{x}_t)\widehat{x}_t^1\right]-\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma(t,\widehat{x}_t)\widehat{x}_t^1\right]\xi(t)-\sigma(t)\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t,\widehat{x}_t)\widehat{x}_t^1\right]\Big]dt\\ \quad +\left[g_x(t)x_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}g(t,\widehat{x}_t)\widehat{x}_t^1\right]+g_v(t)v_t\right]dW_t\\ \quad +\left[\sigma_x(t)x_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma(t,\widehat{x}_t)\widehat{x}_t^1\right]+\sigma_v(t)v_t\right]dY_t\\ \quad +\int_\Theta\left[c_x(t,e)x_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}c(t,e,\widehat{x}_t)\widehat{x}_t^1\right]+c_v(t,e)v_t\right]\widetilde{N}(de,dt),\\ -dy_t^1=\Big[f_x(t)x_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}f(t,\widehat{x}_t)\widehat{x}_t^1\right]+f_y(t)y_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}f(t,\widehat{y}_t)\widehat{y}_t^1\right]\\ \quad +f_z(t)z_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z}f(t,\widehat{z}_t)\widehat{z}_t^1\right]+f_{\bar{z}}(t)\bar{z}_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}}f\left(t,\widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right]\\ \quad +f_r(t)r_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_r}f(t,\widehat{r}_t)\widehat{r}_t^1\right]+f_v(t)v_t\Big]dt\\ \quad -z_t^1dW_t-\bar{z}_t^1dY_t+\int_\Theta r_t^1(e)\widetilde{N}(de,dt),\\ x_0^1=0,\quad y_T^1=\varphi_x(x_T,P_{x_T})x_T^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi(x_T,P_{x_T},\widehat{x}_t)\widehat{x}_T^1\right],\end{cases} \tag{3.7}$$

and a linear SDE

$$\begin{cases} dZ_t^1=\left[Z_t^1\xi(t)+Z_t\xi_x(t)x_t^1+Z_t\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t,\widehat{x}_t)\widehat{x}_t^1\right]\right]dY_t,\\ Z_0^1=0.\end{cases} \tag{3.8}$$

Set $\vartheta=Z^{-1}Z^1$, using Itô's formula, we have

$$\begin{cases} d\vartheta_t=\left[\xi_x(t)x_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t,\widehat{x}_t)\widehat{x}_t^1\right]\right]d\widetilde{W}_t,\\ \vartheta_0=0.\end{cases} \tag{3.9}$$

Next, we introduce the following adjoint equations of McKean–Vlasov type

$$\begin{cases} -dp_t = \Big[b_x(t)p_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b\left(t\right)\widehat{p}_t\right] - \sigma\left(t\right)\xi_x\left(t\right)p_t - \sigma\left(t\right)\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t\right)\widehat{p}_t\right] \\ \qquad -\sigma_x\left(t\right)\xi\left(t\right)p_t - \xi\left(t\right)\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma\left(t\right)\widehat{p}_t\right] + g_x(t)k_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} g\left(t\right)\widehat{k}_t\right] \\ \qquad +\sigma_x(t)\bar{k}_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma\left(t\right)\widehat{\bar{k}}_t\right] + \int_\Theta \left[c_x\left(t,e\right)n_t\left(e\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} c\left(t,e\right)\widehat{n}_t\left(e\right)\right]\right]\pi\left(de\right) \\ \qquad +\xi_x\left(t\right)Q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t\right)\widehat{Q}_t\right] - f_x(t)q_t - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f\left(t\right)\widehat{q}_t\right] + l_x(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t\right)\right]\Big]dt \\ \qquad -k_t dW_t - \bar{k}_t d\widetilde{W}_t - \int_\Theta n_t\left(e\right)\widetilde{N}\left(de,dt\right), \\ dq_t = \Big[f_y(t)q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} f\left(t\right)\widehat{q}_t\right] - l_y\left(t\right) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t\right)\right]\Big]dt \\ \qquad + \Big[f_z\left(t\right)q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} f\left(t\right)\widehat{q}_t\right] - l_z\left(t\right) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t\right)\right]\Big]dW_t \\ \qquad + \Big[f_{\bar{z}}\left(t\right)q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} f\left(t\right)\widehat{q}_t\right] - \xi\left(t\right)q_t - l_{\bar{z}}\left(t\right) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t\right)\right]\Big]d\widetilde{W}_t, \\ \qquad + \int_\Theta \Big[f_r\left(t\right)q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_r} f\left(t\right)\widehat{q}_t\right] - l_r\left(t\right) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_r} l\left(t\right)\right]\Big]\widetilde{N}\left(de,dt\right) \\ p_T = M_x\left(x_T, P_{x_T}\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(\widehat{x}_T, P_{x_T}, x_T\right)\right] \\ \qquad -\varphi_x\left(x_T, P_{x_T}\right)q_t - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{q}_t\right], \\ q_0 = -h_y(y_0, P_{y_0}) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right]. \end{cases} \tag{3.10}$$

It is clear that, under assumptions (**A1**) and (**A2**), there exists a unique $\left(p,k,\bar{k},n\left(\cdot\right),q\right) \in \mathcal{L}_\mathcal{F}^2\left(0,T,\mathbb{R}\right) \times \mathcal{L}_\mathcal{F}^2\left(0,T,\mathbb{R}\right) \times \mathcal{L}_\mathcal{F}^2\left(0,T,\mathbb{R}\right) \times \mathbb{M}^2\left(\left[0,T\right];\mathbb{R}\right) \times \mathcal{L}_\mathcal{F}^2\left(0,T,\mathbb{R}\right)$ satisfying the FBSDE (3.10) of McKean–Vlasov type.

Remark 3.1

Note that the mean-field nature of FBSDE (3.10) comes from the terms involving Fréchet derivatives $\partial_\mu^{P_x} b\left(t\right), \partial_\mu^{P_x} g\left(t\right), \partial_\mu^{P_x}\sigma\left(t\right), \partial_\mu^{P_x}\xi\left(t\right)$ *and* $\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}, \partial_\mu^{P_r}\right)(f,l)$, *which will reduce to a standard BSDE if the coefficients do not explicitly depend on law of the solution.*

Now, we introduce the following BSDE involved in the stochastic maximum principle

$$\begin{cases} -dP_t = l\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, r_t, P_{r_t}, u_t\right)dt \\ \qquad -\bar{Q}_t dW_t - Q_t d\widetilde{W}_t, \\ P_T = M(x_T, P_{x_T}). \end{cases} \tag{3.11}$$

Under assumptions (**A1**) and (**A2**), it is easy to prove that BSDE (3.11) admits a unique strong solution, given by

$$\begin{aligned} P_t &= M(x_T, P_{x_T}) - \int_t^T l\left(s, x_s, P_{x_s}, y_s, P_{y_s}, z_s, P_{z_s}, \bar{z}_s, P_{\bar{z}_s}, r_s, P_{r_s}, v_s\right) ds \\ &\quad + \int_t^T \bar{Q}_s dW_s + \int_t^T Q_s d\widetilde{W}_s. \end{aligned}$$

Let us now, define the Hamiltonian $H : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, associated with the McKean–Vlasov stochastic control problem (3.1)-(3.6) by

$$\begin{aligned} &H(t, x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, r, P_r, v, p, q, k, \bar{k}, n, Q) \\ &= p\left(b\left(t, x, P_x, v\right) - \sigma\left(t, x, P_x, v\right) \xi\left(t, x, P_x\right)\right) - qf\left(t, x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, r, P_r, v\right) \\ &\quad + kg\left(t, x, P_x, v\right) + \bar{k}\sigma\left(t, x, P_x, v\right) + \int_\Theta n_t(e)\, c\left(t, x, P_x, v, e\right) \pi(de) + Q\xi\left(t, x, P_x\right) \\ &\quad + l\left(t, x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, r, P_r, v\right). \end{aligned} \tag{3.12}$$

The main result of this thesis is stated in the following section.

3.2 Necessary and sufficient conditions of optimality

In this section, we prove the necessary and sufficient conditions of optimality for our system of McKean–Vlasov type, satisfied by a partially observed optimal control, assuming that the solution exists. The the proof is based on convex perturbation and some estimates of the state processes of system and observed process.

3.2.1 Necessary conditions for optimal control problem of Mckean–Vlasov FBSDEs with jump

Suppose that u is an optimal control with the optimal trajectory $(x, y, z, \bar{z}, r)$ of FBSDE (3.1). For any $0 \leq \theta \leq 1$ and $v + u \in \mathcal{U}$, we define a perturbed control $u_t^\theta = u_t + \theta v_t$.

Our first result below, is related to the estimate of trajectory $(x, y, z, \bar{z}, r)$ and the observation Z_t.

Lemma 3.2

*Let assumptions (**A1**) and (**A2**) hold. Then, we have*

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|x_t^{\theta}-x_t\right|^2\right]=0, \tag{3.13}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|y_t^{\theta}-y_t\right|^2+\int_0^T\left(\left|z_t^{\theta}-z_t\right|^2+\left|\bar{z}_t^{\theta}-\bar{z}_t\right|^2\right.\right. \tag{3.14}$$
$$\left.\left.+\int_{\Theta}\left|r_t^{\theta}(e)-r_t(e)\right|^2\pi(de)\right)ds\right]=0,$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|Z_t^{\theta}-Z_t\right|^2\right]=0. \tag{3.15}$$

Proof: We first prove (3.13). From standard estimates and by using the Burkholder-Davis-Gundy (BDG) inequality, we get

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t}\left|x_s^{\theta}-x_s\right|^2\right]&\le\mathbb{E}\int_0^t\left|b^{\theta}(s)-b(s)\right|^2ds+\mathbb{E}\int_0^t\left|\sigma^{\theta}(s)\xi^{\theta}(s)-\sigma(s)\xi(s)\right|^2ds\\&+\mathbb{E}\int_0^t\left|g^{\theta}(s)-g(s)\right|^2ds+\mathbb{E}\int_0^t\left|\sigma^{\theta}(s)-\sigma(s)\right|^2ds\\&+\mathbb{E}\int_0^t\int_{\Theta}\left|c^{\theta}(s,e)-c(s,e)\right|^2\pi(de)\,ds,\end{aligned}$$

where

$$\begin{aligned}&\psi\left(s,x_s^{\theta},P_{x_s^{\theta}},u_s^{\theta}\right)=\psi^{\theta}(s),\ \text{ for }\psi=b,g,\sigma,\\&c\left(s,x_{s_-}^{\theta},P_{x_{s_-}^{\theta}},u_s^{\theta},e\right)=c^{\theta}(s,e).\end{aligned}$$

Then,

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t}\left|x_s^{\theta}-x_s\right|^2\right]&\le\mathbb{E}\int_0^t\left|b^{\theta}(s)-b(s)\right|^2ds+\mathbb{E}\int_0^t\left|\sigma^{\theta}(s)\left(\xi^{\theta}(s)-\xi(s)\right)\right|^2ds\\&+\mathbb{E}\int_0^t\left|\xi(s)\left(\sigma^{\theta}(s)-\sigma(s)\right)\right|^2ds+\mathbb{E}\int_0^t\left|g^{\theta}(s)-g(s)\right|^2ds\\&+\mathbb{E}\int_0^t\left|\sigma^{\theta}(s)-\sigma(s)\right|^2ds+\mathbb{E}\int_0^t\int_{\Theta}\left|c^{\theta}(s,e)-c(s,e)\right|^2\pi(de)\,ds.\end{aligned}$$

From assumptions (**A1**) and (**A2**), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}\left|x_t^{\theta}-x_t\right|^2\right]&\le C_T\mathbb{E}\int_0^t\left[\left|x_s^{\theta}-x_s\right|^2+\left|\mathbb{D}_2\left(P_{x_s^{\theta}},P_{x_s}\right)\right|^2\right]ds\\&+C_T\theta^2\mathbb{E}\int_0^t\left|v_s\right|^2ds.\end{aligned} \tag{3.16}$$

Recall that for the 2-Wasserstein metric $\mathbb{D}_2(\cdot,\cdot)$, we obtain

$$\begin{aligned}\mathbb{D}_2\left(P_{x_s^\theta}, P_{x_s}\right) &= \inf\left\{\left[\mathbb{E}\left|\widetilde{x}_s^\theta - \widetilde{x}_s\right|^2\right]^{\frac{1}{2}}, \text{ for all } \widetilde{x}^\varepsilon, \widetilde{x} \in \mathbb{L}^2(\mathcal{F};\mathbb{R}),\right. \\ &\qquad \left. \text{with } P_{x_s^\theta} = \mathbb{P}_{\widetilde{x}_s^\theta} \text{ and } \mathbb{P}_{x_s} = \mathbb{P}_{\widetilde{x}_s}\right\}, \\ &\leq \left[\mathbb{E}\left|x_s^\theta - x_s\right|^2\right]^{\frac{1}{2}}. \end{aligned} \tag{3.17}$$

From (3.16), (3.17), and Definition 2.1, we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|x_t^\theta - x_t\right|^2\right] \leq C_T\mathbb{E}\int_0^t \sup_{r\in[0,s]}\left|x_r^\theta - x_r\right|^2 ds + M_T^2\theta^2.$$

Then, from Gronwall's Lemma, the result follows immediately by letting ε go to zero.

Next, we prove (3.15). By applying Itô's formula to $\left|y_t^\theta - y_t\right|^2$ and taking expectation, we get

$$\begin{aligned} & \mathbb{E}\left|y_t^\theta - y_t\right|^2 + \mathbb{E}\int_t^T\left|z_s^\theta - z_s\right|^2 ds + \mathbb{E}\int_t^T\left|\bar{z}_s^\theta - \bar{z}_s\right|^2 ds + \mathbb{E}\int_t^T\int_\Theta\left|r_s^\theta(e) - r_s(e)\right|^2\pi(de)\,ds \\ = \;& \mathbb{E}\left|\varphi\left(x_T^\theta, P_{x_T^\theta}\right) - \varphi\left(x_T, P_{x_T}\right)\right|^2 + 2\mathbb{E}\int_t^T\left(y_s^\theta - y_s\right)\left[f^\theta(s) - f(s)\right] ds, \end{aligned}$$

where

$$f\left(s, x_s^\theta, P_{x_s^\theta}, y_s^\theta, P_{y_s^\theta}, z_s^\theta, P_{z_s^\theta}, \bar{z}_s^\theta, P_{\bar{z}_s^\theta}, r_s^\theta, P_{r_s^\theta}, u_s^\theta\right) = f^\theta(s).$$

For each $\varepsilon > 0$, and from Young's inequality, we have

$$\begin{aligned} & \mathbb{E}\left|y_t^\theta - y_t\right|^2 + \mathbb{E}\int_t^T\left|z_s^\theta - z_s\right|^2 ds + \mathbb{E}\int_t^T\left|\bar{z}_s^\theta - \bar{z}_s\right|^2 ds + \mathbb{E}\int_t^T\int_\Theta\left|r_s^\theta(e) - r_s(e)\right|^2\pi(de)\,ds \\ \leq \;& \mathbb{E}\left|\varphi\left(x_T^\theta, P_{x_T^\theta}\right) - \varphi\left(x_T, P_{x_T}\right)\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T\left|y_s^\theta - y_s\right|^2 ds + \varepsilon\mathbb{E}\int_t^T\left|f^\theta(s) - f(s)\right|^2 ds. \end{aligned}$$

By applying the Lipschitz conditions on the coefficients φ, f with respect to x, y, z, μ and v, we obtain

$$\begin{aligned} & \mathbb{E}\left|y_t^\theta - y_t\right|^2 + \mathbb{E}\int_t^T\left|z_s^\theta - z_s\right|^2 ds + \mathbb{E}\int_t^T\left|\bar{z}_s^\theta - \bar{z}_s\right|^2 ds + \mathbb{E}\int_t^T\int_\Theta\left|r_s^\theta(e) - r_s(e)\right|^2\pi(de)\,ds \\ \leq \;& \frac{1}{\varepsilon}\mathbb{E}\int_t^T\left|y_s^\theta - y_s\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T\left[\left|y_s^\theta - y_s\right|^2 + \left|\mathbb{D}_2\left(P_{y_s^\theta}, P_{y_s}\right)\right|^2\right] ds \\ & + C\varepsilon\mathbb{E}\int_t^T\left[\left|z_s^\theta - z_s\right|^2 + \left|\mathbb{D}_2\left(P_{z_s^\theta}, P_{z_s}\right)\right|^2\right] ds \\ & + C\varepsilon\mathbb{E}\int_t^T\left[\left|\bar{z}_s^\theta - \bar{z}_s\right|^2 + \left|\mathbb{D}_2\left(P_{\bar{z}_s^\theta}, P_{\bar{z}_s}\right)\right|^2\right] ds \\ & + C\varepsilon\mathbb{E}\int_t^T\int_\Theta\left[\left|r_s^\theta(e) - r_s(e)\right|^2 + \left|\mathbb{D}_2\left(P_{r_s^\theta}, P_{r_s}\right)\right|^2\right]\pi(de)\,ds + \alpha_t^\theta. \end{aligned} \tag{3.18}$$

Here α_t^θ is given by

$$\alpha_t^\theta = \mathbb{E}\left|\varphi\left(x_T^\theta, P_{x_T^\theta}\right) - \varphi\left(x_T, P_{x_T}\right)\right|^2 + C\varepsilon\mathbb{E}\left[\int_t^T \left|x_s^\theta - x_s\right|^2 + \left|\mathbb{D}_2\left(P_{x_s^\theta}, P_{x_s}\right)\right|^2\right] ds + C\varepsilon\theta^2.$$

Recall that for the 2-Wasserstein metric $\mathbb{D}_2(\cdot,\cdot)$, and by invoking (3.13) and sending θ to 0, we get $\lim\limits_{\theta\to 0}\alpha_t^\theta = 0$. Now, we take $\varepsilon = \dfrac{1}{2C}$ and replacing in (3.18), we obtain

$$\begin{aligned}&\mathbb{E}\left|y_t^\theta - y_t\right|^2 + \frac{1}{2}\mathbb{E}\int_t^T \left|z_s^\theta - z_s\right|^2 ds + \frac{1}{2}\mathbb{E}\int_t^T \left|\bar{z}_s^\theta - \bar{z}_s\right|^2 ds \\ &+\frac{1}{2}\mathbb{E}\int_t^T \int_\Theta \left|r_s^\theta(e) - r_s(e)\right|^2 \pi(de)\, ds \leq 2C\mathbb{E}\int_t^T \left|y_s^\theta - y_s\right|^2 ds + \frac{1}{2}\mathbb{E}\int_t^T \left|y_s^\theta - y_s\right|^2 ds + \alpha_t^\theta.\end{aligned}$$

Finally, applying Gronwall's lemma and letting θ goes to 0, we obtain the estimate (3.15).

Now, we proceed to estimate (3.15). Applying Itô's formula to $\left|Z_t^\theta - Z_t\right|^2$ and taking expectation, we get

$$\mathbb{E}\left|Z_t^\theta - Z_t\right|^2 \leq C\int_0^t \left|Z_s^\theta - Z_s\right|^2 ds + C\beta_t^\theta, \tag{3.19}$$

where β_t^θ is given by

$$\beta_t^\theta = \mathbb{E}^u \int_0^t \left|\xi\left(s, x_s^\theta, P_{x_s^\theta}\right) - \xi\left(s, x_s, P_{x_s}\right)\right|^2 ds.$$

Also, from assumptions (**A1**) and (**A2**), we have $\lim\limits_{\theta\to 0}\beta_t^\theta = 0$.

The proof of (3.15) follows directly by using Gronwall's lemma and sending θ to 0. ■

Lemma 3.3

*Under the assumptions (**A1**) and (**A2**), the following estimations holds*

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{x}_t^\theta\right|^2\right] = 0, \tag{3.20}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{y}_t^\theta\right|^2 + \int_0^T \left(\left|\widetilde{z}_t^\theta\right|^2 + \left|\widetilde{\bar{z}}_t^\theta\right|^2 + \int_\Theta \left|\widetilde{r}_t^\theta(e)\right|^2 \pi(de)\right) dt\right] = 0, \tag{3.21}$$

$$\mathbb{E}\int_0^T \left|\widetilde{Z}_t^\theta\right|^2 dt = 0. \tag{3.22}$$

Proof: We start by proving the first limit. For notational ease, we introduce the following notations.

For $t \in [0,T]$, $\theta > 0$, we set

$$\begin{aligned}
\widetilde{x}_t^\theta &= \theta^{-1}\left(x_t^\theta - x_t\right) - x_t^1, & \widetilde{y}_t^\theta &= \theta^{-1}\left(y_t^\theta - y_t\right) - y_t^1,\\
\widetilde{z}_t^\theta &= \theta^{-1}\left(z_t^\theta - z_t\right) - z_t^1, & \widetilde{\bar{z}}_t^\theta &= \theta^{-1}\left(\bar{z}_t^\theta - \bar{z}_t\right) - \bar{z}_t^1,\\
\widetilde{Z}_t^\theta &= \theta^{-1}\left(Z_t^\theta - Z_t\right) - Z_t^1, & \widetilde{r}_t^\theta(\cdot) &= \theta^{-1}\left(r_t^\theta(\cdot) - r_t(\cdot)\right) - r_t^1(\cdot).
\end{aligned}$$

We denote by

$$\begin{aligned}
\widetilde{x}_t^{\lambda,\theta} &= x_t + \lambda\theta\left(\widetilde{x}_t^\theta + x_t^1\right), & \widetilde{z}_t^{\lambda,\theta} &= z_t + \lambda\theta\left(\widetilde{z}_t^\theta + z_t^1\right),\\
\widetilde{y}_t^{\lambda,\theta} &= y_t + \lambda\theta\left(\widetilde{y}_t^\theta + y_t^1\right), & \widetilde{\bar{z}}_t^{\lambda,\theta} &= \bar{z}_t + \lambda\theta\left(\widetilde{\bar{z}}_t^\theta + \bar{z}_t^1\right),\\
\gamma_t^{\lambda,\theta} &= \left(\tilde{x}_t^{\lambda,\theta}, P_{\tilde{x}_t^{\lambda,\theta}}, u_t^\theta\right). & \widetilde{r}_t^{\lambda,\theta}(e) &= r_t(e) + \lambda\theta\left(\widetilde{r}_t^\theta(e) + r_t^1(e)\right),
\end{aligned}$$

First, we have

$$\begin{cases}
d\tilde{x}_t^\theta = \left([b_t^x - \sigma_t^x\xi_t - \sigma_t\xi_t^x]\,\widetilde{x}_t^\theta + [b_t^{\mu,x} - \sigma_t\xi_t^{\mu,x} - \xi_t\sigma_t^{\mu,x}] + \alpha_1^\theta\right)dt + \left(g_t^x\widetilde{x}_t^\theta dt + g_t^{\mu,x} + \alpha_2^\theta\right)dW_t\\
\qquad + \left(\sigma_t^x\widetilde{x}_t^\theta dt + \sigma_t^{\mu,x} + \alpha_3^\theta\right)dY_t + \int_\Theta \left(c_t^x(e)\,\widetilde{x}_t^\theta dt + c_t^{\mu,x}(e) + \alpha_4^\theta\right)\widetilde{N}_t(de,dt),\\
\tilde{x}_0^\theta = 0,
\end{cases} \tag{3.23}$$

where

$$\begin{aligned}
b_t^x &= \int_0^1 b_x\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, & b_t^{\mu,x} &= \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}_t^\theta}\right]d\lambda,\\
\sigma_t^x &= \int_0^1 \sigma_x\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, & \sigma_t^{\mu,x} &= \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \sigma\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}_t^\theta}\right]d\lambda,\\
\xi_t^x &= \int_0^1 \xi_x\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, & \xi_t^{\mu,x} &= \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \xi\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}_t^\theta}\right]d\lambda,\\
g_t^x &= \int_0^1 g_x\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, & g_t^{\mu,x} &= \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} g\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}_t^\theta}\right]d\lambda,\\
c_t^x &= \int_0^1 c\left(t,\gamma_t^{\lambda,\theta},e\right)d\lambda, & c^{\mu,x} &= \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} c\left(t,\gamma_t^{\lambda,\theta},e,\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}_t^\theta}\right]d\lambda,
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1^\theta \; = \; & \int_0^1 \left[b_x\left(t,\gamma_t^{\lambda,\theta}\right) - b_x(t)\right]d\lambda x_t^1\\
& -\xi_t\int_0^1 \left[\sigma_x\left(t,\gamma_t^{\lambda,\theta}\right) - \sigma_x(t)\right]d\lambda x_t^1 - \sigma_t\int_0^1 \left[\xi_x\left(t,\gamma_t^{\lambda,\theta}\right) - \xi_x(t)\right]d\lambda x_t^1\\
& + \int_0^1 \left[b_v\left(t,\gamma_t^{\lambda,\theta}\right) - b_v(t)\right]d\lambda v_t - \xi_t\int_0^1 \left[\sigma_v\left(t,\gamma_t^{\lambda,\theta}\right) - \sigma_v(t)\right]d\lambda v_t\\
& + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} b\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} b\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda\\
& -\xi_t\int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} \sigma\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} \sigma\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda\\
& -\sigma_t\int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} \xi\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} \xi\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda,
\end{aligned}$$

$$\begin{aligned}
\alpha_2^\theta &= \int_0^1 \left[g_x\left(t,\gamma_t^{\lambda,\theta}\right)-g_x(t)\right]d\lambda x_t^1+\int_0^1\left[g_v\left(t,\gamma_t^{\lambda,\theta}\right)-g_v(t)\right]d\lambda v_t\\
&\quad+\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x}g\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)-\partial_\mu^{P_x}g\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda,\\
\alpha_3^\theta &= \int_0^1 \left[\sigma_x\left(t,\gamma_t^{\lambda,\theta}\right)-\sigma_x(t)\right]d\lambda x_t^1+\int_0^1\left[\sigma_v\left(t,\gamma_t^{\lambda,\theta}\right)-\sigma_v(t)\right]d\lambda v_t\\
&\quad+\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x}\sigma\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)-\partial_\mu^{P_x}\sigma\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda,\\
\alpha_4^\theta &= \int_0^1 \left[c_x\left(t,\gamma_t^{\lambda,\theta},e\right)-c_x(t,e)\right]d\lambda x_t^1+\int_0^1\left[c_v\left(t,\gamma_t^{\lambda,\theta},e\right)-c_v(t,e)\right]d\lambda v_t\\
&\quad+\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x}c\left(t,\gamma_t^{\lambda,\theta},e,\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)-\partial_\mu^{P_x}c\left(t,e,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda.
\end{aligned}$$

Noting that under assumptions (**A1**) and (**A2**), we get

$$\lim_{\theta\to 0}\mathbb{E}\left[\left|\alpha_1^\theta\right|^2+\left|\alpha_2^\theta\right|^2+\left|\alpha_3^\theta\right|^2+\left|\alpha_4^\theta\right|^2\right]=0.$$

Applying Itô's formula to $\left|\widetilde{x}_t^\theta\right|^2$, we have

$$\begin{aligned}
\mathbb{E}\left|\widetilde{x}_t^\theta\right|^2 &= 2\mathbb{E}\int_0^T\widetilde{x}_t^\theta\left(\left[b_t^x-\sigma_t^x\xi_t-\sigma_t\xi_t^x\right]\widetilde{x}_t^\theta+\left[b_t^{\mu,x}-\sigma_t\xi_t^{\mu,x}-\xi_t\sigma_t^{\mu,x}\right]+\alpha_1^\theta\right)dt\\
&\quad+\mathbb{E}\int_0^T\left|g_t^x\widetilde{x}_t^\theta+g_t^{\mu,x}+\alpha_2^\theta\right|^2dt+\mathbb{E}\int_0^T\left|\sigma_t^x\widetilde{x}_t^\theta+\sigma_t^{\mu,x}+\alpha_3^\theta\right|^2dt\\
&\quad+\mathbb{E}\int_0^T\left|c_t^x(e)\,\widetilde{x}_t^\theta+c_t^{\mu,x}(e)+\alpha_4^\theta\right|^2\pi(de)dt\\
&\leq C\mathbb{E}\int_0^T\left|\widetilde{x}_t^\theta\right|^2dt+\int_0^T\mathbb{E}\left[\left|\alpha_1^\theta\right|^2+\left|\alpha_2^\theta\right|^2+\left|\alpha_3^\theta\right|^2\right]dt.
\end{aligned}$$

Finally, estimate (3.20) now follows easily from the Gronwall inequality.

Let $\left(\widetilde{y}_t^\theta,\widetilde{z}_t^\theta,\widetilde{\bar{z}}_t^\theta,\widetilde{r}_t^\theta(e)\right)$ be the solution of the following BSDE

$$\begin{cases}
d\widetilde{y}_t^\theta=\left[f_t^x\widetilde{x}_t^\theta+f_t^{\mu,x}+f_t^y\widetilde{y}_t^\theta+f_t^{\mu,y}+f_t^z\widetilde{z}_t^\theta+f_t^{\mu,z}+f_t^{\bar{z}}\widetilde{\bar{z}}_t^\theta+f_t^{\mu,\bar{z}}+f_t^r\widetilde{r}_t^\theta+f_t^{\mu,r}+\Upsilon_t^\theta\right]dt\\
\qquad+\widetilde{z}_t^\theta dW_t+\widetilde{\bar{z}}_t^\theta dY_t+\int_\Theta\widetilde{r}_t^\theta(e)\,\widetilde{N}(de,dt),\\
\widetilde{y}_T^\theta=\theta^{-1}\left[\varphi\left(x_T^\theta,P_{x_T^\theta}\right)-\varphi\left(x_T,P_{x_T}\right)\right]-\varphi_x\left(x_T,P_{x_T}\right)x_T^1-\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(x_T,P_{x_T},\widehat{x}_T\right)\widehat{x}_T^1\right],
\end{cases}$$

where $\tilde{x}_t^\theta$ satisfies SDE (3.23), and

$$\begin{aligned}
f_t^\rho &= -\int_0^1 f_\rho\left(t,\chi_t^{\lambda,\theta}\right)d\lambda,\ \text{ for }\rho=x,y,z,\bar{z},r,\\
\chi_t^{\lambda,\theta} &= \left(\widetilde{x}_t^{\lambda,\theta},P_{\widetilde{x}_t^{\lambda,\theta}},\widetilde{y}_t^{\lambda,\theta},P_{\widetilde{y}_t^{\lambda,\theta}},\widetilde{z}_t^{\lambda,\theta},P_{\widetilde{z}_t^{\lambda,\theta}},\widetilde{\bar{z}}_t^{\lambda,\theta},P_{\widetilde{\bar{z}}_t^{\lambda,\theta}},\widetilde{r}_t^{\lambda,\theta}(e),P_{\widetilde{r}_t^{\lambda,\theta}(e)},u_t^{\lambda,\theta}\right),\\
f_t^{\mu,\rho} &= -\int_0^1\widehat{\mathbb{E}}\left[\partial_\mu^{P_\rho}f\left(t,\chi_t^{\lambda,\theta},\widehat{\widetilde{\rho}_t^{\lambda,\theta}}\right)\widehat{\widetilde{\rho}_t^\theta}\right]d\lambda,\ \text{ for }\rho=x,y,z,\bar{z},\\
f_t^{\mu,r} &= -\int_0^1\widehat{\mathbb{E}}\left[\partial_\mu^{P_r}f\left(t,\chi_t^{\lambda,\theta},\widehat{\widetilde{r}_t^{\lambda,\theta}(e)}\right)\widehat{\widetilde{r}_t^\theta(e)}\right]d\lambda,
\end{aligned}$$

and Υ_t^θ is given by

$$\begin{aligned}
\Upsilon_t^\theta &= \int_0^1 \left[f_x\left(t,\chi_t^{\lambda,\theta}\right)-f_x(t)\right]d\lambda x_t^1+\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x}f\left(t,\chi_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)-\partial_\mu^{P_x}f\left(t,\chi_t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda\\
&+\int_0^1 \left[f_y\left(t,\chi_t^{\lambda,\theta}\right)-f_y(t)\right]d\lambda y_t^1+\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_y}f\left(t,\chi_t^{\lambda,\theta},\widehat{\widetilde{y}_t^{\lambda,\theta}}\right)-\partial_\mu^{P_y}f\left(t,\chi_t,\widehat{\widetilde{y}}_t\right)\right)\widehat{y}_t^1\right]d\lambda\\
&+\int_0^1 \left[f_z\left(t,\chi_t^{\lambda,\theta}\right)-f_z(t)\right]d\lambda z_t^1+\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_z}f\left(t,\chi_t^{\lambda,\theta},\widehat{\widetilde{z}_t^{\lambda,\theta}}\right)-\partial_\mu^{P_z}f\left(t,\chi_t,\widehat{\widetilde{z}}_t\right)\right)\widehat{z}_t^1\right]d\lambda\\
&+\int_0^1 \left[f_{\bar z}\left(t,\chi_t^{\lambda,\theta}\right)-f_{\bar z}(t)\right]d\lambda \bar z_t^1+\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_{\bar z}}f\left(t,\chi_t^{\lambda,\theta},\widehat{\widetilde{\bar z}_t^{\lambda,\theta}}\right)-\partial_\mu^{P_x}f\left(t,\chi_t,\widehat{\widetilde{\bar z}}_t\right)\right)\widehat{\bar z}_t^1\right]d\lambda\\
&+\int_0^1 \left[f_r\left(t,\chi_t^{\lambda,\theta},e\right)-f_r(t,e)\right]d\lambda r_t^1(e)+\int_0^1\left[f_v\left(t,\chi_t^{\lambda,\theta}\right)-f_v(t)\right]d\lambda v_t\\
&+\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_r}f\left(t,\chi_t^{\lambda,\theta},\widehat{\widetilde{r}_t^{\lambda,\theta}(e)}\right)-\partial_\mu^{P_x}f\left(t,\chi_t,\widehat{\widetilde{r}_t(e)}\right)\right)\widehat{r}_t^1(e)\right]d\lambda.
\end{aligned}$$

Due the fact that $f_t^x, f_t^{\mu,x}, f_t^y, f_t^{\mu,y}, f_t^z, f_t^{\mu,z}, f_t^{\bar z}, f_t^{\mu,\bar z}, f_t^r$ and $f_t^{\mu,r}$ are continuous, we have

$$\lim_{\theta\to 0}\mathbb{E}\left|\Upsilon_t^\theta\right|^2=0. \tag{3.24}$$

Appying Itô's formula to $\left|\widetilde{y}_t^\theta\right|^2$, we have

$$\begin{aligned}
&\mathbb{E}\left|\widetilde{y}_t^\theta\right|^2+\mathbb{E}\int_t^T\left|\widetilde{z}_s^\theta\right|^2ds+\mathbb{E}\int_t^T\left|\widetilde{\bar z}_s^\theta\right|^2ds+\mathbb{E}\int_t^T\int_\Theta\left|\widetilde{r}_s^\theta(e)\right|^2\pi(de)ds\\
=\ &\mathbb{E}\left|\widetilde{y}_T^\theta\right|^2+2\mathbb{E}\int_t^T\widetilde{y}_s^\theta\left(f_s^x\widetilde{x}_s^\theta+f_s^{\mu,x}+f_s^y\widetilde{y}_s^\theta+f_s^{\mu,y}+f_s^z\widetilde{z}_s^\theta\bar z\widetilde{\bar z}_s^\theta+f_s^{\mu,\bar z}+f_s^r\widetilde{r}_s^\theta(e)+f_s^{\mu,r}+\Upsilon_s^\theta\right)ds.
\end{aligned}$$

By Young's inequality, for each $\varepsilon>0$, we get

$$\begin{aligned}
&\mathbb{E}\left|\widetilde{y}_t^\theta\right|^2+\mathbb{E}\int_t^T\left|\widetilde{z}_s^\theta\right|^2ds+\mathbb{E}\int_t^T\left|\widetilde{\bar z}_s^\theta\right|^2ds+\mathbb{E}\int_t^T\int_\Theta\left|\widetilde{r}_s^\theta(e)\right|^2\pi(de)ds\\
\leq\ &\mathbb{E}\left|\widetilde{y}_T^\theta\right|^2+\frac{1}{\varepsilon}\mathbb{E}\int_t^T\left|\widetilde{y}_s^\theta\right|^2ds\\
&+\varepsilon\mathbb{E}\int_t^T\left|\left(f_s^x\widetilde{x}_s^\theta+f_s^{\mu,x}+f_s^y\widetilde{y}_s^\theta+f_s^{\mu,y}+f_s^z\widetilde{z}_s^\theta+f_s^{\mu,z}+f_s^{\bar z}\widetilde{\bar z}_s^\theta+f_s^{\mu,\bar z}+f_s^r\widetilde{r}_s^\theta+f_s^{\mu,r}+\Upsilon_s^\theta\right)\right|^2ds
\end{aligned}$$

$$\begin{aligned}
\leq\ &\mathbb{E}\left|\widetilde{y}_T^\theta\right|^2+\frac{1}{\varepsilon}\mathbb{E}\int_t^T\left|\widetilde{y}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^x\widetilde{x}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T|f_s^{\mu,x}|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^y\widetilde{y}_s^\theta\right|^2ds\\
&+C_\varepsilon\mathbb{E}\int_t^T|f_s^{\mu,y}|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^z\widetilde{z}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T|f_s^{\mu,z}|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^{\bar z}\widetilde{\bar z}_s^\theta\right|^2ds\\
&+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^{\mu,\bar z}\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^r\widetilde{r}_s^\theta(e)\right|^2\pi(de)ds+C_\varepsilon\mathbb{E}\int_t^T|f_s^{\mu,r}|^2\pi(de)ds.
\end{aligned}$$

By the boundedness of $f_t^x, f_t^{\mu,x}, f_t^y, f_t^{\mu,y}, f_t^z, f_t^{\mu,z}, f_t^{\bar z}, f_t^{\mu,\bar z}, f_t^r$ and $f_t^{\mu,r}$, we obtain

$$\begin{aligned}
&\mathbb{E}\left|\widetilde{y}_t^\theta\right|^2+\mathbb{E}\int_t^T\left|\widetilde{z}_s^\theta\right|^2ds+\mathbb{E}\int_t^T\left|\widetilde{\bar z}_s^\theta\right|^2ds+\mathbb{E}\int_t^T\int_\Theta\left|\widetilde{r}_s^\theta(e)\right|^2\pi(de)ds\\
\leq\ &\left(\frac{1}{\varepsilon}+C_\varepsilon\right)\mathbb{E}\int_t^T\left|\widetilde{y}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|\widetilde{z}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|\widetilde{\bar z}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\int_\Theta\left|\widetilde{r}_s^\theta(e)\right|^2\pi(de)ds\\
&+\mathbb{E}\left|\widetilde{y}_T^\theta\right|^2+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^x\widetilde{x}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|\Upsilon_s^\theta\right|^2ds.
\end{aligned}$$

Hence, in view of (3.20), (3.24), the fact that $f_t^x, f_t^{\mu,x}$ are continuous and bounded, by Gronwall's inequality, we obtain (3.21).

Now, we proceed to prove (3.22). It is plain to check that $\widetilde{Z}_t^\theta$ satisfies the following equality

$$d\widetilde{Z}_t^\theta = \left[\widetilde{Z}_t^\theta \xi\left(t, x_t^\theta, P_{x_t^\theta}\right) + \bar{\Upsilon}_t^\theta\right] dY_t + Z_t\left[\xi_t^x \widetilde{x}_t^\theta + \xi_t^{\mu,x}\right] dY_t,$$

where

$$\begin{aligned} \xi_t^x &= \int_0^1 \xi_x\left(t, \widetilde{x}_t^{\lambda,\theta}, P_{\widetilde{x}_t^{\lambda,\theta}}\right) d\lambda, \\ \xi_t^{\mu,x} &= \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t, \widetilde{x}_t^{\lambda,\theta}, P_{\widetilde{x}_t^{\lambda,\theta}}, \widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}_t^\theta}\right] d\lambda, \end{aligned}$$

and $\bar{\Upsilon}_t^\theta$ is given by

$$\begin{aligned} \bar{\Upsilon}_t^\theta &= Z_t \int_0^1 \left[\xi_x\left(t, \widetilde{x}_t^{\lambda,\theta}, P_{\widetilde{x}_t^{\lambda,\theta}}\right) - \xi_x(t)\right] d\lambda x_t^1 \\ &\quad + Z_t \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x}\xi\left(t, \widetilde{x}_t^{\lambda,\theta}, P_{\widetilde{x}_t^{\lambda,\theta}}, \widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x}\xi\left(t, \widetilde{x}_t, P_{\widetilde{x}_t}, \widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right] d\lambda \\ &\quad + Z_t^1\left[\xi\left(t, x_t^\theta, P_{x_t^\theta}\right) - \xi(t)\right]. \end{aligned}$$

Taking into account the fact that ξ_t^x and $\xi_t^{\mu,x}$ are continuous, we deduce

$$\lim_{\theta \to 0} \mathbb{E}\left|\bar{\Upsilon}_t^\theta\right|^2 = 0. \tag{3.25}$$

Then, applying Itô's formula to $\left|\widetilde{Z}_t^\theta\right|^2$ and taking expectation, we have

$$\mathbb{E}\left|\widetilde{Z}_t^\theta\right|^2 \leq C\mathbb{E}\int_0^T \left|\widetilde{Z}_t^\theta\right|^2 dt + C\mathbb{E}\int_0^T \left|\widetilde{x}_t^\theta\right|^2 dt + C\mathbb{E}\int_0^T |\xi_t^{\mu,x}|^2 dt + C\mathbb{E}\int_0^T \left|\bar{\Upsilon}_t^\theta\right|^2 dt.$$

Finally, by Gronwall's inequality, estimates (3.20) and recall to the Wasserstein metric, the above convergence result (3.22) holds. ■

Since u is an optimal control, then, we have the following lemma.

Lemma 3.4

*Let assumptions (**A1**) and (**A2**) hold. Then, we have the following variational inequality*

$$\begin{aligned}
0 \leq\; & \mathbb{E}\left[Z_T M_x\left(x_T, P_{x_T}\right) x_T^1 + Z_T \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}, \widehat{x}_T\right) \widehat{x}_T^1\right]\right] \\
& + \mathbb{E}\left[Z_T^1 M\left(x_T, P_{x_T}\right) + h_y\left(y_0, P_{y_0}\right) y_0^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(y_0, P_{y_0}, \widehat{y}_0)\widehat{y}_0^1\right]\right] \\
& + \mathbb{E}\int_0^T \left[Z_t^1 l(t) + Z_t\left(l_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right]\right) + Z_t\left(l_y(t)y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t, \widehat{y}_t\right)\widehat{y}_t^1\right]\right)\right. \\
& + Z_t\left(l_z(t)z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{z}_t\right)\widehat{z}_t^1\right]\right) + Z_t\left(l_{\bar{z}}(t)\bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t, \widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right]\right) \\
& \left. + Z_t\left(l_r(t)r_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_r} l\left(t, \widehat{r}_t\right)\widehat{r}_t^1\right]\right) + Z_t l_v(t) v_t\right] dt.
\end{aligned} \tag{3.26}$$

Proof: Using 3.3 and Taylor expansion, we have

$$\begin{aligned}
0 &\leq \frac{1}{\theta}\left[J\left(u_t^\theta\right) - J\left(u_t\right)\right] \\
&= \frac{1}{\theta}\mathbb{E}\left[Z_T^\theta M\left(x_T^\theta, P_{x_T^\theta}\right) - Z_T M\left(x_T, P_{x_T}\right)\right] \\
&\quad + \frac{1}{\theta}\mathbb{E}\left[h\left(y_0^\theta\right) - h\left(y_0\right)\right] \\
&\quad + \frac{1}{\theta}\mathbb{E}\int_0^T \left[Z_t^\theta l^\theta(t) - Z_t l(t)\right] dt \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where $l^\theta(t) = l\left(t, x_t^\theta, P_{x_t^\theta}, y_t^\theta, P_{y_t^\theta}, z_t^\theta, P_{z_t^\theta}, \bar{z}_t^\theta, P_{\bar{z}_t^\theta}, r_t^\theta, P_{r_t^\theta}, u_t^\theta\right)$.

Then, from the results of (3.20), (3.21) and (3.22), we derive

$$\begin{aligned}
I_1 &= \frac{1}{\theta}\mathbb{E}\left[Z_T^\theta M\left(x_T^\theta, P_{x_T^\theta}\right) - Z_T M\left(x_T, P_{x_T}\right)\right] \\
&= \frac{1}{\theta}\mathbb{E}\left[\left(Z_T^\theta - Z_T\right) M\left(x_T^\theta, P_{x_T^\theta}\right)\right] \\
&\quad + \frac{1}{\theta}\mathbb{E}\left[Z_T \int_0^1 M_x\left(x_T + \lambda\left(x_T^\theta - x_T\right), P_{x_T + \lambda\left(\widehat{x}_T^\theta - \widehat{x}_T\right)}\right)\left(x_T^\theta - x_T\right) d\lambda\right] \\
&\quad + \frac{1}{\theta}\mathbb{E}\left[Z_T \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T + \lambda\left(\widehat{x}_T^\theta - \widehat{x}_T\right), P_{x_T + \lambda\left(\widehat{x}_T^\theta - \widehat{x}_T\right)}, \widehat{x}_T\right)\left(\widehat{x}_T^\theta - \widehat{x}_T\right)\right] d\lambda\right] \\
&\longrightarrow \mathbb{E}^u\left[\vartheta_T M\left(x_T, P_{x_T}\right)\right] + \mathbb{E}^u\left[\left(M_x\left(x_T, P_{x_T}\right)\right) x_T^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}, \widehat{x}_T\right)\widehat{x}_T^1\right]\right].
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
I_2 &= \frac{1}{\theta}\mathbb{E}\left[h\left(y_0^\theta, P_{y_0^\theta}\right) - h\left(y_0, P_{y_0}\right)\right] \\
&= \frac{1}{\theta}\mathbb{E}\left[\int_0^1 h_y\left(y_0 + \lambda\left(y_0^\theta - y_0\right), P_{y_0+\lambda\left(\widehat{y}_0^\theta - \widehat{y}_0\right)}\right)\left(y_0^\theta - y_0\right) d\lambda\right] \\
&\quad + \frac{1}{\theta}\mathbb{E}\left[\int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h\left(y_0 + \lambda\left(\widehat{y}_0^\theta - \widehat{y}_0\right), P_{y_0+\lambda\left(\widehat{y}_0^\theta - \widehat{y}_0\right)}, \widehat{y}_0\right)\left(\widehat{y}_0^\theta - \widehat{y}_0\right)\right] d\lambda\right] \\
&\longrightarrow \mathbb{E}^u\left[\left(h_y\left(y_0, P_{y_0}\right)\right) y_0^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(y_0, P_{y_0}, \widehat{y}_0)\widehat{y}_0^1\right]\right],
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \frac{1}{\theta}\mathbb{E}\left[\int_0^T \left(Z_t^\theta l^\theta(t) - Z_t l(t)\right) dt\right] \\
&\longrightarrow \mathbb{E}^u \int_0^T \left[\vartheta_t l(t) + l_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right] + l_y(t)y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t, \widehat{y}_t\right)\widehat{y}_t^1\right]\right. \\
&\quad + l_z(t)z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{z}_t\right)\widehat{z}_t^1\right] + l_{\bar{z}}(t)\bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right] \\
&\quad \left. + l_r(t)r_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_r} l\left(t, \widehat{r}_t\right)\widehat{r}_t^1\right] + l_v(t)v_t\right] dt.
\end{aligned}
$$

Then, the variational inequality (3.26) can be rewritten as

$$
\begin{aligned}
0 &\leq \mathbb{E}^u\left[M_x\left(x_T, P_{x_T}\right)x^1(T) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}, \widehat{x}_T\right)\widehat{x}_T^1\right]\right] \\
&\quad + \mathbb{E}^u\left[\vartheta_T M\left(x_T, P_{x_T}\right) + h_y\left(y_0, P_{y_0}\right)y^1(0) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(y_0, P_{y_0}, \widehat{y}_0)\widehat{y}_0^1\right]\right] \qquad (3.27) \\
&\quad + \mathbb{E}^u \int_0^T \left[\vartheta_t l(t) + l_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right] + l_y(t)y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t, \widehat{y}_t\right)\widehat{y}_t^1\right]\right. \\
&\quad + l_z(t)z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{z}_t\right)\widehat{z}_t^1\right] + l_{\bar{z}}(t)\bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t, \widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right] \\
&\quad \left. + l_r(t)r_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_r} l\left(t, \widehat{r}_t\right)\widehat{r}_t^1\right] + l_v(t)v_t\right] dt.
\end{aligned}
$$

■

The second main result of this thesis is the following Theorem.

Theorem 3.5 (*Partial necessary conditions of optimality*)

*Let assumptions (**A1**) and (**A2**) hold. Let $(x, y, z, \bar{z}, r, u)$ be an optimal solution of our partially observed optimal control problem. Then, there are $(p, q, k, \bar{k}, n(\cdot))$ and $(P, \bar{Q}, Q)$ of $\mathbb{F}-$adapted processes that satisfy (3.10), (3.11) respectively, and that for all $v \in \mathcal{U}$, we have*

$$\mathbb{E}^u \left[H_v(t)(v_t - u_t) / \mathcal{F}_t^Y \right] \geq 0, a.e, a.s, \tag{3.28}$$

where the Hamiltonian function

$$H(t) = H\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, r_t, P_{r_t}, u_t, p_t, q_t, k_t, \bar{k}_t, n_t(\cdot), Q_t\right),$$

is defined by (3.12).

Proof: Applying Itô's formula to $p_t x_t^1$ and $q_t y_t^1$ such that,

$$\begin{aligned} q_0 &= -h_y(y_0, P_{y_0}) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right], \\ p_T &= M_x(x_T, P_{x_T}) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M(\widehat{x}_T, P_{x_T}, x_T)\right] \\ &\quad -\varphi_x(x_T, P_{x_T}) q_T - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \varphi(\widehat{x}_T, P_{x_T}, x_T) \widehat{q}_T\right], \end{aligned}$$

and using Fubini's theorem, we get

$$\begin{aligned} \mathbb{E}^u\left[p_T x_T^1\right] &= \mathbb{E}^u \int_0^T \Big[p_t(b_v(t) - \sigma_v(t)\xi(t)) v_t + \bar{k}_t \sigma_v(t) v_t + k_t g_v(t) v_t \\ &\quad + \int_\Theta n_t(e) c_v(t,e) \pi(de) v_t\Big] dt \\ &\quad + \mathbb{E}^u \int_0^T x_t^1 \left[f_x(t) q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f(t) \widehat{q}_t\right] - l_x(t) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l(t)\right]\right] dt \\ &\quad - \mathbb{E}^u \int_0^T x_t^1 \left[\xi_x(t) Q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \xi(t) \widehat{Q}_t\right]\right] dt, \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} &\mathbb{E}^u\left[q_T y_T^1\right] + \mathbb{E}^u\left[h_y(y_0, P_{y_0}) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right]\right] \\ = &-\mathbb{E}^u \int_0^T q_t \left[f_v(t) v_t + f_x(t) x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f(t, \widehat{x}_t) \widehat{x}_t^1\right]\right] dt \\ &-\mathbb{E}^u \int_0^T y_t^1 \left[l_y(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l(t)\right]\right] dt - \mathbb{E}^u \int_0^T z_t^1 \left[l_z(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l(t)\right]\right] dt \quad (3.30) \\ &-\mathbb{E}^u \int_0^T \bar{z}_t^1 \left[l_{\bar{z}}(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l(t)\right]\right] dt - \mathbb{E}^u \int_0^T r_t^1 \left[l_r(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_r} l(t)\right]\right] dt. \end{aligned}$$

Now, applying Itô's formula to $\vartheta_t P_t$ and using also Fubini's theorem, we have

$$\begin{aligned} \mathbb{E}^u\left[\vartheta_T M(x_T)\right] &= -\mathbb{E}^u \int_0^T \vartheta_t l(t) dt \\ &\quad + \mathbb{E}^u \int_0^T Q_t \left[\xi_x(t) x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \xi(t, \widehat{x}_t) \widehat{x}_t^1\right]\right] dt. \end{aligned} \tag{3.31}$$

From Eqs. (3.29), (3.30), and (3.31), we obtain

$$\begin{aligned}
&\mathbb{E}^u\left[M_x\left(x_T,P_{x_T}\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}M\left(x_T,P_{x_T}\right)\right]\right]\\
&+\mathbb{E}^u\left[h_y(y_0,P_{y_0})+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}h(\widehat{y}_0,P_{y_0},y_0)\right]+\vartheta_T M(x_T)\right]\\
=\;&\mathbb{E}^u\int_0^T\left[p_t\left[b_v\left(t\right)-\sigma_v\xi(t)\right]v_t+\bar{k}_t\sigma_v\left(t\right)v_t+k_tg_v\left(t\right)v_t+\int_\Theta n_t\left(e\right)c_v\left(t,e\right)v_t-q_tf_v\left(t\right)v_t\right]dt\\
&-\mathbb{E}^u\int_0^T\vartheta_tl\left(t\right)dt-\mathbb{E}^u\int_0^T x_t^1\left[l_x(t)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}l\left(t\right)\right]\right]dt \qquad (3.32)\\
&-\mathbb{E}^u\int_0^T y_t^1\left[l_y\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}l\left(t\right)\right]\right]dt-\mathbb{E}^u\int_0^T z_t^1\left[l_z\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z}l\left(t\right)\right]\right]dt\\
&-\mathbb{E}^u\int_0^T \bar{z}_t^1\left[l_{\bar{z}}\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}}l\left(t\right)\right]\right]dt-\mathbb{E}^u\int_0^T r_t^1\left[l_r\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_r}l\left(t\right)\right]\right]dt,
\end{aligned}$$

thus

$$\begin{aligned}
&\mathbb{E}^u\left[M_x\left(x_T,P_{x_T}\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}M\left(x_T,P_{x_T}\right)\right]\right]\\
&+\mathbb{E}^u\left[h_y(y_0,P_{y_0})+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}h(\widehat{y}_0,P_{y_0},y_0)\right]+\vartheta_T M(x_T)\right]\\
=\;&\mathbb{E}^u\int_0^T H_v\left(t\right)v_t-\mathbb{E}^u\int_0^T l_v(t)v_tdt-\mathbb{E}^u\int_0^T\vartheta_tl\left(t\right)dt-\mathbb{E}^u\int_0^T x_t^1\left[l_x(t)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}l\left(t\right)\right]\right]dt\\
&-\mathbb{E}^u\int_0^T y_t^1\left[l_y\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}l\left(t\right)\right]\right]dt-\mathbb{E}^u\int_0^T z_t^1\left[l_z\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z}l\left(t\right)\right]\right]dt\\
&-\mathbb{E}^u\int_0^T \bar{z}_t^1\left[l_{\bar{z}}\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}}l\left(t\right)\right]\right]dt-\mathbb{E}^u\int_0^T r_t^1\left[l_r\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_r}l\left(t\right)\right]\right]dt.
\end{aligned}$$

This together with the variational inequality (3.27) imply (3.28), the proof is then completed. ■

3.2.2 Sufficient conditions for optimal control problem of Mckean–Vlasov FBSDEs with jump

In what follows, we will prove that, under some additional convexity conditions, the above necessary condition of partially observed optimal control in 3.5 is also sufficient. A function $\phi:\mathbb{R}\times Q_2\left(\mathbb{R}\right)\rightarrow\mathbb{R}$ is convex if, for every $\left(x^u,P_x^u\right),\left(x^v,P_x^v\right)\in\mathbb{R}\times Q_2\left(\mathbb{R}\right),$

$$\phi\left(x^v,P_x^v\right)-\phi\left(x^u,P_x^u\right)\geq\phi_x\left(x^u,P_x^u\right)\left(x^v-x^u\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\phi\left(x^u,P_x^u\right)\left(x^v-x^u\right)\right].$$

For this, we need an additional assumption condition (**A3**) as follows:

Assumption (A3)

1. The functions M, h are convex in (x, P_x) and (y, P_y) respectively.
2. $H\left(t, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, p_t^u, q_t^u, k_t^u, \bar{k}_t^u, n_t^u(\cdot), Q_t^u\right)$

is convex in $(x^u, P_x^u, y^u, P_{y^u}, z^u, P_{z^u}, \bar{z}^u, P_{\bar{z}^u}, r^u, P_r^u, u)$ for $a.e.\ t \in [0,T],\ P-a.s.$

$$\begin{aligned} H^v(t) - H^u(t) \geq\ & H_x^u(t)(x^v - x^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} H^u(t)(\widehat{x}^v - \widehat{x}^u)\right] \\ & + H_y^u(t)(y^v - y^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} H^u(t)(\widehat{y}^v - \widehat{y}^u)\right] \\ & + H_z^u(t)(z^v - z^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} H^u(t)(\widehat{z}^v - \widehat{z}^u)\right] \\ & + H_{\bar{z}}^u(t)(\bar{z}^v - \bar{z}^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} H^u(t)\left(\widehat{\bar{z}}^v - \widehat{\bar{z}}^u\right)\right] \\ & + H_r^u(t)(r^v - r^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_r} H^u(t)(\widehat{r}^v - \widehat{r}^u)\right], \end{aligned}$$

where

$$\begin{aligned} H^v(t) &= H\left(t, x^v, P_x^v, y^v, P_{y^v}, z^v, P_{z^v}, \bar{z}^v, P_{\bar{z}^v}, r^v, P_r^v, v, p^u, q^u, k^u, \bar{k}^u, n^u(\cdot), Q^u\right), \\ H^u(t) &= H\left(t, x^u, P_x^u, y^u, P_{y^u}, z^u, P_{z^u}, \bar{z}^u, P_{\bar{z}^u}, r^u, P_r^u, u, p^u, q^u, k^u, \bar{k}^u, n^u(\cdot), Q^u\right). \end{aligned}$$

Now, we can prove the sufficient conditions of optimality for our control problem of McKean–Vlasov FBSDEs with jumps, which is the third main result of this paper.

Theorem 3.6 (*Partial sufficient conditions of optimality*)

Suppose ($\boldsymbol{A1}$),($\boldsymbol{A2}$) and ($\boldsymbol{A3}$) hold. Let Z^v be $\mathcal{F}_t^Y$-adapted, $u \in \mathcal{U}$ be an admissible control, and $(x, y, z, \bar{z}, r)$ be the corresponding trajectories. Let $\left(p, k, \bar{k}, n(\cdot), q\right)$ and $\left(P, Q, \bar{Q}\right)$ satisfy (3.10) and (3.11), respectively. Moreover, the Hamiltonian H is convex in $(x, P_x, y, P_y, , z, P_z, \bar{z}, P_{\bar{z}}, r, P_r, v)$, and

$$\mathbb{E}^u\left[H_v(t)(v_t - u_t)/\mathcal{F}_t^Y\right] \geq 0, a.\mathbb{E}, a.s, .$$

Then u is a partial observed optimal control for the problem (3.1) − (3.6) subject to (3.4).

Proof. For any $v \in \mathcal{U}$, we have

$$\begin{aligned} J(v) - J(u) =\ & \mathbb{E}\left[Z_T^v M\left(x_T^v, P_{x_T^v}\right) - Z_T^u M\left(x_T^u, P_{x_T^u}\right)\right] \\ & + \mathbb{E}\left[h\left(y_0^v, P_{y_0^v}\right) - h\left(y_0^u, P_{y_0^u}\right)\right] \\ & + \mathbb{E}\int_0^T \left(Z_t^v l^v(t) - Z_t^u l^u(t)\right) dt, \end{aligned}$$

where

$$\begin{aligned} l^{v}(t) &= l\left(t, x_{t}^{v}, P_{x_{t}^{v}}, y_{t}^{v}, P_{y_{t}^{v}}, z_{t}^{v}, P_{z_{t}^{v}}, \bar{z}^{v}, P_{\bar{z}^{v}}, r^{v}, P_{r^{v}}, v_{t}\right), \\ l^{u}(t) &= l\left(t, x_{t}^{u}, P_{x_{t}^{u}}, y_{t}^{u}, P_{y_{t}^{u}}, z_{t}^{u}, P_{z_{t}^{u}}, \bar{z}^{u}, P_{\bar{z}^{u}}, r^{u}, P_{r^{u}}, u_{t}\right). \end{aligned}$$

By the convexity property of M and h, we get

$$\begin{aligned} \mathbb{E}\left[Z_{T}^{v} M\left(x_{T}^{v}, P_{x_{T}^{v}}\right) - Z_{T}^{u} M\left(x_{T}^{u}, P_{x_{T}^{u}}\right)\right] &\geq \mathbb{E}[(Z_{T}^{v} - Z_{T}^{u}) M\left(x_{T}^{u}, P_{x_{T}^{u}}\right)] \\ &+\mathbb{E}^{u}[M_{x}\left(x_{T}^{u}, P_{x_{T}^{u}}\right)(x_{T}^{v} - x_{T}^{u})] \qquad (3.33) \\ &+\mathbb{E}^{u}[\widehat{\mathbb{E}}\left[\partial_{\mu}^{P_{x}} M\left(x_{T}^{u}, P_{x_{T}^{u}}\right)\right](x_{T}^{v} - x_{T}^{u})]. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbb{E}\left[h\left(y_{0}^{v}, P_{y_{0}^{v}}\right) - h\left(y_{0}^{u}, P_{y_{0}^{u}}\right)\right] &\geq \mathbb{E}\left[h_{y}\left(y_{0}^{u}, P_{y_{0}^{u}}\right)(y_{0}^{v} - y_{0}^{u}))\right] \\ &+\mathbb{E}\left[\widehat{\mathbb{E}}\left[\partial_{\mu}^{P_{y}} h\left(y_{0}^{u}, P_{y_{0}^{u}}\right)\right](y_{0}^{v} - y_{0}^{u})\right], \qquad (3.34) \end{aligned}$$

and

$$\mathbb{E}\int_{0}^{T}\left(Z_{t}^{v} l^{v}(t) - Z_{t}^{u} l^{u}(t)\right) dt = \mathbb{E}\int_{0}^{T} Z_{t}^{v}\left(l^{v}(t) - l^{u}(t)\right) dt + \mathbb{E}\int_{0}^{T}\left(Z_{t}^{v} - Z_{t}^{u}\right) l^{u}(t)\, dt. \qquad (3.35)$$

From (3.33), (3.34) and (3.35), we can write

$$\begin{aligned} J(v) - J(u) &\geq \mathbb{E}^{u}[M_{x}\left(x_{T}^{u}, P_{x_{T}^{u}}\right)(x_{T}^{v} - x_{T}^{u})] + \mathbb{E}^{u}[\widehat{\mathbb{E}}\left[\partial_{\mu}^{P_{x}} M\left(x_{T}^{u}, P_{x_{T}^{u}}, \widehat{x}_{T}^{u}\right)\right](x_{T}^{v} - x_{T}^{u})] \\ &+\mathbb{E}\left[h_{y}\left(y_{0}^{u}, P_{y_{0}^{u}}\right)(y_{0}^{v} - y_{0}^{u}))\right] + \mathbb{E}\left[\widehat{\mathbb{E}}\left[\partial_{\mu}^{P_{y}} h\left(y_{0}^{u}, P_{y_{0}^{u}}, \widehat{y}_{0}^{u}\right)\right](y_{0}^{v} - y_{0}^{u}))\right] \\ &+\mathbb{E}\int_{0}^{T} Z_{t}^{v}\left(l^{v}(t) - l^{u}(t)\right) dt + \mathbb{E}\int_{0}^{T}\left(Z_{t}^{v} - Z_{t}^{u}\right) l^{u}(t)\, dt \\ &+\mathbb{E}\left[(Z_{T}^{v} - Z_{T}^{u})\left(\int_{0}^{T} l^{u}(t)\, dt + M\left(x_{T}^{u}, P_{x_{T}^{u}}\right)\right)\right]. \end{aligned}$$

Noting that

$$\begin{aligned} &q_{0} = -h_{y}(y_{0}, P_{y_{0}}) - \widehat{\mathbb{E}}\left[\partial_{\mu}^{P_{y}} h(\widehat{y}_{0}, P_{y_{0}}, y_{0})\right], \\ &p_{T} = M_{x}\left(x_{T}, P_{x_{T}}\right) + \widehat{\mathbb{E}}\left[\partial_{\mu}^{P_{x}} M\left(\widehat{x}_{T}, P_{x_{T}}, x_{T}\right)\right] \\ &\qquad -\varphi_{x}\left(x_{T}, P_{x_{T}}\right) q_{T} - \widehat{\mathbb{E}}\left[\partial_{\mu}^{P_{x}} \varphi\left(\widehat{x}_{T}, P_{x_{T}}, x_{T}\right) \widehat{q}_{T}\right], \end{aligned}$$

we have

$$\begin{aligned} J(v) - J(u) &\geq \mathbb{E}^{u}[p_{T}^{u}\left(x_{T}^{v} - x_{T}^{u}\right)] + \mathbb{E}^{u}\left[\varphi_{x}\left(x_{T}, P_{x_{T}}\right) q_{T}\left(x_{T}^{v} - x_{T}^{u}\right)\right] \\ &+\mathbb{E}^{u}\widehat{\mathbb{E}}\left[\partial_{\mu}^{P_{x}} \varphi\left(\widehat{x}_{T}, P_{x_{T}}, x_{T}\right) \widehat{q}_{T}\left(x_{T}^{v} - x_{T}^{u}\right)\right] - \mathbb{E}\left[q_{0}^{u}\left(y_{0}^{v} - y_{0}^{u}\right)\right] \\ &+\mathbb{E}^{u}\int_{0}^{T}\left(l^{v}(t) - l^{u}(t)\right) dt + \mathbb{E}\left[(Z_{T}^{v} - Z_{T}^{u})\left(\int_{0}^{T} l^{u}(t)\, dt + M\left(x_{T}^{u}, P_{x_{T}^{u}}\right)\right)\right]. \end{aligned}$$

Then, we can write

$$\begin{aligned} J(v)-J(u) \quad &\geq \quad \mathbb{E}^u[p_T^u\,(x_T^v-x_T^u)]+\mathbb{E}^u[q_T^u\,(y_T^v-y_T^u)] \\ & -\mathbb{E}\,[q_0^u\,(y_0^v-y_0^u)]+\mathbb{E}^u\int_0^T [l^v\,(t)-l^u\,(t)]\,dt \\ & +\mathbb{E}\left[(Z_T^v-Z_T^u)\left(\int_0^T l^u\,(t)\,dt+M\left(x_T^u,P_{x_T^u}\right)\right)\right]. \end{aligned}$$

Now, applying Ito's formula respectively to $p_t^u\,(x_t^v-x_t^u)\,,q_t^u\,(y_t^v-y_t^u)$ and $P_t^u\,(Z_t^v-Z_t^u)$, and by taking expectations, we get

$$\begin{aligned} J(v)-J(u) \quad &\geq \quad \mathbb{E}^u\int_0^T (H^v\,(t)-H^u\,(t))\,dt \\ & -\mathbb{E}^u\int_0^T H_x^u\,(t)\,(x_t^v-x_t^u)\,dt-\mathbb{E}^u\int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}H^u\,(t)\right](x_t^v-x_t^u)\,dt \\ & -\mathbb{E}^u\int_0^T H_y^u\,(t)\,(y_t^v-y_t^u)\,dt-\mathbb{E}^u\int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_y}H^u\,(t)\right](y_t^v-y_t^u)\,dt \\ & -\mathbb{E}^u\int_0^T H_z^u\,(t)\,(z_t^v-z_t^u)\,dt-\mathbb{E}^u\int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_z}H^u\,(t)\right](z_t^v-z_t^u)\,dt \\ & -\mathbb{E}^u\int_0^T H_{\bar z}^u\,(t)\,(\bar z_t^v-\bar z_t^u)\,dt-\mathbb{E}^u\int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar z}}H^u\,(t)\right](\bar z_t^v-\bar z_t^u)\,dt \\ & -\mathbb{E}^u\int_0^T H_r^u\,(t)\,(r_t^v-r_t^u)\,dt-\mathbb{E}^u\int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_r}H^u\,(t)\right](r_t^v-r_t^u)\,dt. \end{aligned}$$

By the convexity of the functional H in $(x,P_x,y,P_y,z,P_z,\bar z,P_{\bar z},r,P_r,v)$, we have

$$\begin{aligned} J(v)-J(u) \quad &\geq \quad \mathbb{E}^u\int_0^T H_v\,(t)\,(v_t-u_t)\,dt \\ &= \quad \mathbb{E}\int_0^T Z_t^u\mathbb{E}\left[H_v\,(t)\,(v_t-u_t)\,/\mathcal{F}_t^Y\right]dt. \end{aligned}$$

Since $Z_t^u\geq 0$, and using condition (3.28), we have

$$J(v)-J(u)\geq 0,$$

i.e., u is a partially observed optimal control. □

3.3 Partially observed Linear-Quadratic control problem of McKean-Vlasov FBSDEs

In this section, we will consider a partially observed linear-quadratic control problem. We find an explicit expression of the corresponding optimal control by applying the results obtained in Sect. 3. Consider a partially observed 1-dimensional linear quadratic control problem:

Minimize the expected quadratic cost function

$$\begin{aligned} J(v(\cdot)) &= \mathbb{E}^u \int_0^T \left[L_t^1 x_t^2 + L_t^2 (\mathbb{E}[x_t])^2 + L_t^3 y_t^2 + L_t^4 (\mathbb{E}[y_t])^2 + L_t^5 v_t^2 \right] dt \\ &\quad + \mathbb{E}^u \left[M_1 x_T^2 + M_2 (\mathbb{E}[x_T])^2 + h_t y_0^2 \right], \end{aligned} \tag{3.36}$$

subject to

$$\begin{cases} dx_t = \left(A_t^1 x_t + A_t^2 \mathbb{E}[x_t] + A_t^3 v_t - B_t^2 \gamma_t \right) dt + B_t^1 dW_t + B_t^2 dY_t + \int_\Theta C_t \widetilde{N}(de, dt), \\ -dy_t = \Big(D_t^1 x_t + D_t^2 \mathbb{E}[x_t] + D_t^3 y_t + D_t^4 \mathbb{E}[y_t] + D_t^5 z_t + D_t^6 \mathbb{E}[z_t] + D_t^7 \overline{z}_t + D_t^8 \mathbb{E}[\overline{z}_t] \\ \qquad + D_t^9 r_t + D_t^{10} \mathbb{E}[r_t] + D_t^{11} v_t \Big) dt - z_t dW_t - \overline{z} dY_t - \int_\Theta r_t(e) \widetilde{N}(de, dt), \\ x(0) = x_0, \ y_T = \phi_1 x_T + \phi_2 \mathbb{E}[x_T], \end{cases} \tag{3.37}$$

and

$$\begin{cases} dY_t = \gamma_t dt + d\widetilde{W}_t \\ Y_0 = 0, \end{cases} \tag{3.38}$$

where

$$\begin{aligned} & A_t^1 x_t + A_t^2 \mathbb{E}[x_t] + A_t^3 v_t = b\left(t, x_t^v, P_{x_t^v}, v_t\right), \\ & B_t^1 = g\left(t, x_t^v, P_{x_t^v}, v_t\right), \\ & B_t^2 = \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right), \\ & C_t = c\left(t, x_{t_-}^v, P_{x_{t_-}^v}, v_t, e\right), \\ & \gamma_t = \xi\left(t, x_t^v, P_{x_t^v}\right), \end{aligned}$$

and

$$\begin{aligned} f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \overline{z}_t^v, P_{\overline{z}_t^v}, r_t^v, P_{r_t^v}, v_t\right) &= D_t^1 x_t + D_t^2 \mathbb{E}[x_t] + D_t^3 y_t + D_t^4 \mathbb{E}[y_t] \\ &\quad + D_t^5 z_t + D_t^6 \mathbb{E}[z_t] + D_t^7 \overline{z}_t + D_t^8 \mathbb{E}[\overline{z}_t] \\ &\quad + D_t^9 r_t + D_t^{10} \mathbb{E}[r_t] + D_t^{11} v_t. \end{aligned}$$

Here, all the coefficients $A^1(\cdot), A^2(\cdot), A^3(\cdot), B^1(\cdot), B^2(\cdot), C(\cdot), \gamma(\cdot), D^i(\cdot)$ are bounded and deterministic functions for $i = 1, \cdots, 11$, $L^j(\cdot)$ is positive function and bounded for $j = 1, 2, 3, 4, 5, 6$, and $M_1(\cdot), M_2(\cdot), h(\cdot)$ are positive constants. Then for any $v \in \mathcal{U}$, Eqs. (3.37) and (3.38) have unique solutions, respectively. Now, we introduce

$$Z_t = \exp\left\{\int_0^t \gamma_s dY_s - \frac{1}{2}\int_0^t |\gamma_s|^2 ds\right\},$$

which is the unique $\mathcal{F}_t^Y$-adapted solution of the SDE:

$$\begin{cases} dZ_t = Z_t\gamma_t dY_t, \\ Z_0 = 1, \end{cases}$$

and we define the probability measure P^v by $dP^v = Z_t^v dP$.

In this setting, the Hamiltonian function is defined as

$$\begin{aligned} &H(t, x, y, z, \bar{z}, r, v, p, q, k, \bar{k}, n, Q) \\ &= p\left(A_t^1 x_t + A_t^2 \mathbb{E}[x_t] + A_t^3 v_t - B_t^2\gamma_t\right) - q\left(D_t^1 x_t + D_t^2 \mathbb{E}[x_t] + D_t^3 y_t + D_t^4 \mathbb{E}[y_t] + D_t^5 z_t\right. \\ &+ D_t^6 \mathbb{E}[z_t] + D_t^7 \bar{z}_t + D_t^8 \mathbb{E}[\bar{z}_t] + D_t^9 r_t + D_t^{10} \mathbb{E}[r_t] + \left.D_t^{11} v_t\right) + kB_t^1 + \bar{k}B_t^2 \\ &+ \int_\Theta n_t(e) C_t \pi(de) + Q\gamma_t + L_t^1 x_t^2 + L_t^2 (\mathbb{E}[x_t])^2 + L_t^3 y_t^2 + L_t^4 (\mathbb{E}[y_t])^2 + L_t^5 v_t^2. \end{aligned} \tag{3.39}$$

Further due to Eqs. (3.10) and (3.11), the corresponding adjoint equations will be given by

$$\begin{cases} -dP_t = \left(L_t^1 x_t^2 + L_t^2 (\mathbb{E}[x_t])^2 + L_t^3 y_t^2 + L_t^4 (\mathbb{E}[y_t])^2 + L_t^5 v_t^2\right) dt \\ \qquad -\bar{Q}_t dW_t - Q_t d\widetilde{W}_t, \\ P_T = M(x_T, P_{x_T}), \end{cases} \tag{3.40}$$

and

$$\begin{cases} -dp_t = \left[A_t^1 p_t + A_t^2 \mathbb{E}[p_t] - D_t^1 q_t - D_t^2 \mathbb{E}[q_t] + 2L_t^1 x_t + 2L_t^2 \mathbb{E}[x_t]\right] dt \\ \qquad -k_t dW_t - \bar{k}_t d\widetilde{W}_t - \int_\Theta n_t(e)\widetilde{N}(de, dt), \\ dq_t = \left(D_t^3 q_t + D_t^4 \mathbb{E}[q_t] - 2L_t^3 y_t - 2L_t^4 \mathbb{E}[y_t]\right) dt + \left(D_t^5 q_t + D_t^6 \mathbb{E}[q_t]\right) dW_t \\ \qquad + \left[D_t^7 q_t + D_t^8 \mathbb{E}[q_t]\right] d\widetilde{W}_t + \int_\Theta \left(D_t^9 q_t + D_t^{10} \mathbb{E}[q_t]\right) \widetilde{N}(de, dt) \\ p_T = 2M_1 x_T + 2M_2 \mathbb{E}[x_T] \\ \qquad -\phi_1 x_T - \phi_2 \mathbb{E}[x_T], \\ q_0 = -2h_t y_0. \end{cases} \tag{3.41}$$

According to Theorem 3.4, the necessary condition for optimality (3.28) will be

$$\mathbb{E}^u\left[p_t A_t^3 - q_t D_t^{11} + 2L_t^5 u_t/\mathcal{F}_t^Y\right] = 0. \ \ a.s.a.e.$$

If $u(\cdot)$ is partial observed optimal control, then

$$u_t = -\frac{1}{2L_t^5}\left(A_t^3\mathbb{E}^u\left[p_t/\mathcal{F}_t^Y\right] - D_t^{11}\mathbb{E}^u\left[q_t/\mathcal{F}_t^Y\right]\right). \tag{3.42}$$

Finally, for the sufficient conditions, let $u \in \mathcal{U}$ be a candidate to be optimal control. We suppose that $\left(\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{\bar{z}}, \widetilde{r}\right)$ is the solution to the FBSDE (3.37) corresponding to u, and $\left(P, \bar{Q}, Q\right), (p, q, k, \bar{k}, n(\cdot))$ are the solution corresponding to Eqs. (3.40) and (3.41) respectively. It's easy to verify that the functional H is convex in $(x, y, z, \bar{z}, r)$. So, if u satisfies (3.42) and the condition (3.28). Then by applying 3.6, we can check that u is an optimal control of our partially observed linear-quadratic control problem of McKean–Vlasov type.

Conclusion

Throughout this PhD dissertation, we have investigated partially observed optimal control problem of McKean– Vlasov FBSDEs driven by Poisson random measure and an independent Brownian motion. Using the derivatives with respect to probability law and combining Girsanov's theorem with the classical convex variation technique, we have obtained the main results of this thesis which are the necessary and sufficient conditions of optimality. As an illustration, the theoretical results are applied to partially observed linear–quadratic control problems with jumps.

Many interesting problems remain open. For example, study the stochastic maximum principle for these control problems for a non convex control domain.

Appendix

Proposition A1

Lemma A1 (*Integration by parts formula for jumps processes*)

Suppose that the processes $x_1(t)$ and $x_2(t)$ are given by: for $j = 1, 2,\ t \in [s, T]$:

$$\begin{cases} dx_j(t) = f(t, x_j(t), u(t))\, dt + \sigma(t, x_j(t), u(t))\, dW(t) \\ \qquad\qquad + \int_\Theta g\left(t, x_j(t^-), u(t), e\right) N(de, dt), \\ x_j(s) = 0. \end{cases}$$

Then we get

$$\begin{aligned} \mathbb{E}(x_1(T)x_2(T)) = \mathbb{E}&\left[\int_s^T x_1(t)dx_2(t) + \int_s^T x_2(t)dx_1(t)\right] \\ &+ \mathbb{E}\int_s^T \sigma^*(t, x_1(t), u(t))\, \sigma(t, x_2(t), u(t))\, dt \\ &+ \mathbb{E}\int_s^T \int_\Theta g^*(t, x_1(t), u(t), e)\, g(t, x_2(t), u(t), e)\, \pi(de)dt. \end{aligned}$$

See Framstad et al., ([16]) for the detailed proof of the above Lemma. **Theorem (Burkholder-Davis-Gundy inequality)**

Let $(X_t)_{t\geq 0}$be a continuous local martingale defined on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ satisfying the usual conditions. Let $p > 0$. So there are two constants c_p and C_p, $0 < c_p < C_p < +\infty$ such that

$$1)\quad c_p \|X^*_\infty\|_p \leq \left\|[X, X]^{1/2}_\infty\right\|_p \qquad \text{et} \qquad 2)\quad \left\|[X, X]^{1/2}_\infty\right\|_p \leq C_p \|X^*_\infty\|_p$$

where $X^*_t = \sup\{|X_s| \ / \ 0 \leq s \leq t\}$ and $[X^n, X^n]_k = \sum_{l=1}^{k}\left(X^n_l - X^n_{l-1}\right)^2$.
If the martingale is not continuous, inequalities 1) and 2) remains valid only if $p \geq 1$. For more, see onat, P. [39].

Proof. See for $p \in (1, \infty)$ Burkholder [7]. For $p \in (0, 1]$ Burkholder and Gundy [8], and for the case $p = 1$ of (BDG) see Davis [14].

lemma (Gronwall's lemma) (see Pachpatte [43])

Let $X(t)$ and $f(t)$ be nonnegative continuous functions on $0 \leq t \leq T$, for which the inequality

$$X(t) \leq c + \int_0^t f(s) X(s)\, ds, \qquad t \in [0, T]$$

holds, where $c \geq 0$ is a constant. Then

$$X(t) \leq c \exp\left(\int_0^t f(s)\, ds\right), \qquad t \in [0, T].$$

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