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Presented by

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Backward SDEs and Applications to Optimal Control Problems.

Under The Supervision Of Professor

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Dedication

♦ To my beloved father, to my first role model, to the one who gave me without limits, to the one who preferred me over himself. Father, my pen cannot express my feelings toward you, for my emotions are too great to be written on paper. May Allah have mercy on you and grant you Paradise, my father.

♦ To my first love and eternal friend, Mom, who took care of me until I grew up.

♦To my wife and companion in the struggle of life, who patiently supported me to continue my research.

♦To my son, because of whom I stand and fight despite all obstacles, to my son, who illuminated my life once again and planted hope within me. I love you, Mohamed Mouataz.

♦To my brother and sister, who made every effort to help me.

♦To my grandmother, who raised me and always prayed for me.

♦To my friends, especially Redouane BEN HAMED and Soheib ELHAMEL, who stood by me through thick and thin.

♦To all my relatives and dear teachers.

I dedicate this graduation thesis for my doctorate in mathematics to you.

El Mountasar Billah BOUHADJAR

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ملخص

في هذه الرسالة، نقوم بالتفصيل في جانبين متميزين، إحداهما نظري والأخر تطبيقي. يركز الجانب النظري لبحثنا على فحص المعادلات التفاضلية التراجعية العشوائية التي يقودها كل من عملية بواسون وحركة براونية مستقلة، والتي تُرمز إليها بإختصار بـ آم. ت. ع. بْ. يُظهر المولد نموًا لوغاريتميًا في كل من متغير الحالة والمتغير "z" مع الاحتفاظ بالتواصل ليبشيتز فيما يتعلق بمكون القفز.

تقوم دراستنا بتأسيس وجود وتمييز الحلول بشكل منهجي داخل فضاءات الوظائف المناسبة. وعلاوة على ذلك، نخفف من شرط ليبشيتز على مكون بواسون، مما يسمح للمولد بتجسيد نموٍ لوغاريتمي فيما يتعلق بجميع المتغيرات. وفي خطوة إضافية، نستخدم تحولًا تناسبيًا لاظهار العلاقة بين حلول م. ت. ع. ب ذي نمو تربعي في المتغير "z" و م. ت. ع. ب ذو نمو لوغاريتمي في كل من "y" و "z" . بالإضافة إلى ذلك، نقدم دراسة حول المبدأ الأقصى، خاصة في السيناريوهات الخالية من مكون القفز.

من الناحية العملية، يتحول تركيزنا إلى تنفيذ الشراكات العامة والخاصة (سنختصرها ب : ش. ع. خ) ، التي ظهرت كنهج واعد لإدارة مشاريع وخدمات البنية التحتية العامة بكفاءة. ومع ذلك، يتعثر نجاح عقود ش. ع. خ في كثير من الأحيان بتحديات مثل عدم التماثل في المعلومات والمخاطر المعنوية. لتحسين اتخاذ القرارات في ش. ع. خ، تركز هذه الرسالة على تطبيق تقنيات التحكم العشوائي، مع مراعاة تأثير عامل الغموض "κ" في العقد بين الطرف الرئيسي والوكيل.

من خلال الاستفادة من أطُر رياضية دقيقة، بما في ذلك المعادلات التفاضلية التراجعية العشوائية ذات بعد واحد، وتقنيات التحكم العشوائي، وتقنيات التوقف الأمثل، يقدم هذا البحث رؤى قيمة وحلول عملية للتخفيف من التأثيرات الضارة لعدم التماثل في المعلومات والغموض والديناميات المستمرة في عقود ش. ع. خ.

تشتق هذه الدراسة من معادلة هاملتونجاكوبيبيلمان لعدم المساواة التبارزية المتعلقة بالدالة العامة للقطاع العام، مما يقدم أساساً قوياً لتحسين اتخاذ القرارات في ش. ع. خ.

بالإضافة إلى ذلك، في هذا العمل، يتم إجراء دراسة عددية باستخدام أساليب الفروق المحددة وخوارزمية هاورد لتقريب الإيجار والجهد الأمثل تحت تأثير عدم اليقين. تظهر التحليل العددي تأثير عدم اليقين على اتخاذ القرارات ونتائج المشروع في عقود ش. ع. خ.

بصفة عامة، تقدم هذه الرسالة إسهامات هامة في الحقلين النظري والتطبيقي. أولاً، نقوم بتأسيس وجود وفرادة م. ت. ع. ب مع مولد يسمح بنمو لوغاريتمي. وعلاوة على ذلك، نستكشف الاتصال بين هذه م. ت. ع. ب و م. ت. ع. ب التربعية. ثانياً، نتعمق في مبدأ بونترياجين القصوى لهذا النوع من المعادلات التفاضلية التراجعية العشوائية، خاصة بدون مكون القفز. وأخيراً، نقدم مجال ش. ع. خ. من خلال تحسين اتخاذ القرارات.

Résumé

Dans cette thèse, nous explorons deux facettes distinctes, l'une théorique et l'autre pratique. L'aspect théorique de notre recherche se concentre sur l'examen des équations différentielles stochastiques rétrogrades pilotées à la fois par un processus de Poisson et un mouvement brownien qui est indépendant, succinctement désignées comme EDSRSs. Le générateur présente une croissance logarithmique à la fois dans la variable d'état et le processus z, tout en conservant la continuité de Lipschitz en ce qui concerne la composante de saut.

Notre étude établit systématiquement la présence et la distinction des solutions dans des espaces fonctionnels appropriés. De plus, nous relâchons la condition de Lipschitz sur la composante de Poisson, permettant au générateur de manifester une croissance logarithmique concernant toutes les variables. Faisant un pas supplémentaire, nous utilisons une transformation exponentielle pour établir un parallèle entre les solutions d'une EDSRS caractérisée par une croissance quadratique dans la variable z et une EDSRS présentant une croissance logarithmique avec à la fois y et z. De plus, nous plongeons dans une discussion sur le principe du maximum, spécifiquement dans des scénarios dépourvus de la composante de saut.

Du côté pratique, notre attention se tourne vers la mise en œuvre des Partenariats Public-Privé (PPPs), qui se sont révélés être une approche prometteuse pour gérer efficacement les projets et services d'infrastructure publique. Cependant, le succès des contrats PPP est souvent entravé par des défis tels que l'asymétrie de l'information et le risque moral. Pour optimiser la prise de décision dans les PPPs, cette thèse se concentre sur l'application de techniques de contrôle stochastique, en tenant compte de l'effet du facteur d'ambiguïté κ dans le contrat entre le principal et l'agent.

En exploitant des cadres mathématiques rigoureux, comprenant des EDSRs en une dimension, du contrôle stochastique, et des techniques d'arrêt optimal, cette recherche offre des perspectives précieuses et des solutions pratiques pour atténuer les effets adverses de l'asymétrie de l'information, de l'ambiguïté et de la dynamique en temps continu dans les PPP.

Cette étude dérive l'inégalité variationnelle de Hamilton-Jacobi-Bellman (HJBVI) liée à la fonction de valeur publique, offrant une base solide pour l'optimisation de la prise de décision dans les PPP.

De plus, dans ce travail, une étude numérique est réalisée en utilisant des méthodes de différences finies et l'algorithme de Howard pour approximer le loyer optimal et l'effort sous l'impact de l'incertitude. L'analyse numérique démontre l'impact de l'incertitude sur la prise de décision et les résultats des projets dans les contrats PPP.

Dans l'ensemble, cette thèse apporte d'importantes contributions aux domaines théorique et appliqué. Tout d'abord, nous établissons l'existence et l'unicité des EDSRSs avec un générateur permettant une croissance logarithmique. De plus, nous explorons le lien entre ces EDSRSs et les EDSRSs quadratiques. Ensuite, nous plongeons dans le principe du maximum de Pontryagin pour ces types de EDSRs, spécifiquement sans la composante de saut. Enfin, nous faisons progresser le domaine des Partenariats Public-Privé (PPPs) en optimisant la prise de décision.

Abstract

In this thesis, we delve into two distinct facets, one theoretical and the other practical. The theoretical aspect of our investigation centers on the examination of backward stochastic differential equations driven by both a Poisson process and an independent Brownian motion succinctly denoted as BSDEJs. The generator showcases logarithmic growth in both the state variable and the process z while retaining Lipschitz continuity concerning the jump component.

Our study systematically establishes the presence and distinctiveness of solutions within appropriate functional spaces. Furthermore, we loosen the Lipschitz condition on the Poisson component, allowing the generator to manifest logarithmic growth concerning all variables. Taking an additional stride, we utilize an exponential transformation to draw a parallel between solutions of a BSDEJ characterized by quadratic growth in the z-variable and a BSDEJ exhibiting logarithmic growth with both y and z. Additionally, we delve into a discussion on the maximum principle, specifically in scenarios devoid of the jump component.

On the practical side, our focus shifts to the implementation of Public-Private Partnerships (PPPs), which have emerged as a promising approach for efficiently managing public infrastructure projects and services. However, the success of PPP contracts is often hindered by challenges such as information asymmetry and moral hazard. To optimize decision-making in PPPs, this thesis focuses on the application of stochastic control techniques, taking into account the effect of the ambiguity factor κ in the contract between the principal and the agent.

By leveraging rigorous mathematical frameworks, including one-dimensional BSDEs, techniques in stochastic control, and optimizing stopping times, this research provides valuable insights and practical solutions to mitigate the adverse effects of information asymmetry, ambiguity, and continuous-time dynamics in PPPs.

This study derives the HJB Variational Inequality (HJBVI) associated with the public value function, offering a solid foundation for decision-making optimization in PPPs.

Additionally, this work conducts a numerical study using finite difference methods and the Howard algorithm to approximate the optimal rent and effort under uncertainty. The numerical analysis demonstrates the impact of uncertainty on decision-making and project outcomes in PPP contracts.

Overall, this thesis significantly contributes to the theoretical and applied fields. Firstly, we establish the existence and uniqueness of BSDEJs with a generator allowing for logarithmic growth. Furthermore, we explore the connection of these BSDEJs with quadratic BSDEJs. Secondly, we delve into the Pontryagin maximum principle for these types of BSDEs, specifically without the jump component. Finally, we advance the field of Public-Private Partnerships (PPPs) by optimizing decision-making.

Keywords : *Public Private Partnership, Moral Hazard, Knightian Uncertainty, BSDEs, stochastic control, Maximum principle, logarithmic growth, Poisson random measure, Dynamic Programming Principle, optimal stopping, Hamilton Jacobi Bellman variational inequality, Howard algorithm.*

List of publications

This work has been the subject of one scientific article published in an international journal :

1) **Bouhadjar, E.M.B.**, Khelfallah, N., & Eddahbi, M. (2024). One-Dimensional BSDEs with Jumps and Logarithmic Growth. Axioms, 13(6), 354.
2) **Bouhadjar, E. M. B.**, & Mnif, M. (2023). Public-private partnerships contract under moral hazard and ambiguous information. Stochastics and Dynamics, 23(04), 2350031.
3) **Bouhadjar, E.M.B.**, Almualim, A., Khelfallah, N., & Eddahbi, M. (2024). Maximum Principle for BSDEs with Locally Lipschitz and Logarithmic Growth. Advances in Continuous and Discrete Models (Submitted).

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Chapter 1

Introduction

1.1 Backward Stochastic Differential Equations

Backward Stochastic Differential Equations (BSDEs) represent a vibrant and relatively recent domain within stochastic analysis, gaining momentum since the early 1990s. Extensively explored for their profound connections to various stochastic mathematical challenges, such as those in mathematical finance, differential games, optimal control, and partial differential equations (PDEs), BSDEs have garnered widespread interest. They are positioned squarely within the realm of stochastic analysis.

Let's present the form of a BSDE. Consider a time interval $[0,T]$, W a fixed Brownian motion within a standard filtered probability space $(\Omega,\mathcal{F}=(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$. The filtration $\mathcal{F}$ is assumed to be the augmented filtration of W :

$$\begin{cases} dY_r = -f(r,Y_r,Z_r)\,dr + Z_r dW_r \\ Y_T = \zeta \end{cases}$$

Or, in the same way :

$$Y_s = \zeta + \int_s^T f(r,Y_r,Z_r)ds - \int_s^T Z_r dW_r \tag{1.1.1}$$

The parameters involved are :

- ζ : $\mathcal{F}_T$-measurable random variable that signifies the terminal condition.
- f : A measurable function incorporating variables t, ω, y, z, with the ω dependence typically implied rather than explicitly stated. This function is commonly referred to as the generator or driver.

A sought-after solution to the BSDE is an adapted stochastic process $(Y_t, Z_t)_{t\in[0,T]}$ to the filtration $\mathcal{F}$. The persistent question regarding the solution of (1.1.1) is focused on discerning the conditions that lead to the solution's existence, uniqueness, stability, and regularity. Researchers remain actively engaged in the pursuit of minimizing assumptions that guarantee these properties.

Let's offer a concise yet selective overview of the evolution of BSDE theory. The emergence of Linear BSDEs in 1973 within stochastic control theory, as identified by Bismut [23] in the equation governing the adjoint process, set the stage for a pivotal development. However, the groundbreaking work of Pardoux and Peng in their influential paper [78] marked the systematic commencement of the study of BSDEs. They demonstrated the existence and uniqueness of BSDEs, establishing crucial results under the following classical Assumption :

- Integrability condition : For every $y, z \in \mathbb{R}$, the function $f(., y, z)$ is a progressively measurable process satisfying :

$$\mathbb{E}\left[|\zeta|^2\right] < \infty, \ \mathbb{E}\left[\int_0^T |f(s,0,0)|^2 ds\right] < \infty.$$

- Lipschitz condition : There exists a constant $C_f > 0$ s.t. for any s, ω,

$$\forall (y_1, z_1, y_2, z_2) \ \ |f(s, y_1, z_1) - f(s, y_2, z_2)| \leq C_f \left(|y_1 - y_2| + |z_1 - z_2|\right) \ ds \otimes d\mathbb{P} \ a.e.$$

Theorem 1.1.1. *[Pardoux-Peng [78]] Under the above* ***Assumption****, the BSDE* (1.1.1) *has a unique solution* $(Y_t, Z_t)_{t\leq T}$*, such that :*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T |Z_t|^2 \, dt\right] < \infty.$$

This groundbreaking work gained widespread recognition across diverse fields, including mathematical finance [45], finance and insurance [42], insurance reserve [40], optimal control theory [81], as well as stochastic differential games and stochastic control [52–54]. These contributions are closely linked to partial differential equations (PDEs) [17, 80, 82]. Conversely, subsequent research was the first to showcase BSDEs with random terminal time.

Due to the diverse applications of BSDEs, considerable efforts have been made to relax assumptions on the generator f and/or the final condition. Noteworthy outcomes have been achieved for high-dimensional BSDEs with local Lipschitz assumptions on the driver, as evidenced in [6, 13, 28, 32, 58]. Despite extensive study of real-valued BSDEs, primarily relying on a specific comparison theorem, most works concentrate on scenarios where the generator exhibits at most a linear growth concerning y and grows either linearly or quadratically in z. This facilitates the establishment of solutions under conditions of square integrability (or even integrability) for the terminal datum, as exemplified in [60, 63, 64]. For further literature on quadratic growth in z (referred to as QBSDE), one can refer to [7, 9, 10, 18, 27, 48].

Another avenue of research in the theory of BSDEs explores equations driven by a combination of a Poisson random measure and Brownian motion (in short BSDEJs), pioneered by Tang and Li [90], where BSDEJ has the following form :

$$Y_t = \zeta + \int_t^T f(s, Y_s, Z_s, U_s(\cdot))ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R}^*} U_s(e)\widetilde{N}(ds, de) \tag{1.1.2}$$

where,

- $N(ds, de)$ is a Poisson random measure.
- $\widetilde{N}(ds, de) = N(ds, de) - \nu(de)$ is the compensated Poisson random measure.
- ν is a σ-finite measure on $\mathbb{R}^*$.

Tang and Li [90] establish the existence and uniqueness of solutions under the classical Assumption, where the solution is a triplet $(Y_s, Z_s, U_s)_{s\in[0,T]}$ of progressively measurable processes satisfy (1.1.2) and,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T \left(|Z_t|^2 + \int_\Gamma |U_t(e)|^2 \nu(de)\right) dt\right] < \infty.$$

Various other studies have delved into this area, including [1, 2, 69, 84, 96].

1.2 Stochastic control

Stochastic control theory has emerged as a dynamic field of mathematics since its intensive development in the late 1950s and early 1960s. Its applications to management and finance problems gained momentum in the 1970s, notably with Merton's seminal paper on portfolio selection [71]. Subsequently, numerous authors extended Merton's model and results, including [39, 74, 98]. Two principal and widely used approaches in solving stochastic optimal control problems are the Dynamic programming principle and Pontryagin's maximum principle. The first is abbreviated as DPP, which was introduced by Bellman in the 1950s [20, 21]. Bellman's contributions revolutionized the field by providing a powerful tool for optimizing sequential decision-making under uncertainty. Dynamic programming has since become a cornerstone of stochastic control, enabling the formulation and solution of complex optimization problems. Stochastic control problems often involve the analysis of systems described by stochastic differential equations (SDEs) or stochastic partial differential equations (SPDEs) [73]. These equations capture the stochastic dynamics of the systems and allow for the incorporation of random

disturbances, providing a realistic representation of real-world phenomena affected by uncertain factors.
Over the years, researchers have developed sophisticated mathematical techniques and computational tools to tackle the challenges posed by stochastic control problems. Approaches such as optimal, adaptive, and robust control have been extensively explored [19]. Furthermore, machine learning and reinforcement learning methodologies have opened up new avenues for addressing stochastic control problems [89].
Stochastic control finds relevance in various domains, including finance for portfolio optimization, option pricing, and risk management [66]. It also extends to energy management, robotics, healthcare, and many other fields. For a comprehensive and in-depth exploration of the discussed topics, I recommend referring to the following references, which provide detailed discussions and further insights [19–21, 39, 66, 71, 73, 74, 89, 98].

The inception of the maximum principle, attributed to Pontryagin and his research team in the 1960s, marks a significant milestone in the realm of optimal control theory. This principle asserts that the optimal control, in conjunction with the optimal state trajectory, necessitates addressing the (extended) Hamiltonian system and adhering to a maximum condition associated with the Hamiltonian function. Pontryagin initially formulated the maximum principle for deterministic problems, drawing inspiration from classical calculus of variations.

The extension of the maximum principle to stochastic control problems was pioneered by Kushner and Schweppe in their seminal work [62]. This extension presented a unique challenge, as the adjoint equation transformed into a stochastic differential equation (SDE) with terminal conditions. Unlike deterministic differential equations, reversing time is not a straightforward solution due to the adaptation requirement of the control process and the solution to the SDE with respect to the filtration. Bismut resolved this complication by introducing conditional expectations and deriving the solution to the adjoint equation through the martingale representation theorem.

Several notable contributions in this area include [8, 15, 26, 33, 68], among others.

1.3 The Power of the Hamilton-Jacobi-Bellman Variational Inequality and Verification Theorem in Optimal Control

The HJB variational inequality (in short, HJBVI) and the associated verification theorem are fundamental concepts in the field of stochastic control. They provide a powerful framework for analyzing and solving optimal control problems under uncertainty.
The HJB variational inequality is a key mathematical equation that characterizes the value

function associated with an optimal control problem. It arises in dynamic programming approaches and encapsulates the optimality conditions for the control policy. The HJB variational inequality incorporates the system dynamics, the control actions, and the stochastic nature of the environment.
The verification theorem, often called the HJB equation verification theorem, establishes the connection between the solution of the HJB variational inequality and the optimal control policy. It states that if a function satisfies the HJB variational inequality, then it is the value function of the corresponding optimal control problem. The verification theorem provides a crucial theoretical result that identifies and verifies optimal control policies.

Numerous researchers have made significant contributions to the study of the HJB variational inequality and the verification theorem. Notably, the following references have played pivotal roles in shaping the field :

- W. H. Fleming and H. M. Soner [49] provide a comprehensive introduction to the theory of viscosity solutions and their applications to stochastic control problems.
- M. Bardi and I. Capuzzo-Dolcetta [16] offer a thorough treatment of the theory of viscosity solutions and its use in solving Hamilton-Jacobi-Bellman equations.
- M. G. Crandall et al. [34] provide a comprehensive overview of viscosity solutions theory, encompassing the HJB variational inequality.
- J. Yong and X. Y. Zhou [97] present a detailed exposition of stochastic control theory, with a specific focus on Hamiltonian systems and the HJB equation.

These seminal works offer valuable insights and lay the mathematical foundations for the study of the HJBVI and the verification theorem. In this thesis, we delve into the HJBVI, which characterizes the public value function, and explore the associated verification theorem within the context of stochastic control problems.

1.4 Principal-Agent Problem

In this section, we delve into the intriguing realm of the principal-agent problem and closely examine the complexities surrounding the pursuit of an optimal contract between two distinct parties. Within this dynamic, we have the principal, who plays a pivotal role, and the agent, who assumes a complementary position. The principal extends a contractual proposal, granting the agent the freedom to exercise their agency by accepting or rejecting the offer. It is worth noting that once a decision is made, both parties are bound by their choice and have no recourse to reverse it.
Once the agent willingly accepts the contract, they are obliged to put forth a specific effort,

denoted as 'a,' as a condition of the agreement. Meanwhile, the principal is driven by two primary objectives in this scenario :

- The principal's first objective is to ensure the agent's acceptance of the contract. This is commonly referred to as a reservation constraint, which establishes the minimum value 'x' that must be met or exceeded for the agent to decline the contract. The principal strives to set terms and conditions that entice the agent to willingly agree to the contractual arrangement.
- Additionally, the principal aims to maximize the profits or benefits derived from the contract. The principal seeks to extract the utmost advantage and financial gain from the contractual relationship through careful strategizing and design. This entails optimizing various aspects of the agreement to secure the most favorable outcomes and maximize the principal's returns.

Therefore, the objective is to construct a contract, denoted as 'Γ', that maximizes the principal's utility while ensuring a minimum value for the agent. The optimal contract for the principal varies depending on whether or not they observe the agent's efforts (known as the first-best and second-best scenarios). The utility of the agent is denoted as U. Within the literature, three contract types are commonly distinguished based on the level of information, a distinction we will briefly summarize below for clarity.

First-best :

In the first-best scenario, also known as Risk Sharing, the principal and agent share the same information and collaborate on risk allocation. The principal holds bargaining power, dictating the contract and the agent's actions. This transforms the issue into a stochastic control problem for the principal, who simultaneously determines the contract and actions. Denoted by c for the contract, a for the action, and using U_P as the principal's utility function, the problem involves observing the agent's effort. The goal is to collectively distribute the risk, making it a single-individual problem, where the principal selects both the contract and effort, adhering to the reservation constraint.

The principal's problem can be delineated as follows : The contract, denoted as c, is contingent upon X^a, which stands for the project's social worth and is commonly referred to as the output in the context of principal-agent literature. X^a, in turn, depends on the effort exerted by the agent.

$$\begin{aligned} V_p &= \sup_{c,a} V_p(a,c) \\ &= \sup_{c,a} [U_p(a,c) - c(X^a)]. \end{aligned}$$

Under the following reservation constraint :

$$\mathbb{E}[U_A(c,a) - h(a)] \geq \underline{x},$$

where h represents the cost of effort, a strictly convex function.

To further analyze the problem, we incorporate a Lagrange multiplier denoted as λ and shift our focus to examining the unconstrained problem.

$$\sup\left\{\mathbb{E}[U_p(a,c)+\lambda U_A(a,c)-\lambda h(a)-\lambda\underline{x}\right\}.$$

Several notable works have contributed to this topic [4, 24, 29].

Second best :
In this situation, we encounter a scenario where the principal cannot observe the actions performed by the agent. As a result, there is typically a loss in expected utility for the principal, and she can only achieve the second-best reward or outcome. There are many real-world examples where the principal cannot deduce the agent's actions, either because the cost of monitoring the agent is prohibitively high or simply impossible.
Due to the presence of actions that cannot be observed or contracted, the principal cannot directly dictate the actions that align with their preferences. Instead, when offering a contract c, she must be aware of the action $a=a(c)$ that would be optimal for her to choose. Consequently, the principal faces the challenge of designing incentives to indirectly influence the agent in selecting certain actions by providing an appropriate contract. Since he can undertake actions that may not be in the best interest of the principal, this situation is commonly referred to as a moral hazard, where he may lack moral constraints or face conflicting interests.
For a given contrat c, we get the best answer $a^*(c)$ of the agent :

$$\sup_a \mathbb{E}[U_A(c,a)-h(a)].$$

We solve the principal's problem.

$$\sup_c \mathbb{E}\left[U_p(X^a)-c(X^a)\right].$$

Subject to the specified reservation constraint

$$\mathbb{E}\left[U_A(c,a^*(c))-h(a^*(c))\right]\geq\underline{x}.$$

The phenomenon of moral hazard has received considerable attention in the context of discrete-time models. However, Holmström and Milgrom [55] pioneered addressing this issue in a continuous-time framework, considering a finite horizon and a terminal payment. This study attracted considerable attention from authors, with notable contributions, as evidenced by references [36, 56, 70, 86, 87, 93]. In a different context, the authors applied the principal-agent problem framework to the energy sector (see [3, 46]).
In the literature, several studies have explored the concept of continuous payment with an infinite or random horizon. One of the seminal works in this area [85, 92]. However, some works consider the framework of Poisson processes, where the agent's action influences the process's

jump intensity, exemplified by studies such as [22, 77]. Hu et al. [57] addressed a moral hazard problem with multiple principals and a single agent. Since the agent is constrained to work for only one principal at a time, they studied a switching problem from one principal to another, where the switching time is modeled as a random time characterized by a Poisson process. The agent influences the random switching time by controlling the intensity of the Poisson process. In the case of an infinite number of principals, they used a mean-field formulation.

Third Best

Third-best is the case where the principal does not have perfect knowledge of the agent's characteristics (such as wealth, risk aversion, etc.). Important characteristics remain concealed. This kind of problem has been explored in previous works by [30, 37, 88]. However, this thesis will not extensively discuss this type of contract, as the primary focus is on tackling 'second-best' problems.

1.5 Ambiguity

Principal-agent problems under moral hazard have been extensively studied in economics. The common assumption is that the principal knows the probability distribution governing the agent's effort. However, in real-world scenarios, she often faces uncertainty and ambiguity regarding this probability, introducing the need to consider multiple objective probability measures. The literature has provided preliminary insights into uncertainty, particularly in the context of dominated sets, utilizing objective reference probability measures such as drift uncertainty as explored by Gilboa and Schmeidler [50]. Ambiguity, also known as Knightian uncertainty, holds significant relevance in economic problems, a concept initially introduced by Knight [59]. This notion plays a crucial role in economic contracts, reflecting the inherent inaccuracy of available information and its impact on decision-making.

Building upon Knight's work, subsequent researchers have further explored the relationship between ambiguity and decision-making under uncertainty. Ellsberg [47] and Gilboa and Schmeidler [50] contributed to this line of inquiry by examining the concept of multiple priors within a static framework. Chen and Epstein [31] expanded on these ideas by extending the framework to an intertemporal setting, introducing the concept of κ-ignorance to characterize Knightian uncertainty, where κ represents the ignorance parameter. As the value of κ increases, decision-makers find themselves in increasingly ambiguous situations.

Addressing the implications of ambiguity in specific scenarios, Dumav and Riedel [43] investigated a moral hazard problem over a random horizon involving continuous payments. Within this context, the principal and the agent establish a contractual relationship based on unobservable effort, generating output under conditions of ambiguity. In contrast to Sannikov [85], Dumav and Riedel [43] proposed a model that maps efforts to sets of probability distributions, enabling

them to characterize the optimal contract under ambiguous information. Additionally, Mastrolia and Possamaï [70] explored a scenario where both the agent and the principal faced uncertainty regarding the volatility of the output. Their analysis specifically focused on the case with finite maturity.
In our work, we aim to investigate the impact of the κ factor on the problem of the main agent under moral hazard, specifically examining its effect on the drift. To analyze this effect. Through this approach, we address the problem faced by the agent under the worst possible scenario. Within this context, our objective is to determine the optimal response denoted as a^*, given a specific contract c.

This thesis comprises four chapters. The content in Chapters 2, 3, and 4 presents distinct self-contained research findings that can be perused independently. The initial chapter provided a comprehensive introduction to the central themes of the thesis. The rest of the chapters are outlined below :

Chapter 2 : "One-dimensional Backward Stochastic Differential Equations with Jumps and Logarithmic Growth"

Motivation and Outline : Chapter 2 addresses fundamental questions about the existence and uniqueness of a BSDE involving a Poisson random measure and an independent Brownian motion, commonly abbreviated as BSDEJ. This work derived from the work of Bahllali et al. [14], where they studied a BSDE without the Jump Part and proved its existence and uniqueness under the logarithmic growth condition. This study is underscored by several pivotal factors that command our attention :

- Will the existence and uniqueness be achieved in the presence of the Poisson random measure ?
- What characterizes the auxiliary structure in this BSDEJ class, and to what extent can we systematically formulate assumptions regarding this auxiliary BSDEJ ?

These qualities have been proven under two key assumptions. First, we present pivotal lemmas that lay the foundation for our main result. In the first assumption, the generator exhibits logarithmic growth in both the state variable and the Brownian component while maintaining Lipschitz continuity with respect to the jump component. The first assumption's robustness is validated by including a concrete example. In the second assumption, we also relax the Lipschitz condition on the Poisson component, allowing the generator to exhibit logarithmic growth concerning all variables. Taking a step further, we employ an exponential transformation to establish an equivalence between solutions of a BSDEJ exhibiting a quadratic growth in the z-variable and a BSDEJ showing logarithmic growth for y and z.

Upcoming Challenges :

- Establishing a robust connection with PDEs, drawing upon the foundational insights provided by Bahlali et al. [14], particularly in their exploration of PDEs excluding the "Jump" component and emphasizing its significance in theoretical physics. The Markovian form of BSDE(1.1.1) is related to the following semilinear PDE,

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + u \ln(|u|) = 0 \text{ on } (0,\infty) \times \mathbb{R}^d, \\ u(0^+) = \varphi > 0. \end{cases}$$

- Application of these findings in financial markets. This is particularly relevant for modeling scenarios where asset price dynamics exhibit characteristics of proportional growth, such as valuing growth options. Furthermore, a comprehensive numerical study will be conducted to validate theoretical propositions and glean practical insights.

Chapter 3 : "Optimal Control of BSDEs with Logarithmic Growth Condition : Exploring the Maximum Principle"

Motivation and Outline : In Chapter 3, we examine a stochastic control problem tied to a BSDE that is locally Lipschitz continuous. This equation's generator satisfies a logarithmic growth condition.

This research spotlights the contributions of Azizi and Khelfallah [5]. Our attention is directed towards critical aspects :

- Can we ensure the existence and uniqueness of the SDE, as defined later, in the presence of a logarithmic growth condition to establish a well-posedness problem ?
- Is there flexibility in relaxing the assumptions presented in their work ?

Not constrained by the necessity of convexity within the control domain, we derive a necessary and sufficient condition for optimality applicable across all optimal controls. A local Lipschitz stochastic differential equation and a Hamiltonian subject to a maximum condition delineate these criteria. Our initial focus involves proving, under specific conducive conditions, the existence of a singular solution to the resultant adjoint equation. Employing an approximation methodology on the coefficients, we introduce a class of control problems characterized by global Lipschitz coefficients. This framework enables the derivation of a stochastic maximum principle, facilitating the pursuit of near optimality within these approximated systems. Subsequently, we seamlessly transition back to the initial control problem through a judicious limit-taking process.

Upcoming Challenges :

- Relax the boundedness assumption on the terminal condition and investigate the effects on the solution and stability.
- Generalize the model to a mean-field approach and provide examples of applications such as economics or game theory. This will demonstrate the flexibility of the framework.

Chapter 4 : "Public-Private Partnerships contract under moral hazard and Knightian uncertainty with random horizons"

Motivation and Outline : The final chapter 4 delves into the complexities of the principal-agent problem within the context of Public-Private Partnerships (PPPs) under moral hazard and Knightian uncertainty, all while considering the variable time horizons inherent in long-term PPP contracts. In this chapter, we navigate the intricate interplay of stochastic control and optimal stopping problems within the framework of ambiguous information. The key highlights that have captured our attention in this work :

- When formulating this problem mathematically, it is imperative to establish the existence and uniqueness of BSDE representing the agent's objective function. Can we provide proof or reference supporting this assertion ?
- What challenges are anticipated when formulating the principal's objective function, particularly in employing dynamic programming techniques ?
- To what extent does uncertainty impact the contract ? Is the effect predominantly positive or negative ?
- Will this effectiveness be evident in the figures depicting the agent's effort, rent, and value ?

We adopt a Stackelberg model, wherein the public entity pays rent to the agent, and the latter's acceptance of the contract depends on exceeding a pre-specified reservation constraint. The agent, in response, optimizes its effort under the worst-case scenario. Moreover, the public retains the authority to halt the contract prematurely on a random date, providing compensation to the agent in the process.

In this work, we adopt the 'weak approach', we show that the dynamics of the consortium's objective function are intrinsically connected to a solution of a BSDE problem with a random horizon.

Subsequently, we transform our problem into standard stochastic control and optimal stopping problems, culminating in deriving the HJBVI associated with the public value function. This endeavor leads us to a verification theorem and the eventual characterization of optimal contracts.

Upcoming Challenges :

- Expanding the scope of the contract involves considering contracts between a principal and multiple agents, whether they entail employing all agents simultaneously or individually. The latter scenario adds complexity due to the issue of switching between agents, which impacts motivation.
- Allowing agents more autonomy, like the ability to exit contracts, could improve motivation. However, this could be detrimental and needs balancing through appropriate cost functions.
- Addressing the aforementioned challenges through two main approaches :
  - Creating inter-agent impacts : Exploring how interactions between agents, whether positive or negative, introduce challenges and dynamics into the system.
  - Isolating and comparing impacts : Examining the effects of isolating the interactions between agents and comparing the outcomes with the first scenario where such

interactions are present to determine the optimal decision-making strategy.

Chapter 2

One-dimensional Backward Stochastic Differential Equations with Jumps and Logarithmic Growth

2.1 Introduction and Notations

Pardoux and Peng [78] initially introduced the concept of Backward Stochastic Differential Equations without the jump component, denoted briefly as BSDEs. They established the existence and uniqueness of BSDEs, assuming the Lipschitz continuity condition on the BSDE's generator w.r.t. both (y, z). Additionally, they assumed that the terminal value is square integrable. This result gained widespread recognition across various fields, including mathematical finance [45], finance and insurance [42], insurance reserve [40], optimal control theory [81] as well as stochastic differential games and stochastic control [52–54]. These findings are strongly connected to partial differential equations (PDEs) [17, 80, 82]. In contrast, the latter contributions were the first to demonstrate BSDEs with random terminal time.
Given the diverse applications of BSDEs, there has been a concerted effort to relax assumptions on the generator f and/or the final condition. Notably, limited results were established for high dimensional BSDEs with local Lipschitz assumptions on the driver, as shown in [6, 13, 28, 32, 58]. While real-valued BSDEs have been extensively studied, predominantly relying on a comparison theorem, most works focus on cases where the generator grows at most linearly w.r.t. y and grows either linearly or quadratically in z. This enables the establishment of solutions under

conditions of square integrability (or even integrability) for the terminal datum, as illustrated in [60, 63, 64].

In situations where the generator exhibits a quadratic growth in z (referred to as QBSDE), the existence of solutions hinges upon the requirement for either boundedness or, minimally, exponential integrability of the terminal value. This requirement is demonstrated in various works, such as [18, 27, 48]. It is noteworthy, however, that recent advancements, highlighted in [7, 9, 10], have identified a substantial class of QBSDEs for which solutions exist under the sole condition of a square-integrable terminal datum.

Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ stands for the σ -algebra generated by two key processes : a real-valued Wiener process W_t and a real-valued Poisson random measure $N(ds,de)$ defined on $[0,T]\times\Gamma$, where $\Gamma = \mathbb{R}^*$. Furthermore, we introduce $\widetilde{N}(ds,de)$ as the compensator of N, defined by :

$$\widetilde{N}(ds,de) = N(ds,de) - \nu(de)ds,$$

Here, ν is a σ-finite measure on Γ, equipped with its Borel field $\mathcal{B}(\Gamma)$. Notably, $\widetilde{N}$ serves as a martingale with a zero mean, referred to as the compensated Poisson random measure.

We now direct our attention to the central focus of this research endeavor. Specifically, we investigate solutions denoted as $(Y,Z,U) := (Y_t, Z_t, U_t(e))_{0\le t\le T, e\in\Gamma}$ for a BSDEJ(ζ, f). The following dynamics govern the evolution of these solutions

$$Y_t = \zeta + \int_t^T f(s,Y_s,Z_s,U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_\Gamma U_s(e)\widetilde{N}(ds,de) \tag{2.1.1}$$

The investigation initiated by Tang and Li [90] marked a pioneering achievement in the study of BSDEJ of type (2.1.1). They showed in this work the existence and uniqueness of solutions for such equations subject to Lipschitz conditions. In a closely related context, [96] studied a class of real-valued BSDEs featuring Poisson jumps and random time horizons. They proved the existence of at least one solution for BSDEs characterized by a driver exhibiting linear growth.

Subsequently, [84] extended these discoveries by proving the existence but not the uniqueness of solutions for BSDEs with jumps. They considered continuous coefficients that satisfy an extended linear growth condition in their extension. They also generalized this result to situations where the generators are either left- or right-continuous.

In recent developments, [1, 69] have presented examples that strengthen the relationship between a certain class of quadratic BSDEJs and conventional BSDEJs featuring continuous drivers. Moreover, [2] made an important contribution by proving the well-posedness of solutions under local Lipschitz conditions, with special emphasis on the Brownian motion component. They also showed the existence of one and only one solution for a class of nonlinear variants of the backward Kolmogorov equation.

It is important to note that all of the above results were formulated for one-dimensional BSDEs. In another paper, [44] dealt with the study of a multidimensional Markovian BSDEJ and showed that the adapted solution of the BSDEJ can be expressed by a given Poisson process and deterministic functions. Furthermore, they established the existence of solutions for these equations, assuming that their generators are either continuous w.r.t. y and z and Lipschitz in u or continuous in all their variables and adhere to standard linear growth assumptions. Bahlali and El Asri [11] investigated situations where the generator of the BSDEs is bounded by $(|z|\sqrt{|\ln|z||})$. They also considered the terminal value, assuming it to be merely $\mathbb{L}^p$-integrable, with $p > 2$. However, the extension of this condition was recently explored by [14], who supposed that the drift is dominated $(|y||\ln|y|| + |z|\sqrt{|\ln|z||})$. Additionally, [2, 76] studied BSDEs associated with jump Markov processes, with the latter presenting a proof under assumptions different from those considered in the present study.

In this work, we proceed according to the following methodology. We establish the existence and uniqueness of the solution for BSDEJs whose generators show a growth described by a logarithmic function of the type $(|y||\ln|y|| + |z|\sqrt{|\ln|z||})$ but keeping the linear growth condition in u. Initially, we present a priori estimates for solutions of BSDEs, followed by presenting the main result. This makes the content of Section 2.2. Section 2.3 extends the logarithmic growth condition for BSDEJs by relaxing the Lipschitz condition on the jump coefficient. Section 2.4 demonstrates the equivalence of previously obtained solutions through an exponential transformation. Finally, Section 2.5 provides the conclusion of our work.

2.1.1 Notation and Preliminaries

For a specified $T \geq 0$, the following notation is employed :

- $\mathcal{P}$: represents the predictable σ-field on $[0,T] \times \Omega$.
- $\widetilde{\Omega}$: is defined as $[0,T] \times \Omega \times \Gamma$.
- $\mathcal{E} := \mathcal{B}(\Gamma)$.
- $\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{E}$ denotes the predictable σ-algebra on $\widetilde{\Omega}$

In the subsequent sections of this work, we shall introduce useful functional spaces : For $m \geq 1$:

- $\mathcal{S}^m([s,t];\mathbb{R})$: the space of $\mathbb{R}$-valued adapted càdlàg processes Y such that

$$\|Y\|_{\mathcal{S}}^m = \mathbb{E}\big[\sup_{s\leq r\leq t} |Y_r|^m\big] < \infty.$$

- $\mathcal{S}^\infty([s,t];\mathbb{R})$: the space of $\mathbb{R}$-valued adapted càdlàg processes Y such that

$$\|Y\|_{\mathcal{S}^\infty} = \operatorname*{ess\,sup}_{s\leq r\leq t} |Y_r| < \infty.$$

— $\mathbb{H}^m([s,t];\mathbb{R})$: the space of $\mathbb{R}$-valued predictable processes satisfying

$$\mathbb{E}[\int_s^t |Z_r|^m\, dr] \ < \ \infty.$$

— $L^2(\Gamma,\mathcal{E},\nu;\mathbb{R})$: the space of Borelian functions $\ell:\Gamma\to\mathbb{R}$ such that

$$\|\ell\|_\nu = \Big(\int_\Gamma |\ell(e)|^2\,\nu(de)\Big)^{1/2} \ < \ \infty.$$

— $\mathbb{L}^m([s,t],\nu;\mathbb{R})$: the set of the processes $U:\widetilde{\Omega}\to\mathbb{R}$ is $\widetilde{\mathcal{P}}$-measurable and

$$\mathbb{E}\Big[\int_s^t \|U_r\|_\nu^m\, dr\Big] \ < \ \infty.$$

2.2 Existence and Uniqueness of Solutions

In this section, we establish the foundational assumption that forms the basis of our analysis, providing a framework for subsequent developments. This assumption is pivotal for exploring solutions to the BSDEJ Equation (2.1.1). We then introduce preliminary estimates of the solution and delineate key lemmas crucial for establishing both the existence and uniqueness of solutions.

Assumption 2.2.1.

(A.1) *Assume that $\mathbb{E}[|\zeta|^{\mu_T+1}]$ is finite, where $\mu_t := e^{\theta t}$ for all $t\in[0,T]$ and θ is a sufficiently large positive constant.*

(A.2) *(i) f is continuous in (y,z) and Lipschitz with respect to u (t,ω)-a.e.*

(ii) There exist constants c_0, c_1, c_2, C_{Lip}, and a positive process ϑ such that

$$\int_0^T \mathbb{E}[\vartheta_s^{\mu_s+1}]ds < +\infty.$$

Additionally, for every t,ω,y,z,u,u_1,u_2 :

$$|\, f(t,\omega,y,z,u)\,| \le \vartheta_t + g_{1,c_2}(y) + g_{2,c_0}(z) + c_1\|u\|_\nu,$$

and

$$|f(t,\omega,y,z,u_1) - f(t,\omega,y,z,u_2)| \le C_{Lip}\|u_1-u_2\|_\nu,$$

where $g_{1,c_2}(y) = c_2|y||\ln|y||$ and $g_{2,c_0}(z) = c_0|z|\sqrt{|\ln|z||}$.

(A.3) *There exists a sequence of real numbers $(A_N)_{N>1}$ along with constants $M_2\in\mathbb{R}_+$, $r>0$, satisfying :*

(i) *For every integer $N > 1$, we have $1 < A_N \leq N^r$.*

(ii) $\lim_{N\to\infty} A_N = \infty$.

(iii) *For any natural number $N \in \mathbb{N}$, and every y_1, y_2, z_1, z_2, u such that : $|y_1|$, $|y_2|$, $|z_1|$, $|z_2|$, $\|u\|_\nu \leq N$, the following holds :*

$$\begin{aligned}&(y_1-y_2)\big(f(t,\omega,y_1,z_1,u)-f(t,\omega,y_2,z_2,u)\big)\\&\leq M_2\,\Big(\,|\,y_1-y_2\,|^2\ln(A_N)+|\,y_1-y_2\,|\,|z_1-z_2|\sqrt{\ln(A_N)}+\frac{\ln(A_N)}{A_N}\,\Big).\end{aligned}$$

Definition 2.2.2. *A solution to the BSDEJ(ζ, f) is a triplet*

$$(Y,Z,U)\in\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{L}^2([0,T],\nu;\mathbb{R})$$

that satisfies Equation (2.1.1).

2.2.1 Technical Lemmas

This subsection introduces four technical lemmas needed in the sequel. More precisely, the first three are crucial in proving the results of the next subsection.

Lemma 2.2.3. *Let y, $z \in \mathbb{R}$ such that $|y| > e$. For any positive constant C_1, there exists another positive constant C_2 such that the following inequality holds :*

$$C_1|y||z|\sqrt{|\ln|z||}\leq\frac{|z|^2}{2}+C_2|y|^2\ln|y|. \tag{2.2.1}$$

Proof: We consider two cases based on the relationship between $|y|$ and $|z|$.

Case 1 : $|z| \leq |y|$

In this case, we have $1 < |y|\ln|y|$ and $\ln|z|\ \mathbb{1}_{\{|z|>1\}} \leq \ln|y|\ \mathbb{1}_{\{|z|>1\}}$, thus :

$$\begin{aligned}C_1|z||y|\sqrt{-\ln|z|}\ \mathbb{1}_{\{|z|\leq1\}}&\leq e^{-\frac{1}{2}}\frac{C_1}{\sqrt{2}}|y|\\&\leq e^{-\frac{1}{2}}\frac{C_1}{\sqrt{2}}|y|^2\ln|y|\\&\leq\frac{|z|^2}{2}+e^{-\frac{1}{2}}\frac{C_1}{\sqrt{2}}|y|^2\ln|y|,\end{aligned}$$

and

$$\begin{aligned}C_1|z||y|\sqrt{\ln|z|}\ \mathbb{1}_{\{|z|>1\}}&\leq\frac{|z|^2}{2}+2C_1^2|y|^2\ln|z|\ \mathbb{1}_{\{|z|>1\}}\\&\leq\frac{|z|^2}{2}+2C_1^2|y|^2\ln|y|.\end{aligned}$$

The inequality (2.2.1) becomes

$$C_1|y||z|\sqrt{|\ln|z||} \leq \frac{|z|^2}{2} + C_2|y|^2\ln|y|,$$

where $C_2 = 2C_1^2 \vee e^{-\frac{1}{2}}\frac{C_1}{\sqrt{2}}$. Therefore, the inequality holds in this case.

Case 2 : $|z| > |y|$

Let us set $a = \frac{|z|}{|y|} > 1$. Since $|y| \geq e$, we have $|z| = a|y| > e$. Using this substitution, the inequality becomes

$$\begin{aligned} C_1|y||z|\sqrt{\ln|z|} &\leq C_1 a|y|^2\left(\sqrt{\ln(a)} + \sqrt{\ln|y|}\right) \\ &\leq |y|^2\left(\frac{a^2}{4} + C_1^2\ln|y| + C_1 a\sqrt{\ln(a)}\right); \end{aligned}$$

the latter inequality was derived from Young's inequality. Moreover, we have

$$\frac{|z|^2}{2} + C_2|y|^2\ln|y| = \left(\frac{a^2}{2} + C_2\ln|y|\right)|y|^2.$$

We obtain the desired result by showing that

$$\frac{a^2}{4} + C_1 a\sqrt{\ln(a)} + C_1^2\ln|y| \leq \frac{a^2}{2} + C_2\ln|y|.$$

Let $r = \max\{z \geq 1 : 4C_1\sqrt{\ln(z)} - z = 0\}$, and let us introduce the function h, defined as $h : t \in \mathbb{R}_+ \longrightarrow h(t) := 4C_1\sqrt{\ln(t)} - t$. We denote by $r_0 = \arg\max_{t>0} h(t)$; it follows that $r_0\sqrt{\ln(r_0)} = 2C_1$

There are two sub-cases to consider :

Sub-Case 1 : If $C_1 \geq \frac{r_0}{4\sqrt{\ln(r_0)}}$, then r is well defined. If $a \geq r$, then $C_1 a\sqrt{\ln(a)} \leq \frac{a^2}{4}$, and if $1 < a < r$, then since $|y| \geq e$, we have

$$C_1 a\sqrt{\ln(a)} \leq C_1 r\sqrt{\ln(r)} = C_1\frac{r^2}{4} \leq C_2 \leq C_2\ln|y|.$$

Sub-Case 2 : If $C_1 < \frac{r_0}{4\sqrt{\ln(r_0)}}$, since $2C_1 = r_0\sqrt{\ln(r_0)}$, then $r_0 < e^{\frac{1}{2}}$, which implies that $C_1 < \sqrt{2}e^{\frac{1}{2}}$. Therefore,

$$C_1 a\sqrt{\ln(a)} < \sqrt{2}e^{\frac{1}{2}}a\sqrt{\ln(a)} \quad < \quad \frac{a^2}{4} + 11 \quad < \quad \frac{a^2}{4} + 11\ln|y|, \quad \text{since } |y| > e.$$

Therefore, the inequality holds true in all cases, which completes the proof. □

Lemma 2.2.4. *For $p \in (0, \infty)$ and $x, y \in \mathbb{R}$, the following inequality holds :*

$$\int_0^1 (1-a)|x + ay|^p da \geq 3^{-(1+p)}|x|^p.$$

Proof: Let $y=0$. In this case, the integral simplifies to $\int_0^1(1-a)|x|^p da=\frac{1}{2}|x|^p$. Thus, we consider the scenario where $y\neq 0$ and define $a_0:=\frac{2|x|}{3|y|}$. For any $a\in[0,a_0]\cup[2a_0,\infty)$, it holds that

$$\frac{1}{3}|x| \;\le\; ||x|-a|y|| \;\le\; |x+ay|.$$

We proceed by analyzing three distinct cases :

(1) **Case 1 :** $1\le a_0$. In this case, we have

$$\int_0^1(1-a)|x+ay|^p da\ge\Big(\frac{1}{3}|x|\Big)^p\int_0^1(1-a)da=\frac{1}{2}\Big(\frac{1}{3}|x|\Big)^p.$$

(2) **Case 2 :** $\frac{1}{2}\le a_0<1$. Here, we observe

$$\begin{aligned}\int_0^1(1-a)|x+ay|^p da\ge\int_0^{\frac{1}{2}}(1-a)|x+ay|^p da\ge\Big(\frac{1}{3}|x|\Big)^p\int_0^{\frac{1}{2}}(1-a)da\\ =\frac{3}{8}\Big(\frac{1}{3}|x|\Big)^p.\end{aligned}$$

(3) **Case 3 :** $a_0<\frac{1}{2}$. In this scenario, we have

$$\begin{aligned}\int_0^1(1-a)|x+ay|^p da&\ge\Big(\frac{1}{3}|x|\Big)^p\Big(\int_0^{a_0}(1-a)da+\int_{2a_0}^1(1-a)da\Big)\\ &=\Big(\frac{1}{3}|x|\Big)^p\Big(\frac{3}{2}a_0^2-a_0+\frac{1}{2}\Big)\;\ge\;\frac{1}{3}\Big(\frac{1}{3}|x|\Big)^p.\end{aligned}$$

□

Lemma 2.2.5. *Let (Y,Z,U) be a solution to the BSDEJ (2.1.1). Under (**A.1**) and (**A.2**), there exists a positive constant C such that*

$$\begin{aligned}&\mathbb{E}\Big[|Y_t|^{\mu_t+1}+\int_t^T\mu_s(\mu_s+1)|Y_s|^{\mu_s-1}(|Z_s|^2+\|U_s\|_\nu^2)ds\Big]\\ &\le C\Big(1+\mathbb{E}[|\zeta|^{\mu_T+1}]+(\mu_T+1)^{\mu_T}\int_0^T\mathbb{E}[\vartheta_s^{\mu_s+1}]ds\Big).\end{aligned}$$

Proof: Set $u(t,x):=|x|^{\mu_t+1}$ and $\mathrm{sgn}(x):=-\mathbb{1}_{\{x\le 0\}}+\mathbb{1}_{\{x>0\}}$, then $u_t(t,x)=\theta\mu_t\ln|x||x|^{\mu_t+1}$, $u_x(t,x)=(\mu_t+1)|x|^{\mu_t}\mathrm{sgn}(x)$ and $u_{xx}(t,x)=\mu_t(\mu_t+1)|x|^{\mu_t-1}$. By utilizing Itô's formula to $u(t,Y_t)$

$$
\begin{aligned}
u(T,Y_T) =& u(t,Y_t)+\int_t^T u_s(s,Y_s)ds+\int_t^T u_x(s,Y_{s-})dY_s+\int_t^T u_{xx}(s,Y_{s-})d\langle Y\rangle_s\\
&+\sum_{t\le s\le T}(u(s,Y_s)-u(s,Y_{s-})-u_x(s,Y_{s-})\Delta Y_s)\\
=&u(t,Y_t)+\int_t^T u_s(s,Y_s)ds+\int_t^T u_x(s,Y_{s-})dY_s+\int_t^T u_{xx}(s,Y_s)|Z_s|^2ds\\
&+\int_t^T\int_\Gamma (u(s,Y_{s-}+U_s(e))-u(s,Y_{s-})-u_x(s,Y_{s-})U_s(e))\,N(ds,de)\\
=&u(t,Y_t)+\int_t^T u_s(s,Y_s)ds+\int_t^T u_{xx}(s,Y_s)|Z_s|^2ds\\
&-\int_t^T u_x(s,Y_{s-})f(s,Y_s,Z_s,U_s)ds\\
&+\int_t^T u_x(s,Y_s)Z_sdW_s+\int_t^T\int_\Gamma (u(s,Y_{s-}+U_s(e))-u(s,Y_{s-}))\widetilde{N}(ds,de)\\
&+\int_t^T\int_\Gamma (u(s,Y_{s-}+U_s(e))-u(s,Y_{s-})-u_x(s,Y_{s-})U_s(e))\,\nu(de)ds.
\end{aligned}
\tag{2.2.2}
$$

Setting

$$
\begin{aligned}
\Xi_t &= \int_0^t u_x(s,Y_s)Z_sdW_s+\int_0^t\int_\Gamma (u(s,Y_{s-}+U_s(e))-u(s,Y_{s-}))\widetilde{N}(ds,de)\\
&=\int_0^t(\mu_s+1)|Y_s|^{\mu_s}\mathrm{sgn}(Y_s)Z_sdW_s+\int_0^t\int_\Gamma \left(|Y_{s-}+U_s(e)|^{\mu_s+1}-|Y_{s-}|^{\mu_s+1}\right)\widetilde{N}(ds,de)
\end{aligned}
$$

For $n\ge 0$, define the stopping time τ_n as follows :

$$
\tau_n:=\inf\left\{0\le t\le T:\int_0^t\Big((\mu_s+1)|Y_s|^{\mu_s}Z_s\Big)^2ds\ \vee\ \int_0^t\int_\Gamma \left(|Y_{s-}+U_s(e)|^{\mu_s+1}-|Y_{s-}|^{\mu_s+1}\right)^2\nu(de)ds\ \ge\ n\right\}.
$$

Taking $t=t\wedge\tau_n$ and $T=T\wedge\tau_n$ in the equality (2.2.2), we obtain

$$
\begin{aligned}
&|Y_{t\wedge\tau_n}|^{\mu_{t\wedge\tau_n}+1}+\frac{1}{2}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)\mu_s|Y_s|^{\mu_s-1}|Z_s|^2ds+\int_{t\wedge\tau_n}^{T\wedge\tau_n}\theta\mu_s|Y_s|^{\mu_s+1}\ln|Y_s|ds\\
=&|Y_{T\wedge\tau_n}|^{\mu_{T\wedge\tau_n}+1}+\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s}f(s,Y_s,Z_s,U_s)ds\\
&-\int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma\Big(|Y_{s-}+U_s(e)|^{\mu_s+1}-|Y_{s-}|^{\mu_s+1}-(\mu_s+1)\,|Y_{s-}|^{\mu_s}\,\mathrm{sgn}(Y_{s-})U_s(e)\Big)\nu(de)ds\\
&+\Xi_{t\wedge\tau_n}-\Xi_{T\wedge\tau_n}
\end{aligned}
\tag{2.2.3}
$$

By **Assumption** (**A.2**)-(ii)

$$
\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s}f(s,Y_s,Z_s,U_s)ds\le I_1+I_2+I_3+I_4,
$$

where

$$\begin{aligned} I_1 &:= \int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)\vartheta_s |Y_s|^{\mu_s} ds, \\ I_2 &:= c_2\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s+1}|\ln|Y_s||ds, \\ I_3 &:= c_0\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s}|Z_s|\sqrt{|\ln|Z_s||}ds, \\ I_4 &:= c_1\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s}\|U_s\|_\nu ds. \end{aligned}$$

Estimation of I_1 : Young's inequality yields ($|ab| \le \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$, for $p := \mu_s+1$ and $q := \frac{\mu_s+1}{\mu_s}$) leads to

$$(\mu_s+1)\vartheta_s|Y_s|^{\mu_s} \le (\mu_s+1)^{\mu_s}\vartheta_s^{\mu_s+1} + \frac{\mu_s}{\mu_s+1}|Y_s|^{\mu_s+1}$$

Hence,

$$\begin{aligned} I_1 &\le \int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)^{\mu_s}\vartheta_s^{\mu_s+1} ds + \int_{t\wedge\tau_n}^{T\wedge\tau_n} \frac{\mu_s}{\mu_s+1}|Y_s|^{\mu_s+1} ds \\ &\le \int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)^{\mu_s}\vartheta_s^{\mu_s+1} ds + \int_{t\wedge\tau_n}^{T\wedge\tau_n} |Y_s|^{\mu_s+1} ds \\ &\le (\mu_T+1)^{\mu_T}\int_0^T \vartheta_s^{\mu_s+1} ds + \int_0^T |Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>e\}} ds + Te^{\mu_T+1} \\ &\le (\mu_T+1)^{\mu_T}\int_0^T \vartheta_s^{\mu_s+1} ds + \int_0^T |Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}} ds + Te^{\mu_T+1}. \end{aligned}$$

Estimation of I_2 : Due to the presence of $|\ln|y||$, we split the integral of I_2 into two parts :

$$\begin{aligned} I_2 &\le c_2\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s}(-|Y_s|\ \ln|Y_s|)\ \mathbb{1}_{\{|Y_s|\le 1\}} ds \\ &\quad + c_2\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}} ds \\ &\le c_2 e^{-1}\int_0^T (\mu_s+1)ds + c_2\int_{t\wedge\tau_n}^{T\wedge\tau_n} (\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}} ds. \end{aligned}$$

Estimation of I_3 : Using Lemma 2.2.3, there exists a constant $c_3 > 0$ such that

$$c_0|y||z|\sqrt{|\ln|z||}\ \mathbb{1}_{\{|y|>e\}} \le \frac{1}{4}|z|^2\ \mathbb{1}_{\{|y|>e\}} + c_3|y|^2\ln|y|\ \mathbb{1}_{\{|y|>e\}}.$$

We have

$$|z|\sqrt{|\ln|z||} \le e^{-\frac{1}{2}}\frac{1}{\sqrt{2}} + |z|^{\frac{3}{2}}\ \mathbb{1}_{\{|z|>1\}} \tag{2.2.4}$$

Thus,

$$c_0|y||z|\sqrt{|\ln|z||}\ \mathbb{1}_{\{|y|\leq e\}} \leq c_0 e^{\frac{1}{2}}\frac{1}{\sqrt{2}} + c_0 e|z|^{\frac{3}{2}}\ \mathbb{1}_{\{|z|>1\}}\ \mathbb{1}_{\{|y|\leq e\}}$$
$$\leq \frac{1}{4}|z|^2\ \mathbb{1}_{\{|y|\leq e\}} + \widetilde{c}_0,$$

where the last inequality is obtained by Young's inequality (for $p=\frac{4}{3}$ and $q=4$) and $\widetilde{c}_0 = c_0 e^{\frac{1}{2}}\frac{1}{\sqrt{2}} + 3^3\frac{(c_0 e)^4}{4}$. Therefore,

$$I_3 \leq \widehat{C}_1 + \frac{1}{4}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Z_s|^2|Y_s|^{\mu_s-1}ds + c_3\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>e\}}ds$$
$$\leq \widehat{C}_1 + \frac{1}{4}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Z_s|^2|Y_s|^{\mu_s-1}ds + c_3\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}}ds,$$

where $\widehat{C}_1 = \widetilde{c}_0(\frac{\mu_T-1}{\theta}+T)e^{\mu_T-1}$.

Estimation of I_4 : We observe that we can derive for any small $\rho\in(0,\frac{2}{3^{\mu_T}}]$

$$c_1|y|\|u\|_\nu \leq c_1^2\frac{1}{\rho}|y|^2 + \frac{\rho}{4}\|u\|_\nu^2$$
$$\leq c_1^2\frac{1}{\rho}e^2 + c_1^2\frac{1}{\rho}|y|^2\ln|y|\ \mathbb{1}_{\{|y|>e\}} + \frac{\rho}{4}\|u\|_\nu^2;$$

therefore,

$$I_4 \leq \widehat{C}_2 + \frac{c_1^2}{\rho}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>e\}}ds + \frac{\rho}{4}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s-1}\|U_s\|_\nu^2ds$$
$$\leq \widehat{C}_2 + \frac{c_1^2}{\rho}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}}ds + \frac{\rho}{4}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s-1}\|U_s\|_\nu^2ds,$$

where $\widehat{C}_2 = \frac{c_1^2}{\rho}(\frac{\mu_T-1}{\theta}+T)e^{\mu_T+1}$. It remains to estimate

$$I_5 := -\int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma\Big(|Y_s+U_s(e)|^{\mu_s+1} - |Y_s|^{\mu_s+1} - (\mu_s+1)\,|Y_s|^{\mu_s}\,\mathrm{sgn}(Y_s)U_s(e)\Big)\nu(de)ds.$$

By Taylor's formula and Lemma 2.2.4, we have

$$|y+u|^{\mu_s+1} - |y|^{\mu_s+1} - (\mu_s+1)|y|^{\mu_s}\mathrm{sgn}(y)u$$
$$= \mu_s(\mu_s+1)u^2\int_0^1(1-a)\,|y+au|^{\mu_s-1}\,da \geq \mu_s(\mu_s+1)u^2 3^{-\mu_s}|y|^{\mu_s-1}.$$

Therefore,

$$I_5 \leq -\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s(\mu_s+1)3^{-\mu_s}|Y_s|^{\mu_s-1}\int_\Gamma|U_s(e)|^2\nu(de)ds$$
$$= -\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s(\mu_s+1)3^{-\mu_s}|Y_s|^{\mu_s-1}\|U_s\|_\nu^2ds.$$

Since $3^{-\mu_s} \geq 3^{-\mu_T}$ and $\mu_s \geq 1$, then $\frac{\rho}{2} \leq \mu_s 3^{-\mu_s}$, which implies that

$$\begin{aligned}I_4+I_5 \leq& \hat{C}_2+\frac{c_1^2}{\rho}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}}ds\\ &-\frac{1}{2}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s(\mu_s+1)3^{-\mu_s}|Y_s|^{\mu_s-1}\|U_s\|_\nu^2ds.\end{aligned}$$

and

$$\frac{1}{4}\int_{t\wedge\tau_n}^{T\wedge\tau_n}(\mu_s+1)(1-\mu_s)|Y_s|^{\mu_s-1}|Z_s|^2ds \leq 0.$$

Moreover, for $\theta \geq 2(\frac{c_1^2}{\rho}+c_2+c_3)+1$, we have $1+(\mu_s+1)(\frac{c_1^2}{\rho}+c_2+c_3-\theta\mu_s) \leq 0$, which yields to

$$\begin{aligned}&-\theta\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s|Y_s|^{\mu_s+1}\ln|Y_s|ds+\int_{t\wedge\tau_n}^{T\wedge\tau_n}\Big(1+(\mu_s+1)\Big(\frac{c_1^2}{\rho}+c_2+c_3\Big)\Big)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}}ds\\ &=\int_{t\wedge\tau_n}^{T\wedge\tau_n}\Big(1+(\mu_s+1)(\frac{c_1^2}{\rho}+c_2+c_3-\theta\mu_s)\Big)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}}ds\\ &+\theta\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s|Y_s|^{\mu_s+1}(-\ln|Y_s|)\ \mathbb{1}_{\{|Y_s|\leq 1\}}ds\\ &\leq \theta\sup_{0<a\leq 1}a(-\ln(a))\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s ds\ =\ \theta e^{-1}\int_0^T\mu_s ds.\end{aligned}$$

By Equation (2.2.3) and the preceding result, and noting that for any $0 \leq s \leq T$, $3^{-\mu_T} \leq 3^{-\mu_s}$, it becomes evident that

$$\begin{aligned}&|Y_{t\wedge\tau_n}|^{\mu_{t\wedge\tau_n}+1}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s(\mu_s+1)|Y_s|^{\mu_s-1}(\frac{1}{4}|Z_s|^2+\frac{3^{-\mu_T}}{2}\|U_s\|_\nu^2)ds\\ \leq&|Y_{T\wedge\tau_n}|^{\mu_{T\wedge\tau_n}+1}+(\mu_T+1)^{\mu_T}\int_0^T\vartheta_s^{\mu_s+1}ds-\Xi_{T\wedge\tau_n}+\Xi_{t\wedge\tau_n}+\hat{C}+C_1.\end{aligned}\tag{2.2.5}$$

where $C_1 = 2e^{-1}(\mu_T-1)+c_2Te^{-1}$ and $\hat{C} = \hat{C}_1+\hat{C}_2+Te^{\mu_T+1}$. Thus, we obtain

$$\begin{aligned}&\mathbb{E}\Big[|Y_{t\wedge\tau_n}|^{\mu_{t\wedge\tau_n}+1}+\int_{t\wedge\tau_n}^{T\wedge\tau_n}\mu_s(\mu_s+1)|Y_s|^{\mu_s-1}(|Z_s|^2+\|U_s\|_\nu^2)ds\Big]\\ \leq& C\mathbb{E}\Big[1+|Y_{T\wedge\tau_n}|^{\mu_{T\wedge\tau_n}+1}+(\mu_T+1)^{\mu_T}\int_0^T\vartheta_s^{\mu_s+1}ds\Big].\end{aligned}$$

By Fatou's lemma, we can pass to the limit as $n \to \infty$. Consequently, the desired result follows. □

Lemma 2.2.6. *Let (**A.1**), (**A.2**)-(ii) be satisfied. Then, there exists a positive constant* $C(T,\alpha,c_0,c_1,c_2)$ *such that*

$$\int_0^T\mathbb{E}[|f(s,Y_s,Z_s,U_s)|^{\frac{2}{\alpha}}]ds \leq \widetilde{K}_1,$$

where $1<\alpha<2$, *and*

$$\widetilde{K}_1 := C(T,\alpha,c_0,c_1,c_2)\left(1+\int_0^T \mathbb{E}\big[\vartheta_s^2+|Y_s|^{\mu_s+1}+|Z_s|^2+\|U_s\|_\nu^2\big]ds\right).$$

Proof: Letting $\alpha\in(1,2)$, we have

$$\begin{aligned}|y||\ln|y|| &\le e^{-1}+|y|\ln|y|\ \mathbb{1}_{\{|y|>1\}}\\ &= e^{-1}+\frac{1}{\alpha-1}|y|\ln|y|^{\alpha-1}\ \mathbb{1}_{\{|y|>1\}}\\ &\le e^{-1}+\frac{1}{\alpha-1}|y|^{\alpha}\ \mathbb{1}_{\{|y|>1\}},\end{aligned}$$

$$\begin{aligned}|z|\sqrt{|\ln|z||} &\le \frac{e^{-\frac{1}{2}}}{\sqrt{2}}+|z|\sqrt{|\ln|z||}\ \mathbb{1}_{\{|z|>1\}}\\ &= \frac{e^{-\frac{1}{2}}}{\sqrt{2}}+\frac{1}{\sqrt{2(\alpha-1)}}|z|\sqrt{\ln|z|^{2(\alpha-1)}}\ \mathbb{1}_{\{|z|>1\}}\\ &\le \frac{e^{-\frac{1}{2}}}{\sqrt{2}}+\frac{1}{\sqrt{2(\alpha-1)}}|z|^{\alpha}\ \mathbb{1}_{\{|z|>1\}},\end{aligned}$$

and

$$\vartheta_t+c_1\|u\|_\nu \le 1+c_1+\vartheta_t^{\alpha}+c_1\|u\|_\nu^{\alpha}.$$

Therefore, by (**A.2**)-(ii),

$$\begin{aligned}|f(s,\omega,y,z,u)| &\le \vartheta_s+c_2|y||\ln|y||+c_0|z|\sqrt{|\ln|z||}+c_1\|u\|_\nu\\ &\le \widetilde{c}(1+\vartheta_s^{\alpha}+|y|^{\alpha}+|z|^{\alpha}+\|u\|_\nu^{\alpha}),\end{aligned}$$

where $\widetilde{c}$ is a positive constant depending on c_0,c_1,c_2, and α. For any $p\ge 1$, $n\in\mathbb{N}$ with $n\ge 2$ and $(b_i)_{i\in\mathbb{N}}\in\mathbb{R}_+$, we have

$$(\sum_{i=1}^n b_i)^p \le n^{p-1}\sum_{i=1}^n b_i^p.$$

Thus,

$$\begin{aligned}|f(s,\omega,y,z,u)|^{\frac{2}{\alpha}} &\le \widetilde{c}^{\frac{2}{\alpha}}(1+\vartheta_s^{\alpha}+|y|^{\alpha}+|z|^{\alpha}+\|u\|_\nu^{\alpha})^{\frac{2}{\alpha}}\\ &\le \widetilde{c}^{\frac{2}{\alpha}}5^{\frac{2-\alpha}{\alpha}}(1+\vartheta_s^2+|y|^2+|z|^2+\|u\|_\nu^2).\end{aligned}$$

Since $|y|^2\le 1+|y|^{\mu_s+1}$, we can derive a positive constant $C(T,\alpha,c_0,c_1,c_2)$, such that

$$\int_0^T \mathbb{E}\big[|f(s,Y_s,Z_s,U_s)|^{\frac{2}{\alpha}}\big]ds \le C(T,\alpha,c_0,c_1,c_2)\left(1+\int_0^T \mathbb{E}\big[\vartheta_s^2+|Y_s|^{\mu_s+1}+|Z_s|^2+\|U_s\|_\nu^2\big]ds\right).$$

□

2.2.2 A Priori Estimates

This subsection aims to give some prior estimates for the solutions of BSDEJ (2.1.1). These estimates establish bounds on the solutions, ensuring that if the solutions exist, they will belong to some appropriate spaces.

Lemma 2.2.7. *Consider a solution (Y,Z,U) to the BSDEJ (2.1.1). Additionally, assume that the pair (ζ,f) satisfies conditions (**A.1**) and (**A.2**). In this context, we establish the existence of a universal constant $C(T,c_0,c_1,c_2)$, as follows :*

(i) $\mathbb{E}\Big[\sup_{t\in[0,T]}|Y_t|^{\mu_t+1}\Big] \leq \widetilde{K}_2.$

(ii) $\int_0^T \mathbb{E}\big[|Z_s|^2+\|U_s\|_\nu^2\big]ds \leq \widetilde{K}_3,$

where

$$\widetilde{K}_2 := C\,(T,c_0,c_1,c_2)\Big(1+\mathbb{E}[|\zeta|^{\mu_T+1}]+\int_0^T \mathbb{E}[\vartheta_s^{\mu_s+1}]ds\Big),$$

$$\widetilde{K}_3 := C\,(T,c_0,c_1,c_2)\Big(1+T\widetilde{K}_2+\mathbb{E}[|\zeta|^2]+\int_0^T \mathbb{E}[\vartheta_s^2]ds\Big).$$

Proof:

We begin by proving assertion (i), which relies on Lemma 2.2.5.

For $n\geq 0$, define the stopping time $\widetilde{\tau}_n$ as follows :

$$\widetilde{\tau}_n := \inf\{s\geq 0\ :\ |Y_s|^{\mu_s+1}>n\}.$$

By taking the same steps as in the previous proof of Lemma 2.2.5, we obtain the inequality (2.2.5) for $\widetilde{\tau}_n$

$$\begin{aligned}|Y_{t\wedge\widetilde{\tau}_n}|^{\mu_{t\wedge\widetilde{\tau}_n}+1}&+\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n}\mu_s(\mu_s+1)|Y_s|^{\mu_s-1}\big(\frac{1}{4}|Z_s|^2+\frac{3^{-\mu_T}}{2}\|U_s\|_\nu^2\big)ds\\ \leq |Y_{T\wedge\widetilde{\tau}_n}|^{\mu_{T\wedge\widetilde{\tau}_n}+1}&+(\mu_T+1)^{\mu_T}\int_0^T\vartheta_s^{\mu_s+1}ds-\Xi_{T\wedge\widetilde{\tau}_n}+\Xi_{t\wedge\widetilde{\tau}_n}+C,\end{aligned}$$

where C is a generic positive constant that may vary. Thus, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\leq t\leq T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}\Big]\leq& C\Big(1+\mathbb{E}\Big[|Y_{T\wedge\widetilde{\tau}_n}|^{\mu_T+1}+(\mu_T+1)^{\mu_T}\int_0^T\vartheta_s^{\mu_s+1}ds\Big]\Big)\\&+\mathbb{E}\Big[\sup_{0\leq t\leq T\wedge\widetilde{\tau}_n}\Big|\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n}d\Xi_s\Big|\Big].\end{aligned}\tag{2.2.6}$$

Consider the following inequality, which holds for any non-negative a, $b\geq 0$ and $p>1$,

$$|a^p-b^p|\leq p(a\vee b)^{p-1}|a-b|.$$

Therefore,

$$||Y_{s-}+U_s(e)|^{\mu_s+1}-|Y_{s-}|^{\mu_s+1}| \leq (\mu_s+1)(|Y_{s-}+U_s(e)|\vee|Y_{s-}|)^{\mu_s}|U_s(e)|,$$

clearly, $\sup_{0\leq t\leq T\wedge\widetilde{\tau}_n}|Y_{t-}|^{\mu_t+1}\leq \sup_{0\leq t\leq T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}$ and since $Y_s=Y_{s-}+U_s(e)$, then,

$$\begin{aligned}
&||Y_{s-}+U_s(e)|^{\mu_s+1}-|Y_{s-}|^{\mu_s+1}|^2\\
&\leq (\mu_s+1)^2(|Y_{s-}+U_s(e)|\vee|Y_{s-}|)^{2\mu_s}|U_s(e)|^2\\
&\leq (\mu_s+1)^2\sup_{0\leq t\leq T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}(|Y_{s-}+U_s(e)|\vee|Y_{s-}|)^{\mu_s-1}|U_s(e)|^2,
\end{aligned}$$

Moreover, we have $(\mu_s+1)^2<3\mu_s(\mu_s+1)$. Applying Burkholder–Davis–Gundy inequality to $\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} d\Xi_s$, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq t\leq T\wedge\widetilde{\tau}_n}\Big|\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} d\Xi_s\Big|\Big]\\
&\leq C\mathbb{E}\Big[\Big(\int_0^{T\wedge\widetilde{\tau}_n}(\mu_s+1)^2|Y_s|^{2\mu_s}|Z_s|^2ds\Big)^{\frac{1}{2}}\Big]\\
&\quad +C\mathbb{E}\Big[\Big(\int_0^{T\wedge\widetilde{\tau}_n}\int_\Gamma\Big(|Y_{s-}+U_s(e)|^{\mu_s+1}-|Y_{s-}|^{\mu_s+1}\Big)^2N(ds,de)\Big)^{\frac{1}{2}}\Big]\\
&\leq C\mathbb{E}\Big[\sup_{0\leq t\leq T\wedge\widetilde{\tau}_n}|Y_t|^{\frac{\mu_t+1}{2}}\Big(\int_0^{T\wedge\widetilde{\tau}_n}(\mu_s+1)^2|Y_s|^{\mu_s-1}|Z_s|^2ds\Big)^{\frac{1}{2}}\Big]\\
&\quad +C\mathbb{E}\Big[\sup_{0\leq t\leq T\wedge\widetilde{\tau}_n}|Y_t|^{\frac{\mu_t+1}{2}}\Big(\int_0^{T\wedge\widetilde{\tau}_n}\int_\Gamma(\mu_s+1)^2(|Y_{s-}+U_s(e)|\vee|Y_{s-}|)^{\mu_s-1}|U_s(e)|^2N(ds,de)\Big)^{\frac{1}{2}}\Big]\\
&\leq \mathbb{E}\Big[\frac{1}{2}\sup_{0\leq t\leq T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}+C\int_0^T(\mu_s+1)^2|Y_s|^{\mu_s-1}|Z_s|^2ds\Big]\\
&\quad +C\mathbb{E}\Big[\int_0^T\int_\Gamma(\mu_s+1)^2(|Y_{s-}+U_s(e)|\vee|Y_{s-}|)^{\mu_s-1}|U_s(e)|^2N(ds,de)\Big]
\end{aligned}$$

The last inequality is derived from Young's inequality ($ab\leq\frac{1}{2}a^2+\frac{1}{2}b^2$), and the terms can be controlled as follows :

$$
\begin{aligned}
&= \mathbb{E}\Big[\frac{1}{2}\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}+C\int_0^T(\mu_s+1)^2|Y_s|^{\mu_s-1}|Z_s|^2ds\Big]\\
&\quad +C\mathbb{E}\Big[\int_0^T\int_\Gamma(\mu_s+1)^2|Y_s|^{\mu_s-1}|U_s(e)|^2\nu(de)ds\Big]\\
&= \mathbb{E}\Big[\frac{1}{2}\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}+C\int_0^T(\mu_s+1)^2|Y_s|^{\mu_s-1}|Z_s|^2ds\Big]\\
&\quad +C\mathbb{E}\Big[\int_0^T(\mu_s+1)^2|Y_s|^{\mu_s-1}\|U_s\|_\nu^2ds\Big]\\
&\le \mathbb{E}\Big[\frac{1}{2}\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}+3C\int_0^T\mu_s(\mu_s+1)|Y_s|^{\mu_s-1}|Z_s|^2ds\Big]\\
&\quad +3C\mathbb{E}\Big[\int_0^T\mu_s(\mu_s+1)|Y_s|^{\mu_s-1}\|U_s\|_\nu^2ds\Big]\\
&\le \frac{1}{2}\mathbb{E}\Big[\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}\Big]+C\mathbb{E}\Big[1+|\zeta|^{\mu_T+1}+(\mu_T+1)^{\mu_T}\int_0^T\vartheta_s^{\mu_s+1}ds\Big],
\end{aligned}
$$

the last inequality is derived from Lemma 2.2.5. Observing that for any $n\ge 0$ we have $\widetilde{\tau}_n\le\widetilde{\tau}_{n+1}$, then $\sup_{0\le t\le T\wedge\widetilde{\tau}_n}|Y_t|^{\mu_t+1}\le\sup_{0\le t\le T\wedge\widetilde{\tau}_{n+1}}|Y_t|^{\mu_t+1}$. Consequently, by (2.2.6) and by using the monotone convergence theorem, we obtain

$$
\mathbb{E}[\sup_{0\le t\le T}|Y_t|^{\mu_t+1}]\le C\Big(1+\mathbb{E}[|\zeta|^{\mu_T+1}]+(\mu_T+1)^{\mu_T}\int_0^T\mathbb{E}[\vartheta_s^{\mu_s+1}]ds\Big).
$$

This ends the proof of assertion (i).

We now advance to establish assertion (ii). The application of Itô's formula reveals that

$$
\begin{aligned}
|Y_0|^2+\int_0^T(|Z_s|^2+\|U_s\|_\nu^2)ds+\Xi_T &= |\zeta|^2+2\int_0^T Y_sf(s,Y_s,Z_s,U_s)ds\\
&\le |\zeta|^2+2\int_0^T|Y_s|\left(\vartheta_s+g_{1,c_2}(Y_s)\right)ds\\
&\quad +2\int_0^T|Y_s|\big(g_{2,c_0}(Z_s)+c_1\|U_s\|_\nu\big)ds,
\end{aligned}
$$

where $\Xi_t=2\int_0^tY_sZ_sdW_s+\int_0^t\int_\Gamma\left(2Y_{s-}U_s(e)+|U_s(e)|^2\right)\widetilde{N}(ds,de)$

For any given $\varepsilon>0$, we have

$$
\begin{aligned}
|y|^2|\ln|y||&\le -|y|\ln|y|\ \mathbb{1}_{\{|y\le 1\}}+|y|^{2+\varepsilon}\ \mathbb{1}_{\{|y|>1\}}\\
&\le e^{-1}+|y|^{2+\varepsilon},
\end{aligned}
$$

and

$$
|y|^2 \quad\le\quad |y|^{2+\varepsilon}\ \mathbb{1}_{\{|y|>1\}}+1.
$$

Furthermore, by Lemma 2.2.3 and employing Young's inequality, we can derive a positive constant $\widetilde{c}$, such that

$$2c_0|y||z|\sqrt{|\ln|z||}\ \mathbb{1}_{\{|y|>e\}} \leq \left(\frac{|z|^2}{2}+\widetilde{c}|y|^{2+\varepsilon}\right)\ \mathbb{1}_{\{|y|>e\}}.$$

On the other hand, according to (2.2.4)

$$\begin{aligned}2c_0|y||z|\sqrt{|\ln|z||}\ \mathbb{1}_{\{|y|\leq e\}} &\leq 2c_0e^{\frac{1}{2}}\frac{1}{\sqrt{2}}+2c_0e|z|^{\frac{3}{2}}\ \mathbb{1}_{\{|z|>1\}}\ \mathbb{1}_{\{|y|\leq e\}}\\ &\leq \frac{1}{2}|z|^2\ \mathbb{1}_{\{|y|\leq e\}}+\widetilde{c}_0,\end{aligned}$$

where $\widetilde{c}_0 = c_0\sqrt{2}e^{\frac{1}{2}}+4(c_0e)^4\left(\frac{3}{2}\right)^3$. By Young's inequality, we have

$$2c_1|y|\|u\|_\nu\ \mathbb{1}_{\{|y|>1\}} \leq \left(\frac{\|u\|_\nu^2}{2}+2c_1^2|y|^{2+\varepsilon}\right)\ \mathbb{1}_{\{|y|>1\}},$$

$$2c_1|y|\|u\|_\nu\ \mathbb{1}_{\{|y|\leq 1\}} \leq \frac{\|u\|_\nu^2}{2}\ \mathbb{1}_{\{|y|\leq 1\}}+2c_1^2.$$

and

$$2|y|\vartheta \leq \vartheta^2+|y|^{2+\varepsilon}\ \mathbb{1}_{\{|y|>1\}}+1.$$

Therefore,

$$\begin{aligned}\int_0^T \mathbb{E}\big[|Z_s|^2+\|U_s\|_\nu^2\big]ds \ &\leq\ \widetilde{C}(T,c_0,c_1,c_2)\Big(\widehat{C}+\mathbb{E}\Big[|\zeta|^2+\int_0^T\vartheta_s^2ds+\int_0^T|Y_s|^{2+\varepsilon}ds\Big]\Big)\\ &\leq\ \widetilde{C}(T,c_0,c_1,c_2)\Big(\widehat{C}+\mathbb{E}\Big[|\zeta|^2+\int_0^T\vartheta_s^2ds+T\sup_{0\leq t\leq T}|Y_t|^{2+\varepsilon}\Big]\Big).\end{aligned}$$

By selecting ε as μ_s-1, setting $t=0$, and defining $C(T,c_0,c_1,c_2)=\widetilde{C}(T,c_0,c_1,c_2)(\widehat{C}\vee 1)$, we obtain the desired result.

□

The first lemma that follows allows for a localization procedure introduced to establish solutions' existence and uniqueness. The second one provides a prior estimate for the approximating solutions and guarantees that these solutions do not diverge. The proofs for these lemmas can be performed and adapted to our setting similarly as outlined in [14].

Lemma 2.2.8. *There exists (f_n), a sequence of functions, satisfying :*

(i) For every n, the functions f_n are bounded and exhibit global Lipschitz continuity with respect to (y,z,u) for a.e. t and $\mathbb{P}$ -a.s.

(ii) $\sup_n|f_n(t,\omega,y,z,u)| \leq \vartheta_t+g_{1,c_2}(y)+g_{2,c_0}(z)+c_1\|u\|_\nu$.

(iii) For each N, $\rho_N(f_n - f) \longrightarrow 0$ as $n \longrightarrow \infty$, where

$$\rho_N(f) = \mathbb{E}\Big[\int_0^T \sup_{|y|,|z|,\|u\|_\nu \leq N} |f(s,y,z,u)| ds\Big].$$

Lemma 2.2.9. *Consider f and ζ as defined in Lemma 2.2.7. Let (f_n) denote the sequence of functions associated with f by Lemma 2.2.8. Let (Y^n, Z^n, U^n) represent the solution to the BSDEJ(ζ, f_n). Consequently, we have :*

(a) $\sup_n \mathbb{E}[\int_0^T \|U_s^n\|_\nu^2 ds] \leq K_1$.
(b) $\sup_n \mathbb{E}[\sup_{0\leq t\leq T} |Y_t^n|^{\mu_T+1}] \leq K_2$.
(c) $\sup_n \mathbb{E}[\int_0^T |Z_s^n|^2 ds] \leq K_3$.
(d) $\sup_n \mathbb{E}[\int_0^T |f_n(s, Y_s^n, Z_s^n, U_s^n)|^{\frac{2}{\alpha}} ds] \leq K_4$.

where K_1, K_2, K_3, and K_4 are constants independent of n.

2.2.3 Some Convergence Results

This subsection establishes estimates between two potential solutions. This analysis is essential for demonstrating the existence of solutions and understanding the properties of these solutions in the context of the study on one-dimensional BSDEs with logarithmic growth. Moving forward, we use the notation $\hat{h}_s^{n,m}$ to represent the difference between h_s^n and h_s^m for any given quantities.

Proposition 2.2.10. *For every $R \in \mathbb{N}$, $\beta \in (1, 3-\alpha)$, $0 < \delta < \frac{\beta-1}{2M_2^2 + C_{Lip}^2} \min(\frac{1}{2}, \frac{\kappa}{r\beta})$ and $\varepsilon > 0$, there exists $N_0 > R$ such that for all $N > N_0$ and $S \leq T$:*

$$\limsup_{n,m\to+\infty} \mathbb{E}\Big[\sup_{(S-\delta)^+\leq t\leq S} |\hat{Y}_t^{n,m}|^\beta + \int_{(S-\delta)^+}^S \frac{(|\hat{Z}_s^{n,m}|^2 + \|\hat{U}_s^{n,m}\|_\nu^2)}{(|\hat{Y}_s^{n,m}|^2 + \Lambda_R)^{\frac{2-\beta}{2}}} ds\Big]$$
$$\leq \varepsilon + \frac{\ell}{\beta - 1} e^{C_N \delta} \limsup_{n,m\to+\infty} \mathbb{E}\big[|\hat{Y}_S^{n,m}|^\beta\big].$$

Here , $\Lambda_R = \sup\{(A_N)^{-1}, N \geq R\}$, $C_N := \frac{\beta}{\beta-1}(2M_2^2 + C_{Lip}^2)\ln(A_N)$, and ℓ is a positive constant. The definition of κ can be found below.

Lemma 2.2.11. *Assuming that the conditions of Proposition 2.2.10 are met, and defining φ_t as $|\hat{Y}_t^{n,m}|^2 + (A_N)^{-1}$, and $\kappa := 3 - \alpha - \beta$, we can establish the following result for any $C > 0$:*

$$e^{Ct}\varphi_t^{\frac{\beta}{2}} + C\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}} ds + \widetilde{\mathbb{M}}_t \leq e^{CS}\varphi_S^{\frac{\beta}{2}} - \frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds$$
$$-\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2 ds$$
$$+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}\ |\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds$$
$$+\mathbb{M}_t + J_{1,t} + J_{2,t} + J_{3,t} + J_{4,t},$$

where

$$\begin{aligned}
\widetilde{\mathbb{M}}_t &:= \int_t^S \int_\Gamma e^{Cs}(\varphi_s^{\frac{\beta}{2}} - \varphi_{s-}^{\frac{\beta}{2}})\widetilde{N}(ds,de), \\
\mathbb{M}_t &:= -\beta \int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}\hat{Z}_s^{n,m} dW_s, \\
J_{1,t} &:= \beta e^{CS}\frac{1}{N^\kappa}\int_t^S \varphi_s^{\frac{\beta-1}{2}}\Phi^\kappa(s)|f_n(s,Y_s^n,Z_s^n,U_s^n) - f_m(s,Y_s^m,Z_s^m,U_s^n)|ds, \\
J_{2,t} &:= \beta e^{CS}[4N^2 + A_1]^{\frac{\beta-1}{2}}\left[\int_t^S \sup_{|y|,|z|,\|u\|_\nu \le N} |(f_n - f)(s,y,z,u)|ds \right. \\
&\qquad \left. + \int_t^S \sup_{|y|,|z|,\|u\|_\nu \le N} |(f_m - f)(s,y,z,u)|ds\right], \\
J_{3,t} &:= \beta M_2 \int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\left(\varphi_s \ln(A_N) + \sqrt{\ln(A_N)}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\right) ds, \\
J_{4,t} &:= \beta C_{Lip}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}|\|\hat{U}_s^{n,m}\|_\nu ds,
\end{aligned}$$

and $\Phi(s) = |Y_s^n| + |Y_s^m| + |Z_s^n| + |Z_s^m| + \|U_s^n\|_\nu + \|U_s^m\|_\nu$.

Proof: Let $C > 0$. For any positive integer N, we define the function $u(s,y)$ as

$$u(s,y) = e^{Cs}(\theta(y))^{\frac{\beta}{2}},$$

where $\theta(y) := y^2 + (A_N)^{-1}$; this yields the following partial derivatives :

$$\begin{aligned}
u_s(s,y) = Cu(s,y); \quad u_y(s,y) = \beta e^{Cs} y(\theta(y))^{\frac{\beta}{2}-1}, \\
u_{yy}(s,y) = \beta e^{Cs}(\theta(y))^{\frac{\beta}{2}-1} + \beta(\beta-2)e^{Cs}y^2(\theta(y))^{\frac{\beta}{2}-2}.
\end{aligned}$$

Since $1 < \beta < 2$, we can establish that

$$u_{yy}(s,y) \ge \beta(\beta-1)e^{Cs}(\theta(y))^{\frac{\beta}{2}-1}.$$

Consequently, for all $s \in [0,T]$, we obtain, by Taylor expansion, that

$$\begin{aligned}
&u(s,\hat{Y}_s^{n,m}) - u(s,\hat{Y}_{s-}^{n,m}) - \hat{U}_s^{n,m}(e)u_y(s,\hat{Y}_{s-}^{n,m}) \\
&= |\hat{U}_s^{n,m}(e)|^2 \int_0^1 (1-a)u_{yy}(s, a\hat{U}_s^{n,m}(e) + \hat{Y}_{s-}^{n,m})da \\
&\ge \beta(\beta-1)e^{Cs}|\hat{U}_s^{n,m}(e)|^2 \int_0^1 (1-a)\left(\theta(a\hat{U}_s^{n,m}(e) + \hat{Y}_{s-}^{n,m})\right)^{\frac{\beta}{2}-1} da.
\end{aligned}$$

Since $0 \le a \le 1$, we have

$$\begin{aligned}
\theta(a\hat{U}_s^{n,m}(e) + \hat{Y}_{s-}) &= |a\hat{U}_s^{n,m}(e) + \hat{Y}_{s-}^{n,m}|^2 + (A_N)^{-1} \\
&= |a(\hat{Y}_{s-}^{n,m} + \hat{U}_s^{n,m}(e)) + (1-a)\hat{Y}_{s-}^{n,m}|^2 + (A_N)^{-1} \\
&\le (|\hat{Y}_{s-}^{n,m}| \vee |\hat{Y}_s^{n,m}|)^2 + (A_N)^{-1}.
\end{aligned}$$

Given that $\frac{\beta}{2}-1$ is negative, hence

$$\left(\theta(a\hat{U}_s^{n,m}(e)+\hat{Y}_{s-}^{n,m})\right)^{\frac{\beta}{2}-1} \geq \left((|\hat{Y}_{s-}^{n,m}|\vee|\hat{Y}_s^{n,m}|)^2+(A_N)^{-1}\right)^{\frac{\beta}{2}-1}.$$

Therefore,

$$\begin{aligned}
&u(s,\hat{Y}_s^{n,m})-u(s,\hat{Y}_{s-}^{n,m})-\hat{U}_s^{n,m}(e)u_y(s,\hat{Y}_{s-}^{n,m}) \\
\geq&\beta(\beta-1)e^{Cs}|\hat{U}_s^{n,m}(e)|^2\int_0^1(1-a)\left(\theta(a\hat{U}_s^{n,m}(e)+\hat{Y}_{s-}^{n,m})\right)^{\frac{\beta}{2}-1}da \\
\geq&\beta\frac{(\beta-1)}{2}e^{Cs}|\hat{U}_s^{n,m}(e)|^2\left((|\hat{Y}_{s-}^{n,m}|\vee|\hat{Y}_s^{n,m}|)^2+(A_N)^{-1}\right)^{\frac{\beta}{2}-1}.
\end{aligned} \tag{2.2.7}$$

Applying Itô's formula to $u(t,Y_t)$ reveals that

$$\begin{aligned}
&e^{Ct}\varphi_t^{\frac{\beta}{2}}+C\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds \\
=&e^{CS}\varphi_S^{\frac{\beta}{2}}+\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}\left(f_n(s,Y_s^n,Z_s^n,U_s^n)-f_m(s,Y_s^m,Z_s^m,U_s^m)\right)ds \\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2ds \\
&-\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}\hat{Z}_s^{n,m}dW_s \\
&-\int_t^S e^{Cs}\int_\Gamma\left(\left(|\hat{Y}_{s-}^{n,m}+\hat{U}_s^{n,m}(e)|^2+(A_N)^{-1}\right)^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}}-\beta\varphi_{s-}^{\frac{\beta}{2}-1}\hat{Y}_{s-}^{n,m}\hat{U}_s^{n,m}(e)\right)N(ds,de) \\
&-\beta\int_t^S e^{Cs}\int_\Gamma\varphi_{s-}^{\frac{\beta}{2}-1}\hat{Y}_{s-}^{n,m}\hat{U}_s^{n,m}(e)\tilde{N}(ds,de).
\end{aligned}$$

By (2.2.7), we can reformulate the jump components as follows :

$$\begin{aligned}
&-\beta\int_t^S\int_\Gamma e^{Cs}\varphi_{s-}^{\frac{\beta}{2}-1}\hat{Y}_{s-}^{n,m}\hat{U}_s^{n,m}(e)\tilde{N}(ds,de) \\
&-\int_t^S\int_\Gamma e^{Cs}\left(\varphi_s^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}}-\beta\varphi_{s-}^{\frac{\beta}{2}-1}\hat{Y}_{s-}^{n,m}\hat{U}_s^{n,m}(e)\right)N(ds,de) \\
=&-\int_t^S\int_\Gamma e^{Cs}\left(\varphi_s^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}}-\beta\varphi_{s-}^{\frac{\beta}{2}-1}\hat{Y}_{s-}^{n,m}\hat{U}_s^{n,m}(e)\right)\nu(de)ds \\
&-\int_t^S\int_\Gamma e^{Cs}(\varphi_s^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}})\tilde{N}(ds,de) \\
\leq&-\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\|\hat{U}_s^{n,m}\|_\nu^2\left((|\hat{Y}_{s-}^{n,m}|\vee|\hat{Y}_s^{n,m}|)^2+(A_N)^{-1}\right)^{\frac{\beta}{2}-1}ds \\
&-\int_t^S\int_\Gamma e^{Cs}(\varphi_s^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}})\tilde{N}(ds,de) \\
\leq&-\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2ds-\int_t^S\int_\Gamma e^{Cs}(\varphi_s^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}})\tilde{N}(ds,de).
\end{aligned}$$

Therefore,

$$e^{Ct}\varphi_t^{\frac{\beta}{2}}+C\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds+\widetilde{\mathbb{M}}_t\leq e^{CS}\varphi_S^{\frac{\beta}{2}}-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds$$
$$-\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2ds$$
$$+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2ds$$
$$+\mathbb{M}_t+\acute{J}_{1,t}+\acute{J}_{2,t}+\acute{J}_{3,t}+\acute{J}_{4,t}+\acute{J}_{5,t},$$

where

$$\widetilde{\mathbb{M}}_t:=\int_t^S\int_\Gamma e^{Cs}(\varphi_s^{\frac{\beta}{2}}-\varphi_{s-}^{\frac{\beta}{2}})\widetilde{N}(ds,de),$$

and

$$\mathbb{M}_t:=-\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}\hat{Z}_s^{n,m}dW_s,$$

are $\mathbb{F}$-martingales, and

$$\acute{J}_{1,t}:=\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}\left(f_n(s,Y_s^n,Z_s^n,U_s^n)-f_m(s,Y_s^m,Z_s^m,U_s^n)\right)\mathbb{1}_{\{\Phi(s)>N\}}ds,$$
$$\acute{J}_{2,t}:=\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}\left(f_n(s,Y_s^n,Z_s^n,U_s^n)-f(s,Y_s^n,Z_s^n,U_s^n)\right)\mathbb{1}_{\{\Phi(s)\leq N\}}ds,$$
$$\acute{J}_{3,t}:=\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}\left(f(s,Y_s^n,Z_s^n,U_s^n)-f(s,Y_s^m,Z_s^m,U_s^n)\right)\mathbb{1}_{\{\Phi(s)\leq N\}}ds,$$
$$\acute{J}_{4,t}:=\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\hat{Y}_s^{n,m}\left(f(s,Y_s^m,Z_s^m,U_s^n)-f_m(s,Y_s^m,Z_s^m,U_s^n)\right)\mathbb{1}_{\{\Phi(s)\leq N\}}ds,$$
$$\acute{J}_{5,t}:=\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||f_m(s,Y_s^m,Z_s^m,U_s^n)-f_m(s,Y_s^m,Z_s^m,U_s^m)|ds,$$

with the shorthand $\Phi(s)=|Y_s^n|+|Y_s^m|+|Z_s^n|+|Z_s^m|+\|U_s^n\|_\nu+\|U_s^m\|_\nu$. By using the fact that $|\hat{Y}_s^{n,m}|\leq\varphi_s^{\frac{1}{2}}$ and $\Phi(s)>N$, a simple computation shows that $\acute{J}_{1,t}\leq J_{1,t}$ and $\acute{J}_{2,t}+\acute{J}_{4,t}\leq J_{2,t}$. Finally, the inequalities $\acute{J}_{3,t}\leq J_{3,t}$ and $\acute{J}_{5,t}\leq J_{4,t}$ can be directly derived from **Assumption** (**A.3**)-(iii) and the Lipschitz condition with respect to u. □

Lemma 2.2.12. *Under **Assumption** of Proposition 2.2.10, we have*

$$-C_{N,1}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2ds$$
$$-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds+J_{3,t}$$
$$\leq-\beta\frac{(\beta-1)}{4}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds.$$

Proof: The expression involving the process $(\hat{Z}_s^{n,m})$ in Proposition 2.2.10

$$-\frac{C_{N,1}}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2ds$$
$$-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds+\beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}ds.$$

We have $|\hat{Y}_s^{n,m}|^2\le\varphi_s:=|\hat{Y}_s^{n,m}|^2+(A_N)^{-1}$, since $\beta\frac{(2-\beta)}{2}>0$, therefore

$$\begin{aligned}
&-\frac{C_{N,1}}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2ds\\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds+\beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}ds\\
\le&-\frac{C_{N,1}}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\varphi_s ds-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds\\
&+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds+\beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}ds\\
=&\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\Big(-\frac{C_{N,1}}{2}\varphi_s-\beta\frac{(\beta-1)}{2}|\hat{Z}_s^{n,m}|^2+\beta M_2|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}\Big)ds.
\end{aligned}$$

If we choose $C_{N,1}:=\beta\frac{2M_2^2}{\beta-1}\ln(A_N)$, then

$$\begin{aligned}
&-\frac{C_{N,1}}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2ds\\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds+\beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}ds\\
\le&\,\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\Big(-\frac{M_2^2}{\beta-1}\varphi_s\ln(A_N)-\frac{(\beta-1)}{2}|\hat{Z}_s^{n,m}|^2+M_2|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}\Big)ds\\
\le&\,\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\Big(-\frac{M_2^2}{\beta-1}\varphi_s\ln(A_N)-\frac{(\beta-1)}{2}|\hat{Z}_s^{n,m}|^2+M_2\sqrt{\varphi_s}|\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}\Big)ds.
\end{aligned}$$

The final inequality is derived from the fact that $|\hat{Y}_s^{n,m}|\le\sqrt{\varphi_s}$. We utilize Young's inequality $(ab\le\frac{a^2}{2\varepsilon}+\frac{\varepsilon b^2}{2})$ by selecting $a=A|y|$, $b=z$, and $\varepsilon=\frac{\beta-1}{2}$.

$$A|y||z|-\frac{1}{\beta-1}A^2|y|^2-\frac{(\beta-1)}{2}|z|^2\le-\frac{\beta-1}{4}|z|^2.$$

For $A:=M_2\sqrt{\ln(A_N)}$, $y:=\sqrt{\varphi_s}$ and $z:=|\hat{Z}_s^{n,m}|$, therefore

$$\begin{aligned}
&-\frac{C_{N,1}}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2ds\\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds+\beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}ds\\
\le&-\beta\frac{(\beta-1)}{4}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds.
\end{aligned}\tag{2.2.8}$$

□

Proof: of Proposition 2.2.10. We define the constant C in Lemma 2.2.11 as $C_N := C_{N,1} + C_{N,2}$, where $C_{N,1} := \frac{2M_2^2\beta}{\beta-1}\ln(A_N)$ and $C_{N,2} := \frac{C_{Lip}^2\beta}{\beta-1}\ln(A_N)$. Additionally, let $\gamma := \delta C_N(\ln(A_N))^{-1}$. We will examine the following quantity :

$$\begin{aligned}&-C_N\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds + \beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2 ds\\&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2 ds - \beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2 ds\\&+J_{3,t}+J_{4,t}.\end{aligned}$$

The control of the expression involving the process $(\hat{Z}_s^{n,m})$ has been postponed in Lemma 2.2.12. We direct our attention to the expression encompassing the norm $\|\hat{U}_s^{n,m}\|_\nu$.

By applying Young's inequality and setting $C_{N,2} = \beta\frac{C_{Lip}^2}{\beta-1}\ln(A_N)$ for sufficiently large A_N (i.e., $A_N \geq e$), we obtain the following result :

$$\begin{aligned}&-C_{N,2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds + \beta C_{Lip}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}|\,\|\hat{U}_s^{n,m}\|_\nu ds\\&-\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2 ds\\\leq&-\beta\frac{\beta-1}{4}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2 ds.\end{aligned}\tag{2.2.9}$$

Based on Lemma 2.2.5 and employing Burkholder–Davis–Gundy's inequality and Hölder's inequality, while taking into account the relationship $\frac{\beta-1}{2}+\frac{\kappa}{2}+\frac{\alpha}{2}=1$ as well as the inequalities (2.2.8) and (2.2.9), we obtain a positive universal constant ℓ such that, for all $\delta > 0$, the following inequality universally holds :

$$\begin{aligned}&\mathbb{E}\Big[\sup_{(S-\delta)^+\leq t\leq S}\big[e^{C_N t}\varphi_t^{\frac{\beta}{2}}\big]\Big] + \mathbb{E}\Big[\int_{(S-\delta)^+}^S e^{C_N s}\varphi_s^{\frac{\beta}{2}-1}(|\hat{Z}_s^{n,m}|^2 + \|\hat{U}_s^{n,m}\|_\nu^2)ds\Big]\\\leq&\frac{\ell}{\beta-1}e^{C_N\delta}\Big\{\mathbb{E}[\varphi_S^{\frac{\beta}{2}}] + \frac{\beta}{N^\kappa}\Big[\mathbb{E}\int_0^T\varphi_s ds\Big]^{\frac{\beta-1}{2}}\Big[\mathbb{E}\int_0^T\Phi^2(s)ds\Big]^{\frac{\kappa}{2}}\\&\qquad\times\Big[\mathbb{E}\int_0^T|f_n(s,Y_s^n,Z_s^n,U_s^n) - f_m(s,Y_s^m,Z_s^m,U_s^n)|^{\frac{2}{\alpha}}ds\Big]^{\frac{\alpha}{2}}\\&\qquad+\beta[4N^2+A_1]^{\frac{\beta-1}{2}}\mathbb{E}\Big[\int_0^T\sup_{|y|,|z|,\|u\|_\nu\leq N}|f_n(s,y,z,u)-f(s,y,z,u)|ds\\&\qquad\qquad+\int_0^T\sup_{|y|,|z|,\|u\|_\nu\leq N}|f_m(s,y,z,u)-f(s,y,z,u)|ds\Big]\Big\}.\end{aligned}$$

Utilizing Lemmas 2.2.8 and 2.2.9, for any $N > R$:

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{(S-\delta)^+\le t\le S}|\hat{Y}^{n,m}_t|^\beta+\mathbb{E}\int_{(S-\delta)^+}^{S}\frac{(|\hat{Z}^{n,m}_s|^2+\|\hat{U}^{n,m}_s\|^2_\nu)}{(|\hat{Y}^{n,m}_s|^2+A_R)^{\frac{2-\beta}{2}}}ds\Big]\\
&\le\frac{\ell}{\beta-1}e^{C_N\delta}\mathbb{E}[|\hat{Y}^{n,m}_S|^\beta]+\frac{\ell}{\beta-1}\frac{A_N^\gamma}{(A_N)^{\frac{\beta}{2}}}\\
&\quad+\frac{4\ell}{\beta-1}\beta K_4^{\frac{\alpha}{2}}(4TK_2+TA_R)^{\frac{\beta-1}{2}}(8TK_2+16K_1+16K_3)^{\frac{\kappa}{2}}\frac{A_N^\gamma}{(A_N)^{\frac{\kappa}{r}}}\\
&\quad+\frac{2\ell}{\beta-1}e^{C_N\delta}\beta[2N^2+A_1]^{\frac{\beta-1}{2}}[\rho_N(f_n-f)+\rho_N(f_m-f)].
\end{aligned}$$

Given $\delta<\frac{\beta-1}{2M_2^2+C_{Lip}^2}\min\left(\frac{1}{2},\frac{\kappa}{r\beta}\right)$, we can derive

$$\lim_{N\to\infty}\left(\frac{A_N^\gamma}{(A_N)^{\frac{\beta}{2}}}+\frac{A_N^\gamma}{(A_N)^{\frac{\kappa}{r}}}\right)=0.$$

To complete the proof of Proposition 2.2.10, we commence by taking the limits as n, m approach their respective limits $+\infty$, $+\infty$ followed by a subsequent limit as N tends to infinity, in accordance with assertion (iii) of Lemma 2.2.8. □

2.2.4 The Main Result

The primary focus of this work is to investigate the existence and the uniqueness results of solutions for BSDEJ (2.1.1) under **Assumption** 2.2.1.

Theorem 2.2.13. *Under **Assumption** 2.2.1, Equation* (2.1.1) *admits one and only one solution* (Y,Z,U) *in* $\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{L}^2([0,T],\nu;\mathbb{R})$.

Proof: **Existence**. By applying Proposition 2.2.10 successively with $S=T$, $S=(T-\delta)^+$, $S=(T-2\delta)^+\ldots$ and utilizing the Lebesgue dominated convergence theorem, we can show that for any $\beta\in(1,3-\alpha)$, the following holds :

$$\limsup_{n,m\to+\infty}\mathbb{E}\Big[\sup_{0\le t\le T}|\hat{Y}^{n,m}_t|^\beta+\int_0^T\frac{(|\hat{Z}^{n,m}_s|^2+\|\hat{U}^{n,m}_s\|^2_\nu)}{(|\hat{Y}^{n,m}_s|^2+A_R)^{\frac{2-\beta}{2}}}ds\Big]=0.$$

Through the application of the Cauchy–Schwarz inequality, we derive

$$\begin{aligned}
\mathbb{E}\Big[\int_0^T(|\hat{Z}^{n,m}_s|+\|\hat{U}^{n,m}_s\|_\nu)ds\Big]&\le\sqrt{2}\Big(\mathbb{E}\Big[\int_0^T\frac{(|\hat{Z}^{n,m}_s|^2+\|\hat{U}^{n,m}_s\|^2_\nu)}{(|\hat{Y}^{n,m}_s|^2+A_R)^{\frac{2-\beta}{2}}}ds\Big]\Big)^{\frac{1}{2}}\\
&\quad\times\Big(\mathbb{E}\Big[\int_0^T\Big(|\hat{Y}^{n,m}_s|^2+A_R\Big)^{\frac{2-\beta}{2}}ds\Big]\Big)^{\frac{1}{2}}.
\end{aligned}$$

It is evident from Lemma 2.2.9 that

$$\left(\mathbb{E}\Big[\int_0^T \Big(|\hat{Y}_s^{n,m}|^2 + \Lambda_R\Big)^{\frac{2-\beta}{2}} ds\Big]\right)^{\frac{1}{2}} \;<\; \infty.$$

Consequently,

$$\lim_{n,m\to+\infty} \mathbb{E}\Big[\sup_{0\le t\le T} |\hat{Y}_t^{n,m}|^\beta + \int_0^T (|\hat{Z}_s^{n,m}| + \|\hat{U}_s^{n,m}\|_\nu) ds\Big] = 0.$$

Thus, there exists (Y,Z,U) that satisfies

$$\mathbb{E}\Big[\sup_{0\le t\le T} |Y_t|^\beta + \int_0^T (|Z_s| + \|U_s\|_\nu) ds\Big] < \infty,$$

and

$$\lim_{n\to+\infty} \mathbb{E}\Big[\sup_{0\le t\le T} |Y_t^n - Y_t|^\beta + \int_0^T (|Z_s^n - Z_s| + \|U_s^n - U_s\|_\nu) ds\Big] = 0.$$

Specifically, a sub-sequence denoted as (Y^n, Z^n, U^n) exists, such that

$$\lim_{n\to+\infty} (|Y_t^n - Y_t| + |Z_t^n - Z_t| + \|U_t^n - U_t\|_\nu) \;=\; 0 \quad a.e.\ (t,\omega). \tag{2.2.10}$$

We still need to establish the convergence in probability of the following term :

$$\int_0^T (f_n(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s, Z_s, U_s)) ds,$$

as n approaches ∞. The initial step is applying the triangular inequality, which yields

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T |f_n(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s, Z_s, U_s)| ds\Big] \\
&\le \mathbb{E}\Big[\int_0^T |f_n(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s^n, Z_s^n, U_s^n)| ds\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T |f(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s, Z_s, U_s)| ds\Big].
\end{aligned}$$

Utilizing Hölder's inequality and the following inequality,

$$\mathbb{1}_{\{|Y_s^n| + |Z_s^n| + \|U_s^n\|_\nu \ge N\}} \le \frac{(|Y_s^n| + |Z_s^n| + \|U_s^n\|_\nu)^{2-\alpha}}{N^{2-\alpha}},$$

we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T |(f_n - f)(s, Y_s^n, Z_s^n, U_s^n)| ds\Big] \\
&\le \mathbb{E}\Big[\int_0^T |(f_n - f)(s, Y_s^n, Z_s^n, U_s^n)| \ \mathbb{1}_{\{|Y_s^n| + |Z_s^n| + \|U_s^n\|_\nu < N\}} ds\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T |(f_n - f)(s, Y_s^n, Z_s^n, U_s^n)| \frac{(|Y_s^n| + |Z_s^n| + \|U_s^n\|_\nu)^{2-\alpha}}{N^{2-\alpha}} \ \mathbb{1}_{\{|Y_s^n| + |Z_s^n| + \|U_s^n\|_\nu \ge N\}} ds\Big] \\
&\le \rho_N(f_n - f) + \frac{4K_4^{\frac{\alpha}{2}} (TK_2 + K_1 + K_3)^{1-\frac{\alpha}{2}}}{N^{2-\alpha}}.
\end{aligned}$$

The last inequality is obtained from Lemmas 2.2.8 and 2.2.9. Taking the limit successively first with respect to n and then to N in the preceding inequality, we arrive at

$$\lim_n \mathbb{E}\Big[\int_0^T |f_n(s,Y_s^n,Z_s^n,U_s^n)-f(s,Y_s^n,Z_s^n,U_s^n)|ds\Big]=0.$$

Considering the limit (2.2.10) and the continuity of the function f with respect to (y,z,u) for all $t\in[0,T]$, we obtain

$$\lim_n |f(s,Y_s^n,Z_s^n,U_s^n)-f(s,Y_s,Z_s,U_s)|=0. \quad a.e.\ (t,\omega).$$

Furthermore, Lemma 2.2.6 and the conditions (a-c) outlined in Lemma 2.2.9 affirm the uniform integrability of the sequence

$$|f(s,Y_s^n,Z_s^n,U_s^n)-f(s,Y_s,Z_s,U_s)|.$$

As a result :

$$\lim_{n\to\infty}\int_0^T \mathbb{E}|f(s,Y_s^n,Z_s^n,U_s^n)-f(s,Y_s,Z_s,U_s)|ds=0.$$

Consequently, the BSDE (2.1.1) has a solution in $\mathcal{S}^\beta([0,T];\mathbb{R})\times\mathbb{H}^1([0,T];\mathbb{R})\times\mathbb{L}^1([0,T],\nu;\mathbb{R})$. Taking account of Lemma 2.2.7, we conclude that it belongs to $\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{L}^2([0,T],\nu;\mathbb{R})$. This achieves the proof of the existence part. □

Proof: Uniqueness. Consider two solutions (Y,Z,U) and (Y',Z',U') to the BSDEJ (2.1.1). Drawing from the proof of Proposition 2.2.10, it can be demonstrated that for every $R>2$,

$$\beta\in(1,3-\alpha),\quad \delta<\frac{\beta-1}{2M_2^2+C_{\text{Lip}}^2}\min\Big(\frac{1}{2},\frac{\kappa}{r\beta}\Big)\quad\text{and}\quad \varepsilon>0,$$

there is an $N_0>R$, for all subsequent $N>N_0$ and each $S\le T$:

$$\begin{aligned}&\mathbb{E}\Big[|Y_t-Y_t'|^\beta\Big]+\mathbb{E}\Big[\int_{(S-\delta)^+}^S\Big(|Z_s-Z_s'|^2+\|U_s-U_s'\|_\nu^2\Big)\Big(|Y_s-Y_s'|^2+\Lambda_R\Big)^{\frac{\beta-2}{2}}ds\Big]\\ &\le\varepsilon+\frac{\ell}{\beta-1}e^{C_N\delta}\mathbb{E}\Big[|Y_S-Y_S'|^\beta\Big].\end{aligned}$$

We successively set $S=T$, followed by updating S as $S=(T-\delta)^+$, and so on. Thus, the BSDEJ (2.1.1) has a unique solution $(Y,Z,U)\in\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{L}^2([0,T],\nu;\mathbb{R})$. □

Example 2.2.1. *Let $g(t,\omega,y,z):=\vartheta_t+c_2|y||\ln|y||+c_0|z|\sqrt{|\ln(|z|)|}+\|u\|_\nu$. Clearly, g satisfies **(A.2)**, so we will now verify that **(A.3)** holds true :*

Indeed, letting $g_{1,c_2}(y) := c_2|y||\ln|y||$; $g_{2,c_0}(z) := c_0|z|\sqrt{|\ln|z||}$, *we have*

$$g(t,\omega,y_1,z_1,u) - g(t,\omega,y_2,z_2,u) = g_{1,c_2}(y_1) - g_{1,c_2}(y_2) \\ + g_{2,c_0}(z_1) - g_{2,c_0}(z_2).$$

We shall examine the function g_{1,c_2} *under the following conditions :*

$$0 \leq |y_1|, |y_2| \leq \frac{1}{N} \quad \text{and} \quad \frac{1}{N} \leq |y_1|, |y_2| \leq N.$$

Additionally, we will analyze g_{2,c_0} *across various cases :*

$$\begin{cases} 0 \leq |z_1|,\ |z_2| \leq \frac{1}{N}, & \frac{1}{N} \leq |z_1|,\ |z_2| \leq 1-\tilde{\epsilon}, \\ 1-\tilde{\epsilon} \leq |z_1|,\ |z_2| \leq 1+\tilde{\epsilon}, & 1+\tilde{\epsilon} \leq |z_1|,\ |z_2| \leq N, \end{cases}$$

where $\tilde{\epsilon} \in (0,1)$ *is small enough, and* N *is sufficiently large.*

Clearly, in the first case ($|y|,\ |z| \leq \frac{1}{N}$), *the two functions satisfy **(A.3)**,*

$$|g_{1,c_2}(y_1) - g_{1,c_2}(y_2) + g_{2,c_0}(z_1) - g_{2,c_0}(z_2)| \leq |g_{1,c_2}(y_1)| + |g_{1,c_2}(y_2)| + |g_{2,c_0}(z_1)| + |g_{2,c_0}(z_2)| \\ \leq \max(c_0,c_2)\frac{4}{N}\ln(N).$$

The mean value theorem, applied in the second term, implies the following :

$$|g_{1,c_2}(y_1) - g_{1,c_2}(y_2) + g_{2,c_0}(z_1) - g_{2,c_0}(z_2)| \leq |g_{1,c_2}(y_1) - g_{1,c_2}(y_2)| + |g_{2,c_0}(z_1) - g_{2,c_0}(z_2)| \\ \leq \max(c_0,c_2)\Big(|y_1 - y_2|\ln(N) + |z_1 - z_2|\sqrt{\ln(N)}\Big).$$

Applying the mean value theorem again, we can prove the remaining cases for the function g_{2,c_0}*. Therefore, **(A.3)** holds for* $A_N = N$*.*

Further examples can be found in [12].

2.3 Generalized Logarithmic Growth Condition for BSDEs with Jumps

Now, we examine a distinct BSDE with jumps from the one in (2.1.1), introducing different assumptions for the generator of the next BSDEJ :

$$Y_t = \zeta + \int_t^T f\big(s, Y_s, Z_s, \textstyle\int_\Gamma U_s(e)\nu(de)\big)ds - \int_t^T Z_s dW_s - \int_t^T \int_\Gamma U_s(e)\tilde{N}(ds,de). \tag{2.3.1}$$

Assumption 2.3.1.

(A.1)' *Assume that* $\mathbb{E}[|\zeta|^{\mu_T+1}]$ *is finite, where* $\mu_t := e^{\theta t}$ *for all* $t \in [0,T]$ *and* θ *is a sufficiently large positive constant.*

(A.2)' *(i) For almost all (t,ω), the function f is continuous with respect to (y,z,u).*

(ii) There exists a positive process ϑ such that

$$\int_0^T \mathbb{E}[\vartheta_s^{\mu_s+1}]ds < +\infty.$$

Additionally, for every t, y, z, and u,

$$|f(t,y,z,\int_\Gamma u(e)\nu(de))| \leq \vartheta_t + g_{1,c_2}(y) + g_{2,c_0}(z) + g_{3,c_1}(u),$$

where $g_{3,c_1}(u) = c_1\|u\|_\nu\sqrt{|\ln\|u\|_\nu|}$, c_0, c_1 and c_2 are positive constants.

(A.3)' *There exists a real-valued sequence $(A_N)_{N>1}$ and constants $M_2 \in \mathbb{R}_+$, $r > 0$ such that*

(i) For every integer $N > 1$, we have $1 < A_N \leq N^r$.

(ii) $\lim_{N\to\infty} A_N = \infty$.

(iii) For every $N \in \mathbb{N}$, and every y_1, y_2, z_1, z_2, u_1, u_2 such that $|y_1|$, $|y_2|$, $|z_1|$, $|z_2|$, $\|u_1\|_\nu$, $\|u_2\|_\nu \leq N$, we have

$$\begin{aligned}
&(y_1 - y_2)(f(t,\omega,y_1,z_1,\int_\Gamma u_1(e)\nu(de)) - f(t,\omega,y_2,z_2,\int_\Gamma u_2(e)\nu(de))) \\
\leq M_2 &\Big(|y_1 - y_2|^2 \ln(A_N) + |y_1 - y_2| \sqrt{\ln(A_N)}(|z_1 - z_2| + \|u_1 - u_2\|_\nu) \\
&+ \frac{\ln(A_N)}{A_N} \Big).
\end{aligned}$$

By following the steps outlined in the previous proofs, we can obtain a unique solution for BSDEJ (2.3.1) in which the transaction with u becomes proportionally identical to the transaction with z.

The previous lemmas maintain their validity while adhering to (2.3.1) and **Assumption** 2.3.1. Therefore, we will provide concise proofs, building upon the earlier derivations.

Proof: of Lemma 2.2.7 under **Assumption** 2.3.1. Consider a solution (Y,Z,U) to (2.3.1), and assume that conditions $(\mathbf{A.1})'$ and $(\mathbf{A.2})'$ are satisfied. We define the sign function $\text{sgn}(x)$ as follows : $\text{sgn}(x) = -1$ for $x \leq 0$ and $\text{sgn}(x) = +1$ for $x > 0$. We can apply Itô's formula to obtain

$$
\begin{aligned}
\left|Y_{t\wedge\widetilde{\tau}_n}\right|^{\mu_{t\wedge\widetilde{\tau}_n}+1} \leq &\left|Y_{T\wedge\widetilde{\tau}_n}\right|^{\mu_{T\wedge\widetilde{\tau}_n}+1} + \int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} (\mu_s+1)^{\mu_s}\vartheta_s^{\mu_s+1}ds + C_2 \\
&-\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} \theta\mu_s|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}}ds \\
&+\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} |Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>e\}}ds \\
&+\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} (\mu_s+1)|Y_s|^{\mu_s}\left(\frac{1}{2}g_{1,c_2}(Y_s)\ \mathbb{1}_{\{|Y_s|>1\}} + g_{2,c_0}(Z_s)\right)ds \\
&+\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} (\mu_s+1)|Y_s|^{\mu_s}\left(\frac{1}{2}g_{1,c_2}(Y_s)\ \mathbb{1}_{\{|Y_s|>1\}} + g_{3,c_1}(U_s)\right)ds \\
&-\frac{1}{2}\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} (\mu_s+1)\mu_s|Z_s|^2\mid Y_s\mid^{\mu_s-1}ds \\
&-\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} \mu_s(\mu_s+1)3^{-\mu_s}|Y_s|^{\mu_s-1}\|U_s\|_\nu^2 ds + \Xi_{t\wedge\widetilde{\tau}_n} - \Xi_{T\wedge\widetilde{\tau}_n}.
\end{aligned}
$$

By Lemma 2.2.3, we have

$$
c_0|y||z|\sqrt{|\ln|z||}\ \mathbb{1}_{\{|y|>e\}} \leq \frac{|z|^2}{4}\ \mathbb{1}_{\{|y|>e\}} + c_3|y|^2\ln|y|\ \mathbb{1}_{\{y>e\}},
$$

and

$$
c_1|y|\,\|u\|_\nu\sqrt{|\ln\|u\|_\nu|}\ \mathbb{1}_{\{|y|>e\}} \leq \frac{\rho}{4}\|u\|_\nu^2\ \mathbb{1}_{\{|y|>e\}} + c_4|y|^2\ln|y|\ \mathbb{1}_{\{|y|>e\}}.
$$

Utilizing Young's inequality, we obtain

$$
c_0|y||z|\sqrt{|\ln|z||}\ \mathbb{1}_{\{|y|\leq e\}} \leq \frac{|z|^2}{4}\ \mathbb{1}_{\{|y|\leq e\}} + \widetilde{c}_0,
$$

and

$$
c_1|y|\|u\|_\nu\sqrt{|\ln\|u\|_\nu|}\ \mathbb{1}_{\{|y|\leq e\}} \leq \frac{\rho}{4}\|u\|_\nu^2\ \mathbb{1}_{\{|y|\leq e\}} + \widetilde{c}_1.
$$

where $\widetilde{c}_0 = c_0 e^{\frac{1}{2}}\frac{1}{\sqrt{2}} + 3^3\frac{(c_0 e)^4}{4}$, $\widetilde{c}_1 = c_1 e^{\frac{1}{2}}\frac{1}{\sqrt{2}} + 3^3\frac{(c_1 e)^4}{4}$. For $\theta \geq 2(c_2+c_3+c_4)+1$ we have $-\theta\mu_s + (c_2+c_3+c_4)(\mu_s+1)+1 \leq 0$, thus,

$$
\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n} (-\theta\mu_s + (\mu_s+1)(c_2+c_3+c_4)+1)|Y_s|^{\mu_s+1}\ln|Y_s|\ \mathbb{1}_{\{|Y_s|>1\}}ds \leq 0.
$$

Thus, employing the same steps as outlined above, we can determine a general constant C such that

$$
\mathbb{E}[\sup_{0\leq t\leq T\wedge\widetilde{\tau}_n}|Y_{t\wedge\widetilde{\tau}_n}|^{\mu_{t\wedge\widetilde{\tau}_n}+1}] \leq C\mathbb{E}\Big[1 + |Y_{T\wedge\widetilde{\tau}_n}|^{\mu_{T\wedge\widetilde{\tau}_n}+1} + (\mu_T+1)^{\mu_T}\int_{t\wedge\widetilde{\tau}_n}^{T\wedge\widetilde{\tau}_n}\vartheta_s^{\mu_{s+1}}ds\Big],
$$

The monotone convergence theorem enables us to obtain the assertion (i).

Since

$$2c_1|y|\ \|u\|_\nu\sqrt{|\ln\|u\|_\nu|}\ \mathbb{1}_{\{|y|>e\}}\leq\frac{\|u\|_\nu^2}{2}\ \mathbb{1}_{\{|y|>e\}}+\widetilde{c}|y|^{2+\varepsilon},$$

and

$$2c_1|y|\ \|u\|_\nu\sqrt{|\ln\|u\|_\nu|}\ \mathbb{1}_{\{|y|\leq e\}}\leq\frac{\|u\|_\nu^2}{2}\ \mathbb{1}_{\{|y|\leq e\}}+\widetilde{c}_1,$$

where $\widetilde{c}_1=c_1\sqrt{2}e^{\frac{1}{2}}+4(c_1e)^4\left(\frac{3}{2}\right)^3$, we easily verify the validity of (ii). □

In what follows, we state a lemma concerning the stability result for the solution of BSDEJ (2.3.1). The proof follows the same steps as Lemma 3.5 in [14].

Lemma 2.3.2. *There exists a sequence of functions (f_n) with the following properties :*

(i) For each n, f_n is bounded and globally Lipschitz in (y,z,u) a.e. t and $\mathbb{P}$-a.s.ω.

(ii) Moreover, for all n, we have $\mathbb{P}$-a.s., a.e. $t\in[0,T]$:

$$\sup_n\left|f_n(t,\omega,y,z,\textstyle\int_\Gamma u(e)\nu(de))\right|\leq\vartheta_t+g_{1,c_2}(y)+g_{2,c_0}(z)+g_{3,c_1}(u).$$

(iii) Additionally, for every N, as n tends to infinity, the quantity $\rho_N(f_n-f)$ converges to 0, where

$$\rho_N(f)=\mathbb{E}\Big[\int_0^T\sup_{|y|,|z|,\|u\|_\nu\leq N}\left|f_n(s,\omega,y,z,\textstyle\int_\Gamma u(e)\nu(de))\right|ds\Big].$$

Proposition 2.3.3. *Proposition 2.2.10, which establishes the estimate between two solutions, maintains its validity within this section despite variations in the values of δ and C, as presented in the*

subsequent lemma.

Lemma 2.3.4. *Assuming that $C:=C_N:=3\beta\frac{M_2^2}{\beta-1}\ln(A_N)$ and $\delta<\frac{\beta-1}{3M_2^2}\min(\frac{1}{2},\frac{\kappa}{r\beta})$, for any $S\leq T$ we have*

$$\begin{aligned}e^{Ct}\varphi_t^{\frac{\beta}{2}}+C\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds+\widetilde{\mathbb{M}}_t\leq e^{CS}\varphi_S^{\frac{\beta}{2}}&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds\\&-\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2ds\\&+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2ds\\&+\mathbb{M}_t+\tilde{J}_{1,t}+\tilde{J}_{2,t}+\tilde{J}_{3,t},\end{aligned}$$

where

$$\begin{aligned}\tilde{J}_{1,t}:=\beta e^{CS}\frac{1}{N^\kappa}\int_t^S\varphi_s^{\frac{\beta-1}{2}}\Phi^\kappa(s)\Big|&f_n(s,Y_s^n,Z_s^n,\textstyle\int_\Gamma U_s^n(e)\nu(de))\\&-f_m(s,Y_s^m,Z_s^m,\textstyle\int_\Gamma U_s^m(e)\nu(de))\Big|ds,\end{aligned}$$

$$\tilde{J}_{2,t}:=J_{2,t};\quad\tilde{J}_{3,t}:=J_{3,t}+\beta M_2\sqrt{\ln(A_N)}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}|\|\hat{U}_s^{n,m}\|_\nu ds,$$

and $\Phi(s)=|Y_s^n|+|Y_s^m|+|Z_s^n|+|Z_s^m|+\|U_s^n\|_\nu+\|U_s^m\|_\nu$.

Proof: of Proposition 2.3.3. The proof closely aligns with the methodology employed in establishing Lemma 2.2.11. Let $C := C_N := 3\beta\frac{M_2^2}{\beta-1}\ln(A_N)$ and $\gamma := 3\delta\beta\frac{M_2^2}{\beta-1}$.

As presented in Lemma 2.3.4, it is obvious that

$$\begin{aligned}
&-\frac{C_N}{3}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2ds\\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds+\beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}||\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}ds\\
\leq\ &\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\Big(-\frac{M_2^2}{\beta-1}\varphi_s\ln(A_N)-\frac{(\beta-1)}{2}|\hat{Z}_s^{n,m}|^2+M_2\sqrt{\varphi_s}|\hat{Z}_s^{n,m}|\sqrt{\ln(A_N)}\Big)ds,
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{C_N}{3}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds-\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2ds\\
&+\beta M_2\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Y}_s^{n,m}|\|\hat{U}_s^{n,m}\|_\nu\sqrt{\ln(A_N)}ds\\
\leq\ &\beta\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\Big(-\frac{M_2^2}{\beta-1}\varphi_s\ln(A_N)-\frac{(\beta-1)}{2}\|\hat{U}_s^{n,m}\|_\nu^2\\
&\qquad\qquad+M_2\sqrt{\varphi_s}\|\hat{U}_s^{n,m}\|_\nu\sqrt{\ln(A_N)}\Big)ds.
\end{aligned}$$

Using Young's inequality, it follows that

$$-\frac{1}{\beta-1}M_2^2\ \ln(A_N)a^2-\frac{(\beta-1)}{2}|b|^2+M_2\ ab\sqrt{\ln(A_N)}\leq-\frac{\beta-1}{4}b^2;$$

therefore,

$$\begin{aligned}
&-C_N\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}}ds+\beta\frac{(2-\beta)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-2}|\hat{Y}_s^{n,m}|^2|\hat{Z}_s^{n,m}|^2ds\\
&-\frac{\beta}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}|\hat{Z}_s^{n,m}|^2ds-\beta\frac{(\beta-1)}{2}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}\|\hat{U}_s^{n,m}\|_\nu^2ds+J_{3,t}\\
\leq\ &-\beta\frac{(\beta-1)}{4}\int_t^S e^{Cs}\varphi_s^{\frac{\beta}{2}-1}(|\hat{Z}_s^{n,m}|^2+\|\hat{U}_s^{n,m}\|_\nu^2)ds.
\end{aligned}$$

Based on the preceding lemmas, for any $N>R$ we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{(S-\delta)^+\leq t\leq S}|\hat{Y}_t^{n,m}|^\beta+\mathbb{E}\int_{(S-\delta)^+}^S\frac{(|\hat{Z}_s^{n,m}|^2+\|\hat{U}_s^{n,m}\|_\nu^2)}{(|\hat{Y}_s^{n,m}|^2+\Lambda_R)^{\frac{2-\beta}{2}}}ds\Big]\\
\leq\ &\frac{\ell}{\beta-1}e^{C_N\delta}\mathbb{E}[|\hat{Y}_S^{n,m}|^\beta]+\frac{\ell}{\beta-1}\frac{A_N^\gamma}{(A_N)^{\frac{\beta}{2}}}\\
&+\frac{4\ell}{\beta-1}\beta K_4^{\frac{\alpha}{2}}(4TK_2+T\Lambda_R)^{\frac{\beta-1}{2}}(8TK_2+16K_1+16K_3)^{\frac{\kappa}{2}}\frac{A_N^\gamma}{(A_N)^{\frac{\kappa}{r}}}\\
&+\frac{2\ell}{\beta-1}e^{C_N\delta}\beta(2N^2+\Lambda_1)^{\frac{\beta-1}{2}}\left[\rho_N(f_n-f)+\rho_N(f_m-f)\right].
\end{aligned}$$

Since $\delta < \frac{\beta-1}{3M_2^2}\min(\frac{1}{2},\frac{\kappa}{r\beta})$, we proceed by taking limits for n and m, followed by a limit as N approaches infinity, in accordance with the statement (iii) of Lemma 2.3.2, and we obtain the desired result. $\square$

Theorem 2.3.5. *Under* ***Assumption*** *2.3.1 Equation* (2.3.1) *has a unique solution* (Y,Z,U) *in* $\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{L}^2([0,T],\nu;\mathbb{R})$.

To prove the above theorem, we utilize Proposition 2.3.3 and follow similar steps in the proof of the existence and uniqueness parts of Theorem 2.2.13.

2.4 The Relationship Between BSDEJs and QBSDEJs

We present a supplementary BSDEJ, explicitly formulated through the exponential transformation of the initial problem. This formulation facilitates establishing a connection between the solution of the auxiliary BSDEJ and that of the original BSDEJ (ζ,g). Subsequently, we will demonstrate an application to quadratic BSDEJs.

Lemma 2.4.1 (General exponential transformation)**.** *We assume that either* (ζ,g) *or* $(\widetilde{\zeta},\widetilde{g})$ *satisfies the first* ***Assumption*** *2.2.1. Let* $h\in\mathbb{L}^1(\mathbb{R})$ *a measurable function and* $[u]_h(y)$, $J_u^h(y)$ *two operators, defined as*

$$[u]_h(y):=\int_\Gamma \frac{\Psi(y+u(e))-\Psi(y)-\Psi'(y)u(e)}{\Psi'(y)}\nu(de),$$
$$J_u^h(y):=\int_\Gamma (\Psi^{-1}(y+u(e))-\Psi^{-1}(y)-(\Psi^{-1})'(y)u(e))\nu(de),$$

where Ψ *is defined for every* $x\in\mathbb{R}$ *as*

$$\Psi(x)=\int_0^x \exp\left(2\int_0^y h(t)dt\right)dy.$$

The triplet (Y,Z,U) *is a solution to the BSDEJ* (ζ,g) *if and only if the triplet* $(\widetilde{Y},\widetilde{Z},\widetilde{U})$ *is a solution to the BSDEJ* $(\widetilde{\zeta},\widetilde{g})$, *where*

$$\widetilde{Y}_t = \Psi(Y_t),\quad \widetilde{\zeta} = \Psi(\zeta),\quad \widetilde{Z}_t=\Psi'(Y_t)Z_t,\quad \widetilde{U}_t(e) = \Psi(Y_{t-}+U_t(e))-\Psi(Y_{t-}),$$

and

$$(\Psi^{-1})'(\widetilde{y})\widetilde{g}(t,\widetilde{y},\widetilde{z},\widetilde{u})=g\big(t,\Psi^{-1}(\widetilde{y}),\widetilde{z}(\Psi^{-1})'(\widetilde{y}),\Psi^{-1}(\widetilde{y}+\widetilde{u})-\Psi^{-1}(\widetilde{y})\big)$$
$$-\widetilde{z}^2h(\Psi^{-1}(\widetilde{y}))\big((\Psi^{-1})'(\widetilde{y})\big)^2+J_{\widetilde{u}}^h(\widetilde{y}).$$

Clearly, Ψ is bi-Lipschitz with $\Psi(0)=0$, guaranteeing the preservation of the same spaces for primary BSDEJs and their auxiliary counterparts, i.e., (Y,Z,U) and $(\widetilde{Y},\widetilde{Z},\widetilde{U})$ in $\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{L}^2([0,T],\nu;\mathbb{R})$. The proof proceeds through a series of steps analogous to those outlined in Lemma 2.4.4.

Example 2.4.1. *Consider ζ satisfying condition (**A.1**), and let $g(t,y,z,u)$ be a continuous function with respect to (y,z,u). The function is defined as follows :*

$$\begin{aligned}g(t,y,z,u)=\frac{1}{\Psi'(y)}&\Big[\Psi(y)|\ln|\Psi(y)||+z\Psi'(y)\sqrt{|\ln|z\Psi'(y)||}\\&+\|\Psi(y+u(e))-\Psi(y)\|_\nu\Big]+h(y)|z|^2+[u]_h(y),\end{aligned}$$

*where Ψ is defined as in the previous Lemma 2.4.1. Using its result, it becomes evident that the BSDEJ (ζ,g) is equivalent to the BSDEJ $(e^\zeta,\widetilde{y}|\ln|\widetilde{y}||+\widetilde{z}\sqrt{|\ln|\widetilde{z}||}+\|\widetilde{u}\|_\nu)$, whose generator satisfies **Assumption** 2.2.1, and ensures the existence and uniqueness of the solution for both BSDEJ. Furthermore, (Y,Z,U), $(\widetilde{Y},\widetilde{Z},\widetilde{U})$ in $\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{L}^2([0,T],\nu;\mathbb{R})$.*

Proposition 2.4.2. *Assuming that **Assumption** 2.2.1 holds and further supposing that ζ and $(\vartheta_t)_{0\le t\le T}$ are bounded, then there exists C_T such that*

- $\sup_{t\in[0,T]}|Y_t|\le C_T$.
- $\mathbb{E}[\int_0^T(|Z_s|^2+\|U_s\|_\nu^2)ds]\le C_T$.

Proof: By utilizing Itô's formula and employing the same step as in the proof of Lemma 2.2.7, we obtain

$$|Y_t|^{\mu_t+1}\le C+|\zeta|^{\mu_T+1}+\int_0^T(\mu_s+1)^{\mu_s+1}\vartheta_s^{\mu_s+1}ds+\mathbb{M}_t,$$

where

$$\mathbb{M}_t:=-\int_t^T(\mu_s+1)|Y_s|^{\mu_s}\operatorname{sgn}(Y_s)Z_sdW_s-\int_t^T\int_\Gamma\left(|Y_s|^{\mu_s+1}-|Y_{s-}|^{\mu_s+1}\right)\widetilde{N}(ds,de).$$

We obtain the first result by taking the conditional expectation. We attain the desired outcome by building upon the first result and condition (ii) in Lemma 2.2.7. □

Let $\lambda>0$ and $t\in[0,T]$. Consider the following BSDEJ :

$$\begin{aligned}Y_t=&\zeta+\int_t^T\left(g(s,Y_s,Z_s,U_s)+\frac{\lambda}{2}|Z_s|^2+[U_s]_\lambda\right)ds\\&-\int_t^TZ_sdW_s-\int_t^T\int_\Gamma U_s(e)\widetilde{N}(ds,de),\end{aligned}\tag{2.4.1}$$

where

$$[u]_\lambda=\frac{1}{\lambda}\int_\Gamma(e^{\lambda u(e)}-\lambda u(e)-1)\nu(de).$$

Assumption 2.4.3.

(A.4) *(i) The function g is continuous in (y,z) and Lipschitz with respect to u for almost all (t,ω).*

(ii) There exist constants c_0, c_1, c_2, and C_{Lip}, as well as a bounded positive process $(\vartheta_t)_{t\geq 0}$, such that for every t, ω, y, z, u, u_1, u_2 :

$$|g(t,y,z,u)| \leq \vartheta_t + c_2|y| + c_0|z|\sqrt{|\ln|\lambda z| + \lambda y|} + \frac{c_1}{\lambda}\int_{\Gamma}(e^{\lambda u(e)}-1)\nu(de),$$

and

$$|g(t,\omega,y,z,u_1) - g(t,\omega,y,z,u_2)| \leq C_{Lip}\|u_1-u_2\|_{\nu}.$$

(A.5) *There exists a real-valued sequence $(A_N)_{N>1}$ and constants $M_2 \in \mathbb{R}_+$, $r>0$ such that*

(i) $\forall\ N>1, \quad 1 < A_N \leq N^r$.

(ii) $\lim_{N\to\infty} A_N = \infty$.

(iii) For every $N\in\mathbb{N}$, and every y_1, y_2, z_1, z_2, u such that for all $|y_1|$, $|y_2| \leq \ln(N)$ $|z_1|, |z_2| \leq 1$, $u \leq \ln(2)$, we have

$$\begin{aligned}
&(e^{\lambda y_1} - e^{\lambda y_2})(e^{\lambda y_1} g(t,\omega,y_1,z_1,u) - e^{\lambda y_2} g(t,\omega,y_2,z_2,u))\\
\leq\ & M_2\Big(\ |\ e^{\lambda y_1} - e^{\lambda y_2}\ |^2 \ln(A_N)\\
&\qquad + |e^{\lambda y_1} - e^{\lambda y_2}||z_1 e^{\lambda y_1} - z_2 e^{\lambda y_2}|\sqrt{\ln(A_N)} + \frac{\ln(A_N)}{A_N}\Big).
\end{aligned}$$

In the following lemma, we utilize the exponential transformation while relaxing the Lipschitz condition through the utilization of $\Psi(x) = e^{\lambda x}$.

Lemma 2.4.4. *If ζ and $(\vartheta_t)_{0\leq t\leq T}$ are bounded and **Assumption** 2.4.3 holds, then, for any $\lambda>0$, the following equivalence holds : there exists a unique solution*

$$(Y,Z,U) \in \mathcal{S}^\infty([0,T];\mathbb{R}) \times \mathbb{H}^2([0,T];\mathbb{R}) \times \mathbb{L}^2([0,T],\nu;\mathbb{R})$$

to the BSDEJ (2.4.1) if and only if the triplet

$$(\widetilde{Y},\widetilde{Z},\widetilde{U}) \in \mathcal{S}^\infty([0,T];\mathbb{R}) \times \mathbb{H}^2([0,T];\mathbb{R}) \times \mathbb{L}^2([0,T],\nu;\mathbb{R})$$

is the unique solution to the BSDEJ$(\widetilde{\zeta},\widetilde{g})$, where

$$\widetilde{Y}_t = e^{\lambda Y_t},\ \widetilde{\zeta} \ = \ e^{\lambda\zeta},\ \widetilde{Z}_t \ = \ \lambda e^{\lambda Y_t} Z_t,\ \widetilde{U}_t \ = \ e^{\lambda Y_{t-}}(e^{\lambda U_t} - 1),$$

and

$$\widetilde{g}(t,\widetilde{y},\widetilde{z},\widetilde{u}) = \lambda\widetilde{y}g\Big(t, \frac{1}{\lambda}\ln(\widetilde{y}), \frac{\widetilde{z}}{\lambda\widetilde{y}}, \frac{1}{\lambda}\ln\Big(1+\frac{\widetilde{u}}{\widetilde{y}}\Big)\Big).$$

Proof: By employing Itô's formula on $\widetilde{Y}_t = e^{\lambda Y_t}$, we derive the following result for all $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\widetilde{Y}_t = \widetilde{\zeta} + \int_t^T \lambda e^{\lambda Y_s} g(s, Y_s, Z_s, U_s) ds$$
$$- \int_t^T \lambda e^{\lambda Y_s} Z_s dW_s - \int_t^T \int_\Gamma e^{\lambda Y_{s-}} (e^{\lambda U_s(e)} - 1) \widetilde{N}(de, ds).$$

With the quantities provided above, we can deduce the following :

$$\widetilde{Y}_t = \widetilde{\zeta} + \int_t^T \widetilde{g}(s, \widetilde{Y}_s, \widetilde{Z}_s, \widetilde{U}_s) ds - \int_t^T \widetilde{Z}_s dW_s - \int_t^T \int_\Gamma \widetilde{U}_s(e) \widetilde{N}(de, ds). \tag{2.4.2}$$

Since the generator g satisfies **Assumption** 2.4.3, then the generator $\widetilde{g}$ fulfills **Assumption** 2.2.1 ; therefore, Theorem 2.2.13 shows that Equation (2.4.2) has a unique solution in $\mathcal{S}^{\mu_T+1}([0,T];\mathbb{R}) \times \mathbb{H}^2([0,T];\mathbb{R}) \times \mathbb{L}^2([0,T],\nu;\mathbb{R})$. Thus, taking account of Proposition 2.4.2, the necessary condition is proved.

Conversely, Itô's formula applied to $\ln(\widetilde{Y}_t)/\lambda$ along with Proposition 2.4.2 lead to the sufficient condition.

It is worth mentioning that the functional spaces are conserved due to Proposition 2.4.2. □

Example 2.4.2. *Assume ζ is bounded, and let*

$$g(t,y,z,u) = c_2|y| + c_0|z|\sqrt{|\ln|\lambda z| + \lambda y|} + \frac{c_1}{\lambda}\|e^{\lambda u} - 1\|_\nu,$$

where c_0, c_1, and c_2 are positive constants. Therefore,

$$\widetilde{g}(t,\widetilde{y},\widetilde{z},\widetilde{u}) = c_2|\widetilde{y}||\ln|\widetilde{y}|| + c_0|\widetilde{z}|\sqrt{|\ln|\widetilde{z}||} + c_1\|\widetilde{u}\|_\nu.$$

*Clearly, the generator $\widetilde{g}$ satisfies **Assumption** 2.2.1. Consequently, according to the preceding Lemma 2.4.4, the BSDEJ(ζ, g) has a unique solution and the BSDEJ$(\widetilde{\zeta}, \widetilde{g})$ has a unique solution.*

Remark 2.4.5 (Quadratic–exponential BSDEJs)**.** *Let $g_1(t,y) = g(t,y,0,0)$, where g is defined as in the previous example. Then, the BSDEJ (2.4.1) transforms into a quadratic–exponential BSDEJ, which has a unique solution.*

For a more extensive examination of quadratic BSDEJs, we refer to [69].

Remark 2.4.6. *The primary BSDEJs discussed in the previous section share the same auxiliary counterpart, consistent with the discussions in this section regarding the suitable space for the jump. In other words, the previously established lemmas hold for the generators $g(s,y,z,\int_\Gamma u(e)\nu(de))$ and $\tilde{g}(s,\tilde{y},\tilde{z},\int_\Gamma \tilde{u}(e)\nu(de))$.*

2.5 Conclusion

Our study addresses fundamental questions concerning the existence and uniqueness of BSDEs whose driving processes are a compensated Poisson random measure and an independent Wiener process. Through rigorous proofs under two sets of assumptions, we first emphasize the significance of a generator by the logarithmic growth in both (y, z)-variables and the Lipschitz continuity with respect to the third variable u. We also included a concrete example that strengthens the validity of our first assumption.

Under Assumption 2, we take a step further by relaxing the Lipschitz condition on u. Here, the generator exhibits logarithmic growth in all variables, adding nuance to our understanding of the problem. Moreover, the introduction of the exponential transformation proves to be a key tool that demonstrates the equivalence between the solutions of the auxiliary BSDEJ and our primary BSDEJ.

Chapter 3

Optimal Control of BSDEs with Logarithmic Growth Condition : Exploring the Maximum Principle

3.1 Introduction

The domain of stochastic optimal control problems is commonly traversed through two primary avenues : Pontryagin's maximum principle and Bellman's dynamic programming. These methodologies necessitate distinct mathematical treatments. Dynamic programming, for instance, aims to derive a second-order partial differential equation known as the Hamilton-Jacobi-Bellman equation, serving as a characterization of the value function.

However, a significant drawback arises when employing this approach : the classical solutions to the Hamilton-Jacobi-Bellman equation are only guaranteed for sufficiently smooth value functions, a condition often unmeet in practical scenarios. Crandall and Lions [35] addressed this limitation by introducing viscosity solutions, wherein (set-valued) sub-derivatives replace conventional derivatives. This innovation empowers dynamic programming with enhanced applicability in real-world situations.

While the maximum principle is extensively employed for solving optimal control problems in deterministic systems, translating theoretical results into practical solutions encounters numerous obstacles. The inherent difficulty lies in explicitly solving the resultant adjoint systems. Some scholars (e.g., [67], [72]) have proposed numerical methods to address such challenges, expanding the applicability of Pontryagin's maximum principle into fields like mathematical finance and economics. Several attempts have been made to relax assumptions on coefficients, facilitating the extension of the stochastic maximum principle to irregular cases.

Mezerdi [26] pioneered this direction by deriving a stochastic maximum principle for a controlled stochastic differential equation (SDE) with a non-smooth drift, leveraging Clarke's generalized gradients and stable convergence of probability measures. Building on this, Bahlali et al. [15] extended the principle to SDEs with Lipschitz coefficients and a non-degenerate diffusion matrix employing Krylov's inequality with uniform ellipticity. In a broader context, Bahlali et al. [8] established a stochastic maximum principle for optimal control over a general class of degenerate diffusion processes, assuming only Lipschitz continuity in state equation coefficients and continuous differentiability in cost functional coefficients. Chighoub et al. [33] further expanded these results to cases where both state equation and cost functional coefficients lack differentiability.

Recent advancements include Xu and Wu's [94] work, where they obtained the existence and uniqueness of mild solutions to mean-field backward stochastic evolution equations in Hilbert spaces under conditions weaker than Lipschitz. They subsequently proved a maximum principle for optimal control problems governed by backward stochastic partial differential equations of mean-field type. Additionally, Orrieri [75] introduced a version of the maximum principle for optimal control in stochastic differential equations driven by multidimensional Wiener processes. Dokuchaev and Zhou [41] derived both necessary and sufficient conditions for optimality in cases where the control domain lacks convexity.

Consider $T > 0$ and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a probability space with completeness, equipped with a filtration that satisfies the usual conditions. On this probability space, we define a one-dimensional Brownian motion $W = (W_t)_{0\leq t\leq T}$. We make the assumption that $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ is the $\mathbb{P}$-augmentation of the natural filtration generated by $(W_t)_{0\leq t\leq T}$. For subsequent discussions, we introduce the following spaces for $p \geq 1$:

- $S^p([0,T],\mathbb{R})$: denotes the set of continuous and $\mathbb{F}$-adapted stochastic processes $\{Y_t : t \in [0,T]\}$, such that $\mathbb{E}[\sup_{0\leq t\leq T} |Y_t|^p] < \infty$.
- $\mathcal{M}^2([0,T],\mathbb{R})$: denotes the set of $\mathbb{F}$-predictable and $\mathbb{R}$-valued processes $\{Z_t : t \in [0,T]\}$, such that $\mathbb{E}\int_0^T |Z_r|^2\, dr < \infty$.
- $\mathbb{L}^p_{loc}(\mathbb{R}_+,\mathbb{R})$: the set of $\mathbb{F}$-adapted processes taking values in $\mathbb{R}$, denoted by $\{X_t : t \geq 0\}$, such that $\int_0^T |X_r|^p\, dr < \infty$ $\mathbb{P}$-a.s for every T.

We consider the following controlled backward stochastic differential equation (BSDE for short) :

$$\begin{cases} dY_t &= f(t,Y_t,Z_t,v_t)dt + Z_t dW_t, \\ Y_T &= \zeta. \end{cases} \tag{3.1.1}$$

Here, f is a function defined on $f : [0,T]\times\mathbb{R}\times\mathbb{R}\times U \longrightarrow \mathbb{R}$. The terminal data ζ is a $\mathcal{F}_T$-adapted random variable. The control variable $(v_t)_{t\geq 0}$ is represented by the process v_t, assumed to be an

$\mathbb{F}$-adapted process taking values in a non-empty subset U of $\mathbb{R}$. The set of all admissible controls is denoted by $\mathcal{U}_{ad}$.

For a given function $g:\mathbb{R}\rightarrow\mathbb{R}$, we define the cost functional of our stochastic control problem as

$$\mathcal{J}(v.)=\mathbb{E}\left[g(Y_0^v)\right] \tag{3.1.2}$$

For ease of notation we denote $h_\theta=\frac{\partial h}{\partial\theta}$ for a given function h and parameter θ.

The objective is to minimize the cost functional (3.1.2) among all admissible controls. Now, the control problem can be formulated as follows :

Problem (A) : Given the cost functional (3.1.2) and the constraint (3.1.1), the objective is to identify an optimal control, denoted as u from the set $\mathcal{U}_{ad}$, that minimizes the specified cost functional.

There exists an extensive body of literature addressing stochastic optimal control problems for BSDEs and Forward-BSDEs (FBSDEs) within the global Lipschitz framework. Azizi and Khelfallah [5] were the first to investigate a stochastic control problem for BSDEs with generators which are local Lipschitz in y and globally Lipschitz in z under the first assumption. In their study, they demonstrated that the generator satisfies specific conditions, which include :

— There exist a constant, $M>0$ such that for all y and z,

$$\langle y,f(t,y,z,v)\rangle\leq M(1+|y|^2+|y||z|),\quad \text{a.e. } t\in[0,T].$$

— There exist two constants, $M>0$, $\kappa\in(0,1)$ and a positive function $\varphi:\mathbb{R}_+\rightarrow\mathbb{R}_+$, such that

$$|f(t,y,z,v)|\leq M(1+\varphi(|y|)+|z|^\kappa \quad \text{a.e. } t\in[0,T].$$

Moreover, they present results under another assumption where the generator is locally Lipschitz with respect to both (y,z) and exhibits linear growth. They establish necessary and sufficient optimality conditions for non-convex control domains, described by a linear local Lipschitz SDE and a maximum condition on the Hamiltonian.

In our context, we relax the Lipschitz condition on the generator of the BSDEs, imposing a logarithmic growth condition in y and linear growth in z in the first assumption. In the second assumption, we require the generator to satisfy the logarithmic growth condition for both y and z, and we employ the Malliavin approach.

The primary challenge we face is with the coefficients in the resulting local Lipschitz linear adjoint equation,

$$\left\{\begin{array}{rcl} -dx_t &=& f_y(t,Y_t,Z_t,u_t)x_tdt+f_z(t,Y_t,Z_t,u_t)x_tdW_t,\\ x_0 &=& g_y(Y_0), \end{array}\right. \tag{3.1.3}$$

which are only locally bounded. Consequently, they are locally Lipschitz on x but do not satisfy the linear growth condition. Given the existing results in the literature, confirmation regarding whether the adjoint equation (3.1.3) admits a unique solution remains elusive.

3.2 Foundational Concepts and Existence Findings

In this section, we will state some basic results related to the BSDEs theory and prove the existence and uniqueness results for one kind of linear SDEs with local Lipschitz coefficients.

Assumption 3.2.1.

(A.1.1) *f and g are continuously differentiable with respect to (y,z) and there exists a positive constant L such that : $|g(y)| \leq L(1+|y|)$.*

(A.1.2) *We posit the existence of a positive constant λ, large enough such that the expected value of $|\zeta|^{e^{\lambda T}+1}$ is finite.*

(A.1.3) *(i) The function f is continuous in (y,z).*

(ii) There exist constants η, c_0, c_1 such that : for every $t \geq 0$, y, $z, u \in U$:

$$|f(t,y,z,u)| \leq \eta + c_0|y||\ln|y|| + c_1|z|.$$

(A.1.4) *There exist a real-valued sequence $(A_N)_{N>1}$ and constants $M_0 \in \mathbb{R}_+$, $r>0$ such that :*

(i) $\forall\ N>1, \quad 1<A_N \leq N^r.$

(ii) $\lim_{N\to\infty} A_N = \infty.$

(iii) For every $N \in \mathbb{N}$, $u \in U$ and every y, y', z, z' such that $|y|$, $|y'|$, $|z|$, $|z'| \leq N$, we have :

$$\begin{aligned}&(y-y')\big(f(t,y,z,u)-f(t,y',z',u)\big)\\ &\leq M_0\big(\,|\,y-y'\,|^2 \ln(A_N) + |y-y'||z-z'|\sqrt{\ln(A_N)}\big).\end{aligned}$$

Remark 3.2.2. *If f satisfies (A.1.1), then it satisfies a local Lipschitz condition, i.e., for all $N \in \mathbb{N}$, there exist two constants $L_{1,N}$, $L_{2,N} > 0$ such that for any $u \in U$ and for those $y, y', z, z' \in \mathbb{R}$ with $\max\{|y|,|y'|,|z|,|z'|\} \leq N$, the following condition holds :*

$$|f(t,y,z,u) - f(t,y',z,u)| \leq L_{1,N}|y-y'|$$
$$|f(t,y,z,u) - f(t,y,z',u)| \leq L_{2,N}|z-z'|$$

Remark 3.2.3. *Assume that f satisfies (A.1.1) and (A.1.4). Consequently, $L_{1,N} = M_0 \ln(A_N)$, $L_{2,N} = M_0\sqrt{\ln(A_N)}$.*

Remark 3.2.4. *If f satisfies (A.1.3), then for every t,y,z and $u\in U$:*

$$|f(t,y,z,u)|\leq \widetilde{\eta}+c_0|y||\ln|y||+c_1|z|\sqrt{|\ln(|z|)|},$$

where $\widetilde{\eta}=\eta+c_1 e$.

The following lemmas establish estimates, guaranteeing the boundedness of both the generator and the solutions. The first two lemmas are thoroughly detailed and proven in [14], with further details provided in [25].

Lemma 3.2.5. *Let (A.1.2) and (A.1.3) be satisfied. Then, there exists a positive constant $C(T,\alpha,\eta,c_0,c_1)$ such that,*

$$\int_0^T \mathbb{E}\left[|f(s,Y_s,Z_s,u_s)|^\alpha\right] ds \leq C(T,\alpha,\eta,c_0,c_1)\left(1+\int_0^T \mathbb{E}\left[|Y_s|^{\mu_s+1}+|Z_s|^2\right] ds\right),$$

where $1<\alpha<2$.

Lemma 3.2.6. *Let $(Y_t,Z_t)_{t\geq 0}$ be the unique solutions of the BSDE (3.1.1). Then, there are two positive constants $C_{T,\eta}$, $C(T,c_0,c_1)$ such that, under Assumption 3.2.1 we have :*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|Y_t|^{e^{\lambda t}+1}\Big]\leq C_{T,\eta}\mathbb{E}\Big[1+|\zeta|^{e^{\lambda T}+1}\Big].$$

$$\int_0^T \mathbb{E}[|Z_s|^2]ds\leq C(T,\eta,c_0,c_1)\mathbb{E}\Big[1+|\zeta|^2+\sup_{0\leq t\leq T}|Y_t|^{e^{\lambda T}+1}\Big].$$

Lemma 3.2.7. *If the assumption of the previous Lemma 3.2.6 holds and if ζ is bounded, we can find constants $C_{1,T}$, $C_{2,T}$ and $C_{3,T}$, which depend on η, such that :*

(i) $\sup_{0\leq t\leq T}|Y_t|^{e^{\lambda t}+1}\leq C_{1,T}$, $\int_0^T\mathbb{E}[|Z_s|^2]ds\leq C_{2,T}$.

(ii) $\int_0^T\mathbb{E}[|f(s,Y_s,Z_s,u_s)|^2]ds\leq C_{3,T}$.

Proof: We derive the following insight from the work of Bahlali et al. [14].

$$|\,Y_t\,|^{e^{\lambda T}+1}\leq \ell(\eta)\Big(1+|\,\zeta\,|^{e^{\lambda T}+1}-\int_t^T(e^{\lambda s}+1)\,|\,Y_s\,|^{e^{\lambda s}}\,sgn(Y_s)Z_s dW_s\Big),$$

and

$$\int_t^T |Z_s|^2\ ds\leq \ell(\eta)\left(1+|\zeta|^2+\sup_{s\in[0,T]}|Y_s|^{e^{\lambda T}+1}+\int_t^T Y_s Z_s\ dW_s\right),$$

where $\ell(\eta)$ is a universal positive constant. We get the assertion (i) by taking the conditional expectation for Y and the expectation for the rest.

By (A.1.3) and assertion (i), and since $|Y_t|\leq 1+|Y_t|^{e^{\lambda t}+1}\leq 1+C_{1,T}$, we have

- $$\int_0^T\mathbb{E}[|f(s,Y_s,Z_s,u_s)|^2]ds\leq \ell(\eta)\left(1+\int_0^T\mathbb{E}\left[|Y_s|^{\mu_s+1}+|Z_s|^2\right]ds\right)\ \leq\ C_{3,T}.$$

□

Theorem 3.2.8. *Let Assumptions (A.1.2)–(A.1.4) hold, then the BSDE* (3.1.1) *has a unique solution* (Y,Z) *in* $S^{e^{\lambda T}+1}([0,T],\mathbb{R})\times\mathcal{M}^2([0,T],\mathbb{R})$.

Under Assumptions (A.1.2)–(A.1.4), the conditions in [14] that guarantee the existence and uniqueness of the BSDE solution are satisfied. Therefore, the preceding theorem is applicable.

It is important to observe that, for any $v\in\mathcal{U}_{ad}$, the functions $f_y(t,\cdot,\cdot,v_t)$ and $f_z(t,\cdot,\cdot,v_t)$ are generally unbounded.

In the subsequent theorem, we establish the existence and uniqueness outcomes for the Stochastic Differential Equation (SDE) given by (3.1.3) up to a potential explosion time.

Theorem 3.2.9. *Assuming that Assumption 3.2.1 is satisfied, we can assert that for any* $v\in\mathcal{U}_{ad}$, *the SDE* (3.1.3) *possesses a unique solution.*

Remark 3.2.10. *The previous theorem cannot guarantee the existence of a global solution but rather only up to an 'explosion time' denoted as*

$$\tau_N^{ex}:=\inf\{t\in[0,T];\ |f_y(t,y,z,u)|\wedge|f_z(t,y,z,u)|\geq N\}.$$

To ensure the existence of a global solution, we incorporate the subsequent additional assumptions.

— $\mathbf{H}_{loc}$: $f_y\in\mathbb{L}^1_{loc}(\mathbb{R}_+,\mathbb{R}),\quad f_z\in\mathbb{L}^2_{loc}(\mathbb{R}_+,\mathbb{R})$.

— $\mathbf{H}_{lin}$: There exists a positive constant $L>0$, such that $\forall\ (y,z,u)\in\mathbb{R}\times\mathbb{R}\times U$:

$$|f_y(t,y,z,u)|\leq L(1+|y|)+\epsilon\ln(|z|+1),\ \text{a.e. } t\in[0,T],$$
$$|f_z(t,y,z,u)|\leq L(1+|y|)+\epsilon\sqrt{\ln(|z|+1)}\ \text{a.e. } t\in[0,T],$$

where ϵ is a sufficiently small positive constant.

Remark 3.2.11. *The assumption* $\boldsymbol{H}_{loc}$ *ensures that for any* $(Y_t,Z_t)_{t\geq0}$ $\mathbb{F}$*-adapted stochastic processes, the SDE* (3.1.3) *has a global solution, while the assumption* $\boldsymbol{H}_{lin}$ *guarantees the global solution under square-integrable* $\mathbb{F}$*-adapted stochastic processes (i.e.,* $(Y_t,Z_t)_{t\geq0}\in\mathbb{L}^2_{loc}([0,T],\mathbb{R})$*).*

3.2.1 Statement of the Control Problem

The purpose of this paper is to deal with the control Problem **(A)** described by the equation (3.1.1) and the cost functional (3.1.2). The controller object is to derive a necessary condition as well as a sufficient condition of optimality. Notice that because the derivatives of f are not bounded, the standard duality technique can not be directly applicable in our setup.

For any $p\geq1$ and $v\in\mathcal{U}_{ad}$, let us first define a family of semi-norms $(\rho^v_{N,p}(f))_{N\in\mathbb{N}}$ by

$$\rho^v_{N,p}(f)=\left(\mathbb{E}\int_0^T\sup_{|y|,|z|\leq N}|f(r,y,z,v_s)|^p\,dr\right)^{\frac{1}{p}}.$$

Lemma 3.2.12. *Let f be a function which satisfies Assumption 3.2.1 and $\boldsymbol{H}_{loc}$ or $\boldsymbol{H}_{lin}$. Then, there exists a sequence of functions f^n such that :*

(i) For each n, f^n is globally Lipschitz in (y,z) -a.e. $t\in[0,T]$.

(ii) For each n, f^n satisfies Assumption 3.2.1.

(iii) For every n, $\rho^v_{n,p}(f^n-f)\to 0$ as $n\to\infty$.

(iv) For every n, $|f^n_y|\leq|f_y|+\frac{c}{n}|f|$, $|f^n_z|\ \leq|f_z|+\frac{c}{n}|f|$ and $\lim\limits_{n\to+\infty}g^n$ (resp. g^n_y) $\to g$ (resp. g_y).

The following paragraphs are dedicated to transforming the initial control Problem **(A)** into a series of control problems characterized by global Lipschitz coefficients. For this purpose, consider any fixed $n\in\mathbb{N}^*$ and $v\in\mathcal{U}_{ad}$. Let $\left(\bar{Y}^n_t,\bar{Z}^n_t\right)_{t\geq 0}$ represent the solution to the controlled BSDE :

$$\begin{cases} d\bar{Y}^n_t &= f^n(t,\bar{Y}^n_t,\bar{Z}^n_t,v_t)dt+\bar{Z}^n_t dW_t,\\ \bar{Y}^n_T &= \zeta. \end{cases} \tag{3.2.1}$$

Furthermore, define

$$\mathcal{J}^n(v.)=\mathbb{E}\left[g^n(\bar{Y}^n_0)\right]. \tag{3.2.2}$$

The subsequent lemma provides estimates that will be employed to establish a relationship between the control problem (3.2.1), (3.2.2) and Problem **(A)**.

Lemma 3.2.13. *Let $(Y_t)_{t\geq 0}$ and $(\bar{Y}^n_t)_{t\geq 0}$ be the solutions of BSDE (3.1.1) and (3.2.1), respectively, corresponding to the control $v\in\mathcal{U}_{ad}$. Then, for any $\alpha\in(1,2)$, $q\in(0,2)$ and any $\beta\in(1,3-\frac{2}{\alpha})$, the following estimates hold :*

(i) $\mathbb{E}[|\bar{Y}^n_t-Y_t|^\beta]\leq K_{n,N}$, and $\mathbb{E}[\int_t^T|\bar{Z}^n_r-Z_r|^q dr]\leq K_{n,N}$.

(ii) $|\mathcal{J}^n(v)-\mathcal{J}(v)|\leq C\varepsilon_{n,N}$,

where $K_{n,N}$ and $\varepsilon_{n,N}$ converge to 0 as n and N tend successively to $+\infty$, here N stands for the radius of the ball $B(0,N)$.

The proof of assertion (i) follows a similar methodology to that of Theorem 2.1 in [14], while assertion (ii) is derived using the approach outlined in [5].

Consider an optimal control u defined as the solution to :

$$\mathcal{J}(u.)=\inf_{v\in\mathcal{U}_{ad}}\mathcal{J}(v.),$$

subject to the constraint (3.1.1). It is crucial to note that u may not be optimal for the perturbed control problem. As Lemma 3.2.13 suggests, there exists a sequence (δ_n) of positive real numbers converging to 0 such that :

$$\mathcal{J}^n(u.) \leq \inf_{v \in U_{ad}} \mathcal{J}^n(v.) + \delta_{n,N}, \quad \delta_{n,N} \;=\; 2C\varepsilon_{n,N}.$$

To facilitate the application of Ekeland's lemma, let us introduce a metric d on $\mathcal{U}_{ad}$. For any two controls $u, v \in \mathcal{U}ad$, the metric d is given by :

$$\hat{d}(u., v.) = \mathbb{P} \otimes dt\{(\omega, t) \in \Omega \times [0, T] : u(\omega, t) \neq v(\omega, t)\},$$

where $\mathbb{P} \otimes dt$ is the product measure of $\mathbb{P}$ with the Lebesgue measure on $[0, T]$. By applying Ekeland's lemma to the continuous cost functional $\mathcal{J}^n(u.)$, we obtain an admissible control u^n satisfying :

$$\hat{d}(u^n_., u.) \leq (\delta_{n,N})^{\frac{1}{2}}$$

and

$$\tilde{\mathcal{J}}^n(u^n_.) \leq \tilde{\mathcal{J}}^n(v.) \quad \text{for any } v \in U_{ad},$$

where

$$\tilde{\mathcal{J}}^n(v.) = \mathcal{J}^n(v.) + (\delta_{n,N})^{\frac{1}{2}} \cdot \hat{d}(v., u^n_.).$$

From the preceding arguments, we can deduce that u^n solves the optimal control problem given by equations (3.2.1) and (3.2.2), but with the modified cost function $\tilde{\mathcal{J}}^n$. For every $n \in \mathbb{N}^*$, consider the pair $(Y^n_t, Z^n_t)_{t \geq 0}$, representing the distinctive solution to the subsequent BSDE under the influence of u^n :

$$\begin{cases} dY^n_t &= f^n(t, Y^n_t, Z^n_t, u^n_t)dt + Z^n_t dW_t, \\ Y^n_T &= \zeta. \end{cases} \tag{3.2.3}$$

Associated with this control problem is the following cost function :

$$\mathcal{J}^n(u^n_.) = \mathbb{E}\left[g^n(Y^n_0)\right]. \tag{3.2.4}$$

Now, we pose the following optimal control problem, denoted as Problem **(B)** : For each integer n, find $u^n \in \mathcal{U}_{ad}$ such that u^n minimizes the cost function (3.2.4) subject to (3.2.3).

In concluding this subsection, we introduce a set of controlled SDEs called adjoint equations. For each integer n, consider the following SDE :

$$\begin{cases} -dx^n_t &= f^n_y(t, Y^n_t, Z^n_t, u^n_t)x^n_t dt + f^n_z(t, Y^n_t, Z^n_t, u^n_t)x^n_t dW_t, \\ x^n_0 &= g^n_y(Y^n_0). \end{cases} \tag{3.2.5}$$

Given that f^n is a globally Lipschitz function, its derivatives f_y^n and f_z^n are bounded. Consequently, the coefficients of SDE (3.2.5) are globally Lipschitz and exhibit linear growth. This implies that, for each integer n, equation (3.2.5) possesses a unique solution.

Furthermore, we define a family of Hamiltonian functions $\mathcal{H}^n : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times U\to\mathbb{R}$ as follows :

$$\mathcal{H}^n(t,y,z,x,u) = xf^n(t,y,z,u) \text{ for each } n\in\mathbb{N}.$$

3.2.2 Preliminary Lemmas

In the following part of this subsection, we aim to consolidate and establish several helpful lemmas. These lemmas are pivotal as they will be utilized in the subsequent section to demonstrate the main results.

Lemma 3.2.14. *Let (f_n) be the sequence of functions associated to f by Lemma 3.2.12 and $(Y_t^n, Z_t^n)_{t\geq 0}$ stands for the solution of equation (3.2.3). Then, there exist constants $\bar{K}_1$, $\bar{K}_2$ and $\bar{K}_3$ such that :*

(i) $\sup_n \mathbb{E}[\sup_{0\leq t\leq T}|Y_t^n|^{e^{\lambda T}+1}]\leq \bar{K}_1$.
(ii) $\sup_n \mathbb{E}[\int_0^T |Z_s^n|^2 ds]\leq \bar{K}_2$.
(iii) $\sup_n \mathbb{E}[\int_0^T |f^n(s,Y_s^n,Z_s^n,u_s^n)|^\alpha ds]\leq \bar{K}_3$,

where $\alpha\in(1,2)$.

The proof of the following Lemma is outlined in [14].

Lemma 3.2.15. *Under Assumption 3.2.1, we have :*

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{t\in[0,T]}|Y_t^n - Y_t|^\beta\Big] = 0. \tag{3.2.6}$$

$$\lim_{n\to\infty}\mathbb{E}\int_0^T |Z_t^n - Z_t|^q\,dt = 0. \tag{3.2.7}$$

Lemma 3.2.16. *Under Assumption 3.2.1 and $\boldsymbol{H}_{lin}$ the following estimates hold*

$$\lim_{n\to\infty}\mathbb{E}\int_0^T |f^n(r,Y_r^n,Z_r^n,u_r^n) - f(r,Y_r,Z_r,u_r)|^{\bar{\alpha}}\,dr = 0. \tag{3.2.8}$$

$$\lim_{n\to\infty}\mathbb{E}\int_0^T \Big|f_y^n(r,Y_r^n,Z_r^n,u_r^n) - f_y(r,Y_r,Z_r,u_r)\Big|^q\,dr = 0. \tag{3.2.9}$$

$$\lim_{n\to\infty}\mathbb{E}\int_0^T|f_z^n(r,Y_r^n,Z_r^n,u_r^n)-f_z(r,Y_r,Z_r,u_r)|^q\,dr=0, \tag{3.2.10}$$

where $q\in(0,2)$ *and* $\bar{\alpha}\in(1,\alpha)$. □

Remark 3.2.17. *To demonstrate the convergence of a sequence* X_n *of random variables in* $\mathbb{L}^p$*, where* $p\geq 1$*, it suffices to establish convergence in probability and ensure that* $\{|X_n|^p,\ n\in\mathbb{N}^*\}$ *is uniformly integrable.*

Proof: [Proof of Lemma 3.2.16 :] Assuming Assumption 3.2.1 and $\mathbf{H}_{lin}$ hold. Drawing from our knowledge and the preceding remark, it is essential to demonstrate the convergence in $\mathbb{L}^1$.

$$\begin{aligned}&\mathbb{E}\int_0^T|f^n(r,Y_r^n,Z_r^n,u_r^n)-f(r,Y_r,Z_r,u_r)|dr\\ \leq\,&\mathbb{E}\int_0^T|f^n(r,Y_r^n,Z_r^n,u_r)-f(r,Y_r,Z_r,u_r)|dr\\ &+\mathbb{E}\int_0^T|f^n(r,Y_r^n,Z_r^n,u_r^n)-f^n(r,Y_r^n,Z_r^n,u_r)|\mathbb{1}_{\{u_r^n\neq u_r\}}dr\end{aligned}$$

Considering the previous derivation in [14], we have :

$$\lim_{n\to\infty}\mathbb{E}\int_0^T|f^n(r,Y_r^n,Z_r^n,u_r)-f(r,Y_r,Z_r,u_r)|dr=0.$$

Holder's inequality yields to

$$\begin{aligned}&\mathbb{E}\int_0^T|f^n(r,Y_r^n,Z_r^n,u_r^n)-f^n(r,Y_r^n,Z_r^n,u_r)|\mathbb{1}_{\{u_r^n\neq u_r\}}dr\\ \leq&\left(\mathbb{E}\int_0^T|f^n(r,Y_r^n,Z_r^n,u_r^n)-f^n(r,Y_r^n,Z_r^n,u_r)|^{\alpha}dr\right)^{\frac{1}{\alpha}}\left(\mathbb{E}\int_0^T\mathbb{1}_{\{u_r^n\neq u_r\}}dr\right)^{1-\frac{1}{\alpha}}\\ \leq&\left(4\bar{K}_3\right)^{\frac{1}{\alpha}}\left(\hat{d}(u_\cdot^n,u_\cdot)\right)^{1-\frac{1}{\alpha}}.\end{aligned}$$

$\hat{d}(u_\cdot^n,u_\cdot)$ approaches 0 as n tends to infinity, thus (3.2.8) is satisfied.

We give the proof of (3.2.9). The proof of (3.2.10) can be performed similarly. Since $|y||\ln|y||\leq e^{-1}+|y|^2$ and for any $n\in\mathbb{N}^*$, $t\in[0,T]$, we have $|Y_t^n|,\ |Z_t^n|\leq n$. Thus by **(H.3)**, we have for any $v\in\mathcal{U}_{ad}$ that,

$$\begin{aligned}\frac{1}{n^2}|f^n(r,Y_r^n,Z_r^n,v_r)|^2&\leq\frac{C}{n^2}(1+\eta^2+|Y_r^n|^4+|Z_r^n|^2)\\ &\leq C(1+\frac{1}{n^2}+\frac{\eta^2}{n^2}+|Y_r^n|^2).\end{aligned}$$

By (i) of Lemma 3.2.14, we get :

$$\sup_n\mathbb{E}\int_0^T\frac{1}{n^2}|f^n(r,Y_r^n,Z_r^n,v_r)|^2dr\leq C, \tag{3.2.11}$$

where C is a universal constant. Using assertion (iv) of Lemma 3.2.12, along with $\mathbf{H}_{lin}$ and (3.2.11), we obtain :

$$\sup_n \mathbb{E}\int_0^T (|f_y^n(r,Y_r^n,Z_r^n,v_r)|^2 + |f_z^n(r,Y_r^n,Z_r^n,v_r)|^2)dr \leq \bar{K}_4. \tag{3.2.12}$$

Let $N > 1$, we put $\Lambda_n^N := \{(r,\omega),\ |Y_r^n| + |Z_r^n| > N\}$ and $\bar{\Lambda}_n^N = \Omega \backslash \Lambda_n^N$, then we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \left| f_y^n(r,Y_r^n,Z_r^n,u_r^n) - f_y(r,Y_r,Z_r,u_r)\right| dr \\
\leq & \mathbb{E}\int_0^T \left| f_y^n(r,Y_r^n,Z_r^n,u_r^n) - f_y^n(r,Y_r^n,Z_r^n,u_r)\right| \mathbb{1}_{\{u_r^n \neq u_r\}} dr \\
&+ \mathbb{E}\int_0^T \left| f_y^n(r,Y_r^n,Z_r^n,u_r) - f_y(r,Y_r^n,Z_r^n,u_r)\right| dr \\
&+ \mathbb{E}\int_0^T |f_y(r,Y_r^n,Z_r^n,u_r) - f_y(r,Y_r,Z_r,u_r)| dr,
\end{aligned}$$

By Schwarz's inequality and $\mathbf{H}_{lin}$, we have :

$$\begin{aligned}
&\mathbb{E}\int_0^T |f_y^n(r,Y_r^n,Z_r^n,u_r^n) - f_y^n(r,Y_r^n,Z_r^n,u_r)| \mathbb{1}_{\{u_r^n \neq u_r\}} dr \\
\leq & 2\mathbb{E}\int_0^T (L(1+|Y_r^n|) + \epsilon \ln(|Z_r^n|+1))\, \mathbb{1}_{\{u_r^n \neq u_r\}} dr \\
\leq & 2L\mathbb{E}\int_0^T (2+|Y_r^n|+|Z_r^n|)\, \mathbb{1}_{\{u_r^n \neq u_r\}} dr \\
\leq & 2L\left(8T + 4\mathbb{E}\int_0^T (|Y_r^n|^2 + |Z_r^n|^2)dr\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T \mathbb{1}_{\{u_r^n \neq u_r\}} dr\right)^{\frac{1}{2}} \\
\leq & 4L\left(2T + T\bar{K}_1 + \bar{K}_2\right)^{\frac{1}{2}} \left(\hat{d}(u_\cdot^n, u_\cdot)\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |f_y^n(r,Y_r^n,Z_r^n,u_r^n) - f_y^n(r,Y_r^n,Z_r^n,u_r)| \mathbb{1}_{\{u_r^n \neq u_r\}} dr = 0.$$

Due to the fact that $\mathbb{1}_{A^N} < \frac{|Y_r^n|+|Z_r^n|}{N}\mathbb{1}_{A^N}$, and by using Schwarz's inequality we obtain :

$$\begin{aligned}
\mathbb{E}\int_0^T \left|(f_y^n - f_y)(r,Y_r^n,Z_r^n,u_r)\right| dr \leq\ & \rho_{N,1}^u(f_y^n - f_y) \\
&+ \frac{2(T\bar{K}_1 + \bar{K}_2)^{\frac{1}{2}}}{N}\left(\mathbb{E}\int_0^T \left|(f_y^n - f_y)(r,Y_r^n,Z_r^n,u_r)\right|^2 dr\right)^{\frac{1}{2}}
\end{aligned}$$

By (3.2.12), we can assert the existence of a positive constant ℓ, such that :

$$\mathbb{E}\int_0^T \left|(f_y^n - f_y)(r,Y_r^n,Z_r^n,u_r)\right| dr \leq \rho_{N,1}^u(f_y^n - f_y) + \ell\Big(\frac{2(T\bar{K}_1 + \bar{K}_2)}{N}(\bar{K}_4)^{\frac{1}{2}}\Big).$$

Taking the limit first with respect to n and then for N, we obtain,

$$\lim_{n\to\infty}\mathbb{E}\int_0^T\left|(f_y^n-f_y)(r,Y_r^n,Z_r^n,u_r)\right|dr=0.$$

Assumption $\mathbf{H}_{lin}$ and Lemma 3.2.14 enable the use of the Lebesgue Dominated Convergence Theorem, which facilitates the demonstration that :

$$\lim_{n\to\infty}\mathbb{E}\int_0^T|f_y(r,Y_r^n,Z_r^n,u_r)-f_y(r,Y_r,Z_r,u_r)|\,dr=0.$$

Hence, (3.2.9) is established. □

Assumption 3.2.18. *The validity of Assumption 3.2.1 in conjunction with $\boldsymbol{H}_{lin}$, along with the constraint that ζ is bounded.*

Lemma 3.2.19. *Assume that Assumption 3.2.18 holds. Let $(Y_t,Z_t)_{t\geq 0}$ (resp. $(Y_t^n,Z_t^n)_{t\geq 0}$) denote the unique solutions of the BSDE (3.1.1) (resp. (3.2.3)). Then, for any $v\in\mathcal{U}_{ad}$ and $p\geq 2$ there exists a universal constant C, such that,*

$$\mathbb{E}\int_0^T(|f|^2+|f_y|^p+|f_z|^p)(r,Y_r,Z_r,v_r)dr\leq C,$$
$$\sup_n\mathbb{E}\int_0^T(|f^n|^2+|f_y^n|^p+|f_z^n|^p)(r,Y_r^n,Z_r^n,v_r)dr\leq C.$$

Proof: By assertion (i) of Lemma 3.2.7, we have Y is bounded. Moreover,

$$\ln(|z|+1)=\frac{p}{2}\ln(|z|+1)^{\frac{2}{p}}\ \leq\ \frac{p}{2}(|z|+1)^{\frac{2}{p}}.$$

Thus $(\ln(|z|+1))^p\leq C(|z|^2+1)$. By $\mathbf{H}_{lin}$ and Lemma 3.2.7 we get :

$$\mathbb{E}\int_0^T(|f|^2+|f_y|^p+|f_z|^p)(r,Y_r,Z_r,v_r)dr\leq C.$$

For any $n\in\mathbb{N}^*$ and $t\in[0,T]$, we have $|Y_t^n|\leq C_{1,T}$. Since $|Z_t^n|\leq n$, Assumption (A.1.3) yields,

$$\frac{1}{n^p}|f^n(r,Y_r^n,Z_r^n,v_r)|^p\ \leq\ C\ \text{and}\ |f^n(r,Y_r^n,Z_r^n,v_r)|^2\ \leq\ C(1+|Z_r^n|^2).$$

Thus, by assertion (iv) of Lemma 3.2.12, assertion (ii) of Lemma 3.2.14 and the previous result, we have :

$$\sup_n\mathbb{E}\int_0^T(|f^n|^2+|f_y^n|^p+|f_z^n|^p)(r,Y_r^n,Z_r^n,v_r)dr\leq C.$$

□

Remark 3.2.20. *If Assumption 3.2.18 holds, then for any $\alpha \in (1,2)$ and $p \geq 2$, Lemma 3.2.16 and Lemma 3.2.19 guarantee the following convergence :*

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |f^n(r,Y_r^n,Z_r^n,u_r^n) - f(r,Y_r,Z_r,u_r)|^\alpha \, dr = 0. \tag{3.2.13}$$

$$\lim_{n\to\infty} \mathbb{E}\int_0^T \left|f_y^n(r,Y_r^n,Z_r^n,u_r^n) - f_y(r,Y_r,Z_r,u_r)\right|^p dr = 0. \tag{3.2.14}$$

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |f_z^n(r,Y_r^n,Z_r^n,u_r^n) - f_z(r,Y_r,Z_r,u_r)|^p \, dr = 0. \tag{3.2.15}$$

Lemma 3.2.21. *Under the fulfillment of Assumptions 3.2.18, the solutions x and x^n to equations (3.1.3) and (3.2.5), respectively, are bounded in the space $S^p([0,T],\mathbb{R})$ for all $p \geq 2$. In other words, there exist two positive constants ℓ_T and $\overline{\ell}_T$ such that :*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |x_t|^p\right] \leq \ell_T,$$
$$\mathbb{E}\left[\sup_{0\leq t\leq T} |x_t^n|^p\right] \leq \overline{\ell}_T \;\forall n \in \mathbb{N}.$$

Proof: Let $p \geq 2$ By Itô's formula, we have $(sgn(x_t)x_t = |x_t|)$:

$$\begin{aligned}
|x_t|^p &\leq |g_y(Y_0)|^p + p\int_0^T |x_s|^p \big(|f_y| + \frac{p-1}{2}|f_z|^2\big)(s,Y_s,Z_s,u_s)ds \\
&\quad + \Big|\int_0^t |x_s|^p f_z(s,Y_s,Z_s,u_s)dW_s\Big| \\
&\leq |g_y(Y_0)|^p + p\int_0^T \sup_{0\leq r\leq s}\{|x_r|^p\}\big(|f_y| + \frac{p-1}{2}|f_z|^2\big)(s,Y_s,Z_s,u_s)ds \\
&\quad + \Big|\int_0^t |x_s|^p f_z(s,Y_s,Z_s,u_s)dW_s\Big|
\end{aligned}$$

By BDG's inequality

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t |x_s|^p f_z(s,Y_s,Z_s,u_s)dW_s\right|\right]\\
&\le 3\mathbb{E}\left[\left(\int_0^T |x_s|^{2p}|f_z(s,Y_s,Z_s,u_s)|^2ds\right)^{\frac{1}{2}}\right]\\
&\le 3\mathbb{E}\left[\left(\int_0^T \sup_{0\le r\le s}\{|x_r|^{2p}\}|f_z(s,Y_s,Z_s,u_s)|^2ds\right)^{\frac{1}{2}}\right]\\
&\le 3\mathbb{E}\left[\sup_{0\le t\le T}\{|x_t|\}^{\frac{p}{2}}\left(\int_0^T \sup_{0\le r\le s}\{|x_r|^{p}\}|f_z(s,Y_s,Z_s,u_s)|^2ds\right)^{\frac{1}{2}}\right]\\
&\le 3\mathbb{E}\left[\sup_{0\le t\le T}\{|x_t|^{\frac{p}{2}}\}\left(\int_0^T \sup_{0\le r\le s}\{|x_r|^{p}\}|f_z(s,Y_s,Z_s,u_s)|^2ds\right)^{\frac{1}{2}}\right]\\
&\le \mathbb{E}\left[\frac{1}{2}\sup_{0\le t\le T}\{|x_t|^{p}\}+\frac{9}{2}\int_0^T \sup_{0\le r\le s}\{|x_r|^{p}\}|f_z(s,Y_s,Z_s,u_s)|^2ds\right],
\end{aligned}$$

the last inequality is obtained by using Young's ($ab\le \frac{1}{6}a^2+\frac{3}{2}b^2$), therefore

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T}|x_t|^p\Big]\le \mathbb{E}\Big[2|g_y(Y_0)|^p+\int_0^T \sup_{0\le r\le s}\{|x_r|^p\}\big(2p|f_y(s,Y_s,Z_s,u_s)|\\
+(p(p-1)+9)|f_z(s,Y_s,Z_s,u_s)|^2\big)ds\Big].
\end{aligned}$$

Gronwall's lemma, yields

$$\mathbb{E}\Big[\sup_{0\le t\le T}|x_t|^p\Big]\le 2\mathbb{E}\left[|g_y(Y_0)|^p\exp\left(\int_0^T \big(2p|f_y|+(p(p-1)+9)|f_z|^2\big)(s,Y_s,Z_s,u_s)ds\right)\right].$$

Since g_y is locally bounded and $Y_0, Y_0^n\le C_{1,T}$ (where $C_{1,T}$ does not depend on n), $g_y(Y_0)$ and $g_y(Y_0^n)$ are bounded. Moreover, by $\mathbf{H}_{lin}$, we have :

$$\mathbb{E}\Big[\sup_{0\le t\le T}|x_t|^p\Big]\le C\mathbb{E}\left[\exp\left(\int_0^T \big(2p\epsilon\ln(|Z_s|+1)+(p(p-1)+9)\epsilon^2\ln(|Z_s|+1)\big)ds\right)\right],$$

where C is a constant that may vary. Since ϵ is sufficiently small therefore $2p\epsilon+(p(p-1)+9)\epsilon^2\le 2$. Thus, by Jensen's inequality, we get :

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T}|x_t|^p\Big]&\le C\mathbb{E}\left[\exp\left(\int_0^T \ln(|Z_s|+1)^2ds\right)\right]\\
&\le C\Big(1+\int_0^T \mathbb{E}[|Z_s|^2]ds\Big)\ =:\ell_T.
\end{aligned}$$

Following the same arguments as previously, and since $\frac{1}{n^p}|f^n(r,Y_r^n,Z_r^n,v_r)|^p\le C$, we have

$$\sup_n\mathbb{E}\Big[\sup_{0\le t\le T}|x_t^n|^p\Big]\le \bar{\ell}_T.$$

□

Lemma 3.2.22. *Let $(x_t)_{t\geq 0}$ and $(x_t^n)_{t\geq 0}$ be respectively the solution of (3.1.3) and (3.2.5), then under Assumption 3.2.18, we have*

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[|x_t^n - x_t|^p\right] = 0, \quad \forall\ p\geq 2. \tag{3.2.16}$$

Proof: Lemma 3.2.21 implies that $\{|x_t^n|^p,\ t\in[0,T],\ n\in\mathbb{N}^*,\ p\geq 2\}$ is uniformly integrable. Based on equations (3.1.3) and (3.2.5), applying Itô's formula, we get :

$$\begin{aligned}|x_t^n - x_t|^2 \leq |g_y^n(Y_0^n) - g_y(Y_0)|^2 &\\ +2\int_0^T |x_r^n - x_r||x_r^n f_y^n(r,Y_r^n,Z_r^n,u_r^n) - x_r f_y(r,Y_r,Z_r,u_r)|dr &\\ +\int_0^T |x_r^n f_z^n(r,Y_r^n,Z_r^n,u_r^n) - x_r f_z(r,Y_r,Z_r,u_r)|^2 dr &\\ -2\int_0^t (x_r^n - x_r)(x_r^n f_z^n(r,Y_r^n,Z_r^n,u_r^n) - x_r f_z(r,Y_r,Z_r,u_r))dW_r.&\end{aligned}$$

By using Young's inequality and taking the expectation, we obtain,

$$\begin{aligned}\mathbb{E}\left[|x_t^n - x_t|^2\right] \leq \mathbb{E}\left[|g_y^n(Y_0^n) - g_y(Y_0)|^2\right] &\\ +2\mathbb{E}\left[\int_0^T |x_r^n - x_r|^2(|f_y^n| + |f_z^n|(r,Y_r^n,Z_r^n,u_r^n)dr\right] &\\ +2\mathbb{E}\left[\int_0^T |x_r^n - x_r||x_r||f_y^n(r,Y_r^n,Z_r^n,u_r^n) - f_y(r,Y_r,Z_r,u_r)|dr\right] &\\ +2\mathbb{E}\left[\int_0^T |x_r|^2|f_z^n(r,Y_r^n,Z_r^n,u_r^n) - f_z(r,Y_r,Z_r,u_r)|^2 dr\right].&\end{aligned}$$

Since for any $n\in\mathbb{N}^*$ and $p\geq 2$, $\mathbb{E}[\sup_{0\leq t\leq T}(|x_t|^p + |x_t^n|^p)] \leq \ell_T + \overline{\ell}_T$. By Hölder's inequality, we get a universal constant C, such that :

$$\begin{aligned}\mathbb{E}\left[|x_t^n - x_t|^2\right] \leq \mathbb{E}\left[|g_y^n(Y_0^n) - g_y(Y_0)|^2\right] + C\gamma^n &\\ +2\mathbb{E}\left[\int_0^T |x_r^n - x_r|^2(|f_y^n| + |f_z^n|^2)(r,Y_r^n,Z_r^n,u_r^n)dr\right],&\end{aligned}$$

where,

$$\begin{aligned}\gamma^n := \mathbb{E}\Big[\int_0^T (|f_y^n(r,Y_r^n,Z_r^n,u_r^n) - f_y(r,Y_r,Z_r,u_r)|^2 &\\ +|f_z^n(r,Y_r^n,Z_r^n,u_r^n) - f_z(r,Y_r,Z_r,u_r)|^4)dr\Big].&\end{aligned}$$

γ^n tend to zero as n approaches infinity, as indicated by (3.2.14) and (3.2.15). Moreover, with the same steps as the proof of Lemma 3.2.21, we can obtain the following :

$$\sup_n \mathbb{E}\Big[\exp\Big(2\int_0^T (|f_y^n| + |f_z^n|^2)(r,Y_r^n,Z_r^n,u_r^n)dr\Big)\Big] \leq C.$$

Establishing the desired result is facilitated by demonstrating the convergence of the initial terms to zero and applying Gronwall's lemma. Since $g_y(Y_0)$, $g_y^n(Y_0)$ and $g_y(Y_0^n)$ are bounded, allowing us to use the Dominated Convergence Theorem. Furthermore, by (iv) of Lemma 3.2.12 and equation (3.2.6), we obtain :

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left|g_y^n(Y_0^n)-g_y(Y_0)\right|^2 &\leq 2\lim_{n\to\infty}\mathbb{E}\left[|g_y^n(Y_0^n)-g_y(Y_0^n)|^2+|g_y(Y_0^n)-g_y(Y_0)|^2\right]\\ &=0.\end{aligned}$$

□

3.3 Optimality : The Maximum Principle

This section aims to derive the necessary optimality conditions for the control problem denoted as (**A**).

3.3.1 Necessary Condition for Optimality

We rely on the following lemma to establish the necessary conditions for optimality, which forms the foundation for our further investigation.

Lemma 3.3.1. *Under the fulfillment of Assumption 3.2.18, we can establish the following :*

$$\lim_{n\to\infty}\mathbb{E}\int_0^T|\Phi^n(r)-\Phi(r)|\,dr=0,$$

where

$$\Phi^n(r)=\left[\mathcal{H}^n(r,Y_r^n,Z_r^n,x_r^n,u_r^n)-\mathcal{H}^n(r,Y_r^n,Z_r^n,x_r^n,v_r)\right],$$

and

$$\Phi(r)=\left[\mathcal{H}(r,Y_r,Z_r,x_r,u_r)-\mathcal{H}(r,Y_r,Z_r,x_r,v_r)\right].$$

Proof: A straightforward computation demonstrates that :

$$\begin{aligned}\mathbb{E}\int_0^T|\Phi^n(r)-\Phi(r)|\,dr &\leq \mathbb{E}\int_0^T|f^n(r,Y_r^n,Z_r^n,u_r^n)x_r^n-f(r,Y_r,Z_r,u_r)x_r|\,dr\\ &+\mathbb{E}\int_0^T|f^n(r,Y_r^n,Z_r^n,v_r)x_r^n-f(r,Y_r,Z_r,v_r)x_r|\,dr\end{aligned}$$

For the sake of simplicity, we denote the first and the second integrals by I_1^n and I_2^n, respectively, and demonstrate their convergence to 0 as n goes to ∞.

By applying Hölder's inequality (for $\bar{\alpha} \in (1,\ \alpha)$, $\underline{\alpha} = \frac{\bar{\alpha}}{\bar{\alpha}-1}$) and utilizing both (3.2.21) and property (iii) from Lemma 3.2.14, we obtain :

$$\begin{aligned}
I_1^n &\leq \left[\mathbb{E}\int_0^T |f^n(r, Y_r^n, Z_r^n, u_r^n)|^{\bar{\alpha}}\, dr\right]^{\frac{1}{\bar{\alpha}}} \left[\mathbb{E}\int_0^T |x_r^n - x_r|^{\underline{\alpha}}\, dr\right]^{\frac{1}{\underline{\alpha}}} \\
&+ \left[\mathbb{E}\int_0^T |x_r|^{\underline{\alpha}}\, dr\right]^{\frac{1}{\underline{\alpha}}} \left[\mathbb{E}\int_0^T |f^n(r, Y_r^n, Z_r^n, u_r^n) - f^n(r, Y_r, Z_r, u_r)|^{\bar{\alpha}}\, dr\right]^{\frac{1}{\bar{\alpha}}} \\
&\leq \bar{K}_3^{\frac{1}{\bar{\alpha}}} \left[\mathbb{E}\int_0^T |x_r^n - x_r|^{\underline{\alpha}}\, dr\right]^{\frac{1}{\underline{\alpha}}} \\
&+ T\ell_T \left[\mathbb{E}\int_0^T |f^n(r, Y_r^n, Z_r^n, u_r^n) - f(r, Y_r, Z_r, u_r)|^{\bar{\alpha}}\, dr\right]^{\frac{1}{\bar{\alpha}}}
\end{aligned}$$

By (3.2.8) and (3.2.16), I_1^n converges to 0 as $n \to \infty$. On the flip side, utilizing similar arguments as presented earlier, it becomes apparent that the limit of I_2^n tends to 0 as n approaches $+\infty$. This concludes the proof. □

The primary result in this paper.

Theorem 3.3.2. *Consider the optimal solution $(Y_t, Z_t, u_t)_{t\geq 0}$ for the initial stochastic control problem. There exists a unique adapted process $(x_t)_{t\geq 0}$ in $\mathcal{S}^2([0,T],\mathbb{R})$, which is the solution to the associated forward stochastic differential equation (3.1.3). This process $(x_t)_{t\geq 0}$ is uniquely characterized by ensuring that the Hamiltonian $\mathcal{H}$ is minimized at the control $(u_t)_{t\geq 0}$, such that*

$$\mathcal{H}(t, Y_t, Z_t, x_t, u_t) = \min_{v\in\mathcal{U}_{ad}} \mathcal{H}(t, Y_t, Z_t, x_t, v_t) \ dt\text{-a.e.},\ \mathbb{P}\text{-a.s.} \tag{3.3.1}$$

Proof: To elucidate the key steps in our proof, we begin by transforming Problem **(A)** into a more manageable Problem **(B)**. Next, we employ the spike variation method to derive the necessary condition for near-optimality while addressing Problem **(B)**. Finally, leveraging Lemma 3.3.1 and taking appropriate limits, we culminate the desired optimality condition (3.3.1).

For each integer n, let $u^n \in \mathcal{U}_{ad}$ be an optimal control for Problem **(B)**, satisfying $\mathcal{J}^n(u_\cdot^n) \leq \inf_{v\in\mathcal{U}_{ad}} \mathcal{J}^n(v_\cdot)$. Denote the solution of BSDE (3.2.3) as $(Y_t^n, Z_t^n)_{t\geq 0}$ corresponding to u^n. Introduce the spike variation :

$$u_t^{n,\theta} = \begin{cases} v \text{ if } t \in [t_0, t_0 + \theta], \\ u_t^n \ \text{ otherwise.} \end{cases}$$

where $0 \leq t_0 \leq T$ is fixed, $\theta > 0$ is sufficiently small, and v is an arbitrary $\mathcal{F}_{t_0}$-measurable random variable.

The inequalities

$$\tilde{\mathcal{J}}^n(u_\cdot^n) \leq \tilde{\mathcal{J}}^n(u_\cdot^{n,\theta})$$

and

$$\hat{d}(u^{n,\theta}_{\cdot}, u^{n}_{\cdot}) \leq \theta$$

imply

$$\mathcal{J}^n(u^{n,\theta}_{\cdot}) - \mathcal{J}^n(u^{n}_{\cdot}) \geq -(\delta_{n,N})^{\frac{1}{2}}\theta. \tag{3.3.2}$$

Utilizing standard arguments (see, for instance, [95]), we can show that the left-hand side of the inequality (3.3.2) is equal to

$$\mathbb{E}\int_{t_0}^{t_0+\theta} [\mathcal{H}^n(r, Y^n_r, Z^n_r, x^n_r, v_r) - \mathcal{H}^n(r, Y^n_r, Z^n_r, x^n_r, u^n_r)]\, dr + o(\theta).$$

Dividing both sides of the inequality (3.3.2) by θ, we obtain

$$-(\delta_{n,N})^{\frac{1}{2}} \leq \frac{1}{\theta}\mathbb{E}\int_{t_0}^{t_0+\theta} [\mathcal{H}^n(r, Y^n_r, Z^n_r, x^n_r, v_r) - \mathcal{H}^n(r, Y^n_r, Z^n_r, x^n_r, u^n_r)]\, dr + \frac{o(\theta)}{\theta}.$$

Applying Lemma 3.3.1 and successively taking limits on n, N, and θ, while considering the arbitrary nature of t_0 in $[0, T]$, yields

$$\mathbb{E}[\mathcal{H}(t, Y_t, Z_t, x_t, v) - \mathcal{H}(t, Y_t, Z_t, x_t, u_t)] \geq 0.$$

Now, let $a \in U$ be a deterministic element, and B be an arbitrary element of the $\sigma-$algebra $\mathcal{F}_t$. Define

$$w_t = a\mathbb{1}_B + u_t\mathbb{1}_{\Omega|B}.$$

The control w satisfies the admissibility criteria. Utilizing the aforementioned inequality with w, we infer

$$\mathbb{E}[\mathbb{1}_B(\mathcal{H}(t, Y_t, Z_t, x_t, a) - \mathcal{H}(t, Y_t, Z_t, x_t, u_t))] \geq 0, \quad \forall\, B \in \mathcal{F}_t,$$

which leads to

$$\mathbb{E}^{\mathcal{F}_t}[\mathcal{H}(t, Y_t, Z_t, x_t, a) - \mathcal{H}(t, Y_t, Z_t, x_t, u_t)] \geq 0$$

The quantity within the conditional expectation is $\mathcal{F}_t$-measurable, and consequently, the result is immediately established. This concludes the proof of the theorem. □

3.3.2 Sufficient Condition of Optimality

This section investigates the extension of a previously established necessary optimality condition (3.3.1) to serve as a sufficient condition under additional assumptions.

Theorem 3.3.3. *Let the mapping $(y,z,u) \mapsto \mathcal{H}(t,y,z,x,u)$ is convex almost everywhere for $t \in [0,T]$, and f satisfies the Lipschitz condition with respect to u. Additionally, assume g is convex. If the previously established necessary optimality condition (3.3.1) is met, then $(u_t)_{t\geq 0}$ is optimal for the Problem **(A)**.*

Proof: Let u satisfy the condition in Equation (3.3.1). Note that u does not necessarily satisfy the necessary condition for optimality for the perturbed control problem (3.2.3) and (3.2.4).

Let B be an arbitrary element of the $\sigma-$algebra $\mathcal{F}_t$. Furthermore, define $\mathcal{I}_n(u)$ as :

$$\mathcal{I}_n(u) = \mathbb{E}[\mathcal{H}^n(t, Y_t^n, Z_t^n, x_t^n, u_t)\mathbb{1}_B].$$

Using convergence results, a simple computation shows that :

$$\mathcal{I}_n(u) = \min_{v\in\mathcal{U}_{ad}} \mathcal{I}_n(v) + \delta_n,$$

where δ_n is a sequence of positive real numbers converging to 0.

Applying Ekeland's variational principle to $\mathcal{I}_n$, there exists an admissible control u^n such that :

$$\mathcal{I}_{n,\delta}(v) = \mathcal{I}_n(v) + \sqrt{\delta_n}\hat{d}(v_., u_.^n),$$

We want to show that u is an optimal control for the original cost function $\mathcal{J}$.

(i) u^n minimizes $\mathcal{I}_{n,\delta}$:

$$\mathcal{I}_{n,\delta}(u^n) \leq \mathcal{I}_{n,\delta}(v), \quad \text{for any } v \in \mathcal{U}_{ad}.$$

(ii) The distance between u^n and u is bounded by :

$$\hat{d}(u_.^n, u_.) \leq \sqrt{\delta_n}.$$

(iii) Following the results from [41] (since f^n is globally Lipschitz and x^n is bounded, $\mathcal{H}^n$ is also globally Lipschitz), we obtain :

$$\mathcal{J}_\delta^n(u_.^n) = \min_{v\in\mathcal{U}_{ad}} \mathcal{J}_\delta^n(v_.),$$

This definition of the modified cost-functional

$$\mathcal{J}_\delta^n(v_.) := \mathcal{J}^n(v_.) + \sqrt{\delta_n}\hat{d}(v_., u_.^n),$$

allows us to conclude that for each admissible control $v \in \mathcal{U}_{ad}$,

$$\mathcal{J}^n(u_.^n) \leq \mathcal{J}^n(v.) + O(\delta_n),$$

where $O(\delta_n)$ represents terms that vanish as δ_n approaches zero.

According to assertion (ii) in Lemma 3.2.13, $\mathcal{J}^n(v.)$ converges to $\mathcal{J}(v.)$ as n tends to infinity. Moreover, we have

$$\begin{aligned}|\mathcal{J}^n(u_.^n) - \mathcal{J}(u.)| &\leq \mathbb{E}[|g^n(Y_0^n) - g(Y_0)|]\\ &\leq \mathbb{E}[|g^n(Y_0^n) - g^n(Y_0)|] + \mathbb{E}[|g^n(Y_0) - g(Y_0)|]\\ &\leq C\mathbb{E}[|Y_0^n - Y_0|] + \mathbb{E}[|g^n(Y_0) - g(Y_0)|]\end{aligned}$$

Since g (respectively, g^n) has linear growth and Y_0 (respectively, Y_0^n) is bounded, this allows us to use the Dominated Convergence Theorem. By assertion (iv) of Lemma 3.2.12 and Lemma 3.2.15, we obtain : $\lim_{n\to+\infty} \mathcal{J}^n(u_.^n) \to \mathcal{J}(u.)$. Thus,

$$\mathcal{J}(u.) = \min_{u\in\mathcal{U}_{ad}} \mathcal{J}(v.),$$

which implies that u is an optimal control for the cost function $\mathcal{J}$. □

Assumption 3.3.4.

(A.3.1) *f and g are continuously differentiable with respect to (y,z) and f is globally Lipschitz with respect to v.*

(A.3.2) *Assume that ζ is bounded and an element of $\mathbb{D}^{1,2}$, and there exist two constants M_1 and M_2 such that, for all $v \in \mathcal{U}_{ad}$, we have :*

$$\int_0^T |D_r v_s| ds \leq M_1, \text{ and } |D_r\zeta| \leq M_2, \ \forall\ r \leq T.$$

(A.3.3) *There exists a positive constants c such that, for every $t,\ y,\ z,\ v \in U$:*

$$|f(t,y,z,v)| \leq c(1 + |y||\ln|y|| + |z|\sqrt{|\ln(|z|)|}).$$

(A.3.4) *There exists a positive constant $L > 0$, such that $\forall\ (y,z,v) \in \mathbb{R}\times\mathbb{R}\times U$:*

$$|f_y(t,y,z,v)| \leq L(1+|y|) + \ln(|z|+1), \text{ -a.e. } t \in [0,T].$$

Theorem 3.3.5. *Assuming conditions (A.3.2) and (A.3.3) hold, the BSDE (3.1.1) possesses at least one solution (Y,Z) in $S^{e^{\lambda T}+1}([0,T],\mathbb{R}) \times \mathcal{M}^2([0,T],\mathbb{R})$.*

The proof follows directly from Theorem 2.2 in [14], as Assumptions (A.3.2) and (A.3.3) imply the conditions (H1) and (H2) in [14]. Therefore, the BSDE (3.1.1) has at least one solution.

Lemma 3.3.6. *If Assumption 3.3.4 holds, we can get constants $C_{1,T}$, $C_{2,T}$ and $C_{3,T}$ such that :*

(i) $\sup_{0\leq t\leq T}|Y_t| \leq C_{1,T}$.

(ii) $\sup_{0\leq t\leq T}|Z_t| \leq C_{2,T}$.

(iii) $\sup_{0\leq t\leq T}|f(t,Y_t,Z_t,v_t)| \leq C_{3,T}$.

Proof: Following the same steps used in the proof of Lemma 3.2.7, we can show that assertion (i) also holds.
We aim to substantiate assertion (ii). Let $N\in\mathbb{N}^*$ and $f^N(t,y,z,v)=f(t,y,z,v)\psi(\frac{z}{N})$, where $\psi(x)=1$ if $|x|\leq 1$ and $\psi(x)=0$ if $|x|\geq 2$. Clearly that f^N satisfies Assumption 3.3.4, thus

$$\begin{cases} dY_t &= f^N(t,Y_t,Z_t,v_t)dt+Z_t dW_t,\\ Y_T &= \zeta, \end{cases}$$

has at least one solution $(Y,Z)\in S^{e^{\lambda T}+1}([0,T],\mathbb{R})\times\mathcal{M}^2([0,T],\mathbb{R})$. Moreover, $\sup_{0\leq t\leq T}|Y_t|\leq C_{1,T}$. According to Proposition 2.2 in [79], we have for all $t\leq T$, Y_t and Z_t are the elements of $\mathbb{D}^{1,2}$. Furthermore, for all $r\in[0,T]$ the pair $(D_rY_t,D_rZ_t)_{t\leq T}$ satisfies,

$$\begin{aligned} D_rY_t = D_r\zeta - &\int_t^T (f_y^N(s,Y_s,Z_s,v_s)D_rY_s + f_z^N(s,Y_s,Z_s,v_s)D_rZ_s)ds\\ &-\int_t^T A_sD_rv_sds - \int_t^T D_rZ_sdW_s\\ D_tY_t = Z_t,& \end{aligned}$$

where A_s is a bounded process, with the bound denoted by a constant M_3 [65]. We define a process $\gamma^{f_z}=(\gamma_t^{f_z})_{0\leq t\leq T}$ as,

$$\gamma_t^{f_z} := \mathcal{E}\left(-\int_0^t f_z^N(s,Y_s,Z_s,v_s)dW_s\right),\ t\in[0,T],\ \mathbb{P}\ a.s.,$$

where $\mathcal{E}$ denotes the stochastic exponential. Since f_z^N is uniformly bounded it follows that, the process $(\gamma_t^{f_z})_{0\leq t\leq T}$ is a martingale process. Moreover, $\mathbb{E}[|\gamma_t^{f_z}|^2]<\infty$. Let $\gamma_t^{f_z} := \frac{d\mathbb{P}^{f_z}}{d\mathbb{P}}|_{\mathcal{F}_t}$, this implies absolute continuity of $\mathbb{P}^{f_z}$ with respect to $\mathbb{P}$ under Girsanov's theorem.

Girsanov's theorem further establishes that :

$$W_t^{f_z} = W_t + \int_0^t f_z^N(s,Y_s,Z_s,v_s)ds,\ \text{for}\ t\in[0,T]$$

is a Brownian motion under $\mathbb{P}^{f_z}$. Therefore, under $\mathbb{P}^{f_z}$ we have

$$\begin{aligned} D_rY_t =& D_r\zeta - \int_t^T \left(f_y^N(s,Y_s,Z_s,v_s)D_rY_s + A_sD_rv_s\right)ds - \int_t^T D_rZ_sdW_s^{f_z} \quad t\leq T, \\ D_rY_t =& 0 \quad r>t. \end{aligned} \tag{3.3.3}$$

Moreover,

$$\mathbb{E}^{\mathbb{P}^{f_z}}\left[\left(\int_0^T |D_r Z_s|^2 ds\right)^{\frac{1}{2}}\right] = \mathbb{E}\left[\gamma_T^{f_z}\left(\int_0^T |D_r Z_s|^2 ds\right)^{\frac{1}{2}}\right]$$
$$\leq \mathbb{E}[|\gamma_T^{f_z}|^2] + \mathbb{E}\left[\int_0^T |D_r Z_s|^2 ds\right] < \infty.$$

By taking the conditional expectation of (3.3.3) and applying Jensen's inequality, we obtain :

$$\begin{aligned}
&|D_r Y_t| \\
\leq & M_2 + \mathbb{E}^{\mathbb{P}^{f_z}}\Big[\int_0^T |A_s D_r v_s| ds + \int_t^T |f_y^N(s, Y_s, Z_s, v_s) D_r Y_s| ds \Big| \mathcal{F}_t\Big] \\
\leq & M_2 + M_1 M_3 + \mathbb{E}^{\mathbb{P}^{f_z}}\Big[\int_t^T |f_y^N(s, Y_s, Z_s, v_s) D_r Y_s| ds \Big| \mathcal{F}_t\Big].
\end{aligned} \tag{3.3.4}$$

Since $\sup_{t\in[0,T]} |Y_t| \leq C_{1,T}$ and ψ guarantees that $|Z_t| \leq N$, there exists a constant $C_{T,N}$ such that $|f_y^N(s, Y_s, Z_s, v_s)| \leq C_{T,N}$. For any $\iota \leq t$, we have :

$$\mathbb{E}^{\mathbb{P}^{f_z}}\Big[|D_r Y_t| \Big| \mathcal{F}_\iota\Big] \leq M_2 + M_1 M_3 + C_{T,N}\int_t^T \mathbb{E}^{\mathbb{P}^{f_z}}\Big[|D_r Y_s| \Big| \mathcal{F}_\iota\Big] ds.$$

Gronwall's Lemma yields to,

$$\mathbb{E}^{\mathbb{P}^{f_z}}\Big[|D_r Y_t| \Big| \mathcal{F}_\iota\Big] \leq (M_2 + M_1 M_3) e^{T C_{T,N}}.$$

For $\iota = t$, we get $|D_r Y_t| \leq (M_2 + M_1 M_3) e^{T C_{T,N}}$; thus, $(D_r Y_t)_{t\geq 0}$ is uniformly bounded. Therefore, we can apply Gronwall's Lemma to (3.3.4) (Theorem 1 in [91]), and we obtain :

$$|D_r Y_t| \leq (M_2 + M_1 M_3)\mathbb{E}^{\mathbb{P}^{f_z}}\Big[\exp\Big(\int_t^T |f_y^N(s, Y_s, Z_s, v_s)| ds\Big) \Big| \mathcal{F}_t\Big].$$

Using (A.3.4) and the boundedness of Y and for $r = t$,

$$\begin{aligned}
|Z_t| &\leq (M_2 + M_1 M_3)\exp(L(1 + C_{1,T}))\mathbb{E}^{\mathbb{P}^{f_z}}\left[\exp\left(\int_t^T \ln(|Z_s| + 1) ds\right) \Big| \mathcal{F}_t\right] \\
&\leq (M_2 + M_1 M_3)\exp(L(1 + C_{1,T}))\mathbb{E}^{\mathbb{P}^{f_z}}\left[\int_t^T (|Z_s| + 1) ds | \mathcal{F}_t\right] \\
&\leq (M_2 + M_1 M_3)\exp(L(1 + C_{1,T}))\left(T + \mathbb{E}^{\mathbb{P}^{f_z}}\left[\int_t^T |Z_s| ds | \mathcal{F}_t\right]\right).
\end{aligned}$$

By taking the conditional expectation with respect to $\mathcal{F}_\iota$, where $\iota \leq t$, we obtain :

$$\mathbb{E}^{\mathbb{P}^{f_z}}[|Z_t| | \mathcal{F}_\iota] \leq (M_2 + M_1 M_3)\exp(L(1 + C_{1,T}))\left(T + \mathbb{E}^{\mathbb{P}^{f_z}}\left[\int_t^T |Z_s| ds | \mathcal{F}_\iota\right]\right).$$

By applying Gronwall's Lemma and then setting $\iota = t$, we obtain

$$\begin{aligned}\sup_{0\leq t\leq T} |Z_t| &\leq (M_2+M_1M_3)T\exp\left(L(1+C_{1,T})\right)\exp\Big((M_2+M_1M_3)T\exp\left(L(1+C_{1,T})\right)\Big)\\ &= (M_2+M_1M_3)T\exp\Big(L(1+C_{1,T})+(M_2+M_1M_3)T\exp\left(L(1+C_{1,T})\right)\Big).\end{aligned}$$

Alternatively, we can use Theorem 1 from [91], as $Z_t = D_tY_t$, and thus it is uniformly bounded. Thus, for any $N \geq C_{2,T} := (M_2+M_1M_3)T\exp\Big(L(1+C_{1,T})+(M_2+M_1M_3)T\exp\left(L(1+C_{1,T})\right)\Big)$, $f^N = f$ and $\sup_{0\leq t\leq T}|Z_t| \leq C_{2,T}$. The assertion (iii) follows directly from (A.3.3) and the preceding assertions. □

Theorem 3.3.7. *Under Assumption 3.3.4, the BSDE (3.1.1) has one solution.*

Proof: Regarding Theorem 3.3.5, the BSDE (3.1.1) has a solution. To prove uniqueness, let (Y,Z), (Y',Z') be two solutions of (3.1.1), then we have :

$$\begin{aligned}Y_t - Y'_t &= -\int_t^T \left(f\left(s,Y_s,Z_s,v_s\right) - f\left(s,Y'_s,Z'_s,v_s\right)\right)ds - \int_t^T \left(Z_s - Z'_s\right)dW_s\\ &= -\int_t^T \left(f\left(s,Y_s,Z_s,v_s\right) - f\left(s,Y'_s,Z_s,v_s\right)\right)ds\\ &\quad -\int_t^T \left(f\left(s,Y'_s,Z_s,v_s\right) - f\left(s,Y'_s,Z'_s,v_s\right)\right)ds - \int_t^T \left(Z_s - Z'_s\right)dW_s.\end{aligned}$$

Since f is locally Lipschitz and according to Lemma 3.3.6, the solutions are bounded, thus there exists a positive constant C_T depends on $C_{1,T}$ and $C_{2,T}$, such that $\forall s \in [0,T]$:

$$|f(s,Y_s,Z_s,v_s) - f(s,Y'_s,Z'_s,v_s)| \leq C_T(|Y_s - Y'_s| + |Z_s - Z'_s|).$$

By taking similar steps as the proof of Lemma 3.3.6, we get

$$\begin{aligned}Y_t - Y'_t = &-\int_t^T \left(f\left(s,Y_s,Z_s,v_s\right) - f\left(s,Y'_s,Z_s,v_s\right)\right)ds\\ &-\int_t^T \left(Z_s - Z'_s\right)d\widetilde{W}_s,\end{aligned}$$

where

$$\widetilde{W}_s = W_s + \int_0^t \left(f\left(s,Y'_s,Z_s,v_s\right) - f\left(s,Y'_s,Z'_s,v_s\right)\right)\left(Z_s - Z'_s\right)^{-1}\mathbb{1}_{\{Z_s \neq Z'_s\}}ds.$$

Moreover, the same arguments yield that for all $t \in [0,T]$: $|Y_t - Y'_t| = 0$. This implies Y and Y' coincide. Intuitively, this should also imply, $Z_t = Z'_t$ for all t. Thus the uniqueness is satisfied. □

These results ensure that the control problem is well-posed. Additionally, the boundedness of f and f_y allows us to leverage the previous control result under Assumption 3.3.4.

Conclusion

This study explored a stochastic optimal control problem for a specific type of controlled BSDE characterized by a local Lipschitz coefficient and a generator with logarithmic growth. The main challenges stemmed from the local Lipschitz nature of the BSDE generator and the adjoint equation, described by a linear SDE, complicating the application of standard duality techniques for solving the control problem. To address these challenges, we introduced certain assumptions to ensure the existence and uniqueness of the associated adjoint process. By employing Ekeland's variational principle, combined with methods of approximation and taking limits, we derived both necessary and sufficient conditions for optimality.

Chapter 4

Public Private Partnerships contract under moral hazard and ambiguous information

4.1 Introduction

Public-private partnerships (PPPs) entail long-term contracts between private entities and public agencies to construct or manage assets or services. In these collaborations, the private consortium assumes significant risks and responsibilities, aiming to enhance the project's societal impact while receiving compensation from the public sector.

PPPs aim to optimize the quality-price ratio of public spending, yet they often face challenges due to information disparities between the parties. This information asymmetry complicates both negotiation and project oversight, particularly as the public may struggle to evaluate the consortium's efforts—a classic Principal-Agent problem compounded by moral hazard.

The seminal work on Principal-Agent problems in continuous time was pioneered by Holmstrom and Milgrom (1987). Their study delved into a Brownian framework where the Agent's exertion solely influences the output process's drift. Moreover, the Agent receives a lump sum payment upon the contract's conclusion, operating within a finite time horizon. Within this context, the Principal is depicted as risk-neutral, while the Agent exhibits risk aversion, characterized by a Constant Absolute Risk Aversion (CARA) utility function. Holmstrom and Milgrom examined a Stackelberg leadership model in their research [55], which involves a sequential decision-making process between the Principal and the Agent. This model is approached in two steps : initially, the Principal devises an optimal contract given a fixed set of terms, considering the anticipated response from the Agent. Subsequently, armed with the Agent's best response, the Principal fine-tunes the contract to optimize her own. This approach is particularly suited for situations characterized by short contract durations, offering insights into the dynamics of Principal-Agent relationships within finite time horizons. To address our PPP-related challenges, adopting a randomized contract horizon, as proposed by Sannikov [85], proves advantageous. Sannikov's extension of the HM model introduces a random time horizon wherein the Principal pays continuous rent to the Agent, deviating from the traditional end-of-contract payment scheme. Employing dynamic programming principles, Sannikov derives the Hamilton-Jacobi-Bellman equation governing the principal value function, enabling the determination of the optimal contract through a verification theorem. This methodology offers a robust solution framework that facilitates the computation of optimal rent and effort levels in a feedback loop and is amenable to numerical approximation by solving the HJB equation. An alternative methodology, as explored by Williams [93] and extensively discussed in the monograph by Cvitanić and Zhang [37], along with various other authors, diverges from the continuous rent framework for finite horizons. This approach leverages the Pontryagin stochastic maximum principle within Brownian Motion-driven models to establish necessary conditions for optimal efforts and contracts, articulated through a fully coupled Forward-Backward Stochastic Differential Equations system. In instances where authors assume Markovian models, sufficient conditions are discerned through the conventional route of employing HJB equations. In Principal-Agent scenarios involving moral hazard, it's commonly assumed that the Principal possesses perfect knowledge of the probability distribution governing the Agent's effort. However, in reality, the Principal often faces uncertainty or ambiguity regarding this probability distribution, necessitating consideration of multiple objective probability measures. Initial inquiries into uncertainty within this context have focused on dominated sets, particularly with respect to an objective reference probability measure, such as drift uncertainty, as explored by Gilboa and Schmeidler [50].

Ambiguity, also known as Knightian uncertainty, has significant economic implications. Coined by Knight [59], this notion plays a pivotal role in economic contracts due to the inherent inaccuracies in available information.

The concepts of risk and ambiguity are distinct. Risk pertains to situations where the probability distribution for each action is known, while ambiguity involves economic decision-making under uncertainty, where multiple probability distributions arise due to imperfect information and cannot be consolidated into a single distribution. Knight [59] initially delineated this difference, which was further explored by Ellsberg [47] and elaborated upon by Gilboa and Schmeidler [50]. They linked ambiguity to the notion of multiple priors in a static framework. Chen and Epstein [31] extended this concept to an intertemporal setting, introducing the concept of 'κ-ignorance' to characterize Knightian uncertainty, where κ represents the level of ignorance. Decision-makers face greater ambiguity when κ is higher.

Dumav and Riedel [43] investigated a moral hazard scenario involving continuous payments and a random horizon, wherein the Principal and the Agent interact under a contractual agreement concerning unobservable effort levels that yield output subject to ambiguity. Unlike Sannikov [85], they developed a model where efforts correspond to sets of probability distributions, ultimately delineating the optimal contract under ambiguous information.

In a related vein, Mastrolia and Possamaï [70] examined a scenario where both the Agent and the Principal face uncertainty regarding the volatility of the output, particularly in finite maturity contexts.

This paper considers a contract between a public entity and a consortium in a continuous time setting. The consortium is trying to improve the project's social value, driven by a one-dimensional Brownian motion. The effort is not observable by the public and is ambiguous. The public must choose a continuous rent to pay the consortium as compensation for its effort. We assume that the effort only affects the drift and not the volatility of the social value. Indeed in a one-dimensional setting, controlling the volatility would imply that the effort is observable, through the quadratic variation of the social value. Our approach is inspired by the seminal paper of Sannikov [85].

In the first step, we establish the Agent's value function under the most possible scenario, demonstrating its satisfaction with a BSDE with a random horizon. Subsequently, we determine the Agent's optimal response. Then, we formulate the public value function as a conventional stochastic control problem, utilizing the Agent's value function as a state variable and the contract alongside the Agent's optimal response as control processes. We derive the HJBVI governing the public value function by leveraging the dynamic programming principle.

We employ the Howard algorithm and finite difference methods to approximate the optimal rent and effort numerically. We obtain the optimal effort and rent through a feedback form. Our numerical results indicate that each increase in the degree of Knightian uncertainty leads to an increase in effort and a decrease in the value function.

In contrast to Dumav and Riedel [43], we suppose that at time t, the social value of the project X is not distributed constant but rather depends on time, which augments the state variable in the HJBVI. We establish a rigorous mathematical framework incorporating BSDEs, stochastic control, and optimal stopping techniques and elucidate the computational procedure for obtaining numerical solutions.

This work is formatted as follows : We outline the difficulties faced by the public and the consortium in Section 4.2, where we also explain the issue formulation process utilizing the weak method. In Section 4.3, the dynamics of the consortium goal function are explained using the BSDE with a random horizon approach, and the incentive-compatible contract under worst-case conditions is determined. We obtain the HJBVI linked to the public value function in 4.4, augmented by a verification theorem. The main focus of Section 4.5 is the numerical analysis of the HJBVI using the Howard method. Finally, Section 4.6 presents the conclusion of our study.

4.2 Problem Statement and Framework

Consider a probability space denoted by $(\Omega, \mathcal{F}, \mathbb{P})$, where a one-dimensional Brownian motion $W = (W_t)_{t\geq 0}$ is defined. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ represent the completed natural filtration of W. Both the Agent and the Principal observe the project's societal significance X_t, which is expressed as,

$$dX_t^x := \sigma(X_t^x)dW_t,\ X_0^x = x,\ t \geq 0\ , \mathbb{P}\ a.s., \tag{4.2.1}$$

where

- $x > 0$ is the project's starting value.
- $\sigma(.)$ represents the operational cost volatility for infrastructure maintenance. Given σ_{max} and σ_{min} as positive constants, the function $\sigma(.)$ is Lipschitz and adheres to the condition $\sigma_{max} > \sigma(.) > \sigma_{min}$.

The project's performance is influenced by the Agent's effort A_t, which alters the distribution of the process W. The following is our definition of the martingale process $\gamma^A = (\gamma_t^A)_{t\geq 0}$:

$$\gamma_t^A := \mathcal{E}\left(\int_0^t \frac{\vartheta(A_s)}{\sigma(X_s^x)} dW_s\right),\ t \geq 0,\ \mathbb{P}\ a.s.,$$

here, $\mathcal{E}$ represents the Doléans-Dade exponential, and ϑ is a function that will be defined later. Assuming that $\mathbb{P}^A$ is comparable to $\mathbb{P}$, which can be recognized by its density, let us consider it as a probability measure on $(\Omega, \mathcal{F})$. The formula is $\gamma_t^A = \frac{d\mathbb{P}^A}{d\mathbb{P}}|_{\mathcal{F}_t}$. Next, we get the following $\mathbb{P}^A$-Brownian motion based on Girsanov's theorem.

$$W_t^A = W_t - \int_0^t \frac{\vartheta(A_s)}{\sigma(X_s^x)} ds,\ \text{for}\ t \geq 0.$$

Consequently, we have under $\mathbb{P}^A$

$$X_t^x = x + \int_0^t \vartheta(A_s)ds + \int_0^t \sigma(X_s^x)dW_s^A,\ t \geq 0,\ \mathbb{P}\ a.s. \tag{4.2.2}$$

In the context of this study, we employ the notation $\mathbb{F}$-$\mathcal{P}r$ to refer to $\mathbb{F}$-progressively measurable processes for the sake of brevity and clarity.

In the framework proposed by Sannikov, the assumption is that when the effort A is determined, the probability measure $\mathbb{P}^A$ remains constant. This implies that the Agent possesses full knowledge regarding the probability distribution governing the evolution of the state process $(X_t)_{t\geq 0}$. However, in practical scenarios, the Agent may encounter uncertainty regarding this probability distribution, prompting the consideration of multiple objective probability measures denoted by $\mathbb{P}^\theta$. To address this uncertainty rigorously, we introduce the parameter θ, which is a process taking values in $\mathbb{R}$ and is responsible for generating a probability measure $\mathbb{P}^\theta$ that is equivalent to $\mathbb{P}^A$. Consequently, we define by $\gamma^\theta := \frac{d\mathbb{P}^\theta}{d\mathbb{P}^A}$ the density of $\mathbb{P}^\theta$ with respect to $\mathbb{P}^A$, expressed as,

$$\gamma_t^\theta := \mathcal{E}\left(\int_0^t \frac{\theta_s}{\sigma(X_s^x)} dW_s^A\right), \ t \geq 0, \ \ \mathbb{P} \ a.s.,$$

Here, $\theta \in \Theta := \{(\theta_s)_{s\geq 0} \mathbb{F} - \mathcal{P}r$ process, $\theta_s \in [-\kappa, \kappa]$ $ds \otimes d\mathbb{P}$ a.e.$\}$, where κ is a positive constant.

The set of different scenarios, referred to as priors, is denoted by $\mathbb{P}^\theta, \theta \in \Theta$. According to Girsanov's theorem, we have the following $\mathbb{P}^\theta$-Brownian motion,

$$W_t^\theta = W_t^A - \int_0^t \frac{\theta_s}{\sigma(X_s^x)} ds, \ \text{for} \ t \geq 0. \tag{4.2.3}$$

Thus, the project's social value under $\mathbb{P}^\theta$ satisfies

$$X_t^x = x + \int_0^t (\vartheta(A_s) + \theta_s)\, ds + \int_0^t \sigma(X_s^x) dW_s^\theta, \ t \geq 0, \ \mathbb{P} \ a.s. \tag{4.2.4}$$

We represent the collection of all $\mathbb{F}$-stopping times as $\mathcal{T}$. Fixing $\hat{p} \in (2, \infty)$, we examine the collection of admissible actions.

$$\mathcal{A}^{\hat{p}} := \left\{ (A_s)_{s\geq 0}, \ \mathbb{F} - \mathcal{P}r \ \text{processes}, \ A_s \geq 0, \ ds \otimes d\mathbb{P} \ a.e. \ \text{ and } \ \sup_{\iota \in \mathcal{T}} \sup_{\theta \in \Theta} \mathbb{E}^{\mathbb{P}} \left[(\gamma_\iota^{A,\theta})^{\hat{p}} \right] < \infty \right\}.$$

The principal observes the project's social value X, but she cannot distinguish between $\int_0^\cdot \vartheta(A_s) ds$ and $\int_0^\cdot \sigma(X_s^x) dW_s^A$, suggesting that she is not directly observing the consortium's work. This presents a moral hazard scenario. We infer that W^θ is not observable by the Principal from equality (4.2.3). She decides how much rent she will provide to the agent in exchange for him supporting the operating expenses and his efforts. At date ι, where ι is a stopping time in $\mathcal{T}$, the public may terminate the contract.

$\Gamma = ((R_t)_t, \iota, \xi)$ is a triplet that represents a contract. $R \geq 0$ is a $\mathbb{F} - \mathcal{P}r$, ι is in the set $\mathcal{T}$, and $\xi \geq 0$ is a $\mathcal{F}_\iota$-measurable random variable. This random variable encapsulates the financial implication associated with the cessation of the contract.

We proceed by delineating the optimization tasks for both the consortium and the public. Initially, we define the functions integral to formulating these optimization problems.

Assumption 4.2.1.
- $\vartheta : [0,\infty) \to [0,\infty)$ *characterizes the incremental impact of the consortium's efforts on the social value. It satisfies the properties of being* C^2*, strictly concave, and increasing. Additionally,* $\vartheta(0)=0$ *and* $\vartheta'(0)>0$*. The supremum norm of* $\frac{\vartheta}{\sigma}$*, denoted as* $\|\frac{\vartheta}{\sigma}\|_\infty := \sup_{a\geq 0, x\in\mathbb{R}} \frac{|\vartheta(a)|}{|\sigma(x)|}$ *and is bounded.*
- *The consortium's utility function,* $U:[0,\infty)\to[0,\infty)$*, is limited, strictly concave, growing, and satisfies* $U(0)=0$ *as well as Inada's constraints* $U'(\infty)=0,\ U'(0)=\infty$*.*
- $h:[0,\infty)\to[0,\infty)$ *represents the cost associated with the effort of the consortium. It is characterized by being* C^2*,* $h(0)=0$*, strictly convex, and monotonically growing.*
- *The consortium's time preference parameter* λ *is greater than the public's, or* δ *(*$\lambda\geq\delta$*), suggesting that the consortium exhibits a higher degree of impatience compared to the public.*

We operate under the assumption that when presented with a contract Γ by the Principal, the consortium responds optimally by determining an effort process A. This setup reflects a Stackelberg leadership model, wherein the Principal acts as the leader by proposing a contract, and the consortium, acting as a follower, responds sequentially.

The consortium's acceptance of the contract is contingent upon the condition that the expected benefits outweigh its reservation value, denoted as $\underline{x}$.
Three actions are taken to solve the public and consortium's concerns. Firstly, we determine $\theta^* = \theta^*(A,\Gamma)$, as a function of contractual parameters Γ and effort allocation A delineates the most probable scenario for the consortium. We remove (A,Γ) to reduce notations. Next, we ascertain the Agent's optimal reaction given $(\theta^*(A,\Gamma),\Gamma)$. The notation $A^*(\theta^*(A,\Gamma),\Gamma)$ represents the answer. We remove $(\theta^*(A,\Gamma),\Gamma)$ to reduce notations.

(1) In the initial step, and with the parameters (A,Γ) at hand, we proceed to determine the worst-case scenario by addressing the following equation :

$$\theta^* \in \arg\min_{\theta\in\Theta} \mathbb{E}^\theta\left[\int_0^\iota e^{-\lambda s}(U(R_s)-h(A_s))ds + e^{-\lambda\iota}U(\xi)\right].$$

The objective function for the Agent, beginning from time t, is defined as follows,

$$J_t^{amb}(\Gamma,A,\theta) := \mathbb{E}^\theta\left[\int_t^\iota e^{-\lambda(s-t)}(U(R_s)-h(A_s))ds + e^{-\lambda(\iota-t)}U(\xi)|\mathcal{F}_t\right],\ \forall t\in[\![0,\iota[\![\quad \mathbb{P}\ a.s.$$

(2) In the subsequent step, we ascertain the consortium's optimal response given the worst-case scenario by resolving :

$$A^* \in \arg\max_{A\in\mathcal{A}_\nu^C} \mathbb{E}^A\left[\int_0^\iota \gamma_s^{\theta^*} e^{-\lambda s}(U(R_s)-h(A_s))ds + \gamma_\iota^{\theta^*} e^{-\lambda\iota}U(\xi)\right],$$

here, $\nu > 0$,

$$\mathcal{A}^C_{\nu-2\lambda} := \{(A_s)_{s\geq 0} \in \mathcal{A}^{\hat{p}},\ \text{s.t. } \mathbb{E}^{\mathbb{P}}[\int_0^{\infty} e^{(\nu-2\lambda)s}|h(A_s)|^2 ds] < \infty$$
$$\text{and } \mathbb{E}^{\mathbb{P}}[\int_0^{\infty} e^{(\nu-2\lambda)s}|\vartheta(A_s)|^2 ds] < \infty\}.$$

The objective function representing the scenario of utmost adversity commencing at time t for the Agent is,

$$J^C_t(\Gamma,A) := \frac{1}{\gamma_t^{\theta^*}}\mathbb{E}^A\left[\int_t^{\iota}\gamma_s^{\theta^*}e^{-\lambda(s-t)}(U(R_s)-h(A_s))ds + \gamma_\iota^{\theta^*}e^{-\lambda(\iota-t)}U(\xi)|\mathcal{F}_t\right],\ \forall t \in [\![0,\iota[\![\ \ \mathbb{P}\ a.s.$$

(3) In the best-case scenario and with the consortium's optimal answer, the public problem is defined by :

$$\sup_{\Gamma\in\mathcal{A}^P_\nu}\ \sup_{\mathbb{P}^{A^*}\in\mathcal{P}} \mathbb{E}^{A^*}\left[\int_0^{\iota}\gamma_s^{\theta^*}e^{-\delta s}(\vartheta(A_s^*)+\theta_s^*-R_s)ds - \gamma_\iota^{\theta^*}e^{-\delta\iota}\xi\right], \tag{4.2.5}$$

subject to the restriction on reservations

$$\mathbb{E}^{A^*}\left[\int_0^{\iota}\gamma_s^{\theta^*}e^{-\lambda s}(U(R_s)-h(A_s^*))ds + \gamma_\iota^{\theta^*}e^{-\lambda\iota}U(\xi)\right] \geq \underline{x},$$

where

$$\mathcal{A}^P_{\nu-2\lambda} := \Big\{((R_s)_{s\geq 0},\iota,\xi) \text{ s.t. } (R_s)_{s\geq 0} \text{ is a } \mathbb{F}-\mathcal{P}r \text{ process},\ R_s \geq 0\ ds\otimes d\mathbb{P}\ a.e.,$$
$$\mathbb{E}^{\mathbb{P}}\left[\int_0^{\iota}e^{(\nu-2\lambda)s}(U(R_s)^2\vee R_s^2)ds\right] < \infty,\ \iota\in\mathcal{T},\xi \text{ non negative } \mathcal{F}_\iota\text{-measurable},$$
$$\text{and } \mathbb{E}^{\mathbb{P}}\left[e^{(\nu-2\lambda)\iota}(|U(\xi)|^2\vee\xi^2)\mathbb{1}_{\{\iota<+\infty\}}\right] < \infty\Big\},$$

and

$$\mathcal{P} = \{\mathbb{P}^{A^*}\sim\mathbb{P},\ A^*\in\mathcal{A}^C_\nu\}.$$

The function representing the objective, initiated from time t, for the Principal, is :

$$J^P_t(\Gamma,A^*) := \frac{1}{\gamma_t^{\theta^*}}\mathbb{E}^{A^*}\left[\int_t^{\iota}\gamma_s^{\theta^*}e^{-\delta(s-t)}(\vartheta(A_s^*)+\theta_s^*-R_s)ds - \gamma_\iota^{\theta^*}e^{-\delta(\iota-t)}\xi|\mathcal{F}_t\right],$$
$$\forall t\in[\![0,\iota[\![\ \ \mathbb{P}\ a.s.$$

4.3 Incentive compatible contracts

This section presents the dynamics guiding the consortium's goal function J^C and attempts to develop contracts that align incentives in the best possible way. The solution to a particular kind of BSDE with a random horizon has the following structure, which we will use to prove the uniqueness of J^C,

$$Y_t = \zeta \mathbb{1}_{\{\iota<+\infty\}} + \int_t^\iota g(s,\omega,X_s,Y_s,Z_s)ds - \int_t^\iota Z_s dW_s, \tag{4.3.1}$$

Previous literature has explored the study of BSDEs with random horizons, as evidenced by Darling and Pardoux [38].

For a predetermined stopping time ι and for some positive constant η, we present the following spaces :

- $\mathcal{M}_\eta(0,\iota;\mathbb{R})$ the set of $\mathbb{F}-\mathcal{P}r$, $\mathbb{R}$- valued processes on $\Omega\times[\![0,\iota]\!]$.
- $\mathcal{H}^2_\eta(0,\iota;\mathbb{R}) = \left\{Z\in\mathcal{M}_\eta(0,\iota;\mathbb{R}) \text{ s.t. } \mathbb{E}\left[\int_0^\iota e^{\eta t}|Z_t|^2\,dt\right] < +\infty\right\}$.
- $\mathcal{S}^2_\eta(0,\iota;\mathbb{R}) = \left\{Y\in\mathcal{M}_\eta(0,\iota;\mathbb{R}) \text{ s.t. } \mathbb{E}\left[\sup_{0\le t\le\iota} e^{\eta t}|Y_t|^2\right] < +\infty\right\}$.

In this work, we make the following assumption :

- ►(H1) For any $x,y,z\in\mathbb{R}$, the function $g(.,.,x,y,z)$ belongs to the space $\mathcal{M}_\eta(0,\iota;\mathbb{R})$ and satisfying the following condition :

$$\mathbb{E}\left[e^{\eta\iota}|\zeta|^2\mathbb{1}_{\{\iota<+\infty\}} + \int_0^\iota e^{\eta s}|g(s,\omega,X_s,0,0)|^2 ds\right] < \infty.$$

- ►(H2) The generator g exhibits Lipschitz continuity with respect to both y and z, denoted by positive constants C_1 and C_2, ensuring that for any s, ω, x,

$$|g(s,\omega,x,y_1,z_1) - g(s,\omega,x,y_2,z_2)| \le C_1|y_1-y_2| + C_2|z_1-z_2| \; ds\otimes d\mathbb{P} \; a.e.$$

Theorem 4.3.1 (Theorem 3.4 and Corollary 4.4.2 in [38])**.** *Under the fulfillment of conditions (H1) and (H2) and the constraint $\eta \ge C_2^2 - 2C_1$, we have :*

Existence and Uniqueness *: The existence of a unique solution (Y,Z) to the BSDE (4.3.1) in the space $\mathcal{H}^2_\eta(0,\iota;\mathbb{R})\times\mathcal{H}^2_\eta(0,\iota;\mathbb{R})$ is guaranteed. Furthermore, it is established that $Y\in\mathcal{S}^2_\eta(0,\iota;\mathbb{R})$.*

Comparison *: Consider two solutions (Y^1,Z^1) and (Y^2,Z^2) of the BSDEs associated with parameters (g^1,ξ,ι) and (g^2,ξ,ι), respectively. Assuming that $g^1(t,\omega,X_t,Y^1,Z^1) \le g^2(t,\omega,X_t,Y^1,Z^1)$ $dt\otimes d\mathbb{P}$ holds almost everywhere, we establish the inequality $Y^1_t \le Y^2_t$ for all $t\in[0,\iota[\![$ almost surely under $\mathbb{P}$.*

Theorem 4.3.1 is a foundational tool in identifying the worst-case scenario for the Agent, establishing incentive-compatible contracts, and elucidating the dynamics of the consortium objective function. Central to the resolution of these initial steps is the application of the comparison theorem.

Proposition 4.3.2. *Suppose $\Gamma \in \mathcal{A}^p_{\nu-2\lambda}$, $A \in \mathcal{A}^C_{\nu-2\lambda}$ and $\theta \in \Theta$. We establish the existence of $Z^{A,\theta} \in \mathcal{H}^2_{\nu-2\lambda}(0,\iota;\mathbb{R})$, such that the dynamics governing the Agent's objective function follow the BSDE with a random horizon, given by the following equation :*

$$\begin{cases} -dJ^{amb}_t(\Gamma,A,\theta) &= \left(-\lambda J^{amb}_t(\Gamma,A,\theta) + U(R_t) + \varphi(A_t, X^x_t, Z^{A,\theta}_t) + \frac{\theta_t}{\sigma(X^x_t)} Z^{A,\theta}_t\right) dt - Z^{A,\theta}_t dW_t, \\ J^{amb}_\iota(\Gamma,A,\theta) &= U(\xi)\mathbb{1}_{\{\iota<+\infty\}}, \end{cases} \tag{4.3.2}$$

and

$$J^{amb}_t(\Gamma,A,\theta) \geq J^C_t(\Gamma,A) := J^{amb}_t(\Gamma,A,\theta^*(Z^{A,\theta})) \ \forall t \in [\![0,\iota[\![\quad \mathbb{P} \ a.s., \tag{4.3.3}$$

where

$$\begin{cases} \varphi(A_t, X^x_t, Z^{A,\theta}_t) &:= -h(A_t) + \frac{\vartheta(A_t)}{\sigma(X^x_t)} Z^{A,\theta}_t, \\ \theta^*(Z^{A,\theta}_t) &:= \arg\min_{\alpha\in[-\kappa,\kappa]} \left(\alpha Z^{A,\theta}_t\right) = -\kappa\, \mathrm{sgn}(Z^{A,\theta}_t). \end{cases}$$

Proof: For any admissible contract $\Gamma \in \mathcal{A}^P_{\nu-2\lambda}$, effort allocation $A \in \mathcal{A}^C_{\nu-2\lambda}$, ambiguity process $\theta \in \Theta$, and for any $t \in [\![0,\iota[\![$, we introduce the following process

$$M_t(\Gamma,A,\theta) := \tilde{J}^{amb}_t(\Gamma,A,\theta) + \int_0^t (\tilde{U}(R_s) - e^{-\lambda s} h(A_s)) ds,$$

Where $\tilde{J}^{amb}$ and $\tilde{U}(R_t)$ represent the discounted quantities, defined respectively as $\tilde{J}^{amb}_t(\Gamma,A,\theta) = e^{-\lambda t} J^{amb}_t(\Gamma,A,\theta)$ and $\tilde{U}(R_t) := e^{-\lambda t} U(R_t), dt \otimes d\mathbb{P}$. By leveraging the definition of J^{amb}_t and employing Bayes' formula, we derive :

$$\begin{aligned} M_t(\Gamma,A,\theta) &= \mathbb{E}^\theta\left[\int_0^\iota (\tilde{U}(R_s) - e^{-\lambda s} h(A_s)) ds + e^{-\lambda\iota}\xi\mathbb{1}_{\{\iota<+\infty\}} |\mathcal{F}_t\right] \\ &= \frac{1}{\gamma^\theta_t}\mathbb{E}^{\mathbb{A}}\left[\gamma^\theta_\iota \left(\int_0^\iota (\tilde{U}(R_s) - e^{-\lambda s} h(A_s)) ds + e^{-\lambda\iota}\xi\mathbb{1}_{\{\iota<+\infty\}}\right) |\mathcal{F}_t\right] \\ &= \frac{1}{\gamma^\theta_t \gamma^A_t}\mathbb{E}^{\mathbb{P}}\left[\gamma^\theta_\iota \gamma^A_\iota \left(\int_0^\iota (\tilde{U}(R_s) - e^{-\lambda s} h(A_s)) ds + e^{-\lambda\iota}\xi\mathbb{1}_{\{\iota<+\infty\}}\right) |\mathcal{F}_t\right]. \end{aligned}$$

A straightforward calculus shows that

$$\gamma^{A,\theta}_t := \gamma^\theta_t \gamma^A_t := \mathcal{E}\left(\int_0^\cdot \left(\frac{\vartheta(A_s)+\theta_s}{\sigma(X^x_s)}\right) dW_s\right)_t$$

This establishes that the process $\gamma_t^{A,\theta} M_t(\Gamma,A,\theta)$ is a $(\mathbb{P},\mathbb{F})$-local martingale. Hence, by the martingale representation theorem, there exists a singular progressively measurable process denoted as χ, s.t.

$$d\gamma_t^{A,\theta} M_t(\Gamma,A,\theta) = \chi_t dW_t.$$

By applying Itô's formula to $\gamma_t^{A,\theta} M_t(\Gamma,A,\theta)$, we derive,

$$\begin{aligned} dM_t(\Gamma,A,\theta) &= \frac{1}{\gamma_t^{A,\theta}}\left(d(\gamma_t^{A,\theta} M_t(\Gamma,A,\theta)) - M_t(\Gamma,A,\theta)d\gamma_t^{A,\theta} - d\left\langle M(\Gamma,A,\theta),\gamma^{A,\theta}\right\rangle_t\right) \\ &= \frac{1}{\gamma_t^{A,\theta}}\left(\left(\chi_t - M_t(\Gamma,A,\theta)\gamma_t^{A,\theta}\frac{\vartheta(A_t)+\theta_t}{\sigma(X_t^x)}\right)dW_t - d\left\langle M(\Gamma,A,\theta),\gamma^{A,\theta}\right\rangle_t\right) \\ &= \left(\frac{\chi_t}{\gamma_t^{A,\theta}} - M_t(\Gamma,A,\theta)\frac{\vartheta(A_t)+\theta_t}{\sigma(X_t^x)}\right)dW_t - \frac{1}{\gamma_t^{A,\theta}} d\left\langle M(\Gamma,A,\theta),\gamma^{A,\theta}\right\rangle_t. \end{aligned} \tag{4.3.4}$$

We proceed to establish the process $\tilde{Z}_\cdot^{A,\theta}$ as outlined below :

$$\tilde{Z}_t^{A,\theta} := \frac{\chi_t}{\gamma_t^{A,\theta}} - M_t(\Gamma,A,\theta)\frac{\vartheta(A_t)+\theta_t}{\sigma(X_t^x)}\, dt\otimes dP \text{ a.e.,} \tag{4.3.5}$$

Subsequently, the quadratic variation of $M(\Gamma,A,\theta)$ and $\gamma^{A,\theta}$ fulfills,

$$d\left\langle M(\Gamma,A,\theta),\gamma^{A,\theta}\right\rangle_t = \tilde{Z}_t^{A,\theta}\gamma_t^{A,\theta}\frac{\vartheta(A_t)+\theta_t}{\sigma(X_t^x)}dt$$

Utilizing equations (4.3.4)-(4.3.5), it follows that,

$$dM_t(\Gamma,A,\theta) = -\tilde{Z}_t^{A,\theta}\frac{\vartheta(A_t)+\theta_t}{\sigma(X_t^x)}dt + \tilde{Z}_t^{A,\theta}dW_t.$$

Based on the definition of $\tilde{J}^{amb}$, we derive the following :

$$\begin{cases} -d\tilde{J}_t^{amb}(\Gamma,A,\theta) &= \left(\tilde{U}(R_t) + \tilde{\varphi}(A_t,X_t^x,\tilde{Z}_t^{A,\theta}) + \tilde{Z}_t^{A,\theta}\dfrac{\theta_t}{\sigma(X_t^x)}\right)dt - \tilde{Z}_t^{A,\theta}dW_t \\ \tilde{J}_\iota^{amb}(\Gamma,A,\theta) &= e^{-\lambda\iota}U(\xi)\mathbb{1}_{\{\iota<+\infty\}}, \end{cases}$$

where $\tilde{\varphi}(A_t,X_t^x,\tilde{Z}_t^{A,\theta}) := -e^{-\lambda\iota}h(A_t) + \dfrac{\vartheta(A_t)}{\sigma(X_t^x)}\tilde{Z}_t^{A,\theta}\ dt\otimes dP$ a.e. Through the application of Itô's formula, we establish that the pair $(J^{amb}, Z^{A,\theta})$ satisfies the BSDE (4.3.2), where the generator is specified as follows : $g(t,\omega,X_t^x,y,z) = -\lambda y + U(R_t) + \varphi(A_t,X_t^x,z) + z\dfrac{\theta_t}{\sigma(X_t^x)}$. Given the integrability assumptions on A and Γ, **Assumption (H1)** is fulfilled. With $\frac{\vartheta}{\sigma}$ bounded, as stipulated in **Assumption** 4.2.1, and θ bounded, while σ is assumed to be bounded from below, the existence of a positive constant K is guaranteed, yielding,

$$\left|\frac{\vartheta(A_t)+\theta_t}{\sigma(X_t^x)}\right| \le K\ dt\otimes d\mathbb{P} \text{ a.e .}$$

Consequently, the generator g associated with the BSDE (4.3.2) exhibits uniform Lipschitz continuity with respect to the variables (y,z), thereby satisfying **Assumption** (**H2**). This ensures the existence of a unique solution $\left(Y, Z^{A,\theta}\right) \in \mathcal{S}^2_{\nu-2\lambda}(0,\iota;\mathbb{R}) \times \mathcal{H}^2_{\nu-2\lambda}(0,\iota;\mathbb{R})$ for the BSDE (4.3.2). Given any $\theta \in \Theta$,

$$Z_t^{A,\theta}\theta_t \geq -\left|Z_t^{A,\theta}\right|\kappa = Z_t^{A,\theta}\theta^*(Z_t^{A,\theta})\ dt \otimes d\mathbb{P} \text{ a.e .}$$

Utilizing Theorem 4.3.1, we deduce,

$$J_t^{amb}(\Gamma, A, \theta) \geq J_t^{amb}(\Gamma, A, \theta^*(Z_t^{A,\theta}))\ \forall t \in [\![0,\iota[\![\quad \mathbb{P} \text{ a.s.} \tag{4.3.6}$$

Hence, inequality (4.3.3) is established. □

The subsequent proposition elucidates the derivation of the BSDE that governs the Agent objective function under the worst-case scenario

Proposition 4.3.3. *Consider $\Gamma \in \mathcal{A}^P_{\nu-2\lambda}$, $A \in \mathcal{A}^C_{\nu-2\lambda}$. Then, the BSDE that satisfies $J^C(\Gamma, A)$, is given by*

$$\begin{cases} -dJ_t^C(\Gamma, A) &= \left(-\lambda J_t^C(\Gamma, A) + U(R_t) + \varphi(A_t, X_t^x, Z_t^A) - \left|Z_t^A\right| \dfrac{\kappa}{\sigma(X_t^x)}\right) dt - Z_t^A dW_t, \\ J_\iota^C(\Gamma, A) &= U(\xi)\mathbb{1}_{\{\iota < +\infty\}}, \end{cases}$$

where φ is specified as per Lemma 4.3.2.

Proof: For $\Gamma \in \mathcal{A}^P_{\nu-2\lambda}$ and $A \in \mathcal{A}^C_{\nu-2\lambda}$, we consider the BSDE

$$\begin{cases} -dY_t &= \left(-\lambda Y_t + U(R_t) + \varphi(A_t, X_t^x, Z_t^A) - \left|Z_t^A\right| \frac{\kappa}{\sigma(X_t^x)}\right) dt - Z_t^A dW_t, \\ Y_\iota &= U(\xi)\mathbb{1}_{\{\iota < +\infty\}}, \end{cases} \tag{4.3.7}$$

Under the fulfillment of **Assumptions** $(H1)$ and $(H2)$, Theorem (4.3.1) guarantees the existence of a unique solution $(Y, Z^A) \in \mathcal{S}^2_{\nu-2\lambda}(0,\iota;\mathbb{R}) \times \mathcal{H}^2_{\nu-2\lambda}(0,\iota;\mathbb{R})$ which resolves BSDE (4.3.7). Concurrently, $J^{amb}(\Gamma, A, \theta^*)$ satisfies,

$$\begin{aligned} &- dJ_t^{amb}(\Gamma, A, \theta^*) \\ &= \left(-\lambda J_t^{amb}(\Gamma, A, \theta^*) + U(R_t) + \varphi(A_t, X_t^x, Z_t^{A,\theta^*}) + \frac{\theta_t^*}{\sigma(X_t^x)} Z_t^{A,\theta^*}\right) dt - Z_t^{A,\theta^*} dW_t \\ &= \left(-\lambda J_t^{amb}(\Gamma, A, \theta^*) + U(R_t) + \varphi(A_t, X_t^x, Z_t^{A,\theta^*}) - |Z_t^{A,\theta^*}| \frac{\kappa}{\sigma(X_t^x)}\right) dt - Z_t^{A,\theta^*} dW_t. \end{aligned}$$

The terminal condition is specified by $J_\iota^{amb}(\Gamma, A, \theta^) = U(\xi)\mathbb{1}_{\iota < +\infty}$. The uniqueness of the solution to the BSDE (4.3.7) entails that $\left(J^C(\Gamma, A) = J^{amb}(\Gamma, A, \theta), Z^A\right)$ solves (4.3.7), thereby establishing the proposition. □

Remark 4.3.4. *Hajjej et al. [51] demonstrated that for any $\Gamma \in \mathcal{A}^P_{\nu-2\lambda}$, the optimal response of the consortium, denoted as $A^* \in \mathcal{A}^C_{\nu-2\lambda}$, can be represented as a deterministic function contingent upon the project's social value and the process Z. This functional relationship is expressed as follows :*

$$A^*_t = A^*(X^x_t, Z^{A,\theta^*}_t) = (\frac{h'}{\vartheta'})^{-1}(\frac{Z^{A,\theta^*}_t}{\sigma(X^x_t)})\mathbb{1}_{\{Z^{A,\theta^*}_t>0\}} \tag{4.3.8}$$

The subsequent proposition elucidates the dynamics of J^C concerning any incentive-compatible contract.

Proposition 4.3.5. *We propose that the generator function, denoted by $g(t,\omega,x,y,z) := -\lambda y + U(R_t) - h(A^*(z)) + z\frac{\vartheta(A^*(z))}{\sigma(x)} - |z|\frac{\kappa}{\sigma(x)}$ fulfills **Assumptions** $(H1)$ and $(H2)$, featuring a Lipschitz coefficient C_2 with $C_2^2 < \nu$. Consequently, the evolution of J^C for any incentive-compatible contract $(\Gamma, A^*(X^x, Z))$ is delineated by the BSDE with a random terminal condition,*

$$\begin{cases} dJ^C_t(\Gamma, A^*(X^x,Z)) & = -\Big(-\lambda J^C_t(\Gamma, A^*(X^x,Z)) + U(R_t) + \varphi(A^*(X^x_t, Z_t), X^x_t, Z_t) \\ & \quad -|Z_t|\frac{\kappa}{\sigma(X^x_t)}\Big)\, dt \quad + Z_t dW_t, \\ J^C_\iota(\Gamma, A^*(X^x,Z)) & = U(\xi)\mathbb{1}_{\{\iota<+\infty\}}. \end{cases}$$

where $A^*(X^x, Z)$ is explicitly defined by equation (4.3.8).

Proof: Let $\Gamma \in \mathcal{A}^P_{\nu-2\lambda}$ and $A^*(X^x, Z) \in \mathcal{A}^C_{\nu-2\lambda}$, we examine the BSDE,

$$\begin{cases} dY_t & = -\Big(-\lambda Y_t + U(R_t) + \varphi(A^*(X^x_t, Z_t), X^x_t, Z_t) - |Z_t|\frac{\kappa}{\sigma(X^x_t)}\Big)\, dt + Z_t dW_t \\ Y_\iota & = U(\xi)\mathbb{1}_{\{\iota<+\infty\}}. \end{cases} \tag{4.3.9}$$

Under the prescribed conditions outlined in the Proposition, specifically the fulfillment of $(H1)$ and $(H2)$, Theorem 4.3.1 ensures the existence of a unique solution $(Y,Z) \in \mathcal{S}^2_{\nu-2\lambda}(0,\iota;\mathbb{R}) \times \mathcal{H}^2_{\nu-2\lambda}(0,\iota;\mathbb{R})$ to the BSDE (4.3.9). Simultaneously, by applying the martingale representation theorem to $(J^C_t(\Gamma, A^*(X^x,Z))_{t\geq 0}$, as elucidated in Lemma 4.3.2, we deduce the existence of a progressively measurable process $(Z^{A^*(X^x,Z)}_t)_t$ such that :

$$\begin{cases} dJ^C_t(\Gamma, A^*(X^x,Z)) & = -\Big(-\lambda J^C_t(\Gamma, A^*(X^x,Z)) + U(R_t) + \varphi(A^*(X^x_t, Z_t), X^x_t, Z^{A^*(X^x,Z)}_t) \\ & \quad -\Big|Z^{A^*(X^x,Z)}_t\Big|\frac{\kappa}{\sigma(X^x_t)}\Big)\, dt \quad + Z^{A^*(X^x,Z)}_t dW_t, \\ J^C_\iota(\Gamma, A^*(X^x,Z)) & = U(\xi)\mathbb{1}_{\{\iota<+\infty\}}. \end{cases}$$

The uniqueness property of the solution entails that

$$Z_t = Z^{A^*(X^x,Z)}_t \ \forall t \in [\![0,\iota[\![\ \mathbb{P} \ a.s.$$

Furthermore, given that $\varphi(A_t, X^x_t, Z_t) \leq \varphi(A^*(X^x_t, Z_t), X^x_t, Z_t) \ \forall t \in [\![0,\iota[\![\ \mathbb{P} \ a.s \ \forall A \in \mathcal{A}^C_\nu$, as per the comparison part in Theorem 4.3.1), it follows that for all $A \in \mathcal{A}^C_{\nu-2\lambda}$

$$J^C_t(\Gamma, A) \leq J^C_t(\Gamma, A^*(X^x,Z)), \ \forall t \in [\![0,\iota[\![\ \mathbb{P} \ a.s.$$

□

Remark 4.3.6. *Hajjej et al. [51] established that the process* $(J_t^C(\Gamma, A^*(X^x, Z)))_{t\geq 0}$ *is a solution to the BSDE characterized by :*

$$\begin{cases} dJ_t^C(\Gamma, A^*(X^x,Z)) &= -\Big(-\lambda J_t^C(\Gamma, A^*(X^x,Z)) + U(R_t) + \varphi(A^*(X_t^x, Z_t), X_t^x, Z_t)\Big)dt + Z_t dW_t, \\ J_\iota^C(\Gamma, A^*(X^x,Z)) &= U(\xi)\mathbb{1}_{\{\iota<+\infty\}}, \end{cases}$$

differs from the one outlined in Proposition 4.3.5. This observation underscores the distinction between the optimal efforts $A^*(X^x, Z)$*, as defined in (4.3.8), under conditions with and without ambiguity. The discrepancy arises from the non-identity between the processes* Z *in the two scenarios.*

Example 4.3.1. *Consider fixed parameters* $\alpha > 0$ *and* $\beta > 0$*. Define* ϑ *and* h *as follows :*

$$\vartheta(x) := 1 - \exp(-\alpha x) \quad \text{and} \quad h(x) := \exp(\beta x) - 1.$$

Upon straightforward calculation, we obtain :

$$A^*(x,z) = \frac{1}{\alpha+\beta}\log\big(\frac{\alpha z}{\beta\sigma(x)}\big)\mathbb{1}_{\{z>\sigma(x)\frac{\beta}{\alpha}\}}.$$

Consequently, the generator of the BSDE (4.3.9) takes the form :

$$\begin{aligned} &g(t,\omega,x,y,z) \\ &= \begin{cases} -\lambda y + U(R_t) + 1 - \big(\frac{\alpha z}{\beta\sigma(x)}\big)^{\frac{\beta}{\alpha+\beta}} + \frac{z}{\sigma(x)} - \big(\frac{\alpha}{\beta}\big)^{\frac{-\alpha}{\alpha+\beta}}\big(\frac{z}{\sigma(x)}\big)^{\frac{\beta}{\alpha+\beta}} - \frac{\kappa}{\sigma(x)}|z| & \text{if } z > \sigma(x)\frac{\beta}{\alpha}, \\ -\lambda y + U(R_t) - \frac{\kappa}{\sigma(x)}|z| & \text{if } z \leq \sigma(x)\frac{\beta}{\alpha}, \end{cases} \end{aligned}$$

it satisfies the Lipschitz condition with respect to both y *and* z*, and the following integrability condition holds :*

$$\mathbb{E}\left[\int_0^\iota e^{(\nu-2\lambda)s}|g(s,\omega,X_s,0,0)|^2 ds\right] = \mathbb{E}\left[\int_0^\iota e^{(\nu-2\lambda)s}|U(R_s)|^2 ds\right] < \infty,$$

Thus, we can conclude that the generator g *fulfills the requirements stated in* $(H1)$ *and* $(H2)$*.*

4.4 Solving the Principal problem

The preceding section provides a comprehensive delineation of incentive-compatible contracts. The objective function governing the consortium's actions under any such contract conforms to a well-defined BSDE with a random termination point. The public's objective is to devise contracts that effectively disclose the consortium's actions, namely, incentive-compatible contracts. Consequently, the Principal's stochastic control problem is reframed as a standard stochastic control scenario, where J^C serves as a state variable, and the contract Γ and the optimal effort $A^*(X,Z)$ serve as control processes. The Principal's value function can be expressed as follows :

$$v(x,y) = \sup_{(R,\iota,A^*(X^x,Z))\in\mathcal{A}_A^P} \mathbb{E}^{A^*(X^x,Z)}\left[\int_0^\iota e^{-\delta s}(\vartheta(A^*(X_s^x,Z_s)) - \kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s)ds - e^{-\delta\iota}U^{-1}(J_\iota^{C,y})\right], \tag{4.4.1}$$

where y represents the initial value of the process $(J_t^{C,y})_{t\geq 0}$, which follows the dynamics described by the following SDE :

$$\begin{cases} dJ_t^{C,y} &= \left(\lambda J_t^{C,y} - U(R_t) - \varphi(A^*(X_t^x, Z_t), X_t^x, Z_t) + |Z_t| \dfrac{\kappa}{\sigma(X_t^x)}\right) dt - Z_t dW_t, \\ J_0^{C,y} &= y \geq \underline{x}, \end{cases} \tag{4.4.2}$$

the set $\mathcal{A}_A^P$ is defined as follows :

$$\begin{aligned} \mathcal{A}_A^P := \Big\{ & (R, \iota, A^*(X^x, Z)) \text{ s.t. } (R_s)_{s\geq 0} \text{ is } \mathbb{F}-\mathcal{P}r \text{ nonnegative process}, \\ & \mathbb{E}^{\mathbb{P}} \left[\int_0^{\iota} e^{(\nu - 2\lambda)s} (U(R_s)^2 \vee R_s^2) ds \right] < \infty, \ \iota \in \mathcal{T}, \ A^*(X^x, Z) \in \mathcal{A}_{\nu-2\lambda}^C \Big\}. \end{aligned}$$

Due to the condition $J_0^{C,y} = y \geq \underline{x}$, the reservation constraint of the Agent is fulfilled. We introduce the DPP, a cornerstone in stochastic control theory. Specifically, we state : For any stopping time $\zeta \in \mathcal{T}$,

$$\begin{aligned} v(x,y) = \sup_{(R,\iota,A^*(X^x,Z)) \in \mathcal{A}_A^P} \mathbb{E}^{A^*(X^x,Z)} \Bigg[& \int_0^{\iota \wedge \zeta} e^{-\delta s} (\vartheta(A^*(X_s^x, Z_s)) - \kappa \mathbb{1}_{\{A^*(X_s^x, Z_s) > 0\}} - R_s) ds \\ & - e^{-\delta \iota} U^{-1}(J_\iota^{C,y}) \mathbb{1}_{\iota < \zeta} + e^{-\delta \zeta} v(X_\zeta^x, J_\zeta^{C,y}) \mathbb{1}_{\zeta \leq \iota} \Bigg]. \end{aligned} \tag{4.4.3}$$

The HJBVI governing the public value function represents an infinitesimal counterpart of the DPP. It elucidates the local dynamics of the value function, elucidating the evolution as the stopping time ζ in (4.4.3) transitions to the initial time.

$$\min \left\{ \delta w(x,y) - \sup_{(r,a) \in \mathbb{R}_+ \times \mathbb{R}_+} \left[\mathcal{L}^{a,r} w(x,y) + \vartheta(a) - \kappa \mathbb{1}_{\{a>0\}} - r \right], w(x,y) + U^{-1}(y) \right\} = 0, \ (x,y) \in \mathbb{R} \times (0,\infty), \tag{4.4.4}$$

where $\mathcal{L}^{a,r}$ is the generator associated with the SDE(4.4.2). It is given by :

$$\begin{aligned} \mathcal{L}^{a,r} w(x,y) := & \frac{1}{2} \sigma^2(x) \frac{\partial w^2(x,y)}{\partial x^2} + \frac{1}{2} (\sigma(x) \frac{h'(a)}{\vartheta'(a)})^2 \mathbb{1}_{\{a>0\}} \frac{\partial w^2(x,y)}{\partial y^2} \\ & + \sigma^2(x) \frac{h'(a)}{\vartheta'(a)} \mathbb{1}_{\{a>0\}} \frac{\partial w^2(x,y)}{\partial x \partial y} + (\vartheta(a) - \kappa \mathbb{1}_{\{a>0\}}) \frac{\partial w(x,y)}{\partial x} \\ & + [\lambda y - U(r) + h(a)] \frac{\partial w(x,y)}{\partial y}. \end{aligned}$$

In the subsequent lemma, we present the boundary condition corresponding to the case where $y = 0$.

Lemma 4.4.1. *The value function v is subject to the boundary condition :*

$$v(x,0) = 0 \text{ for all } x \in \mathbb{R}. \tag{4.4.5}$$

Proof: We begin by considering a fixed $x \in \mathbb{R}$. Initially, the consortium's value function at time 0 is 0. Given that the consortium can achieve non-negative utility without exerting any effort, it follows that $J_t^{C,y} \geq 0$ for all $t \in [\![0, \iota^*[\![$ $\mathbb{P}$ a.s. Since $J_t^{C,y}$ adheres the SDE (4.4.2), ensuring a non-negative solution requires $Z_t = 0$ for all $t \in [\![0, \iota^*[\![$ $\mathbb{P}$ a.s. By the mapping between Z and $A^*(X, Z)$, it implies $A^*(X_t, Z_t) = 0$ for all $t \in [\![0, \iota^*[\![$ $\mathbb{P}$ a.s. From the perspective of the public, it becomes highly advantageous to select $R_t^* = 0$ for all $t \in [\![0, \iota^*[\![$ $\mathbb{P}$ a.s. Given that the drift of the SDE (4.4.2) equals 0, it is optimal for the public to terminate the contract at $\iota^* = 0$ $\mathbb{P}$ a.s. This demonstrates that $v(x, 0) = 0$ for all $x \in \mathbb{R}$. This completes the proof.

□

The forthcoming lemma establishes both upper and lower bounds for the value function v.

Lemma 4.4.2. *For all $(R, 0, A^*(Z)) \in \mathcal{A}_A^P$, the value function $v(x, y)$ satisfies the following inequalities :*

$$-U^{-1}(y) \leq v(x, y) \leq L, \tag{4.4.6}$$

where $L = \frac{\sigma_{max} \|\vartheta/\sigma\|_\infty}{\delta}$.

Proof:

1. **Lower Bound :** By definition of the value function from Equation (4.4.1), we have :

$$v(x, y) \geq -U^{-1}(y)$$

2. **Upper Bound :** Since the continuation cost $J_\iota^{C,y}$ is non-negative almost surely under any probability measure $\mathbb{P}$ and the rent is also non-negative, we can write :

$$v(x, y) \leq \sup_{(R, \iota, A^*(X^x, Z)) \in \mathcal{A}_A^P} \mathbb{E}^{A^*(X^x, Z)} \left[\int_0^{\iota \wedge \zeta} \gamma_s^{\theta^*(Z)} e^{-\delta s} \vartheta(A^*(X_s^x, Z_s)) ds \right]$$

3. **Applying Assumption and Bounding :** By **Assumption** 4.2.1 and the boundedness of $\sigma(X_\cdot^x)$, we can further bound the expression :

$$\begin{aligned} v(x, y) &\leq \sigma_{max} \|\vartheta/\sigma\|_\infty \sup_{(R, \iota, A^*(X^x, Z)) \in \mathcal{A}_A^P} \mathbb{E}^{A^*(X^x, Z)} \left[\int_0^\infty \gamma_s^{\theta^*(Z)} e^{-\delta s} ds \right] \\ &\leq \sigma_{max} \|\vartheta/\sigma\|_\infty \sup_{(R, \iota, A^*(X^x, Z)) \in \mathcal{A}_A^P} \mathbb{E}^{\theta^*(Z)} \left[\int_0^\infty e^{-\delta s} ds \right] \\ &\leq \frac{\sigma_{max} \|\vartheta/\sigma\|_\infty}{\delta} = L \end{aligned}$$

Therefore, we have established the desired bounds for the value function $v(x, y)$. □

The subsequent theorem serves as a verification theorem, pivotal in demonstrating that a smooth solution to the HJBVI equation (4.4.4) indeed corresponds to the value function. This theorem facilitates the determination of optimal control in a feedback form.

Theorem 4.4.3. *Suppose there exist a constant $\hat{b} > 0$ and a continuously differentiable function $w : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying the following properties :*

(i) ***Initial Condition :*** *$w(x,0) = 0$ and $w \in C^2(\mathbb{R} \times [0,\hat{b}))$ satisfying a growth condition (denoted by Eq. (4.4.6)).*

(ii) ***Comparison with Utility :*** *$w > -U^{-1}$ on $\mathbb{R} \times (0,\hat{b})$ and $w = -U^{-1}$ on $\mathbb{R} \times [\hat{b},\infty)$.*

(iii) ***HJB Equation :*** *For all $(x,y) \in \mathbb{R} \times (0,\hat{b})$,*

$$\delta w(x,y) = \sup_{(r,a)\in\mathbb{R}_+\times\mathbb{R}_+} \left\{\mathcal{L}^{a,r} w(x,y) + \vartheta(a) - \kappa \mathbb{1}_{\{a>0\}} - r\right\}.$$

(iv) ***Exit Condition :*** *For all $(x,y) \in \mathbb{R} \times [\hat{b},\infty)$,*

$$\delta(-U^{-1}(y)) \geq \sup_{(r,a)\in\mathbb{R}_+\times\mathbb{R}_+} \left\{\mathcal{L}^{a,r}(-U^{-1}(y)) + \vartheta(a) - \kappa \mathbb{1}_{\{a>0\}} - r\right\},$$

We also assume that Eq. (4.4.7) holds,

$$\sup_{(R,\iota,A^*(Z))\in\mathcal{Y}} \mathbb{E}[|e^{-\delta\iota} U^{-1}(J_\iota^{C,y})|^2] < \infty. \tag{4.4.7}$$

Then, we have :

1. Comparison with value function : For all $(x,y) \in \mathbb{R} \times \mathbb{R}_+$, $w(x,y) \geq v(x,y)$. 2. Verification with Optimal Control : Suppose there exist measurable, non-negative functions (a^*, r^*) defined on $\mathbb{R}_+ \times \mathbb{R}_+$ such that for all $y \in (0,\hat{b})$, the following verification condition holds :

$$\begin{aligned}&\sup_{(r,a)\in\mathbb{R}_+\times\mathbb{R}_+} \left\{\mathcal{L}^{a,r} w(x,y) - \kappa \mathbb{1}_{\{a>0\}} + \vartheta(a) - r\right\} \\ &\qquad = \mathcal{L}^{a^*(x,y),r^*(x,y)} w(x,y) + \vartheta(a^*(x,y)) - \kappa \mathbb{1}_{\{a^*(x,y)>0\}} - r^*(x,y)\end{aligned}$$

Furthermore, consider the following SDE :

$$\begin{aligned}dJ_t^C &= [\lambda J_t^C - U(r^*(X_t^x, J_t^C)) - \varphi(a^*(X_t^x, J_t^C), X_t^x, Z_t) + |Z_t|\kappa]dt + Z_t dW_t \\ J_0^C &= y\end{aligned}$$

If this SDE admits a unique solution $\widehat{J_t^C}$, we define the stopping time ι^* as :

$$\iota^* := \inf\{t \geq 0 \ : \ w(X, \widehat{J_t^C}) \leq -U^{-1}(\widehat{J_t^C})\}. \tag{4.4.8}$$

We further assume that $(r^*(X, \widehat{J^C}), \iota^*, a^*(X, \widehat{J^C}))$ *lies in the set of admissible controls* $\mathcal{A}_A^P$ *and* $\mathbb{E}^{\mathbb{P}}[e^{(\rho-2\lambda)\iota^*}\widehat{J_{\iota^*}^C}^2 \mathbb{1}_{\{\iota^*<\infty\}}] < \infty$.

Then, if w *is a solution of (4.4.4) with boundary condition given by (4.4.5), we have :*

$w = v$, *meaning* w *coincides with the value function* v. ι^* *is an optimal stopping time for the problem (4.4.3).*

Proof:

1. Let $(x, y) \in \mathbb{R} \times \mathbb{R}_+$ and an admissible control $(R, \iota, A^*(X^x, Z)) \in \mathcal{A}_A^P$. If $y = 0$, then from assumption(i), we have $v(x, 0) = w(x, 0) = 0$. We assume that $0 < y$. From (iii) and (iv), we have
$$\delta w(x,y) \geq \sup_{(r,a)\in\mathbb{R}^+\times\mathbb{R}^+} \{\mathcal{L}^{a,r}w(x,y) + \vartheta(a) - \kappa\mathbb{1}_{\{a>0\}} - r\}, \tag{4.4.9}$$
For $n \in \mathbb{N}$, We introduce the stopping time ι_n :
$$\iota_n := \iota \wedge \inf\{t \geq 0, \sigma(X_t^x)|\frac{\partial w(X_t^x, J_t^{C,y})}{\partial x} + \frac{\partial w(X_t^x, J_t^{C,y})}{\partial y}\frac{h'(A^*(X_t^x, Z_t))}{\vartheta'(A^*(X_t^x, Z_t))}\mathbb{1}_{\{A^*(X_t^x, Z_t)>0\}}| \geq n\}.$$
From (i)-(ii), we have w is continuous on $\mathbb{R} \times \mathbb{R}_+$, $w \in C^2(\mathbb{R} \times [0, \hat{b}))$ and $w = -U^{-1} \in C^2([\hat{b}, \infty))$, then w is continuous and piece-wise C^2 on $\mathbb{R} \times \mathbb{R}_+$. Applying the generalized Itô's formula (see Krylov [61], Theorem 2, p. 124) to the process $e^{-\delta t}\gamma_t^{\theta^*(Z)}w(X_t^x, J_t^{C,y})$ between 0 and ι_n, using inequality (4.4.9) and Bayes formula, we obtain :
$$\begin{aligned} &w(x,y) \\ &\geq \mathbb{E}^{A^*(X^x,Z)}[\gamma_{\iota_n}^{\theta^*(Z)}\int_0^{\iota_n} e^{-\delta s}(\vartheta(A^*(X_s^x, Z_s)) - \kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s)ds + e^{-\delta\iota_n}\gamma_{\iota_n}^{\theta^*(Z)}w(X_{\iota_n}^x, J_{\iota_n}^{C,y})] \\ &= \mathbb{E}[\gamma_{\iota_n}^{A^*,\theta^*}\int_0^{\iota_n} e^{-\delta s}(\vartheta(A^*(X_s^x, Z_s)) - \kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s)ds + e^{-\delta\iota_n}\gamma_{\iota_n}^{A^*,\theta^*}w(X_{\iota_n}^x, J_{\iota_n}^{C,y})] \end{aligned} \tag{4.4.10}$$
where $\gamma_{\iota_n}^{A^*,\theta^*} = \gamma_{\iota_n}^{A^*(X^x,Z)}\gamma_{\iota_n}^{\theta^*(Z)}$. In the next step, we show that the sequence
$$\Big(\gamma_{\iota_n}^{A^*,\theta^*}\int_0^{\iota_n} e^{-\delta s}(\vartheta(A^*(X_s^x, Z_s)) - \kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}} - R_s)ds + e^{-\delta\iota_n}\gamma_{\iota_n}^{A^*,\theta^*}w(X_{\iota_n}^x, J_{\iota_n}^{C,y})\Big)_n$$
is uniformly integrable under $\mathbb{P}$. Let $p \in (1, 2)$, we define $p_1 := \frac{2}{2-p}$, $p_2 := \frac{2}{p}$ the conjugate of p_1. We denote by $\hat{p} := pp_1 \in (2, \infty)$. Using Hölder's inequality, and the growth condition

of w, we obtain :

$$\begin{aligned}\mathbb{E}[\Big|e^{-\delta\iota_n}\gamma_{\iota_n}^{A^*,\theta^*}w(X_{\iota_n}^x,J_{\iota_n}^{C,y})\Big|^p] &\leq \mathbb{E}[\Big|Le^{-\delta\iota_n}\gamma_{\iota_n}^{A^*,\theta^*}+e^{-\delta\iota_n}\gamma_{\iota_n}^{A^*,\theta^*}U^{-1}(J_{\iota_n}^{C,y})\Big|^p]\\ &\leq C\mathbb{E}[\Big|\gamma_{\iota_n}^{A^*,\theta^*}\Big|^p+\Big|e^{-\delta\iota_n}\gamma_{\iota_n}^{A^*,\theta^*}U^{-1}(J_{\iota_n}^{C,y})\Big|^p]\\ &\leq C\left(\mathbb{E}[\Big|\gamma_{\iota_n}^{A^*,\theta^*}\Big|^p]+\mathbb{E}[\Big|\gamma_{\iota_n}^{A^*,\theta^*}\Big|^{\hat{p}}]^{\frac{1}{p_1}}\mathbb{E}[\Big|e^{-\delta\iota_n}U^{-1}(J_{\iota_n}^{C,y})\Big|^{pp_2}]^{\frac{1}{p_2}}\right)\\ &\leq C\left(\mathbb{E}[\Big|\gamma_{\iota}^{A^*,\theta^*}\Big|^p]+\mathbb{E}[\Big|\gamma_{\iota}^{A^*,\theta^*}\Big|^{\hat{p}}]^{\frac{1}{p_1}}\mathbb{E}[\Big|e^{-\delta\iota_n}U^{-1}(J_{\iota_n}^{C,y})\Big|^{2}]^{\frac{1}{p_2}}\right),\end{aligned}$$

where C is a generic positive constant that could change from line to line.

From Jensen's inequality, we have $\mathbb{E}[\Big|\gamma_{\iota}^{A^*,\theta^*}\Big|^p]\leq(\mathbb{E}[\Big|\gamma_{\iota}^{A^*,\theta^*}\Big|^{\hat{p}}])^{\frac{1}{p_1}}$. Additionally, Assumption (4.4.7) substantiates this conclusion :

$$\sup_{n\in\mathbb{N}}\mathbb{E}[\Big|e^{-\delta\iota_n}\gamma_{\iota_n}^{A^*,\theta^*}w(X_{\iota_n}^x,J_{\iota_n}^{C,y})\Big|^p]<\infty$$

Using the same arguments as above, we get

$$\begin{aligned}&\mathbb{E}[\Big|\gamma_{\iota_n}^{A^*,\theta^*}\int_0^{\iota_n}e^{-\delta s}(\vartheta(A^*(X_s^x,Z_s))-\kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}}-R_s)ds\Big|^p]\\ &\leq\mathbb{E}[\Big|\gamma_{\iota_n}^{A^*,\theta^*}\Big|^{pp_1}]^{\frac{1}{p_1}}\mathbb{E}[\Big|\int_0^{\iota_n}e^{-\delta s}(\vartheta(A^*(X_s^x,Z_s))-\kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}}-R_s)ds\Big|^{pp_2}]^{\frac{1}{p_2}}\\ &\leq C\mathbb{E}[\Big|\gamma_{\iota}^{A^*,\theta^*}\Big|^{\hat{p}}]^{\frac{1}{p_1}}\mathbb{E}[\int_0^{\infty}e^{-\delta s}ds\int_0^{\iota}e^{-\delta s}(\vartheta(A^*(X_s^x,Z_s))-\kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}}-R_s)^2ds]^{\frac{1}{p_2}}\\ &\leq C\mathbb{E}[\Big|\gamma_{\iota}^{A^*,\theta^*}\Big|^{\hat{p}}]^{\frac{1}{p_1}}\mathbb{E}[\int_0^{\iota}e^{-\delta s}(\vartheta(A^*(X_s^x,Z_s))^2+\kappa^2\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}}+R_s^2)ds]^{\frac{1}{p_2}},\end{aligned}$$

where the second inequality is obtained by using Hölder's inequality and since $\int_0^{\iota_n}e^{-\delta s}ds\leq\int_0^{\infty}e^{-\delta s}ds$ a.s. As $(R,\iota,A^*(X^x,Z))\in\mathcal{A}_A^P$, we have

$$\sup_{n\in\mathbb{N}}\mathbb{E}[\Big|\gamma_{\iota_n}^{A^*,\theta^*}\int_0^{\iota_n}e^{-\delta s}(\vartheta(A^*(X_s^x,Z_s))-\kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}}-R_s)ds\Big|^p]<\infty$$

According to the Theorems A.1.2 and A.1.1 in Pham [83], we have the convergence of the previous sequences in $L^1(\mathbb{P})$. By passing to the limit in (4.4.10), we obtain

$$\begin{aligned}&w(x,y)\\ &\geq\mathbb{E}^{A^*(X^x,Z)}[\gamma_{\iota}^{\theta^*(Z)}\int_0^{\iota}e^{-\delta s}(\vartheta(A^*(X_s^x,Z_s))-\kappa\mathbb{1}_{\{A^*(X_s^x,Z_s)>0\}}-R_s)ds-e^{-\delta\iota}\gamma_{\iota}^{\theta^*(Z)}U^{-1}(J_{\iota}^{C,y})],\end{aligned}$$

and so for all $(x,y)\in\mathbb{R}\times(0,\infty)$, we obtain $w(x,y)\geq v(x,y)$ and then for all $(x,y)\in\mathbb{R}\times\mathbb{R}^+$ we have :

$$w(x,y)\geq v(x,y). \tag{4.4.11}$$

2. We fix $(x,y) \in \mathbb{R}\times(0,\infty)$. Let ι_n be the stopping time given by

$$\iota_n = \iota^* \wedge \inf\{t \geq 0, \sigma(X_t^x)|\frac{\partial w(X_t^x,\hat{J}_t^{C,y})}{\partial x} + \frac{\partial w(X_t^x,\hat{J}_t^{C,y})}{\partial y}\frac{h'(a^*(X_t^x,\hat{J}_t^{C,y}))}{\vartheta'(a^*(X_t^x,\hat{J}_t^{C,y}))}\mathbb{1}_{\{a^*(X_t^x,\hat{J}_t^{C,y})>0\}}| \geq n\}.$$

Since $[\![0,\iota_n[\![\subset [\![0,\iota^*[\![$, then by (ii), $w(X_t^x,\hat{J}_t^{C,y}) > -U^{-1}(\hat{J}_t^{C,y})$ on $[\![0,\iota_n[\![$, and so by (iii), we have :

$$\begin{aligned}&\delta w(X_t^x,\hat{J}_t^{C,y}) - \mathcal{L}^{a^*(X_t^x,\hat{J}_t^{C,y}),r^*(X_t^x,\hat{J}_t^{C,y})} w(X_t^x,\hat{J}_t^{C,y})\\ &- \Big(\vartheta(a^*(X_t^x,\hat{J}_t^{C,y})) - \kappa\mathbb{1}_{\{a^*(X_t^x,\hat{J}_t^{C,y})>0\}} - r^*(X_t^x,\hat{J}_t^{C,y})\Big) = 0 \text{ on } [\![0,\iota_n[\![.\end{aligned}$$

Therefore

$$\begin{aligned}w(x,y) = {}&\mathbb{E}^{a^*(X^x,\hat{J}^{C,y})}[\gamma_{\iota_n}^{\theta^*(Z)}\int_0^{\iota_n} e^{-\delta s}(\vartheta(a^*(X_s^x,\hat{J}_s^{C,y})) - \kappa\mathbb{1}_{\{a^*(X_s^x,\hat{J}_s^{C,y})>0\}} - a^*(X_s^x,\hat{J}_s^{C,y}))ds]\\ &+ \mathbb{E}^{a^*(X^x,\hat{J}^{C,y})}[e^{-\delta\iota_n}\gamma_{\iota_n}^{\theta^*(Z)} w(X_{\iota_n}^x,\hat{J}_{\iota_n}^{C,y})]\end{aligned}$$

Using the same previous steps, we can pass to the limit. From the definition of ι^* and since $\iota_n \longrightarrow \iota^*$ when n goes to infinity we obtain :

$$\begin{aligned}&w(x,y)\\ &= \mathbb{E}^{a^*(X^x,\hat{J}^{C,y})}[\gamma_{\iota^*}^{\theta^*(Z)}\int_0^{\iota^*} e^{-\delta s}(\vartheta(a^*(X_s^x,\hat{J}_s^{C,y})) - \kappa\mathbb{1}_{\{a^*(X_s^x,\hat{J}_s^{C,y})>0\}} - a^*(X_s^x,\hat{J}_s^{C,y}))ds]\\ &- \mathbb{E}^{a^*(X^x,\hat{J}^{C,y})}[e^{-\delta\iota^*}\gamma_{\iota^*}^{\theta^*(Z)} U^{-1}(\hat{J}_{\iota^*}^{C,y})]\\ &\leq v(x,y).\end{aligned}$$

As $w(x,0) = v(x,0)$ for all $x \in \mathbb{R}$, we deduce that for all $(x,y) \in \mathbb{R}\times\mathbb{R}^+$, we have $w(x,y) \leq v(x,y)$. Combining with inequality (4.4.11), we deduce that $w = v$.

□

Remark 4.4.4. *The optimal rent $r^*(x,y)$ depends on U', and the marginal value, $\frac{\partial v(x,y)}{\partial y}$, of the value function $v(x,y)$. Specifically, it is given by :*

$$r^*(x,y) = (U')^{-1}(-\frac{1}{\frac{\partial v(x,y)}{\partial y}})\mathbb{1}_{\frac{\partial v(x,y)}{\partial y}<0}$$

.

The indicator function $\mathbb{1}_{\frac{\partial v(x,y)}{\partial y}<0}$ ensures that the optimal rent is only positive when the marginal value of wealth is negative.

4.5 Numerical study

In this section, we undertake the numerical solution of the HJBVI presented in equation (4.4.4). Initially, we confine the domain to the intervals $[-\bar{x}, \bar{x}]$ and $[0, \bar{y}]$, where $\bar{x}$ and $\bar{y}$ denote empirically determined boundaries. Subsequently, employing finite difference approximations, we discretize equation (4.4.4). Finally, the resulting discrete variational inequality is solved utilizing the Howard algorithm.

4.5.1 Numerical scheme

We utilize the finite difference method to discretize the HJBVI (4.4.4). This method involves replacing the domain with a bounded interval, specifically $[-\bar{x}, \bar{x}]$, $[0, \bar{y}]$. Let Δ_x (resp. Δ_y) be the finite difference step on the state coordinate. The grid points are defined as $x^{\Delta} = (x_i)_{i=1,N}$, where $x_i = -\bar{x} + i\Delta_x$, and $y^{\Delta} = (y_i)_{i=1,N}$, where $y_i = i\Delta_y$. These points form the grid $\Omega_{\Delta} := \{-\bar{x} + i\Delta_x, i = 1, ..., N\} \times \{j\Delta_y, j = 1, ..., N\}$. Next, we approximate the derivatives of the value function v using the following expressions :

$$
\begin{aligned}
\frac{\partial v(x,y)}{\partial x} &\simeq \begin{cases} \frac{v(x+\Delta_x,y)-v(x,y)}{\Delta_x} & \text{if } -\vartheta(a) + \kappa \mathbb{1}_{\{a>0\}} \geq 0, \\ \frac{v(x,y)-v(x-\Delta_x,y)}{\Delta_x} & \text{if not,} \end{cases} \\
\frac{\partial v(x,y)}{\partial y} &\simeq \begin{cases} \frac{v(x,y+\Delta_y)-v(x,y)}{\Delta_y} & \text{if } -\lambda y + U(r) - h(a) \geq 0, \\ \frac{v(x,y)-v(x,y-\Delta_y)}{\Delta_y} & \text{if not,} \end{cases} \\
\frac{\partial v^2(x,y)}{\partial x^2} &\simeq \frac{v(x+\Delta_x,y) - 2v(x,y) + v(x-\Delta_x,y)}{\Delta_x^2}, \\
\frac{\partial v^2(x,y)}{\partial y^2} &\simeq \frac{v(x,y+\Delta_y) - 2v(x,y) + v(x,y-\Delta_y)}{\Delta_y^2}, \\
\frac{\partial v^2(x,y)}{\partial x \partial y} &\simeq \frac{v(x+\Delta_x,y+\Delta_y) - v(x+\Delta_x,y) - v(x,y+\Delta_y) + 2v(x,y)}{2\Delta_x\Delta_y}, \\
&\quad - \frac{v(x-\Delta_x,y) + v(x,y-\Delta_y) - v(x-\Delta_x,y-\Delta_y)}{2\Delta_x\Delta_y}, \\
v(x,0) = 0,\ v(x,\bar{y}) &= -U^{-1}(\bar{y}),\ \frac{\partial v(-\bar{x},y)}{\partial x} = 0,\ \frac{\partial v(\bar{x},y)}{\partial x} = 0.
\end{aligned}
$$

We represent the linear operator by the symbol $\mathcal{L}^{\Delta,a,r}$. This operator takes the form of a block tridiagonal matrix with dimension $(N-1)^2 \times (N-1)^2$
$[\mathcal{L}^{\Delta,(a,r)}]_{k,k-1} = \bar{\mathbb{T}}_k^{\Delta,(a,r)}; [\mathcal{L}^{\Delta,(a,r)}]_{k,k} = D_k^{\Delta,(a,r)}; [\mathcal{L}^{\Delta,(a,r)}]_{k,k+1} = \mathbb{T}_k^{\Delta,(a,r)}$, $k = 1, ..., N-1$;
$\bar{\mathbb{T}}_k^{\Delta,(a,r)}$, $D_k^{\Delta,(a,r)}$, $\mathbb{T}_k^{\Delta,(a,r)}$ are tridiagonal matrices $(N-1) \times (N-1)$ defined by

$[\bar{\mathbb{T}}_k^{\Delta,(a,r)}]_{i,i-1} = \frac{\gamma_{k,i}}{2\Delta_x\Delta_y}; [\bar{\mathbb{T}}_k^{\Delta,(a,r)}]_{i,i} = \frac{\beta_{k,i}^{1,-}}{\Delta_x} + \frac{\alpha_{k,i}^1}{\Delta_x^2} - \frac{\gamma_{k,i}}{2\Delta_x\Delta_y}; [\bar{\mathbb{T}}_k^{\Delta,(a,r)}]_{i,i+1} = 0;$

$[D_k^{\Delta,(a,r)}]_{i,i-1} = \frac{\beta_{k,i}^{2,-}}{\Delta_y} + \frac{\alpha_{k,i}^2}{\Delta_y^2} - \frac{\gamma_{k,i}}{2\Delta_x\Delta_y}; [D_k^{\Delta,(a,r)}]_{i,i} = \delta - \frac{|\beta_{k,i}^1|}{\Delta_x} - \frac{|\beta_{k,i}^2|}{\Delta_y} - 2\left(\frac{\alpha_{k,i}^1}{\Delta_x^2} + \frac{\alpha_{k,i}^2}{\Delta_y^2} - \frac{\gamma_{k,i}}{2\Delta_x\Delta_y}\right);$

$[D_k^{\Delta,(a,r)}]_{i,i+1} = \frac{\beta_{k,i}^{2,+}}{\Delta_y} + \frac{\alpha_{k,i}^2}{\Delta_y^2} - \frac{\gamma_{k,i}}{2\Delta_x\Delta_y}; [\mathbb{T}_k^{\Delta,(a,r)}]_{i,i-1} = 0; [\mathbb{T}_k^{\Delta,(a,r)}]_{i,i} = \frac{\beta_{k,i}^{1,+}}{\Delta_x} + \frac{\alpha_{k,i}^1}{\Delta_x^2} - \frac{\gamma_{k,i}}{2\Delta_x\Delta_y};$

$[\mathbb{T}_k^{\Delta,(a,r)}]_{i,i+1} = \frac{\gamma_{k,i}}{2\Delta_x\Delta_y}$, $i = 1,...,N-1;$

where $\beta_{i,j}^{l,+}(x,y) = \max(\beta_{i,j}^l, 0)$, $\beta_{i,j}^{l,-} = \max(-\beta_{i,j}^l, 0)$, $l = 1,2$, and the parameters γ, β^1, β^2, α^1, α^2 are given as :
$\beta_{i,j}^1 = -\vartheta(a) + \kappa\mathbb{1}_{\{a>0\}}$, $\beta_{i,j}^2 = -\lambda y - h(a) + U(r)$, $\alpha_{i,j}^1 = -\frac{1}{2}\sigma^2(x_i)\mathbb{1}_{a>0}$, $\alpha_{i,j}^2 = -\frac{1}{2}(\sigma(x_i)\frac{h'(a)}{\vartheta'(a)})^2\mathbb{1}_{a>0}$, $\gamma_{i,j} = -\sigma^2(x_i)\frac{h'(a)}{\vartheta'(a)}\mathbb{1}_{a>0}$.
Through this process, we arrive at a system of $(N-1)^2$ linear approximation equations. The vector $V^\Delta = (v_1^\Delta, ..., v_{N-1}^\Delta)$ represents the collection of unknowns associated with this system. Each sub-vector within V^Δ, denoted as $v_i^\Delta = (v_{i,1}^\Delta, ..., v_{i,N-1}^\Delta)$, has dimension $(N-1)$.

$$\min\left[\inf_{(r,a)\in\mathbb{R}_+\times\mathbb{R}_+}[\mathcal{L}^{\Delta,(a,r)}V^\Delta + B^{\Delta,(a,r)}], V^\Delta + U^{-1}(y^\Delta)\right] = 0 \tag{4.5.1}$$

where $U^{-1}(y^\Delta) := (U^{-1}(y_i^\Delta))_{i=1..N-1}$. Each element $U^{-1}(y_i^\Delta)$ is an $\mathbb{R}^{(N-1)}$ vector given by $(U^{-1}(y_i), ..., U^{-1}(y_i))$. Furthermore, we define the vector $B^{\Delta,(a,r)}$ as :

$$B^{\Delta,(a,r)} = (B_1^{\Delta,(a,r)} + \bar{\mathbb{T}}_1^{\Delta,(a,r)}v_1^\Delta, B_2^{\Delta,(a,r)}, ..., B_{N-2}^{\Delta,(a,r)}, B_{N-1}^{\Delta,(a,r)} + A_{N-1}^{\Delta,(a,r)}v_{N-1}^\Delta),$$

such that :

$$B_k^{\Delta,\kappa,(a,r)} = \begin{pmatrix} -\vartheta(a) + \kappa + r + \left[\bar{\mathbb{T}}_k^{\Delta,(a,r)}\right]_{1,0} v_{k-1,0}^\Delta + \left[D_k^{\Delta,(a,r)}\right]_{1,0} v_{k,0}^\Delta \\ -\vartheta(a) + \kappa + r \\ \vdots \\ -\vartheta(a) + \kappa + r \\ -\vartheta(a) + \kappa + r + \left[D_k^{\Delta,(a,r)}\right]_{N-1,N} v_{k,N}^\Delta + \left[\mathbb{T}_k^{\Delta,(a,r)}\right]_{N-1,N} v_{k+1,N}^\Delta \end{pmatrix}.$$

Here, we address the solution of the discrete variational inequality (4.5.1) by employing Howard's algorithm. We detail the application of this algorithm as follows :
The Howard algorithm : It consists in computing iteratively two sequences $((a_{i,j}^n, r_{i,j}^n)_{i,j=1,...N-1})_{n\geq 1}$ and $(V^{\Delta,n})_{n\geq 1}$, as follows :

- Step $2n-1$. To the vector $V^{\Delta,n}$, we associate the strategy

$$(a^n, r^n) \in \arg\min_{a,r}\{\mathcal{L}^{\Delta,(a,r)}V^{\Delta,n} + B^{\Delta,(a,r)}\}.$$

- Step $2n$. From the strategy (a^n, r^n), we compute a partition $(D_1^n \cup D_2^n)$ of $\mathbb{R}_+^2$ defined by

$$\mathcal{L}^{\Delta,(a^n,r^n)} V^{\Delta,n} + B^{\Delta,(a^n,r^n)} \leq V^{\Delta,n} + U^{-1}(y^\Delta), \text{ on } D_1^n,$$
$$\mathcal{L}^{\Delta,(a^n,r^n)} V^{\Delta,n} + B^{\Delta,(a^n,r^n)} > V^{\Delta,n} + U^{-1}(y^\Delta), \text{ on } D_2^n.$$

The solutions $V^{\Delta,n+1}$ is obtained by solving two linear systems

$$\mathcal{L}^{\Delta,(a^n,r^n)} V^{\Delta,n+1} + B^{\Delta,(a^n,r^n)} = 0, \text{ on } D_1^n,$$

and

$$V^{\Delta,n+1} + U^{-1}(y^\Delta) = 0, \text{ on } D_2^n.$$

- If $|V^{\Delta,n+1} - V^{\Delta,n}| \leq \varepsilon$, stop, otherwise, go to step $2n+1$.

The Howard Algorithm involves an iterative procedure aimed at computing two sequences :

A sequence of pairs, denoted as $((a^n_{i,j}, r^n_{i,j})_{i,j=1,\dots,N-1})_{n\geq 1}$.
A sequence of value functions, denoted as $(V^{\Delta,n})_{n\geq 1}$.

The algorithm proceeds through the following steps at each iteration n :

1. ***Minimization Step (Step*** $2n-1$***)*** : At this stage, we associate the vector $V^{\Delta,n}$ with the strategy

$$(a^n, r^n) \in \arg\min_{a,r}\{\mathcal{L}^{\Delta,(a,r)}V^{\Delta n} + B^{\Delta,(a,r)}\}.$$

2. ***Partitioning Step (Step*** $2n$***)*** : Based on the strategy (a^n, r^n), we establish a partition $(D^n_1 \cup D^n_2)$ of $\mathbb{R}^2_+$ defined by

$$\mathcal{L}^{\Delta,(a^n,r^n)}V^{\Delta,n} + B^{\Delta,(a^n,r^n)} \leq V^{\Delta,n} + U^{-1}(y^{\Delta}), \text{ on } D^n_1,$$
$$\mathcal{L}^{\Delta,(a^n,r^n)}V^{\Delta,n} + B^{\Delta,(a^n,r^n)} > V^{\Delta,n} + U^{-1}(y^{\Delta}), \text{ on } D^n_2.$$

3. ***Update Step*** : Two linear systems are solved : The first system ensures the updated value function $V^{\Delta,n+1}$ equals zero within region D^n_1, i.e.

$$\mathcal{L}^{\Delta,(a^n,r^n)}V^{\Delta,n+1} + B^{\Delta,(a^n,r^n)} = 0, \text{ on } D^n_1.$$

The second system ensures the updated value function $V^{\Delta,n+1}$ plus a specific threshold $(U^{-1}(y^{\Delta}))$ equals zero within region D^n_2,

$$V^{\Delta,n+1} + U^{-1}(y^{\Delta}) = 0, \text{ on } D^n_2.$$

4. ***Convergence Check*** : Should the condition $|V^{\Delta,n+1} - V^{\Delta,n}| \leq \varepsilon$ be met, the algorithm halts ; otherwise, it proceeds to step $2n+1$.

4.5.2 Numerical results

In this subsection, we present a numerical analysis to investigate the impact of ambiguity on the optimal effort level and rent within the consortium formation model. We define the key functions involved in the model :

- Impact of effort on social welfare : $\vartheta(x) = 3(1 - e^{-\alpha x})$
- Cost of effort : $h(x) = e^{\beta x} - 1$
- Consortium's utility : $U(x) = x^p$

The corresponding parameters are chosen as $\alpha = 0.1$, $\beta = 0.1$, $p = \frac{1}{3}$ and $\sigma = 2$. Additionally, the preference parameters for the public and consortium are set at $\delta = \lambda = 0.085$.

We numerically explore how varying the ambiguity level (κ) affects the optimal solution. The analysis considers four cases : $\kappa = 0$, 0.25, 0.5, and 1. The initial value for the value function is set to $v(0) = 0$, and the threshold value is $\overline{y} = 0.5$.

Figure 4.1 depicts the value function across the range $[0, \overline{y}]$ for the specific case of $\kappa = 0.5$. Figures 4.2, 4.3, and 4.4 will subsequently present the value function, optimal effort level, and optimal rent, respectively, for different values of κ. Figure 4.5 will illustrate the relationship between the optimal rent and the optimal effort level.

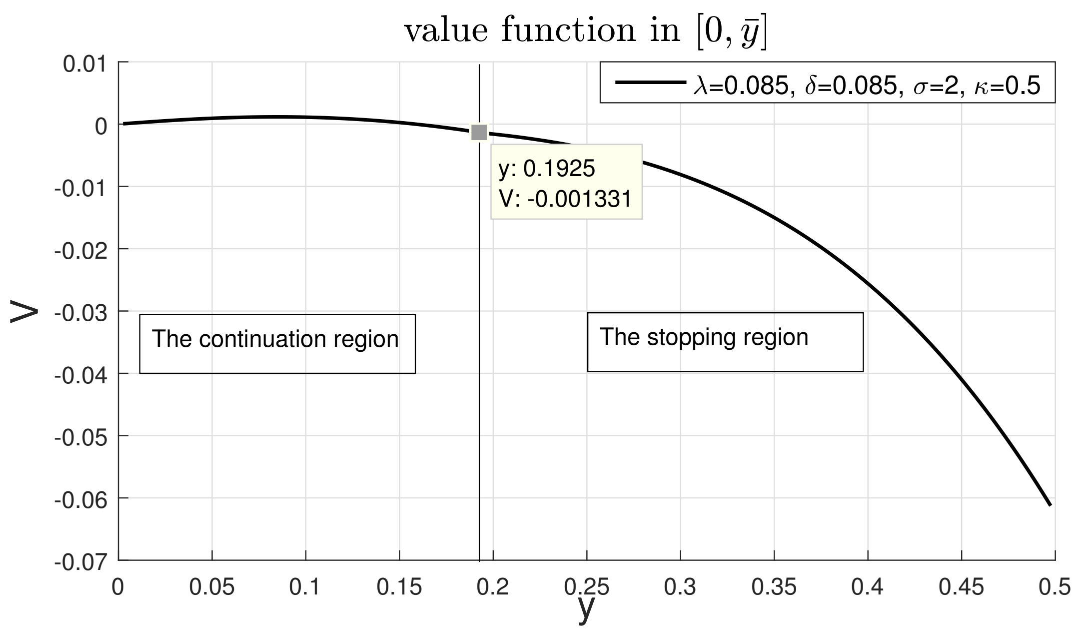

FIGURE 4.1 – Value function across the interval $[0, \bar{y}]$.

We observe in Figure 4.1 that the value function is concave, in accordance with Sannikov [85], Dumav and Riedel [43], and Hajjej, Hillairet and Mnif [51].
Figure 4.2 plots the value function for different values of κ. We observe that the ambiguity influences the continuation region. The higher of κ, the smaller the continuation region. In particular, the value function of the public without ambiguity is higher than the one with ambiguity. Figure 4.3 and 4.4, respectively, plot the optimal effort and the optimal rent on the continuation region for different values of κ. When κ is higher, the consortium makes more effort to compensate the ambiguity and to increase the social welfare. However, at a high level of the consortium's value function, the public pays a higher rent when the ambiguity decreases.

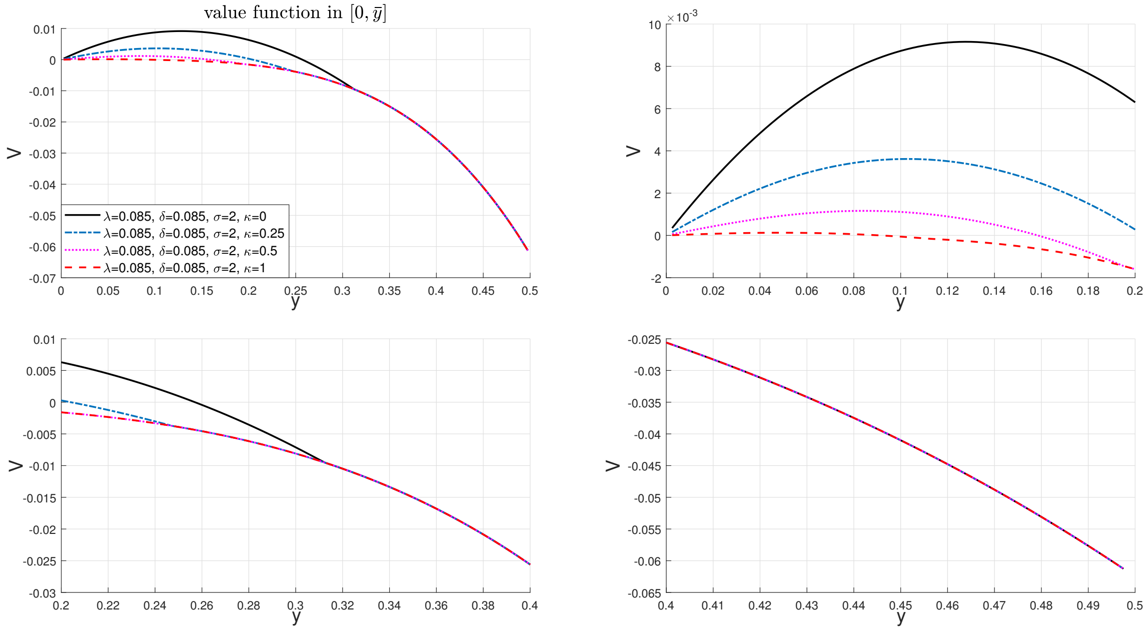

FIGURE 4.2 – Value function for $\sigma = 2$ and different κ

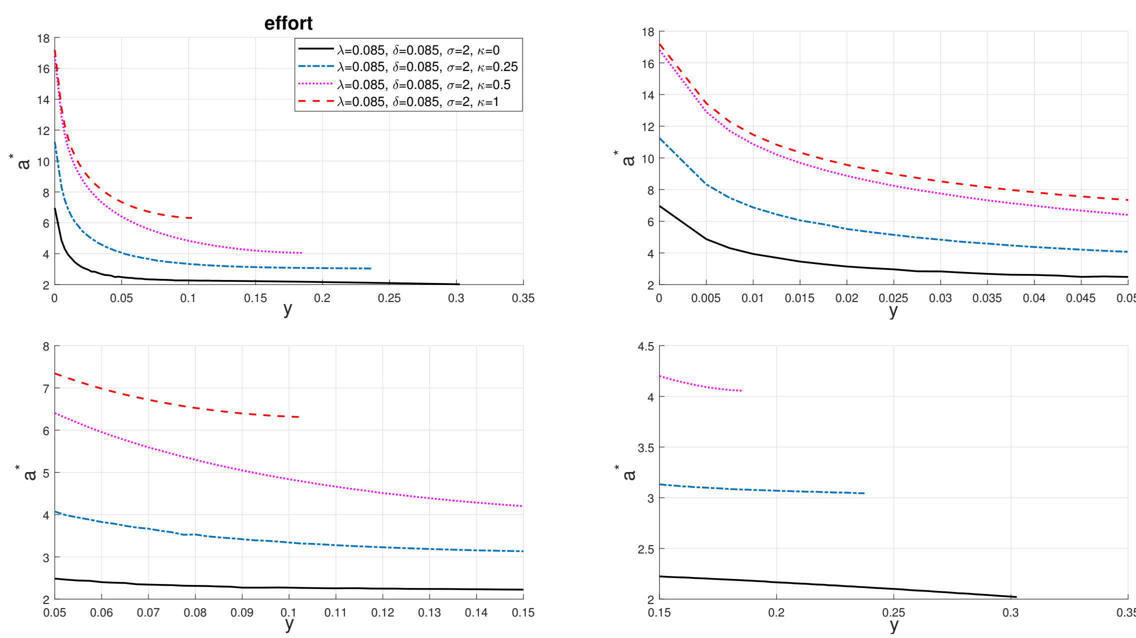

FIGURE 4.3 – Optimal effort for $\sigma = 2$ and different κ.

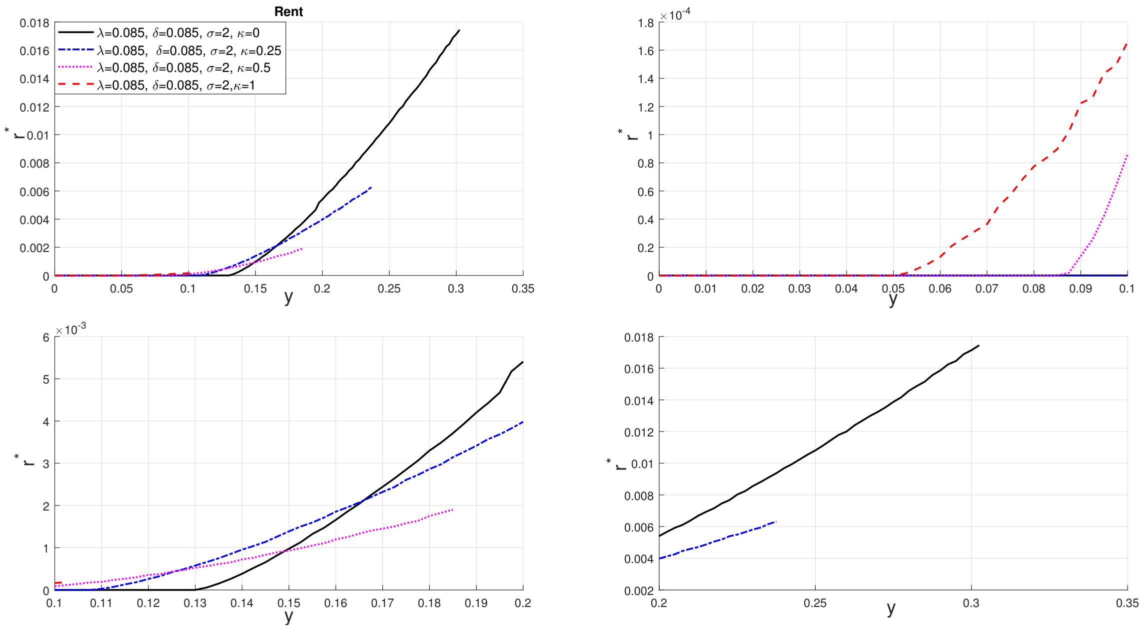

FIGURE 4.4 – Optimal rent for $\sigma = 2$ and different κ

Figure 4.5 shows the relationship between the optimal effort and the optimal rent.

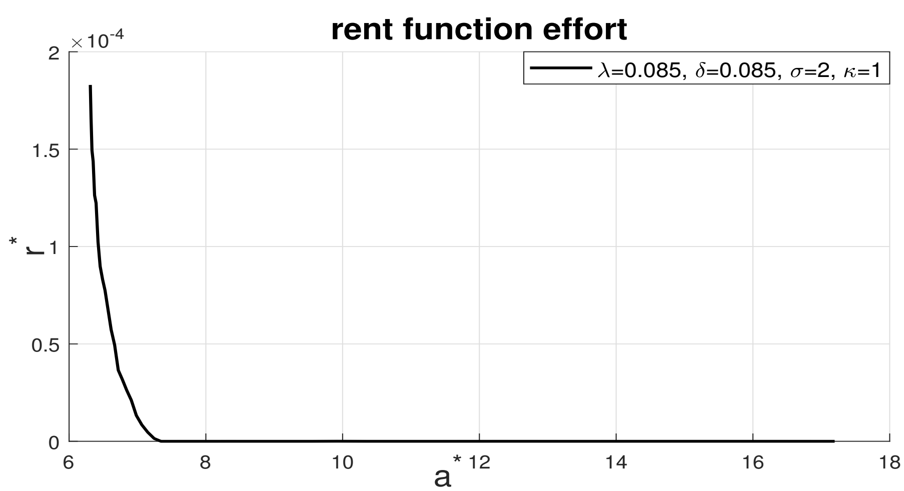

FIGURE 4.5 – Optimal rent versus optimal effort with ambiguity

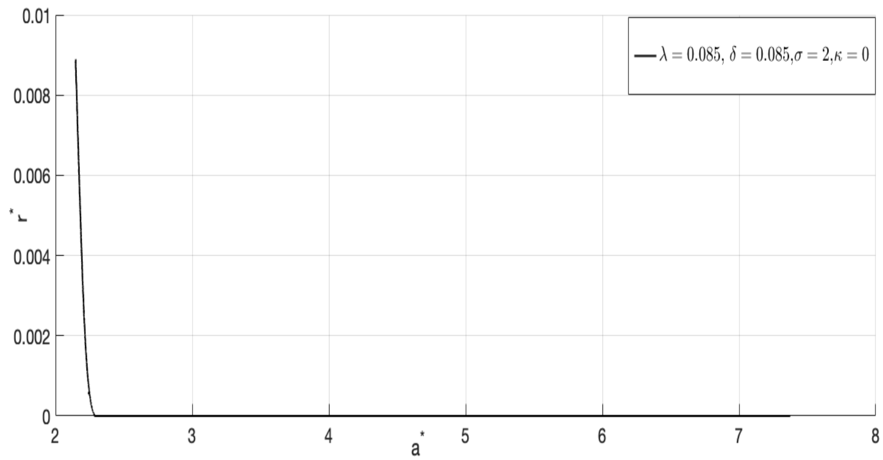

FIGURE 4.6 – Optimal rent versus optimal effort without ambiguity

The optimal rent versus the optimal effort is a decreasing function, which is by the results of Sannikov [85] and Dumav and Riedel [43]. It is more costly for the public to incentivize the consortium to start to work. In fact, for a low level of effort, the Principal pays more to the Agent to make an effort. Figure 4.5 shows that, in the presence of ambiguity, the public stops to pay rent when the level of effort exceeds 7.3. On the other hand, from figure 4.6, we remark that in the absence of ambiguity, the public stops paying rent from level 2.25. From figures 4.5 and 4.6, when the consortium receives a rent, we notice that its level is higher when there is no ambiguity.

4.6 Conlusion

This work investigates the optimal Public Private Partnership (PPP) contract between a public entity and a consortium under the possibility of contract termination by the public. The study addresses a Principal-Agent problem with moral hazard and Knightian uncertainty, characterized by κ-ignorance, and formulates the interaction as a Stackelberg leadership model. Through a series of steps, including worst-case scenario analysis, the agent's best response is derived, and the public's problem is solved by maximizing the social value minus the rent paid. Numerical methods, specifically the Howard algorithm, are employed to compute the public value function, optimal effort, and rent. The results indicate that increased Knightian uncertainty leads to higher optimal effort from the consortium and a reduction in the public's value function. While the model and methods provide valuable insights, future research could benefit from calibration with real-world PPP data to better assist public authorities in contract design.

Conclusion

In summary, this comprehensive study substantially contributes to both the theoretical and applied realms. The theoretical exploration begins by delving into fundamental questions regarding the well-posedness of BSDEs. This includes a driving process involving a Poisson random measure subject to compensation alongside an independent Wiener process. Through rigorous proofs under key assumptions, the study underscores the importance of a generator with logarithmic growth in both (y, z) variables and Lipschitz continuity concerning u. Relaxing Lipschitz conditions in Assumption 2 adds nuance to our understanding, allowing the generator to exhibit logarithmic growth in all variables.

The introduction of the exponential transformation proves instrumental in demonstrating the equivalence between solutions of the auxiliary BSDEJ and the primary BSDEJ. Additionally, a comprehensive discussion on the maximum principle, specifically in scenarios devoid of the jump component, enriches the theoretical landscape.

Simultaneously, the practical front of this research focuses on Public-Private Partnerships (PPPs). Using stochastic control techniques and the HJB Variational Inequality leads to developing a robust solution methodology. This methodology targets enhancing contract design, risk management, and value creation in public infrastructure projects and services. The inclusion of a numerical study employing finite difference methods and the Howard algorithm sheds light on the optimal rent and effort under real-world uncertainty. This emphasizes the importance of addressing Knightian uncertainty in PPP contracts.

This dual-pronged approach, addressing theoretical intricacies before delving into practical challenges, positions this research to have a lasting impact on both the theoretical foundations of BSDEs and stochastic control problems and the implementation of PPPs.

Perspectives

As this research journey unfolds, several promising avenues beckon for further investigation. One crucial aspect deserving of future attention involves venturing into a multidimensional framework by incorporating multiple agents. This extension would include considering scenarios where the efforts of these agents exert influence on both the drift and volatility of social value.

Expanding the scope of this study could entail exploring contracts executed among numerous agents and principals operating under the umbrella of a single company. In this intricate setup, each agent operates under the stewardship of a principal, fostering complex interactions, whether positive or negative. Additionally, each agent retains the prerogative to terminate their contract, albeit subject to an associated penalty.

Upon formulating these problems mathematically, we are bound to encounter theoretical challenges, which add an extra layer of excitement to the research. Among the critical hurdles we may face are the existence and uniqueness of certain types of multidimensional BSDEs and the comparison theorem in the presence of a stochastic control problem. The latter may represent various elements, including agents' actions and compensations. While challenging, these theoretical obstacles contribute to the research's depth and significance.

In conclusion, this thesis lays the foundation for future research pursuits examining dynamic interactions among multiple principals and agents. These potential directions not only promise to enrich our understanding of complex economic interactions but also present theoretical challenges that are central to advancing the field. By connecting these perspectives with the current research on Public-Private Partnerships, the proposed avenues hold the potential for both theoretical insights and practical applications in real-world decision-making scenarios.

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