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Dedication

To my dearest family: with a special mention to my parents, Noureddine and Zineb, who provided me with all the psychological, moral, and material support Throughout my academic journey. My heartfelt prayers go out to them, may they always be under Allah's protection and care, my brothers and sister and their families especially grandchildren: Djoumana, Othman, Imen.

My dear girlfriends throughout my academic path and outside, my colleagues, my professors and my teachers of the Holy Quran.

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"Enjoy your life, focus on what you truly love to do, make it unique to you, then other things will come together smoothly. Your effort put in focus today will become your confidence tomorrow".

Biskra, February 29, 2024

Ikram Hamed

Abstract

This thesis expands upon Pontryagin's stochastic maximum principle to accommodate systems modeled by fractional Brownian motion. we present two research topics. The first centers on an optimal control problem wherein the state equation is driven by fractional Brownian motion, and the cost functional follows a risk-neutral type. Initially, we present the optimal control problem and its underlying dynamics, followed by the convex perturbation method in which the set of admissible controls is convex. Subsequently, we establish both optimality conditions for this model. Finally, we demonstrate our findings through a linear quadratic problem, solving the associated Riccati type equation. The second topic focuses on characterizing optimal control problems within a risk-sensitive framework. The system dynamics are defined using only the backward stochastic differential equations. However, the performance criterion is distinct; instead of directly minimizing costs, we aim to minimize a convex disutility function of the cost. As an initial step, we elucidate the relationship between risk-neutral and risk-sensitive loss functionals. Next, we establish the equivalence between expected exponential utility and quadratic backward stochastic differential equations. Further, we reformulate the risk-sensitive problem into a standard risk-neutral one by introducing an auxiliary term and demonstrate the determination of the adjoint equation. Thus, we derive the stochastic maximum principle using a standard application of risk-neutral results. Finally, we apply these concepts to a control problem with linear quadratic risk sensitivity.

Key words: Fractional Brownian motion, Risk-sensitive control, SDE, SMP.

Résumé

Cette thèse étend le principe du maximum stochastique de Pontryagin pour prendre en compte les systèmes modélisés par le mouvement brownien fractionnaire. Nous présentons deux sujets de recherche. Le premier se concentre sur un problème de contrôle optimal dans lequel l'équation d'état est gouvernée par un mouvement brownien fractionnaire, le fonctionnel de coût est donné de type risque-neutre. Initialement, nous présentons le problème de contrôle optimal et sa dynamique sous-jacente, suivi de la méthode de perturbation convexe dans laquelle l'ensemble des contrôles admissibles est convexe. Ensuite, nous établissons les conditions d'optimalité pour ce modèle. Enfin, nous démontrons nos résultats à travers un problème linéaire quadratique, en résolvant l'équation de type Riccati associée. Le deuxième sujet se concentre sur la caractérisation des problèmes de contrôle optimal dans un cadre risque-sensible. La dynamique du système est définie uniquement à l'aide des équations différentielles stochastiques rétrogrades. Cependant le critère de performance est différent, au lieu de minimiser directement le coût, nous visons à minimiser une fonction de désutilité convexe du coût. Dans un premier temps, nous élucidons la relation entre risque-neutre et risque-sensible. Ensuite, nous établissons l'équivalence entre l'utilité exponentielle attendue et les équations différentielles stochastiques rétrogrades quadratique. De plus, nous reformulons le problème sensible au risque en un problème standard neutre au risque en introduisant un terme auxiliaire et démontrons la détermination de l'équation adjointe. Ainsi, nous dérivons le principe du maximum stochastique. Enfin, nous appliquons ces concepts à un problème de contrôle avec une sensibilité au risque linéaire quadratique.

Mots Clés: mouvement Brownien fractionnaire, Contrôle de risque-sensible, EDS, PMS.

Contents

Symbols and Abbreviations

For readers' convenience, we list the different symbols and abbreviations used in this thesis as follows.

Abbreviations

fBm	:	fractional Brownian motion.
BDSDE	:	backward doubly stochastic differentiale quations.
PDE	:	partial differential equations.
STEM	:	science, technology, engineering and mathematics.
FBSDE	:	forward backward stochastic differential equation.
SDE	:	stochastic differential equation.
SMP	:	stochastic maximum principle.
BDG	:	Burkholder-Davis-Gundy.
BSDE	:	backward stochastic differential equation.
LQ	:	linear quadratic.
a.s.	:	almost surely.
i.e.	:	Namely or that is.

Symbols

$\mathbb{E}$	:	Mathematical expectation.
$\widetilde{\mathbb{E}}$	:	Quasi-conditional expectation.
B^H	:	Fractional Brownian motion.
$\left\langle B^H \right\rangle$	:	Quadratic variation process of B^H.
$(\Omega, \mathcal{F}, \mathbb{P})$	:	Probability space.
$\left(\mathcal{F}_t^H\right)_{t\in[0,T]}$	:	Filtration generated by B^H.
$\mathcal{U}$	:	The set of all admissible controls.
$\lvert\cdot\rvert$	:	Euclidean norm on $\mathbb{R}$.
u	:	Optimal control.
u^θ	:	Perturbed control.
θ	:	Perturbation index.
ε	:	Risk sensitivity index.
v	:	Arbitrary control
$\mathbb{P} \ll \mathbb{Q}$	:	The probability measure $\mathbb{P}$ is absolutely continuous with respect to the measure $\mathbb{Q}$.
$\mathcal{H}$	:	The risk neutral Hamiltonian.
$\mathcal{H}^\varepsilon$	:	The risk sensitive Hamiltonian.

Introduction

Optimization Theory revolves around the task of solving mathematical problems where the goal is to either minimize or maximize the value of a given function. Optimization has emerged as a crucial element in research and learning across diverse fields, extending beyond STEM disciplines to encompass finance, economics, life sciences, genetics, biology, healthcare and population studies. The subject of optimization is frequently deliberated upon due to its broad applicability and significance in various areas of study.

The historical roots of optimization theory trace back to as early as 100 BC, where it was used to calculate the most suitable distance between two points. In the 17th century, Lagrange tackled the "brachistochrone problem" initially posed by Newton in 1699. Lagrange's efforts led to the publication of two papers [55, 56]. The first paper, titled "Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies", was released in 1762. The second paper, "Mécanique Analytique", followed in 1788. Throughout the nineteenth and twentieth centuries, significant advancements were made in the development of optimization theorems and concepts. Nowadays, optimization techniques are applied extensively to enhance the performance of various tasks, with different techniques yielding different performance outcomes. According to Oxford dictionary, optimization is a procedure or technique that can make something ideal and efficient, where a design system or decision that becomes better and better over time using the optimization process could be considered. According to the Cambridge dictionary, optimization is the practice of making something as good or as effective as possible. One thing that all of these frequent definitions have in common is the enhancement of a process, method, design or choice to make something more efficient and more effective. Despite being referred to as a process or methodology, optimization is actually made up of a number of different elements, including decision factors, constraints and objectives, where constraints are specific conditions that must be met for optimization to generate the intended result (its target), whereas variables are the most significant and vital guiding factors.

Take for example the bustling rush hour traffic in major cities like New York, Beijing or Tokyo. Despite its seemingly chaotic nature, the daily movement of commuters along main roads follows stochastic patterns. It's a common expectation that during rush hours when everyone is heading home from work, congestion is bound to occur, leading to gridlock on almost every road. Governments must proactively consider strategies to minimize traffic flow and regulate it to prevent such bottlenecks. Another intriguing scenario arises in finance, it's well-known that salaries in public sectors are typically disbursed on the same day each month, leading employees to withdraw their wages simultaneously. This synchronized action significantly increases the likelihood of a widespread financial crash. If for any reason some employees fail to receive their wages on time, they may resort to strikes, thereby paralyzing the economy further. Governments must address the potential risks associated with cash shortages, bankruptcy or even inflation, which could arise if bank accounts are depleted or if individuals are unable to access their funds.

In the realm of formal and rigorous mathematical frameworks, The programming dynamic principle also called Bellmann's principle and the Pontryagin's maximum principle are the two basic approaches for solving such optimization problems, which we will discuss in detail in the first chapter. In this thesis, we adopt the latter approach, specifically Pontryagin's stochastic maximum principle (also known as the necessary conditions of optimality) for risk-neutral and risk-sensitive control problems associated with dynamics driven by many systems.

The method for solving a linear backward stochastic differential equation (BSDE) which serves as the adjoint process for a stochastic control problem was initially explored by Kushner [52] in 1972, followed by Bismut [11], [12] in 1973, then Bensoussan [8] in 1983 and Haussmann [37] in 1998. Pioneering work on the existence of an adapted solution to a continuous nonlinear BSDE with Lipschitzian coefficient was achieved by Pardoux and Peng in 1990. Subsequently, they expanded upon this theory and its applications in a series of papers [64, 65, 66, 67], under the assumption that coefficients satisfy either globally or locally Lipschitzian conditions, albeit with certain additional requirements.

For the nonlinear backward stochastic differential equations driven by fractional Brownian motion (fBm), Hu and Peng achieved a groundbreaking result regarding the existence and uniqueness of solutions when the Hurst parameter H exceeds one half [44]. Their approach relied on the utilization of Malliavin calculus and the theory of partial differential equations (PDEs). A similar results was presented in another study [48]. Various works have addressed the issue of existence and uniqueness of solutions for stochastic differential equations (SDEs) driven by fBm. For instance, references [62, 78, 83] discuss this topic. In [62], the authors explored the existence and measurability of solutions, while [78] employed approximation techniques and a comparison theorem, subject to certain conditions of linear growth. In [83], Zhu et al. investigated the aspect of continuous dependence on the initial state variable. The exploration of controlled dynamics driven by fBm has been relatively limited in existing literature. Biagini et al. [9] delved into a stochastic maximum principle (SMP) concerning backward doubly stochastic differential equations (BDSDEs) driven by an m-dimensional fBm. Their approach involved utilizing an adjoint BSDE driven by fBm alongside a conventional Brownian motion. Similarly, Han et al. [35] achieved an SMP for a system governed by fBm, particularly when the Hurst parameter exceeds one-half. Meanwhile, Hu and Zhou [45] tackled a linear stochastic control problem associated with fBm characterized by a Hurst parameter less than one-half. Their methodology incorporated the utilization of the Riccati equation, a type of BDSDE driven by both fBm and standard Brownian motion. Additionally, Bender [6] examined the explicit solution of a specific class of linear fractional BSDEs through PDEs, considering the parameter H within the range $]0,1[$. In [13], the authors employed Malliavin calculus to establish a stochastic maximum principle applicable to systems driven by both fractional and standard Wiener processes, specifically addressing BSDE with quadratic growth.

The fBm presents intriguing characteristics like long-range dependence, prompting us to explore controlled dynamics driven by fBm due to its broad applicability. However, fBm doesn't conform to the properties of either a semi-martingale or a Markov process, rendering classical methods like the dynamic programming principle inapplicable. Hence, we turn to the Pontryagin stochastic maximum principle for guidance.

This thesis addresses two primary objectives. The first objective [39] involves solving a stochastic control problem for a forward-backward dynamics propelled by an fBm. Our principal approach to resolving this problem relies on the generalized Itô formula. We leverage variational calculus techniques to address a stochastic optimization problem, employing Pontryagin's SMP. This involves dealing with forward-backward dynamics governed by an fBm with the Hurst parameter $H \in (0,1)$. The system is characterized by a nonlinear forward-backward stochastic differential equation (FBSDE)

$$\begin{cases} dx_t & = & b\left(t, x_t, v_t\right) dt + \sigma\left(t, x_t, v_t\right) dB_t^H, \\ -dy_t & = & f\left(t, x_t, y_t, z_t, v_t\right) dt - z_t dB_t^H. \end{cases}$$

Recently created and developed stochastic analysis for fBm utilizing Malliavin calculus. In this study, we demonstrate many principal results can be reached using simple elementary justifications and calculations. The primary advantage of this approach over those outlined in previous studies is its ability to define the stochastic integral for any value of H from 0 to 1. Moreover, it doesn't necessitate the incorporation of fractional white noise theory, since it relies on the well-established theory applicable to the standard case.

The second objective, as outlined in [40], pertains to the risk-sensitive Pontryagin's SMP applied to BSDE driven by fractional Wiener motion

$$-dy_t = f\left(t, y_t, z_t, \upsilon_t\right) dt - z_t dB_t^H.$$

To address this problem, we adopted the approach developed by Djehiche et al. [23], their contribution can be summarized as follows: they established an SMP for a specific class of risk-sensitive mean-field type control problems. In these problems, the distribution only influences the mean of the state process, implying that the drift, diffusion and terminal cost functions are dependent on the state, control and means of the state process. The necessary and sufficient optimality conditions for risk-sensitive control problems, where the systems are driven by an SDE, were investigated by Lim and Zhou in [57]. Furthermore, Shi and Wu extended this analysis to cases involving nonlinear forward SDEs with jumps, where the set of admissible controls is characterized as convex, as elaborated in [74]. They also applied these principles to finance as discussed in [75]. Pontryagin's SMP applied to risk-sensitive control is discussed in references [19, 20], focusing particularly on systems driven by BSDEs and SDEs, respectively. For a more comprehensive exploration of the risk-sensitive SMP in more general cases, you may find insights in the paper authored by Khallout and Chala [49], this work addresses FBSDE. Additionally, if you are interested in BDSDEs, relevant insights can be found in the paper edited by Hafayed and Chala [32], both of which tackle situations where the admissible control set is convex. In [14], the Malliavin calculus have been employed by the authors to established an intriguing result on the risk-sensitive maximum principle for a BSDE driven by an fBm.

This is a thesis presented for the degree of *Doctorate in Mathematics in the field of probability*, and it is organized as follows:

In the first chapter: we provides background on the Control Problem, stochastic processes, natural filtration and provide the Wiener and fractional Brownian motion processes along with their properties. Additionally, we discussed two prominent methods for analyzing optimal control: Bellman's dynamic programming method and Pontryagin's SMP. Towards the end of this chapter, we review various classes of stochastic control with notable properties for our study, such as admissible control, feedback controls, and relaxed controls....

In the second chapter: We introduce the main tools necessary for our subsequent analysis. by formulate the problem including the Itô-Russo-Vallois stochastic integral with respect to fractional Brownian motion. These tools lay the groundwork for presenting the first major result of this thesis: Stochastic Controls of Fractional Brownian Motion. We establish both necessary and sufficient optimality conditions in the form of a stochastic maximum principle, preceded by presenting preliminary results on the solutions and linearization of the state equations.

In the third chapter: We delves into a risk-sensitive control problem, wherein we aim to optimize a risk-sensitive cost functional for a system driven by a BSDE governed by an fBm. Initially, we establish optimality conditions for risk-neutral controls as a preliminary step. The approach involves utilizing an auxiliary state process, which serves as a solution to certain SDEs, enabling the transformation of our system into one governed by an FBSDE. Finally, we demonstrate our main result through an illustrative linear quadratic example.

Relevant Papers

The content of this thesis was the subject of the following papers by I. Hamed and A. Chala:

1. Title: Stochastic Controls of Fractional Brownian Motion.

- Journal: Random Operators and Stochastic Equations.
- Doi.org/10.1515/rose-2023-2025.
- Volume **32**, Issue **01**, Pages 27–39 (2024).
- Site: https://doi.org/10.1515/rose-2023-2025.

2. Title: Pontryagin's Control Problem of Risk-Sensitive for Fractional Backward Stochastic Differential Equations with Application (Under review)

- Journal: Bulletin of the Korean Mathematical Society.

International Communications

Several communications in the above subjects were done:

1. I. Hamed and A. Chala, "On the estimate solution of fractional stochastic differential equation". *The international symposium on applied mathematics and engineering.* January 21-23, 2022, Biruni University, Istanbul - Turkey.
2. I. Hamed and A. Chala, "Preliminary results of fractional forward-backward stochastic differential system". *The first international workshop on applied mathematics.* December 6-8, 2022, Constantine 1 University, Constantine - Algeria.
3. I. Hamed and A. Chala, "Stochastic Control of backward Differential Equation". *The 7th international conference on Mathematics* (7th-ICOM). July 11-13, 2023, Fatih Sultan Mehmet University, Istanbul - Turkey.

4. I. Hamed and A. Chala, "Stochastic Maximum Principle for a Fractional Stochastic Differential Equation". *The 7eme international workshop on applied mathematics and modelling (*WIMAM'2023). December 13-14, 2023, 8 May 1945 University, Guelma - Algeria.

National Communications

1. I. Hamed and A. Chala, "Results on fractional stochastic differential equation". *The first national applied mathematics seminar* (1st-NAMS'23). May 14-15, 2023, Mohamed Kheider University, Biskra - Algeria.

2. I. Hamed and A. Chala, "Sufficient optimality condition for system driven by fractional Brownian motion". National Conference of Interactive Mathematical Areas (CIMA-1). October 17-18, 2023, Bouira University, Bouira - Algeria.

3. I. Hamed and A. Chala, "Necessary optimality condition for system driven by fractional Brownian motion". The first national conference on differential geometry and Dynamical systems (DGDS 2023). December 19-20, 2023, Relizane University, Relizane - Algeria.

Chapter 1

Basic Notations and Stochastic Control

In this auxiliary chapter, divided into three key sections, We establish the framework that serves as the foundation for our work throughout this thesis. The initial section delves into fractional Brownian motion, where we discuss various concepts and results crucial for proving our findings, including stochastic processes and natural filtration. Additionally, we introduce this process, extending its properties and associated theorems. The second section, we talk about two distinct approaches for addressing optimal control problems. Finally, we wrap up the chapter in the concluding paragraph, where we touch upon some classes of stochastic control.

Furthermore, those interested in further details on the subsequent sections can refer to [10, 31, 53, 61, 68].

1.1 Fractional Brownian motion

Many instances of non semi-martingale processes can be found in gaussian processes. Fractional Brownian motion is one of the most common Gaussian processes, and its covariance function is especially straightforward.

1.1.1 Stochastic processes

Let T be a nonempty index set, and consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω represents the sample space, $\mathcal{F}$ is the sigma-algebra of events, and $\mathbb{P}$ is the probability measure. In the context of control theory, a crucial concept of the stochastic process is defined as a family $X = (X(t), t > 0)$ of random variables from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}^n$.

The function associating each time point t to the appropriate value $X(t, w)$ $(t \to X(t, w))$ for any w in Ω is called a sample path. With this stochastic framework, systems influenced by random variables may be effectively modeled, and capturing their evolution across time in a probabilistic manner.

1.1.2 Natural filtration

Natural filtration (the filtaration generated) plays a crucial role in the domain of stochastic processes, specifically when it comes to the stochastic process $\{X(t) : t \in T\}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denoted by $\mathcal{F}_t^X$, it represents the natural filtration of X and has the formula $\mathcal{F}_t^X = \sigma(X_s,\ 0 \leq s \leq t)$. This filtration retains the information producing up to the time t, created by the whole historical trajectory of the process leading up to that moment, illuminating on the dynamics of the process as it changes and providing a critical foundation for comprehending the interaction of random variables over time.

1.1.3 Brownian motion

Let $(W_t)_{t\geq 0}$ be a process on the space $(\Omega, \mathcal{F}, \mathbb{P})$, We say that the process $(W_t)_{t\geq 0}$ is a standard Brownian motion if:

1)$\mathbb{P}\left[W_0 = 0\right] = 1$.

2) For all $0 \leq s \leq t$, the random variable $W_t - W_s$ follows a normal distribution center with variation $(t-s)$, $\left(W_t - W_s \rightsquigarrow \mathcal{N}(0, t-s)\right)$.

3) The mapping $t \rightarrow W_t$, is continuous $\mathbb{P}$ p.s.

4) $(W_t)_{t\geq 0}$ has independent increments i.e: for any $0 < t_1 < t_2 < ... < t_n$, the variables $W_{t_n} - W_{t_{n-1}}, ..., W_{t_1} - W_{t_0}, W_{t_0}$ are independents.

1.1.4 Fractional Brownian motion

Kolmogorov was first described the fractional Brownian motion in 1940 in [50], under the name Wiener Helix, within a Hilbert space framework, then Yaglom continued to research it in [79]. The term fractional Brownian motion is credited to Mandelbrot and Van Ness, who published in 1968 a stochastic integral formulation of this process in terms of a normal Brownian motion in [59].

Definition 1.1 *An fBm* $B^H = \left(B^H(t) : 0 \leq t \leq T\right)$ *with Hurst index* $H \in]0,1[$, *is a continuous and centered Gaussian process with covariance function*

$$\mathbb{E}[B^H(t)\, B^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \forall s, t \in [0, T].$$

For $H = \frac{1}{2}$ then the fBm is a Brownian motion.

According the above definition, the fBm has the following properties:

- $B^H(0) = 0$ and $\mathbb{E}\left[B^H(t)\right] = 0$ for all $t \geq 0$.

- The process $B^H(t+s) - B^H(s)$ has the same law of $B^H(t)$ for $s, t \geq 0$. (homogeneous increments).
- B^H has continuous trajectories.
- B^H is a Gaussian process and $\mathbb{E}\left[B^H(t)^2\right] = t^{2H}, t \geq 0$, for all $H \in \left]0,1\right[$.
- B^H is a semimartingale if and only if $H = \frac{1}{2}$.
- B^H is a Markov process if and only if $H = \frac{1}{2}$.
- The mbf of the parameter H is order self-similarity H.
- The mbf has a long memory, if $H > \frac{1}{2}$ and it has a short memory if $H < \frac{1}{2}$.

Remark 1.1 *If you are interested in proofs of these properties, you may consult [38].*

The existence of the fBm comes from the general existence theorem of centered Gaussian processes with given covariance functions. (see [69]).

The fBm is divided into two common families according to the value of H and they are: $0 < H < 1/2$ and $1/2 < H < 1$ (if we consider that $H = 1/2$ as an independent case), and there is also another case which takes up the whole value of index H . In our work, we are intrested about this most recent situation.

Mandelbrot named the H parameter "Hurst parameter" after the British hydrologist Harold Edwin Hurst, who made a statistical study of yearly water run-offs of the Nile river. He took into account the values of $\eta_1, ..., \eta_d$ of d-consecutive annual run-offs and their corresponding cumulative value $\Delta_d = \sum_{j=1}^{d} \eta_j$ for the years 662 through 1469. The behavior of the normal values for the amplitude of deviation from the experimental mean, discovered by Harold Edwin Hurst in the case where $H = 0.7$, was almostly cd^H. Furthermore, with $H > 1/2$, the distribution of $\Delta_d = \sum_{j=1}^{d} \eta_j$ was quite similar to that of $d^H \eta_1$. As a result, this phenomenon could not be represented by a process with independent increments; rather, we can be consider the η_j as an increments of an fBm. Mandelbrot coined the term Hurst index as a result of this research (see [46]). There have been several recent approaches introducing integral representation for the fractional Brownian motion. One might see [22, 63, 76], and the references therein for more information on these approaches.

By the approach of Mandelbrot et al. in [59], have proven that the process

$$\begin{aligned} Z(t) &= \frac{1}{\Gamma(H+1/2)} \int_{\mathbb{R}} \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right) dB(s) \\ &= \frac{1}{\Gamma(H+1/2)} \left(\int_{-\infty}^{0} \left[(t-s)^{H-1/2} - (-s)^{H-1/2} \right] dB(s) \right. \\ &+ \left. \int_0^t (t-s)^{H-1/2} dB(s) \right), \end{aligned} \tag{1.1}$$

where $B(t)$ is a standard Brownian motion and Γ represents the gamma function given by $\forall n > 0$, $\Gamma(n) = (n-1)!$, is an fBm with Hurst index $H \in]0,1[$. The integral (1.1) yields the complex fBm if $B(t)$ is changed to a Brownian motion with complex values.

By adhering to [63], we sketch a proof for the representation (1.1). Additional details can be found in [72].

Initially, it is observed that $Z(t)$ constitutes a continuous centered Gaussian process. Therefore, our focus shifts to computing the covariance functions. Throughout the subsequent calculations, we omit the constant $\frac{1}{\Gamma(H+1/2)}$ for the sake of simplicity. The result yields

$$\mathbb{E}\left[Z^2(t)\right] = \int_{\mathbb{R}} \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right)^2 ds.$$

Incorporating the change of variable $s = tu$, we arrive at the following expression:

$$\begin{aligned} \mathbb{E}\left[Z^2(t)\right] &= t^{2H} \int_{\mathbb{R}} \left((1-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right)^2 du \\ &= t^{2H} C(H). \end{aligned}$$

Similarly, we find that

$$\mathbb{E}\left[\left|Z(t)-Z(s)\right|^{2}\right] = \int_{\mathbb{R}}\left((t-u)_{+}^{H-1/2}-(s-u)_{+}^{H-1/2}\right)^{2}ds$$
$$= |t-s|^{2H}C(H).$$

As we know that

$$\mathbb{E}\left[\left|Z(t)-Z(s)\right|^{2}\right] = \mathbb{E}\left[Z^{2}(t)\right]+\mathbb{E}\left[Z^{2}(s)\right]-2\mathbb{E}\left[Z(t)Z(s)\right].$$

Then

$$\mathbb{E}\left[Z(t)Z(s)\right]=\frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).$$

Therefore, we can deduce that $Z(t)$ is an fBm with a Hurst index of H. Now, we present the following generalization of Itô formula.

Theorem 1.1 *(A fractional Itô formula) Let H be in $]0,1[$, assume that $\psi(s,x):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ belongs to $C^{1,2}(\mathbb{R}\times\mathbb{R})$, and assume that the random variables*

$$\psi\left(t,B_t^H\right),\quad \int_0^t\frac{\partial\psi}{\partial s}\left(s,B_s^H\right)ds \text{ and } \int_0^t\frac{\partial^2\psi}{\partial x^2}\left(s,B_s^H\right)s^{2H-1}ds,$$

are square integrable, for all $t\in[0,T]$. Then

$$\psi\left(t,B_t^H\right)=\psi(0,0)+\int_0^t\frac{\partial\psi}{\partial s}\left(s,B_s^H\right)ds+\int_0^t\frac{\partial\psi}{\partial x}\left(s,B_s^H\right)dB_s^H$$
$$+H\int_0^t\frac{\partial^2\psi}{\partial x^2}\left(s,B_s^H\right)s^{2H-1}ds.$$

Remark 1.2 *This formula is formulated in terms of The Wick Itô Skorohod integral (WIS integral) and holds for every $H\in]0,1[$.*

Proof. see [10]. ■

Lemma 1.1 *[25] Let B_t^H be a fractional Brownian motion, and $u(s)$ be a stochastic* *process, For every $T < \infty$, there exists a constant $C(H,T) = HT^{2H-1}$ such that*

$$\mathbb{E}\left[\left(\int_0^T u_t dB_t^H\right)^2\right] \leq C(H,T)\,\mathbb{E}\left[\int_0^T u_t^2 dt\right]. \tag{1.2}$$

Proposition 1.1 *Let η a deterministic continuous function, then*

$$\mathbb{E}\left[\left(\int_0^T \eta_t dB_t^H\right)^2\right] = \mathbb{E}\left[\int_0^T \eta_t^2 d\left\langle B_t^H\right\rangle\right].$$

All distributional uncertainty associated with the fractional Brownian motion is concentrated in $\left\langle B_t^H\right\rangle$. From the above, we can discernible that

$$\mathbb{E}\left[\left(\int_0^T \eta_t dB_t^H\right)^2\right] = \mathbb{E}\left[H\int_0^T \eta_t^2 t^{2H-1} dt\right].$$

Lemma 1.2 *Let ϑ and ϱ are two process, suppose that*

$$\mathbb{E}\left[\int_0^T \left(|\vartheta_i(t)|^2 + |\varrho_i(t)|^2\right) ds\right] < \infty, \text{ for } i = 1,2.$$

If we put: $K(t) = \int_0^t \vartheta_1(s)\,ds + \int_0^t \vartheta_2(s)\,dB_s^H$ and $L(t) = \int_0^t \varrho_1(s)\,ds + \int_0^t \varrho_2(s)\,dB_s^H$. Then we have

$$\begin{aligned}(KL)(t) = &\int_0^t K(s)\,\varrho_1(s)\,ds + \int_0^t K(s)\,\varrho_2(s)\,dB_s^H \\ &+ \int_0^t L(s)\,\vartheta_1(s)\,ds + \int_0^t L(s)\,\vartheta_2(s)\,dB_s^H + \langle K, L\rangle_t\end{aligned}$$

Or in the context of differential notation:$dK(t) = \vartheta_1(t)\,dt + \vartheta_2(t)\,dB_t^H$ and $dL(t) = \varrho_1(t)\,dt + \varrho_2(t)\,dB_t^H$. Then

$$d(KL)(t) = K(t)\,dL(t) + L(t)\,dK(t) + d\langle K, L\rangle_t$$

In this case the quadratic covariation is $\langle K, L\rangle_t = H\int_0^t \vartheta_2(t)\,\varrho_2(t)\,t^{2H-1}dt$.

In the next, We introduce Girsanov's theorems, it plays an important role in the application, especially in economics and optimal control.

Girsanov's type transformation

Girsanov's theorem is used frequently since it transforms a class of processes to Brownian motion with an equivalent probability measure transformation. At first we give the definition of equivalent probability measures.

Definition 1.2 *Let* $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^H\right)_{t\in[0,T]}, \mathbb{Q}_1\right)$, *and* $\mathbb{Q}_2$ *be another probability measure. We say that* $\mathbb{Q}_1$ *is equivalent to* $\mathbb{Q}_2 \mid \mathcal{F}_T$ *if and only if*

$$\mathbb{Q}_1 \mid \mathcal{F}_T \ll \mathbb{Q}_2 \quad \text{and} \quad \mathbb{Q}_2 \mid \mathcal{F}_T \ll \mathbb{Q}_1,$$

and we write $\mathbb{Q}_1 \mid \mathcal{F}_T \sim \mathbb{Q}_2$*. or equivalently if* $\mathbb{Q}_1$ *and* $\mathbb{Q}_2$ *have the same zero sets in* $\mathcal{F}_T$.

The theorem presented below is regarded as an extension of the conventional probability transformation theorem introduced by I. W. Girsanov [28].

Theorem 1.2 *Let* $T \geq 0$, *and let* φ *be a continuous function with* $suup\varphi \subset [0,T]$. *Let* l *be a function with* $suppl \subset [0,T]$, *On the* σ*-algebra* $\mathcal{F}_T^H$, *generated by the process* $\left\{B_s^H : 0 \leq s \leq T\right\}$, *a probability measure* $\mathbb{Q}$ *can be defined as*

$$\frac{\mathbb{Q}}{\mathbb{P}} := \exp\left\{-\int_0^t l\left(s\right) dB_s^H - \frac{1}{2} l^2\left(s\right) H t^{2H-1} dt\right\}.$$

Then $\widetilde{B}_t^H := B_t^H + \int_0^t \varphi\left(s\right) ds, 0 \leq t \leq T$, *is a fractional Brownian motion under* $\mathbb{Q}$.

Proof. The reader can see [10]. ■

1.2 Approaches for resolving optimal control problems.

There are two main techniques for study optimal control: Bellman's dynamic programming method and Pontryagin's maximum principle stochastic.

1.2.1 Dynamic programming method

In this section we are discussing a valuable approach to solving optimal control problems following the dynamic programming technique that was pioneered in the early 1950s by R. Bellmane. This mathematical technique is adept at addressing a sequence of interconnected decisions and can be used to a wide range of various optimization cases particularly those posed by optimal control problems. The major idea behind this method utilized for optimal control is to look at a family of optimal control problems, each with different initial conditions and states. The connections between these problems are clarifies through the Hamilton-Jacobi-Bellman equation (HJB), a nonlinear first-order equation in deterministic cases or a second-order equation in stochastic cases of PDEs. Resolving the HJB equation, whether analytically or numerically, enables the derivation of optimal feedback control by maximizing or minimizing the Hamiltonian or generalized Hamiltonian embedded within the HJB equation which is a process referred to as the authentication technique.

Notably, This approach provides solutions for the complete set of problems , each characterized by unique initial conditions and states. However, the classical dynamic programming approach had a major flaw: it required the HJB problem to admit classical solutions, which implied that the solutions had to be sufficiently smooth (relative to the order of derivatives in the equation). To get over this restriction, Crandall and Lions proposed the so-called viscosity approaches in the early 1980s. This new paradigm is a non-smooth method of solving partial differential equations, fundamentally replacing conventional derivatives with (set-valued) super-/sub-differentials while ensuring the uniqueness of solutions under extremely mild conditions. The HJB equation may also lack classical solutions, particularly in the stochastic case where diffusion can degenerate.

The Bellman principle

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions, Let $W(t)$ be a Brownian motion valued in $\mathbb{R}^d$, denote by A the set of all progressively measurable processes $\{u(t), t \geq 0\}$ valued in $U \subset \mathbb{R}^k$. Also given a positive constant T and a metric space U, for any $(s, y) \in [0, T) \times \mathbb{R}^n$, we consider the following stochastic controlled system

$$\begin{cases} dy(t) = a(t, y(t), u(t))\, dt + b(t, y(t), u(t))\, dW(t), \\ y(0) = y, \end{cases} \tag{1.3}$$

where

$$a : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n,\ \ b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d},$$

be two given functions satisfying, for some constant $C > 0$:

$$\begin{aligned} |a(t, y(t), u(t)) - a(t; x(t), u(t))| + |b(t, y(t), u(t)) - b(t; x(t), u(t))| &\leq C\,|y - x|, \\ |a(t, y(t), u(t))| + |b(t, y(t), u(t))| &\leq (1 + |y(t)|), \end{aligned}$$

this above conditions is to ensure the existence and uniqueness of the solution to SDE 1.3.

We define the cost functional associated with equation 1.3 as following:

$$\mathcal{J}(t, y, u) = \mathbb{E}\left[\int_t^T l(s, y(s), u(s))\, ds + g(y(T))\right], \tag{1.4}$$

where $l : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be also two given functions, we suppose that

$$l\,(s, y, u)\, ds + g\,(y) \leq C\left(1 + |y|^2\right), \tag{1.5}$$

where C is constant.

The quadratic growth condition expressed in equation 1.5 serves to guarantee the well-definedness of the functional $\mathcal{J}$. This condition ensures that the associated cost functional possesses a growth rate proportional to the square of the state variables or related quantities. Dynamic programming stands as a foundational principle in stochastic control theory, and we present a rendition of the stochastic Bellman's optimality principle. We recommend reading papers by Lions [58], Krylov [51], Yong and Zhou [80], Fleming and Soner [26], for in-depth mathematical analysis of this problem. The goal here is to maximize the gain function, leading us to introduce what is known as the value function.

Definition 1.3 *We define the value function of the original Problem*

$$\begin{cases} V\,(s, x) = \inf\limits_{u(.)\in U} \mathcal{J}\,(t, x, u\,(.))\,, \forall\,(s, x) \in [0, T] \times \mathbb{R}^n, \\ V\,(t, x) = g\,(x)\,, \forall x \in \mathbb{R}^n. \end{cases}$$

Assumption 1.1 *It's important to note that for the value function $\mathcal{J}\,(t, x, u\,(.))$ to be well-defined, the functions a, b, l and g must satisfy the following conditions:*

- The functions a, b, l and g are uniformly continuous.

- (U, d) is polish space (complete separable metric space).

Theorem 1.3 *Under the above conditions, for any $(t, y) \in [0, T] \times \mathbb{R}^n$ be given. Then we have*

$$V\,(t, y) = \inf_{u(.)\in U} \mathbb{E}\left[\int_t^{t+h} l\,(s, y(s), u(s))\, ds + V\,((t + h)\,, x\,(t + h))\right], \forall t \leq t + h \leq T. \tag{1.6}$$

Proof. The technical proof of the dynamic programming principle has been studied through various methods. For a detailed understanding, we direct the reader to the work of Yong and Zhou [80]. ■

The Hamilton-Jacobi-Bellman equation

Now, we present the Hamilton-Jacobi-Bellman (HJB) equation, representing the infinitesimal form of the dynamic programming principle.

Proposition 1.2 *Let the conditions in assumption 1.1 hold. Then the value function $V(s,x)$satisfies the following:*

1. $|V(s,x)| \leq M(1+|x|), \forall (t,y) \in [0,T] \times \mathbb{R}^n, M > 0.$
2. $|V(s_1,x_1) - V(s_2,x_2)| \leq M\left\{|x_1 - x_2| + (1+|x_1| \vee |x_2|)|s_1 - s_2|^{\frac{1}{2}}\right\}$, $\forall x_1, x_2 \in \mathbb{R}^n$, $\forall s_1, s_2 \in [0,T]$, $(p \vee q = \max(p,q))$.
3. *If $V \in C^{1,2}([0,T] \times \mathbb{R}^n)$. Then V is a solution of a second-order partial differential equation:*

$$\begin{cases} -v(t) + \sup G(t,y,u,-v_y,-v_{yy}) = 0, (t,y) \in [0,T] \times \mathbb{R}^n, \\ v|_{t=T} = g(y), y \in \mathbb{R}^n, \end{cases} \tag{1.7}$$

where, the function $G(t,y,u,p,P)$ is called the generalized Hamiltonian and is defined as $G(t,y,u,p,P) = \frac{1}{2}tr\left(Pb(t;y,u)b(t;y,u)^T\right) + (p, l(t,s,u)) - l(t,y,u)$.

Viscosity Solutions

Regular solutions, specifically those outside of $C^{1,2}([0,T] \times \mathbb{R}^n)$, are typically not admitted by the HJB equation. To address this shortcoming, Crandall and Lions (1983) [27] introduced viscosity solutions, offering a remedy for the lack of regularity in solutions.

Definition 1.4 *1. A function $v \in C([0,T] \times \mathbb{R}^n)$ is referred to as a viscosity super-solution of equation 1.7 if*

$$v(T,y) \geq g(y), \forall y \in \mathbb{R}^n,$$

and for $\psi \in (C^{1,2} \times \mathbb{R}^n)$, whenever $v - \psi$ attains a local minimum at $(t,y) \in [0,T] \times \mathbb{R}^n$, we have

$$-\psi_t(t,y) + \sup_{u \in U} G(t,y,u,-\psi_y(t,y),-\psi_{yy}(t,y)) \geq 0.$$

2. A function $v \in C([0,T] \times \mathbb{R}^n)$ is referred to as a viscosity sub-solution of equation 1.7 if

$$v(T,y) \leq g(y), \forall y \in \mathbb{R}^n,$$

and for $\psi \in (C^{1,2} \times \mathbb{R}^n)$, whenever $v - \psi$ attains a local maximum at $(t,y) \in [0,T] \times \mathbb{R}^n$, we have

$$-\psi_t(t,y) + \sup_{u \in U} G(t,y,u,-\psi_y(t,y),-\psi_{yy}(t,y)) \leq 0.$$

3. A function $v \in C([0,T] \times \mathbb{R}^n)$is referred to as a viscosity solution of 1.7 if it is both a viscosity supersolution and viscosity sub-solution of equation 1.7.

Theorem 1.4 *Assuming that conditions of assumption 1.1 are satisfied, the value function V is considered a viscosity solution of equation 1.7.*

The classical verification

The traditional verification method involves seeking a smooth solution to the HJB equation and subsequently verifying that this proposed solution aligns with the value function, subject to specific and appropriate conditions. Commonly referred to as a verification theorem, this outcome not only confirms the solution's validity but also yields an optimal control. The specifics of a verification theorem may vary depending on the unique aspects of each problem, dictated by the necessary technical conditions. It is crucial to tailor these conditions to the specific context of the given problem. For more information about classical verification and its theories, we refer the reader to the book Yong & Zhou [80].

1.2.2 Pontryagin's maximum principle

Kushner laid the groundwork for the stochastic maximum principle, and subsequent significant contributions to this field have been made by researchers such as Bensoussan, Peng and others. In the realm of optimization and control problems, the conventional approach involves ensuring that optimal solutions meet the necessary conditions. The proposition here is to employ a precise calculus of variations on the gain function $\mathcal{J}(t, x, .)$, specifically with respect to the control variable. This method aims to derive the essential optimality conditions.

The Maximum Principle introduced by Pontryagin in the 1960s, dictates that the optimal state trajectory should satisfy the Hamilton system and adhere to the maximum condition of a function known as the generalized Hamilton. Generally, solving a Hamilton is expected to be more tractable task compared to solving the original control problem. The Pontryagin Maximum Principle in its original form was developed for deterministic concerns. Following a concept akin to the traditional variance calculus, the fundamental approach entails perturbing the optimal control and utilizing a Taylor expansion for both the state trajectory and the objective functional centered around the optimal control. By transmission a perturbation to zero, any inequality can be derived, and through duality, the maximum principle is expressed in terms of an adjoint variable. During the 1970s, Bensoussan, Bismut, Haussmann and Kushner in [8, 11, 12, 37, 52]. They played a significant role in the extensive development of the initial version of the stochastic maximum principle. However, it's important to note that during that period, the outcomes were primarily derived under the assumption of no control over the coefficient of diffusion. For example, Haussmann and Suo [36] delved into the maximum transformation principle of Girsanov, and this constraint elucidates why this approach is ineffective when dealing with control dependent and degenerate diffusion coefficients, as evidenced in [33, 34].

However, Peng was the one to derive the first version of the stochastic maximum principle, wherein the diffusion coefficient is directly influenced by the control variable, and the control domain is not necessarily convex. Peng achieved this by examining the second-order term in Taylor's expansion of the perturbation method arising from the Itô integral. Through this exploration, he established the maximum principle applicable to potentially degenerating and control-dependent diffusion. Notably, this formulation encompasses not only the first-order adjoint variable but also introduces the second-order adjoint variable. These adjoint variables find their definition in what is now recognized as BSDE. In 1973, Bismut initially proposed linear BSDE. It's worth mentioning that Pardoux and Peng established the uniqueness and existence theorem for solutions of nonlinear BSDE driven by Brownian motion in 1990, subject to the Lipschitz condition. Presently, BSDE theory holds significant importance not only in addressing problems related to stochastic optimal control but also in the broader realm of mathematical science. It plays a crucial role in areas such as hedging and nonlinear pricing theory, particularly in the context of imperfect markets.

The maximum principle

We present an outline of the derivation process for the maximum principle in the context of a deterministic control problem. Within this framework, let's consider the stochastic controlled system $\forall t \in [0, T]$

$$dx_t = a\left(t, x_t, u_t\right) dt; \quad x_0 = \varkappa, \tag{1.8}$$

where $a : [0, T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$.

The action space $\mathcal{U}$ is defined as a subset of the real numbers $\mathbb{R}$, and the fundamental objective is to minimize a cost function structured in the following form:

$$\mathcal{J}\left(u\left(.\right)\right)=\int_{0}^{T}c\left(t,x_{t},u_{t}\right)dt+g\left(x\left(T\right)\right),$$

where $c:[0,T]\times\mathbb{R}\times\mathcal{U}\rightarrow\mathbb{R}$ and $g:\mathbb{R}\rightarrow\mathbb{R}$.

In this context, the function c imposes a running cost, while the function g imposes a terminal cost. For any $u^{*}\in\mathcal{U}$ satisfying $\mathcal{J}\left(u^{*}\left(.\right)\right)=\inf_{u}\mathcal{J}\left(u\left(.\right)\right)$, it is referred to as an optimal control.

We proceed by assuming the existence of an optimal control (t). To derive necessary conditions for optimality, we introduce small perturbations to the optimal control

$$u^{\varepsilon}\left(t\right)=\begin{cases}v & \text{for } \tau-\varepsilon\leq t\leq\tau,\\ u^{*}\left(t\right) & \text{otherwise.}\end{cases}$$

Where u^{ε} is the spike variation of u^{*}

The solution to equation 1.8 with the control $u^{\varepsilon}\left(t\right)$ is represented by $x^{\varepsilon}\left(t\right)$. we put that $x^{*}\left(t\right)$ and $x^{\varepsilon}\left(t\right)$ are equal up to $t=\tau-\varepsilon$, and that

$$\begin{aligned}x^{\varepsilon}\left(\tau\right)-x^{*}\left(\tau\right)&=\left(a\left(\tau,x^{\varepsilon}\left(\tau\right),v\right)-a\left(\tau,x^{*}\left(\tau\right),u^{*}\left(\tau\right)\right)\right)\varepsilon+o(\varepsilon)\\&=\left(a\left(\tau,x^{*}\left(\tau\right),v\right)-a\left(\tau,x^{*}\left(\tau\right),u^{*}\left(\tau\right)\right)\right)\varepsilon+o(\varepsilon)\end{aligned}\tag{1.9}$$

The second equality is valid because $x^{\varepsilon}\left(\tau\right)-x^{*}\left(\tau\right)$ is of order ε. Examining the Taylor expansion of the state with respect to ε, we put $z\left(t\right)=\frac{\partial}{\partial\varepsilon}x^{\varepsilon}\left(t\right)\mid_{\varepsilon=0}$, in other words, the Taylor expansion of $x^{\varepsilon}\left(t\right)$ is expressed as: $x^{\varepsilon}\left(t\right)=x^{*}\left(t\right)+z\left(t\right)\varepsilon+o(\varepsilon)$.

and from 1.9, we get

$$z\left(\tau\right)=a\left(\tau,x^{*}\left(\tau\right),v\right)-a\left(\tau,x^{*}\left(\tau\right),u^{*}\left(\tau\right)\right).\tag{1.10}$$

We will employ duality to derive a more explicit necessary condition. For this purpose, we introduce the adjoint equation: $d\Psi(t) = -a_x(t, x_t^*, u_t^*)\,\Psi(t)\,dt,$ where $t \in [0, T]$, and $\Psi(T) = g_x(x^*(T))$.

Considering the terminal condition for the adjoint equation, we obtain:

$$\Psi(t)\,z(t) = g_x(x^*(T))\,z(T) \geq 0, \text{ for all } 0 \leq t \leq T.$$

Especially, by 1.10: $\Psi(\tau)\left(a(\tau, x^*(\tau), v) - a(\tau, x^*(\tau), u^*(\tau))\right) \geq 0.$

As τ was selected arbitrarily, this is equivalent to:

$$\Psi(t)\,a(t, x^*(t), u^*(t)) = \inf_{v \in \mathcal{U}} \Psi(t)\,a(t, x^*(t), v), \text{ for all } 0 \leq t \leq T.$$

By performing the computations again for this two-dimensional system, we can deduce the necessary condition.

$$H(t, x^*(t), u^*(t), \Psi(t)) = \inf_v H(t, x^*(t), v, \Psi(t)), \text{ for all } 0 \leq t \leq T, \tag{1.11}$$

where H is the Hamiltonian, (sometimes defined with a minus sign, which transforms the minimum condition above into a maximum condition) define as $H(x, u, \Psi) = c(x, u) + \Psi a(x, u)$. The adjoint equation is expressed as:

$$\begin{cases} d\Psi(t) = -\left[c_x(t, x^*(t), u^*(t)) + a_x(t, x^*(t), u^*(t))\,\Psi(t)\right] dt, \\ \Psi(T) = g_x(x^*(T)). \end{cases} \tag{1.12}$$

The Hamiltonian system for our control problem is determined by the minimum condition 1.11 along with the adjoint equation 1.12.

The stochastic maximum principle

Stochastic control is the extension of optimal control, it is recognizes and accommodates uncertainties in the system by incorporating SDE instead of deterministic ones:

$$dx_t = a(t, x_t, u_t)\, dt + b(t, x_t)\, dB_t, \quad t \in [0, T]. \tag{1.13}$$

The functions a and b are deterministic, and the last term represents an Itô integral with respect to a Brownian motion B, defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. In more general cases, the diffusion coefficient b can explicitly depend on the control variable as follow

$$dx_t = a(t, x_t, u_t)\, dt + b(t, x_t, u_t)\, dB_t, \quad t \in [0, T]. \tag{1.14}$$

The optimal control problem we are addressing involves over the control space $U[0, T]$ minimizing the following cost functional $\mathcal{J}(u(.)) = \mathbb{E}\left[\int_0^T c(t, x_t, u_t)\, dt + g(x(T))\right]$. For the case 1.13, the adjoint equation is expressed as the following BSDE:

$$\begin{cases} -d\Psi(t) &= [a_x(t, x^*(t), u^*(t))\, \Psi(t) + b_x(t, x^*(t))\, P(t) + c_x(t, x^*(t), u^*(t))]\, dt, \\ &- P(t)\, dB(t), \\ \Psi(t) &= g_x(x^*(T)). \end{cases} \tag{1.15}$$

A solution to this BSDE is a pair $(\Psi(t), P(t))$ that satisfies 1.15. The Hamiltonian is:

$$H(x, u, \Psi(t), P(t)) = c(t, x, u) + a(t, x, u)\, \Psi(t) + b(t, x)\, P(t),$$

and the maximum principle reads for all $0 \leq t \leq T$,

$$H(t, x^*(t), u^*(t), \Psi(t), P(t)) = \inf_{u\in\mathcal{U}} H(t, x^*(t), u, \Psi(t), P(t)) \quad \mathbb{P}-\text{a.s.}$$

Notably, there is a third case to consider: when the state is described by 1.14, but the action space U is supposed to be convex, it is potential to derive the maximum principle in a local form. This is achieved by employing a convex perturbation of the control instead of a spike variation, as detailed in Bensoussan [7], and the necessary condition for optimality in this case is expressed as follows:

$$\mathbb{E}\left[\int_0^T H_u\left(t, x^*\left(t\right), u^*\left(t\right), \Psi^*\left(t\right), P^*\left(t\right)\right)\left(u - u^*\left(t\right)\right)\right] \geq 0, \quad \text{for all } 0 \leq t \leq T.$$

1.3 Some classes of stochastic controls

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ that is complete.

1.3.1 Relaxed control

The concept is to condense the control space $\mathcal{U}$ by broadening the definition to encompass the space of probability measures on U. The collection of relaxed controls $\mu_t(du)dt$, where μ_t is a probability measure, is the closure under the weak topology of the measures $\delta_{u(t)}(du)dt$ associated with conventional or strict controls. The introduction of this idea of relaxed control is credited to Young in the context of deterministic optimal control problems [81].

In the researchs of Chala [15, 16, 17, 18]. This control type has applied, employing the stochastic maximum principle to fully coupled forward-backward doubly systems and Mean-Field SDEs systems with Poisson jumps, along with its application to the LQ problem.

1.3.2 Random horizon

In the classical problem, the time horizon is predetermined, extending until a deterministic terminal time T. However, in certain practical applications, the time horizon may be subject to randomness, and the cost functional is expressed as follows

$$\mathcal{J}\left(u\left(.\right)\right)=\mathbb{E}\left[h\left(x\left(\tau\right)\right)+\int_{0}^{\tau}h\left(t,x\left(t\right),y\left(t\right),u\left(t\right)\right)dt\right],$$

where τ is a finite random time.

1.3.3 Admissible control

An admissible control is an $\mathcal{F}_t$-adapted process $u(t)$ with values in a Borelian set $A \subset \mathbb{R}^n$, meeting certain conditions specific to the given problem.

1.3.4 Feedback control

We define $v(\cdot)$ as a feedback control if the control $v(\cdot)$ is contingent upon the state variable $X\left(.\right)$. Specifically if $\mathcal{F}_t^X$ represents the natural filtration generated by the process X, then $v(\cdot)$ qualifies as a feedback control if it is adapted to $\mathcal{F}_t^X$.

1.3.5 Optimal control

The objective of the optimal control problem is to minimize (or maximize) a cost function $\mathcal{J}\left(u\right)$ within the set of admissible controls $\mathcal{U}$. The control $u'(\cdot)$ is consider as an optimal control for all $u\left(t\right) \in \mathcal{U}$, if $\mathcal{J}\left(u'\left(t\right)\right)-\mathcal{J}\left(u\left(t\right)\right) \leq 0$ (or $\mathcal{J}\left(u'\left(t\right)\right)-\mathcal{J}\left(u\left(t\right)\right) \geq 0$).

1.3.6 Near-optimal control

Let $\varepsilon > 0$, a control is considered as a near–optimal control (or ε-optimal) if for any control $u\left(.\right) \in \mathcal{U}$ we have

$$\mathcal{J}\left(u^{\varepsilon}\left(t\right)\right)-\mathcal{J}\left(u\left(t\right)\right) \leq \varepsilon,$$

for more details about this type, the reader can see [80].

1.3.7 Ergodic control

In certain stochastic systems, a prolonged duration may reveal a stationary behavior marked by an invariant measure. If such a measure exists, it is derived through the average of states over an extended period. An ergodic control problem involves optimizing a criterion over the long term, considering this invariant measure. The cost functional is expressed as:

$$\lim_{T\to+\infty}\sup\frac{1}{T}\mathbb{E}\left[\int_0^T f\left(x\left(t\right),u\left(t\right)\right)dt\right].$$

For a more comprehensive understanding, the reader can refer to Pham [68].

1.3.8 Robust control

In the problems stated above, we assume that the dynamics of the control system are both known and constant. Robust control theory provides a methodology for evaluating how the performance of a control system is affected by variations in system parameters. This is particularly crucial in engineering systems and has more recently found applications in finance, particularly in connection with the theory of risk measures initiated by Artzner et al. [3].

It has been demonstrated that a coherent risk measure for an uncertain payoff X_T at time T is represented by

$$\rho\left(-X_T\right)=\sup_{Q\in\mathcal{K}}\mathbb{E}^Q\left[X_T\right],$$

where $\mathcal{K}$ is a set of absolutely continuous probability measures concerning the original probability P. In a more general sense, a risk measure can be defined as

$$\rho\left(-X_T\right)=-\inf_{Q\in\mathbb{Q}}\mathbb{E}^Q\left[U\left(-X_T\right)\right],$$

where U is a concave and nondecreasing function. In the realm of finance, these robust optimization problems have been the subject of recent studies in [29] and [73].

1.3.9 Partial observation control problem

Up to this point, the assumption has been that the controller possesses complete observations of the state system. In various practical situations, the controller is often limited to partial observations of the state through additional variables, and the observation process is subject to inherent noise. For instance, in financial models, one might observe the asset price without having complete information about its rate of return and/or volatility. Consequently, portfolio investment decisions are based solely on the available asset price information. This scenario gives rise to a partial observation control problem, which can be generally formulated as follows:

A controlled signal (unobserved) process is governed by the SDE

$$dX_t = \alpha\left(t, x_t, y_t, u_t\right) dt + \beta\left(t, x_t, y_t, u_t\right) dB_t,$$

and an observation process $dY_t = \lambda\left(t, x_t, y_t, u_t\right) dt + \delta\left(t, x_t, y_t, u_t\right) dW_t,$ where $W\left(t\right)$ represents another Brownian motion, potentially correlated with $B\left(t\right)$. The control $u\left(t\right)$ is adapted concerning the filtration generated by the observation $\mathbb{F}^Y = \left(\mathcal{F}_t^Y\right)$.

we find this type of control in works of Lakhdari et al. [1, 2, 54, 60].

1.3.10 Singular control

An admissible control is defined as a pair $(u(\cdot), \xi(\cdot))$ consisting of measurable processes taking values in $A_1 \times A_2$, both are $\mathcal{F}_t$-adapted. Additionally, $\xi(\cdot)$ is required to be of bounded variation, non-decreasing, continuous on the left with right limits and $\xi(0-) = 0$. It is noteworthy that $d\xi(t)$ might be singular concerning Lebesgue measure dt;

hence, $\xi(.)$ is referred to as the singular part of the control, while the process $u(\cdot)$ is its absolutely continuous part.

This control has been demonstrated with stochastic maximum principle in the works of Chala [4, 5, 20], and of Guenane et al. [30].

Chapter 2

Stochastic Maximum Principle for Risk-Neutral Control Problem

In this chapter, we introduce the first main result of this thesis: Stochastic controls of fractional Brownian motion. we aime to solving a stochastic optimization problem for a forward-backward stochastic differential equations (FBSDE) and we extracted it from our work [39]. For all $0 \leq t \leq T$,

$$\begin{cases} dx_t^v &= b\left(t, x_t^v, v_t\right) dt + \sigma\left(t, x_t^v, v_t\right) dB_t^H, \; x_0^v = \varkappa, \\ dy_t^v &= -f\left(t, x_t^v, y_t^v, z_t^v, v_t\right) dt + z_t^v dB_t^H, \; y_T^v = \xi. \end{cases}$$

The cost functional associated with the system is
$\mathcal{J}\left(v\right) = \mathbb{E}\left[l\left(x_T^v\right) + g\left(y_0^v\right) + \int_0^T h\left(t, x_t^v, y_t^v, z_t^v, v_t\right) dt\right].$
we consider the control u as an optimal one if it solve $\mathcal{J}\left(u\right) = \inf_{v \in \mathcal{U}} \mathcal{J}\left(v\right).$

Our goal in this chapter is to derive the necessary as well as sufficient conditions of optimality for this model. We give the results in the form of an SMP. On which the domain of controls is convex. Firstly, we establish these necessary conditions by employing the convex perturbation method. Specifically, if we denote u as an optimal control and v as any arbitrary control, we can create a perturbed control, denoted as $u_t^\theta = u_t + \theta v_t$, for each $t \in [0, T]$, with $\theta > 0$ being sufficiently small, by utilizing some assumptions on the coefficients, we then derive the SMP in the global form.

2.1 Problem formulation

Let T be a positive real number, we define $B^H = \left(B_t^H\right)_{t \geq 0}$, a one-dimensional fractional Brownian motion with Hurst parameter $H \in]0,1[$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\mathcal{N}$ denote the class of $\mathbb{P}-$null sets of $\mathcal{F}$, we provide the space $(\Omega, \mathcal{F}, \mathbb{P})$ by the natural filtration of the fBm B^H, $\mathcal{F}_t^H = \sigma\left(B_s^H, 0 \leq s \leq t\right) \vee \mathcal{N}$ for each $t \in [0, T]$, such that $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^H\right)_{t \geq 0}, \mathbb{P}\right)$ satisfying the usual conditions.

We will work in the following spaces throughout this chapter:

$\mathcal{M}^2\left([0, T], \mathbb{R}\right) := \mathcal{F}_t^H$-adapted processes ϕ such that $\mathbb{E}\left[\int_0^T \left|\phi_t\right|^2 dt\right] < \infty$,

$\mathcal{K}^2\left([0, T], \mathbb{R}\right) := \mathcal{F}_t^H$-adapted processes ϕ such that $\mathbb{E}\left[\int_t^T \left(Hs^{2H-1} - 1\right)\left|\phi_s\right|^2 ds\right] < \infty$, for every $H \in \left]\frac{1}{2}, 1\right[$ when $s \in]1, T]$, and $H \in \left]0, \frac{1}{2}\right[$ when $s \in [t, 1[$, respectively.

Let U be a closed, convex, nonempty subset of $\mathbb{R}$.

Definition 2.1 *An admissible control v is an $\mathcal{F}_t^H-$adapted process assuming values in U, where it satisfies*

$$\mathbb{E}\left[\int_0^T \left|v_t\right|^2 dt\right] < \infty. \tag{2.1}$$

We denote the set of all admissible controls by $\mathcal{U}$, which we suppose convex.

Now, we consider the following FBSDE for all $v \in \mathcal{U}$, $t \in [0, T]$

$$\begin{cases} dx_t^v &= b\left(t, x_t^v, v_t\right) dt + \sigma\left(t, x_t^v, v_t\right) dB_t^H, \ x_0^v = \varkappa, \\ dy_t^v &= -f\left(t, x_t^v, y_t^v, z_t^v, v_t\right) dt + z_t^v dB_t^H, \ y_T^v = \xi. \end{cases} \tag{2.2}$$

This system is associated with a cost functional defined as:

$$\mathcal{J}(v) = \mathbb{E}\left[l\left(x_T^v\right) + g\left(y_0^v\right) + \int_0^T h\left(t, x_t^v, y_t^v, z_t^v, v_t\right) dt\right], \tag{2.3}$$

where

$$\begin{aligned} f, h: \quad [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} &\rightarrow \mathbb{R}, \\ b, \sigma: [0,T] \times \mathbb{R} \times \mathcal{U} &\rightarrow \mathbb{R}, \\ \text{and } l, g: \mathbb{R} &\rightarrow \mathbb{R}. \end{aligned}$$

$\varkappa$ is an $\mathcal{F}_0^H$_adapted random variable and the final condition ξ is an $\mathcal{F}_T^H$-adapted and square integrable random variable.

Note that the integral with respect to the fBm B_t^H is in the Russo-Vallois sense.

The optimal control problem involves minimizing the functional $\mathcal{J}$ over the set $\mathcal{U}$. If $u \in \mathcal{U}$ is an optimal control (solution), then

$$\mathcal{J}(u) = \inf_{v \in \mathcal{U}} \mathcal{J}(v). \tag{2.4}$$

A control that satisfies the conditions (2.2), (2.3) and (2.4) is called optimal. Our aim is to derive both necessary and sufficient optimality conditions for a given control. specifically in the context of stochastic maximum principle performance functional.

Assumption 2.1 *To ensure the well-posedness of the control problem defined by equations* (2.2)*,* (2.3)*, and* (2.4)*, we suppose*

(H1) *There exist constants* $\gamma, \zeta > 0$*, such that for any* $(w,t) \in \Omega \times [0,T]$; $(x_i, u_i) \in \mathbb{R} \times \mathcal{U}$ *and* $(x_i, y_i, z_i, u_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U}$, $i = 1, 2$; *we have*

$$\left|b\left(t, x_1, u_1\right) - b\left(t, x_2, u_2\right)\right|^2 \leq \zeta\left(\left|x_1 - x_2\right|^2 + \left|u_1 - u_2\right|^2\right).$$

$$\left|\sigma\left(t, x_1, u_1\right) - \sigma\left(t, x_2, u_2\right)\right|^2 \leq \zeta\left(\left|x_1 - x_2\right|^2 + \left|u_1 - u_2\right|^2\right).$$

$$\left|f\left(t, x_1, y_1, z_1, u_1\right) - f\left(t, x_2, y_2, z_2, u_2\right)\right|^2 \leq \gamma\left(\left|x_1 - x_2\right|^2 + \left|y_1 - y_2\right|^2 \left|z_1 - z_2\right|^2 + \left|u_1 - u_2\right|^2\right).$$

(H2) *The functions* b *and* σ *are continuously differentiable with respect to* (x, u).

(H3) *The functions* f *and* h *are continuously differentiable with respect to* (x, y, z, u).

(H4) *The derivatives* $b_x, b_u, \sigma_x, \sigma_u, f_x, f_y, f_z, f_u, h_x, h_y, h_z, h_u, b_x^2, f_x^2$ *and* f_z^2 *are bounded.*

(H5) *The functions* l_x *and* g_y *are bounded and continuous.*

Under the above hypothesis **(H1)**-**(H3)**, for all $v \in \mathcal{U}$, the equation (2.2) admits a unique solution. (For further details, the reader is encouraged to refer to these works.[6] and [47]).

Remark 2.1 *As indicated in the previous chapter, this process is a semimartingale if and only if* $H = \frac{1}{2}$, *which is why we will present the next paragraph.*

The Russo & Vallois Integral

The first foundations of a stochastic calculus were established in 1993 by F. Russo and P. Vallois, where they generalizing those of Itô and stratonovich. One advantages of this calculus is that it possible the interpretation of integrals versus processes that aren't always semi-martingales. It is no longer possible to define a stochastic integral with regard to the fBm using the traditional stochastic Itô integration since the fBm fails to satisfy the semimartingale property unless $H = 1/2$. In the meantime, many approaches to explain an integration with regard to the fBm have been used in the literature, including [9], [41] and the references therein. Amongst these approaches, the Russo-Vallois integral described by Russo-Vallois in [70], [71] which we use throught this thesis.

Definition 2.2 *If* $Y_1 = \{Y_1(t), t \in [0, 1]\}$ *and* $Y_2 = \{Y_2(t), t \in [0, 1]\}$ *are two continu-*

ous stochastic processes at 0 *and* 1*, we put*

$$I^{-}(\varepsilon, Y_1, dY_2) = \int_0^1 Y_1(t) \frac{Y_2[(t+\varepsilon) \wedge 1] - Y_2(t)}{\varepsilon} dt,$$
$$I^{+}(\varepsilon, Y_1, dY_2) = \int_0^1 Y_1(t) \frac{Y_2(t) - Y_2[(t-\varepsilon) \vee 0]}{\varepsilon} dt,$$
$$I^{\circ}(\varepsilon, Y_1, dY_2) = \int_0^1 Y_1(t) \frac{Y_2[(t+\varepsilon) \wedge 1] - Y_2[(t-\varepsilon) \vee 0]}{\varepsilon} dt,$$

when the limit in probability exists, we denote by

$$\int_0^1 Y_1 d^{-} Y_2 = \lim_{\varepsilon \to 0} I^{-}(\varepsilon, Y_1, dY_2), \quad \int_0^1 Y_1 d^{+} Y_2 = \lim_{\varepsilon \to 0} I^{+}(\varepsilon, Y_1, dY_2),$$
$$\int_0^1 Y_1 d^{\circ} Y_2 = \lim_{\varepsilon \to 0} I^{\circ}((\varepsilon, Y_1, dY_2).$$

These limits are called forward, backward and symmetric, integral of Y_1 *with respect to* Y_2*, respectively. We can be aware* $I^{\circ}((\varepsilon, Y_1, dY_2) = \frac{I^{+}(\varepsilon, Y_1, dY_2) + I^{-}(\varepsilon, Y_1, dY_2)}{2}$*, and therefore that* $\int_0^1 Y_1 d^{\circ} Y_2 = \frac{1}{2}\left(\int_0^1 Y_1 d^{+} Y_2 + \int_0^1 Y_1 d^{-} Y_2\right)$.

2.2 Primary results

In this chapter, we shall derive a Pontryagin's maximum principle. In particular, we aim to prove the optimality conditions. However, In order to give and prove thses conditions, it is convenient to present some essential lemmas and results, that serve our purpose in the sequel. And since the set $\mathcal{U}$ is convex, the conventional methods for establishing the necessary as well as sufficient conditions involve employing a convex variational perturbation method. More precisely, let u be an optimal control and (x^u, y^u, z^u) the solution of (2.2), controlled by u. Then, for every $t \in [0, T]$, defines the perturbation of the optimal control as:

$$u_t^{\theta} = u_t + \theta v_t, \tag{2.5}$$

where $\theta > 0$ is sufficiently small and v represents any arbitrary element of $\mathcal{U}$, (i.e $v \in \mathcal{U}$) , then $u^\theta \in \mathcal{U}$, i.e. u^θ is an admissible control.

Let x^θ represent the trajectory associated with control $u^\theta \in \mathcal{U}$; x^θ is the trajectory defined as:

$$\begin{cases} dx_t^\theta & = & b\left(t, x_t^\theta, v_t^\theta\right) dt + \sigma\left(t, x_t^\theta, v_t^\theta\right) dB_t^H, \\ dy_t^\theta & = & -f\left(t, x_t^\theta, y_t^\theta, z_t^\theta, v_t^\theta\right) dt + z_t^\theta dB_t^H. \end{cases} \tag{2.6}$$

From (2.4) and the optimality of u we have

$$0 \leq \mathcal{J}\left(u^\theta\right) - \mathcal{J}(u). \tag{2.7}$$

Let $\left(x^\theta, y^\theta, z^\theta\right)$ and (x, y, z) be the trajectories associated with the perturbed control u^θ and the optimal control u , respectively.

Lemma 2.1 *Under Assumption **(H1)** and the definition of the perturbation u^θ, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|x_t^\theta - x_t\right|^2\right] \leq C'\theta^2, \tag{2.8}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|y_t^\theta - y_t\right|^2\right] + \mathbb{E}\left[\int_t^T \left(Hs^{2H-1} - 1\right)\left|z_s^\theta - z_s\right|^2 ds\right] \leq \overline{C}\theta^2. \tag{2.9}$$

Before making the proof of 2.1, for simplicity, we employ abbreviated notations.

Notation 2.1 *For $\tau \in \{u, v\}$, $\phi \in \{f, h\}$, $\Pi \in \{b, \sigma\}$, $F \in \{\phi, \Pi, l, g\}$, $\kappa \in \{x, y, z, u\}$, then*

$$\begin{cases} \Pi^\tau(t) = \Pi(t, x_t, \tau_t) \\ \Pi^\theta(t) = \Pi\left(t, x_t^\theta, u_t^\theta\right) \\ \Pi^{u^\theta}(t) = \Pi\left(t, x_t, u_t^\theta\right) \end{cases}, \quad \begin{cases} \phi^\tau(t) = \phi(t, x_t, y_t, z_t, \tau_t) \\ \phi^\theta(t) = \phi\left(t, x_t^\theta, y_t^\theta, z_t^\theta, u_t^\theta\right) \\ F_\kappa(t) = \dfrac{\partial F}{\partial \kappa}(t, x_t, u_t) \end{cases},$$

$$\begin{cases} \overline{x} = x_s + \lambda\left(x_s^\theta - x_s\right),\ \overline{y} = y_s + \lambda\left(y_s^\theta - y_s\right) \\ \overline{z} = z_s + \lambda\left(z_s^\theta - z_s\right),\ \overline{u} = u_s + \lambda\left(u_s^\theta - u_s\right) \end{cases}, \quad and \quad \begin{cases} \mathcal{A}_s^{\theta,\lambda} = (s, \overline{x}, \overline{y}, \overline{z}, \overline{u}) \\ \mathcal{Q}_s^{\theta,\lambda} = (s, \overline{x}, \overline{u}) \end{cases}.$$

Proof. At first, by substituting the values of x_t^θ and x_t, we derive

$$\begin{aligned} d(x_t^\theta - x_t) = &\left[b^\theta(t) - b^{u^\theta}(t) + b^{u^\theta}(t) - b^u(t)\right] dt \\ &+ \left[\sigma^\theta(t) - \sigma^{u^\theta}(t) + \sigma^{u^\theta}(t) - \sigma^u(t)\right] dB_t^H. \end{aligned}$$

Integrating from 0 to T, then taking the mathematical expectation, applying the Cauchy-Schwarz's inequality and the isometry property, we obtain

$$\begin{aligned} \mathbb{E}\left[\left|x_t^\theta - x_t\right|^2\right] \leq\ & 4T\mathbb{E}\left[\int_0^T \left|b^\theta(t) - b^{u^\theta}(t)\right|^2 dt\right] + 4T\mathbb{E}\left[\int_0^T \left|b^{u^\theta}(t) - b^u(t)\right|^2 dt\right] \\ &+ 4\mathbb{E}\left[\int_0^T \left|\sigma^\theta(t) - \sigma^{u^\theta}(t)\right|^2 Ht^{2H-1} dt\right] \\ &+ 4\mathbb{E}\left[\int_0^T \left|\sigma^{u^\theta}(t) - \sigma^u(t)\right|^2 Ht^{2H-1} dt\right]. \end{aligned}$$

Since b and σ are Lipschitz in (x, u) and from (2.5), we have

$$\begin{aligned} \mathbb{E}\left[\left|x_t^\theta - x_t\right|^2\right] &\leq 4T\zeta\left(\mathbb{E}\left[\int_0^T \left|x_t^\theta - x_t\right|^2 dt\right] + \mathbb{E}\left[\int_0^T \left|u_t^\theta - u_t\right|^2 dt\right]\right) \\ &\quad + 4\zeta HT^{2H-1}\left(\mathbb{E}\left[\int_0^T \left|x_t^\theta - x_t\right|^2 dt\right] + \mathbb{E}\left[\int_0^T \left|u_t^\theta - u_t\right|^2 dt\right]\right) \\ &\leq C_1\left(\mathbb{E}\left[\int_0^T \left|x_t^\theta - x_t\right|^2 dt\right] + \theta^2\mathbb{E}\left[\int_0^T \left|v_t\right|^2 dt\right]\right), \end{aligned}$$

where $C_1 = 2\max\left(4T\zeta, 4\zeta HT^{2H-1}\right)$, and from (2.1), then we have

$$\mathbb{E}\left[\left|x_t^\theta - x_t\right|^2\right] \leq C_1\left(\mathbb{E}\left[\int_0^T \left|x_t^\theta - x_t\right|^2 dt\right] + \theta^2 C_2\right).$$

Using the Gronwall's inequality, we arrive at

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|x_t^\theta - x_t\right|^2\right] \leq \theta^2 C'. \tag{2.10}$$

such that $C' = C_2 \exp\{C_1 T\}$.

In order to prove the second estimation, we apply the Itô formula, we get

$$\begin{aligned} d\left|y_t^\theta - y_t\right|^2 &= 2\left|y_t^\theta - y_t\right|\left(f^u(t) - f^\theta(t)\right)dt \\ &\quad -2\left|y_t^\theta - y_t\right|\left(z_t - z_t^\theta\right)dB_t^H + Ht^{2H-1}\left|z_t^\theta - z_t\right|^2 dt. \end{aligned}$$

Passing the integral from t to T, we find

$$\begin{aligned} \left|y_t^\theta - y_t\right|^2 + \int_t^T Hs^{2H-1}\left|z_s^\theta - z_s\right|^2 ds &= 2\int_t^T \left(y_s^\theta - y_s\right)\left(f^\theta(s) - f^u(s)\right)ds \\ &+ 2\int_t^T \left(y_s^\theta - y_s\right)\left(z_s - z_s^\theta\right)dB_s^H. \end{aligned}$$

Since f is Lipschitz in (x, y, z, u) and from (2.5), it comes

$$\begin{aligned} &\left|y_t^\theta - y_t\right|^2 + \int_t^T Hs^{2H-1}\left|z_s^\theta - z_s\right|^2 ds \\ \leq\ & 2\gamma\int_t^T \left|x_s^\theta - x_s\right|\left|y_s^\theta - y_s\right| ds + 2\gamma\int_t^T \left|y_s^\theta - y_s\right|^2 ds + 2\theta\gamma\int_t^T \left|y_s^\theta - y_s\right|\left|v_s\right| ds \\ +\ & 2\gamma\int_t^T \left|y_s^\theta - y_s\right|\left|z_s^\theta - z_s\right| ds + 2\int_t^T \left(y_s^\theta - y_s\right)\left(z_s - z_s^\theta\right)dB_s^H. \end{aligned}$$

We take the mathematical expectation and apply Young's inequality, we get

$$\begin{aligned} &\mathbb{E}\left[\left|y_t^\theta - y_t\right|^2\right] + \mathbb{E}\left[\int_t^T Hs^{2H-1}\left|z_s^\theta - z_s\right|^2 ds\right] \\ \leq\ & \mathbb{E}\left[\int_t^T \left|x_s^\theta - x_s\right|^2 ds\right] + \gamma^2\mathbb{E}\left[\int_t^T \left|y_s^\theta - y_s\right|^2 ds\right] + 2\gamma\mathbb{E}\left[\int_t^T \left|y_s^\theta - y_s\right|^2 ds\right] \\ +\ & \gamma^2\mathbb{E}\left[\int_t^T \left|y_s^\theta - y_s\right|^2 ds\right] + \theta^2\mathbb{E}\left[\int_t^T \left|v_s\right|^2 ds\right] + \gamma^2\mathbb{E}\left[\int_t^T \left|y_s^\theta - y_s\right|^2 ds\right] \\ +\ & \mathbb{E}\left[\int_t^T \left|z_s^\theta - z_s\right|^2 ds\right], \end{aligned}$$

if we put $\pi_\theta(t) = \mathbb{E}\left[\int_t^T \left(\theta^2 |v_s|^2 + \left|x_s^\theta - x_s\right|^2\right) ds\right]$. Then, it comes

$$\begin{aligned}
&\mathbb{E}\left[\left|y_t^\theta - y_t\right|^2\right] + \mathbb{E}\left[\int_t^T \left(Hs^{2H-1} - 1\right) \left|z_s^\theta - z_s\right|^2 ds\right] \\
\leq\; & (2\gamma + 3\gamma^2)\, \mathbb{E}\left[\int_t^T \left|y_s^\theta - y_s\right|^2 ds\right] + \pi_\theta(t),
\end{aligned}$$

As a direct result from (2.10), there exists some positive real constant C_3, such that $\pi_\theta(t) \leq \theta^2 C_3$.

On the one hand, we have

$$\mathbb{E}\left[\left|y_t^\theta - y_t\right|^2\right] \leq \left(2\gamma + 3\gamma^2\right) \mathbb{E}\left[\int_t^T \left|y_s^\theta - y_s\right|^2 ds\right] + \theta^2 C_3. \tag{2.11}$$

By using the Gronwall's inequality on (2.11), we obtain

$$\mathbb{E}\left[\left|y_t^\theta - y_t\right|^2\right] \leq \theta^2 C_3 \exp\left\{\left(2\gamma + 3\gamma^2\right) T\right\}.$$

Hence, we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|y_t^\theta - y_t\right|^2\right] \leq \theta^2 N, \tag{2.12}$$

such that $N = C_3 \exp\{(2\gamma + 3\gamma^2) T\}$. On the other hand, we note

$$K = Hs^{2H-1} - 1, \tag{2.13}$$

then

$$\begin{aligned}
\mathbb{E}\left[\int_t^T K \left|z_s^\theta - z_s\right|^2 ds\right] &\leq \left(2\gamma + 3\gamma^2\right) \mathbb{E}\left[\int_t^T \left|y_s^\theta - y_s\right|^2 ds\right] + \theta^2 C_3, \\
&\leq \theta^2 \left(N + C_3\right).
\end{aligned} \tag{2.14}$$

If we note $\theta^2 (2N + C_3) = \overline{C}$, then sum up the inequalities (2.12) and (2.14) , we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|y_t^\theta - y_t\right|^2\right] + \mathbb{E}\left[\int_t^T \left(Hs^{2H-1} - 1\right)\left|z_s^\theta - z_s\right|^2 ds\right] \leq \theta^2\overline{C},$$

which ends the proof of lemma 2.1. ■

Lemma 2.2 *We suppose that hypotheses **(H2)** and **(H3)** in assumption* 2.1 *hold. If we put*

$$X_t = \lim_{\theta\to 0}\frac{1}{\theta}\left(x_t^\theta - x_t\right), \quad Y_t = \lim_{\theta\to 0}\frac{1}{\theta}\left(y_t^\theta - y_t\right) \text{ and } Z_t = \lim_{\theta\to 0}\frac{1}{\theta}\left(z_t^\theta - z_t\right).$$

Then X and Y can be written in the following forms

$$\begin{aligned} X_t &= \int_0^t \{b_x(s,x_s,u_s) X_s + b_u(s,x_s,u_s) v_s\} ds \\ &+ \int_0^t \{\sigma_x(s,x_s,u_s) X_s + \sigma_u(s,x_s,u_s) v_s\} dB_s^H, \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} Y_t &= \int_t^T \{f_x(s,x_s,y_s,z_s,u_s) X_s + f_y(s,x_s,y_s,z_s,u_s) Y_s \\ &+ f_z(s,y_s,z_s,u_s) Z_s + f_u(s,y_s,z_s,u_s) v_s\} ds - \int_t^T Z_s dB_s^H. \end{aligned} \tag{2.16}$$

Proof. We use notations 2.1. By applying Taylor's expansion with integral remain of

$b\left(s, x_s^\theta, u_s^\theta\right)$ and $\sigma\left(s, x_s^\theta, u_s^\theta\right)$ at (x,u), and of $f\left(s, x_s^\theta, y_s^\theta, z_s^\theta, u_s^\theta\right)$ at (x,y,z,u), we get

$$\begin{aligned}
\frac{1}{\theta}\left(x_t^\theta - x_t\right) &= \int_0^t \left(\tfrac{b^\theta(s)-b^u(s)}{\theta}\right) ds + \int_0^t \left(\tfrac{\sigma^\theta(s)-\sigma^u(s)}{\theta}\right) dB_s^H \\
&= \int_0^t \int_0^1 b_x\left(\mathcal{Q}_s^{\theta,\lambda}\right)\left(\frac{x_s^\theta - x_s}{\theta}\right) d\lambda ds \\
&+ \int_0^t \int_0^1 b_u\left(\mathcal{Q}_s^{\theta,\lambda}\right)\left(\frac{u_s^\theta - u_s}{\theta}\right) d\lambda ds \\
&+ \int_0^t \int_0^1 \sigma_x\left(\mathcal{Q}_s^{\theta,\lambda}\right)\left(\frac{x_s^\theta - x_s}{\theta}\right) d\lambda dB_s^H \\
&+ \int_0^t \int_0^1 \sigma_u\left(\mathcal{Q}_s^{\theta,\lambda}\right)\left(\frac{u_s^\theta - u_s}{\theta}\right) d\lambda dB_s^H,
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
\frac{1}{\theta}\left(y_t^\theta - y_t\right) &= \int_t^T \left(\frac{f^\theta(s) - \ f^u(s)}{\theta}\right) ds - \int_t^T \left(\frac{z_s^\theta - z_s}{\theta}\right) dB_s^H \\
&= \int_t^T \int_0^1 f_x\left(\mathcal{A}_s^{\theta,\lambda}\right)\left(\frac{x_s^\theta - x_s}{\theta}\right) d\lambda ds \\
&+ \int_t^T \int_0^1 f_y\left(\mathcal{A}_s^{\theta,\lambda}\right)\left(\frac{y_s^\theta - y_s}{\theta}\right) d\lambda ds \\
&+ \int_t^T \int_0^1 f_z\left(\mathcal{A}_s^{\theta,\lambda}\right)\left(\frac{z_s^\theta - z_s}{\theta}\right) d\lambda ds \\
&+ \int_t^T \int_0^1 f_u\left(\mathcal{A}_s^{\theta,\lambda}\right)\left(\frac{u_s^\theta - u_s}{\theta}\right) d\lambda ds - \int_t^T \left(\frac{z_s^\theta - z_s}{\theta}\right) dB_s^H.
\end{aligned} \tag{2.18}$$

We take the limits when $\theta \to 0$, due to the hypothesis which we have indicated, and we apply Lebesgue's bounded convergence theorem, equations (2.17) and (2.18), respectively, become

$$\begin{aligned}
X_t &= \lim_{\theta \to 0} \frac{1}{\theta}\left(x_t^\theta - x_t\right) \\
&= \int_0^t \left\{b_x(s, x_s, u_s) X_s + b_u(s, x_s, u_s) v_s\right\} ds \\
&+ \int_0^t \left\{\sigma_x(s, x_s, u_s) X_s + \sigma_u(s, x_s, u_s) v_s\right\} dB_s^H,
\end{aligned}$$

and

$$\begin{aligned} Y_t &= \lim_{\theta \to 0} \frac{1}{\theta} \left(y_t^\theta - y_t \right) \\ &= \int_t^T f_x \left(s, x_s, y_s, z_s, u_s \right) X_s ds + \int_t^T f_y \left(s, x_s, y_s, z_s, u_s \right) Y_s ds \\ &+ \int_t^T f_z \left(s, x_s, y_s, z_s, u_s \right) Z_s ds + \int_t^T f_u \left(s, x_s, y_s, z_s, u_s \right) v_s ds \\ &- \int_t^T Z_s dB_s^H . \end{aligned}$$

which is the result. ■

Considering lemma 2.1 and lemma 2.2, we arrive at the following

Lemma 2.3 *We suppose that assumptions **(H2)** and **(H4)** hold. We note*

$$\widetilde{x}_t^\theta = \frac{1}{\theta} \left(x_t^\theta - x_t \right) - X_t, \quad \widetilde{y}_t^\theta = \frac{1}{\theta} \left(y_t^\theta - y_t \right) - Y_t \quad \text{and} \quad \widetilde{z}_t^\theta = \frac{1}{\theta} \left(z_t^\theta - z_t \right) - Z_t. \tag{2.19}$$

Then by virtue of lemmas 2.1 and 2.2, we have the convergences

$$\mathbb{E} \left[\sup_{t \in [0,T]} \left| \widetilde{x}_t^\theta \right|^2 \right] \underset{\theta \to 0}{\longrightarrow} 0, \quad \text{and} \tag{2.20}$$

$$\mathbb{E} \left[\sup_{t \in [0,T]} \left| \widetilde{y}_t^\theta \right|^2 \right] + \mathbb{E} \left[\int_t^T \left(H s^{2H-1} - 1 \right) \left| \widetilde{z}_s^\theta \right|^2 ds \right] \underset{\theta \to 0}{\longrightarrow} 0. \tag{2.21}$$

Proof. By using notations 2.1 and the $\widetilde{x}_t^\theta, \widetilde{y}_t^\theta$ and $\widetilde{z}_t^\theta$ formulas given in 2.19, we have

$$\begin{aligned} \mathbb{E} \left| \widetilde{x}_t^\theta \right| &= \mathbb{E} \left| \int_0^t \left(\frac{b^\theta (s) - b^u (s)}{\theta} \right) ds + \int_0^t \left(\frac{\sigma^\theta (s) - \sigma^u (s)}{\theta} \right) dB_s^H - X_t \right| \\ &= \mathbb{E} \left| \int_0^t \int_0^1 b_x \left(\mathcal{Q}_s^{\theta,\lambda} \right) \left(\frac{x_s^\theta - x_s}{\theta} \right) d\lambda ds + \int_0^t \int_0^1 b_u \left(\mathcal{Q}_s^{\theta,\lambda} \right) \left(\frac{u_s^\theta - u_s}{\theta} \right) d\lambda ds \right. \\ &+ \int_0^t \int_0^1 \sigma_x \left(\mathcal{Q}_s^{\theta,\lambda} \right) \left(\frac{x_s^\theta - x_s}{\theta} \right) d\lambda dB_s^H + \int_0^t \int_0^1 \sigma_u \left(\mathcal{Q}_s^{\theta,\lambda} \right) \left(\frac{u_s^\theta - u_s}{\theta} \right) d\lambda dB_s^H \\ &\left. - \int_0^t \left\{ b_x (s) X_s + b_u (s) v_s \right\} ds - \int_0^t \left\{ \sigma_x (s) X_s + \sigma_u (s) v_s \right\} dB_s^H \right| . \end{aligned}$$

Namely

$$\begin{aligned}\mathbb{E}\left|\widetilde{x}_t^\theta\right| = \mathbb{E}\bigg|&\int_0^t\int_0^1 b_x\left(\mathcal{Q}_s^{\theta,\lambda}\right)\left(\widetilde{x}_s^\theta + X_s\right)d\lambda ds + \int_0^t\int_0^1 b_u\left(\mathcal{Q}_s^{\theta,\lambda}\right)v_s d\lambda ds \\ &+\int_0^t\int_0^1 \sigma_x\left(\mathcal{Q}_s^{\theta,\lambda}\right)\left(\widetilde{x}_s^\theta + X_s\right)d\lambda dB_s^H + \int_0^t\int_0^1 \sigma_u\left(\mathcal{Q}_s^{\theta,\lambda}\right)v_s d\lambda dB_s^H \\ &-\int_0^t \left\{b_x(s)X_s + b_u(s)v_s\right\}ds - \int_0^t\left\{\sigma_x(s)X_s + \sigma_u(s)v_s\right\}dB_s^H\bigg|. \end{aligned} \tag{2.22}$$

If we define

$$\begin{aligned}\varpi^\theta = &\int_0^t\int_0^1 b_x\left(\mathcal{Q}_s^{\theta,\lambda}\right)X_s d\lambda ds + \int_0^t\int_0^1 b_u\left(\mathcal{Q}_s^{\theta,\lambda}\right)v_s d\lambda ds \\ &+\int_0^t\int_0^1 \sigma_x\left(\mathcal{Q}_s^{\theta,\lambda}\right)X_s d\lambda dB_s^H + \int_0^t\int_0^1 \sigma_u\left(\mathcal{Q}_s^{\theta,\lambda}\right)v_s d\lambda dB_s^H \\ &-\int_0^t \left\{b_x(s)X_s + b_u(s)v_s\right\}ds - \int_0^t\left\{\sigma_x(s)X_s + \sigma_u(s)v_s\right\}dB_s^H.\end{aligned}$$

Then (2.22) becomes

$$\mathbb{E}\left|\widetilde{x}_t^\theta\right| = \mathbb{E}\left|\int_0^t\int_0^1 b_x\left(\mathcal{Q}_s^{\theta,\lambda}\right)\widetilde{x}_s^\theta d\lambda ds + \int_0^t\int_0^1 \sigma_x\left(\mathcal{Q}_s^{\theta,\lambda}\right)\widetilde{x}_s^\theta d\lambda dB_s^H + \varpi^\theta\right|,$$

Using triangular inequality, we obtain

$$\begin{aligned}\mathbb{E}\left|\widetilde{x}_t^\theta\right|^2 \le\; &6\mathbb{E}\left[\int_0^t\int_0^1 b_x^2\left(\mathcal{Q}_s^{\theta,\lambda}\right)\left|\widetilde{x}_s^\theta\right|^2 d\lambda ds\right] + 6\mathbb{E}\left[\int_0^t\int_0^1 \sigma_x^2\left(\mathcal{Q}_s^{\theta,\lambda}\right)\left|\widetilde{x}_s^\theta\right|^2 d\lambda dB_s^H\right] \\ &+\mathbb{E}\left[\left|\varpi^\theta\right|^2\right].\end{aligned}$$

Since

$$\begin{aligned}\mathbb{E}\left[\int_0^t\int_0^1 \sigma_x^2\left(\mathcal{A}_s^{\theta,\lambda}\right)\left|\widetilde{x}_s^\theta\right|^2 d\lambda dB_s^H\right] &= 0, \\ \text{and } \mathbb{E}\left[\left|\varpi^\theta\right|^2\right] &\underset{\theta\to 0}{\longrightarrow} 0.\end{aligned}$$

Applying the assumptions imposed on the functions b and σ mentioned in **(H2)** and

(H4), get

$$\mathbb{E}\left[\left|\varpi^{\theta}\right|^{2}\right] \underset{\theta\to 0}{\to} 0.$$

Using the Gronwall's inequality, we have

$$\mathbb{E}\left|\widetilde{x}_t^{\theta}\right|^2 \leq \mathbb{E}\left[\left|\varpi^{\theta}\right|^2\right] \exp\left\{T\int_0^1 b_x^2\left(\mathcal{A}_s^{\theta,\lambda}\right)d\lambda\right\} \underset{\theta\to 0}{\to} 0,$$

which yields finally to

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\widetilde{x}_t^{\theta}\right|^2\right] \underset{\theta\to 0}{\to} 0. \tag{2.23}$$

For establishing (2.21) , we apply Itô's formula to $\left|\widetilde{y}_t^{\theta}\right|^2$, we get

$$\begin{aligned}
d\left|\widetilde{y}_t^{\theta}\right|^2 &= 2\left|\widetilde{y}_t^{\theta}\right|\left[\frac{1}{\theta}\left(dy_t^{\theta}-dy_t\right)-dY_t\right]+d\left\langle\widetilde{y}_t^{\theta}\right\rangle \\
&= 2\left|\widetilde{y}_t^{\theta}\right|\left[\left(\frac{f^u(t)-f^{\theta}(t)}{\theta}\right)dt-\left(\frac{z_t-z_t^{\theta}}{\theta}\right)dB_t^H\right] \\
&+ 2\left|\widetilde{y}_t^{\theta}\right|\left[f_x(t,x_t,y_t,z_t,u_t)X_tdt+f_y(t,x_t,y_t,z_t,u_t)Y_tdt\right. \\
&\left.+f_z(t,x_t,y_t,z_t,u_t)Z_tdt+f_u(t,x_t,y_t,z_t,u_t)v_tdt\right] \\
&- 2\left|\widetilde{y}_t^{\theta}\right|Z_tdB_t^H+Ht^{2H-1}\left|\widetilde{z}_t^{\theta}\right|^2dt.
\end{aligned}$$

Integrating from t to T, we have

$$\begin{aligned}
&\left|\widetilde{y}_t^{\theta}\right|^2+\int_t^T Hs^{2H-1}\left|\widetilde{z}_s^{\theta}\right|^2ds \\
&= 2\int_t^T\left|\widetilde{y}_s^{\theta}\right|\left[\left(\frac{f^{\theta}(s)-f^u(s)}{\theta}\right)ds-\left(\frac{z_s^{\theta}-z_s}{\theta}\right)dB_s^H\right] \\
&- 2\int_t^T\left|\widetilde{y}_s^{\theta}\right|\left[f_x(s,x_s,y_s,z_s,u_s)X_s+f_y(s,x_s,y_s,z_s,u_s)Y_s\right. \\
&\left.+f_z(s,x_s,y_s,z_s,u_s)Z_s+f_u(s,x_s,y_s,z_s,u_s)v_s\right]ds+2\int_t^T\left|\widetilde{y}_s^{\theta}\right|Z_sdB_s^H.
\end{aligned}$$

With some Taylor expansion with integral remain of $f\left(s,x_s^{\theta},y_s^{\theta},z_s^{\theta},u_s^{\theta}\right)$ at (x,y,z,u), from (2.19) we get the expressions of $\widetilde{x}_t^{\theta}$, $\widetilde{y}_t^{\theta}$ and $\widetilde{z}_t^{\theta}$, and inject them in the last equality, it comes

$$\begin{aligned}
&\mathbb{E}\left[\left|\widetilde{y}_t^\theta\right|^2\right]+\mathbb{E}\left[\int_t^T Hs^{2H-1}\left|\widetilde{z}_s^\theta\right|^2 ds\right]\\
&=2\mathbb{E}\left[\int_t^T\int_0^1\left|\widetilde{y}_s^\theta\right|\left(f_x\left(\mathcal{A}_s^{\theta,\lambda}\right)\left(\widetilde{x}_s^\theta+X_s\right)+f_y\left(\mathcal{A}_s^{\theta,\lambda}\right)\left(\widetilde{y}_s^\theta+Y_s\right)\right.\right.\\
&\left.\left.+f_z\left(\mathcal{A}_s^{\theta,\lambda}\right)\left(\widetilde{z}_s^\theta+Z_s\right)+f_u\left(\mathcal{A}_s^{\theta,\lambda}\right)v_s\right)d\lambda ds\right]\\
&-2\mathbb{E}\left[\int_t^T\left|\widetilde{y}_s^\theta\right|\left(f_x\left(s,x_s,y_s,z_s,u_s\right)X_s+f_y\left(s,x_s,y_s,z_s,u_s\right)Y_s\right.\right.\\
&\left.\left.+f_z\left(s,x_s,y_s,z_s,u_s\right)Z_s+f_u\left(s,x_s,y_s,z_s,u_s\right)v_s\right)ds\right].
\end{aligned}$$

Applying Young's inequality

$$\begin{aligned}
&\mathbb{E}\left[\left|\widetilde{y}_t^\theta\right|^2\right]+\mathbb{E}\left[\int_t^T Hs^{2H-1}\left|\widetilde{z}_s^\theta\right|^2 ds\right]\\
&\leq\mathbb{E}\left[\int_t^T\int_0^1\left|\widetilde{y}_s^\theta\right|^2 f_x^2\left(\mathcal{A}_s^{\theta,\lambda}\right)d\lambda ds\right]+2\mathbb{E}\left[\int_t^T\int_0^1\left|\widetilde{y}_s^\theta\right|^2 f_y\left(\mathcal{A}_s^{\theta,\lambda}\right)d\lambda ds\right]\\
&+\mathbb{E}\left[\int_t^T\int_0^1\left|\widetilde{y}_s^\theta\right|^2 f_z^2\left(\mathcal{A}_s^{\theta,\lambda}\right)d\lambda ds\right]+\mathbb{E}\left[\int_t^T\int_0^1\left|\widetilde{z}_s^\theta\right|^2 d\lambda ds\right]+\mathbb{E}\left[\rho^\theta\right]\\
&\leq\mathbb{E}\left[\int_t^T\varphi^\theta\left|\widetilde{y}_s^\theta\right|^2 ds\right]+\mathbb{E}\left[\int_t^T\left|\widetilde{z}_s^\theta\right|^2 ds\right]+\mathbb{E}\left[\rho^\theta\right],
\end{aligned}$$

where

$$\begin{aligned}
\varphi^\theta &= \int_0^1\left(f_x^2\left(\mathcal{A}_s^{\theta,\lambda}\right)+2f_y\left(\mathcal{A}_s^{\theta,\lambda}\right)+f_z^2\left(\mathcal{A}_s^{\theta,\lambda}\right)\right)d\lambda,\ \text{and}\\
\rho^\theta &= 2\int_t^T\int_0^1\left|\widetilde{y}_s^\theta\right|\left[f_x\left(\mathcal{A}_s^{\theta,\lambda}\right)X_s+f_y\left(\mathcal{A}_s^{\theta,\lambda}\right)Y_s+f_z\left(\mathcal{A}_s^{\theta,\lambda}\right)Z_s\right.\\
&+ \left.f_u\left(\mathcal{A}_s^{\theta,\lambda}\right)v_s d\lambda ds\right]+\int_t^T\int_0^1\left|\widetilde{x}_s^\theta\right|^2 d\lambda ds\\
&- 2\int_t^T\left|\widetilde{y}_s^\theta\right|\left[f_x\left(s,x_s,y_s,z_s,u_s\right)X_s+f_y\left(s,x_s,y_s,z_s,u_s\right)Y_s\right.\\
&+ \left.f_z\left(s,x_s,y_s,z_s,u_s\right)Z_s+f_u\left(s,x_s,y_s,z_s,u_s\right)v_s\right]ds.
\end{aligned}$$

We recall that K is defined in (2.13), we arrive at

$$\mathbb{E}\left[\left|\widetilde{y}_t^\theta\right|^2\right]+\mathbb{E}\left[\int_t^T K\left|\widetilde{z}_s^\theta\right|^2 ds\right]\leq\mathbb{E}\left[\int_t^T\varphi^\theta\left|\widetilde{y}_s^\theta\right|^2 ds\right]+\mathbb{E}\left[\rho^\theta\right].$$

On the one hand, we have $\mathbb{E}\left[\left|\widetilde{y}_t^\theta\right|^2\right] \leq \mathbb{E}\left[\int_t^T \varphi^\theta \left|\widetilde{y}_s^\theta\right|^2 ds\right] + \mathbb{E}\left[\rho^\theta\right]$.
By using the Gronwall's inequality obtain to: $\mathbb{E}\left[\left|\widetilde{y}_t^\theta\right|^2\right] \leq \mathbb{E}\left[\rho^\theta\right] \exp\left\{\varphi^\theta T\right\}$,
from hypothesis **(H4)**, with Lebesgue bounded convergence theorem, we can simply show that $\mathbb{E}\left[\rho^\theta\right] \underset{\theta\to 0}{\to} 0$, hence, it comes $\mathbb{E}\left[\sup_{t\in[0,T]} \left|\widetilde{y}_t^\theta\right|^2\right] \underset{\theta\to 0}{\to} 0$.
On the other hand

$$\mathbb{E}\left[\int_t^T K \left|\widetilde{z}_s^\theta\right|^2 ds\right] \leq \mathbb{E}\left[\int_t^T \varphi^\theta \left|\widetilde{y}_s^\theta\right|^2 ds\right] + \mathbb{E}\left[\rho^\theta\right] \underset{\theta\to 0}{\to} 0.$$

Summing up the two last above inequalities we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\widetilde{y}_t^\theta\right|^2\right] + \mathbb{E}\left[\int_t^T \left(Hs^{2H-1} - 1\right) \left|\widetilde{z}_s^\theta\right|^2 ds\right] \underset{\theta\to 0}{\to} 0.$$

■

Lemma 2.4 *Under assumptions **(H3)** to **(H5)**, we have the following limits*

$$\frac{1}{\theta}\mathbb{E}\left[l\left(x_T^\theta\right) - l\left(x_T\right)\right] \underset{\theta\to 0}{\to} \mathbb{E}\left[l_x\left(x_T\right) X_T\right], \tag{2.24}$$
$$\frac{1}{\theta}\mathbb{E}\left[g\left(y_0^\theta\right) - g\left(y_0\right)\right] \underset{\theta\to 0}{\to} \mathbb{E}\left[g_y\left(y_0\right) Y_0\right], \tag{2.25}$$
$$\frac{1}{\theta}\mathbb{E}\left[h^\theta\left(t\right) - h\left(t\right)\right] \underset{\theta\to 0}{\to} \mathbb{E}\left[\int_0^T \left(h_x\left(t\right) X_t + h_y\left(t\right) Y_t + h_z\left(t\right) Z_t + h_u\left(t\right) v_t\right) dt\right]. \tag{2.26}$$

Proof. We apply Taylor's expansion with integral remain to the functions $g\left(y_0^\theta\right)$ and $l\left(x_T^\theta\right)$ at y_0 and x_T, respectively, we get

$$l\left(x_T^\theta\right) - l\left(x_T\right) = \int_0^1 l_x\left(x_T + \lambda\theta\left(\widetilde{x}_T^\theta + X_T\right)\right)\theta\left(\widetilde{x}_T^\theta + X_T\right) d\lambda,$$

and

$$g\left(y_0^\theta\right) - g\left(y_0\right) = \int_0^1 g_y\left(y_0 + \lambda\theta\left(\widetilde{y}_0^\theta + Y_0\right)\right)\theta\left(\widetilde{y}_0^\theta + Y_0\right) d\lambda.$$

Then we have

$$\begin{aligned}\frac{1}{\theta}\mathbb{E}\left[l\left(x_T^\theta\right)-l\left(x_T\right)\right]&=\mathbb{E}\left[\int_0^1 l_x\left(x_T+\lambda\theta\left(\widetilde{x}_T+X_T\right)\right)\widetilde{x}_T^\theta d\lambda\right]\\&+\mathbb{E}\left[\int_0^1 l_x\left(x_T+\lambda\theta\left(\widetilde{x}_T+X_T\right)\right)X_T d\lambda\right],\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\theta}\mathbb{E}\left[g\left(y_0^\theta\right)-g\left(y_0\right)\right]&=\mathbb{E}\left[\int_0^1 g_y\left(y_0+\lambda\theta\left(\widetilde{y}_0^\theta+Y_0\right)\right)\widetilde{y}_0^\theta d\lambda\right]\\&+\mathbb{E}\left[\int_0^1 g_y\left(y_0+\lambda\theta\left(\widetilde{y}_0^\theta+Y_0\right)\right)Y_0 d\lambda\right].\end{aligned}$$

Since l_x and g_y bounded and continuous (from **(H5)**), and from Cauchy-Schwarz's inequality, then we apply Lebesgue's bounded convergence theorem, and from (2.20) and (2.21) , we have

$$\mathbb{E}\left[\int_0^1 l_x\left(x_T+\lambda\theta\left(\widetilde{x}_T^\theta+X_T\right)\right)\widetilde{x}_T^\theta d\lambda\right]\underset{\theta\to 0}{\to}0,$$

and

$$\mathbb{E}\left[\int_0^1\left[g_y\left(y_0+\lambda\theta\left(\widetilde{y}_0^\theta+Y_0\right)\right)\widetilde{y}_0^\theta d\lambda\right]\right]\underset{\theta\to 0}{\to}0.$$

On the other hand, we have

$$\mathbb{E}\left[\int_0^1 l_x\left(x_T+\lambda\theta\left(\widetilde{x}_T^\theta+X_T\right)\right)X_T d\lambda\right]\underset{\theta\to 0}{\to}\mathbb{E}\left[l_x\left(x_T\right)X_T\right],$$

and

$$\mathbb{E}\left[\int_0^1 g_y\left(y_0+\lambda\theta\left(\widetilde{y}_0^\theta+Y_0\right)\right)Y_0 d\lambda\right]\underset{\theta\to 0}{\to}\mathbb{E}\left[g_y\left(y_0\right)Y_0\right].$$

Therefore

$$\frac{1}{\theta}\mathbb{E}\left[l\left(x_T^\theta\right)-l\left(x_T\right)\right]\underset{\theta\to 0}{\to}\mathbb{E}\left[l_x\left(x_T\right)X_T\right],$$

and

$$\frac{1}{\theta}\mathbb{E}\left[g\left(y_0^\theta\right)-g\left(y_0\right)\right]\underset{\theta\to 0}{\longrightarrow}\mathbb{E}\left[g_y\left(y_0\right)Y_0\right].$$

Analogously, we find

$$\begin{aligned}h^\theta(t)-h(t)=&\int_0^1 h_x\left(\mathcal{A}_t^{\theta,\lambda}\right)\left(x_t^\theta-x_t\right)d\lambda+\int_0^1 h_y\left(\mathcal{A}_t^{\theta,\lambda}\right)\left(y_t^\theta-y_t\right)d\lambda\\&+\int_0^1 h_z\left(\mathcal{A}_t^{\theta,\lambda}\right)\left(z_t^\theta-z_t\right)d\lambda+\int_0^1 h_u\left(\mathcal{A}_t^{\theta,\lambda}\right)\left(u_t^\theta-u_t\right)d\lambda.\end{aligned}$$

By taking the mathematical expectation, using (2.5) and (2.19), we obtain

$$\begin{aligned}\frac{1}{\theta}\mathbb{E}\left[h^\theta(t)-h(t)\right]=&\,\mathbb{E}\left[\int_0^1 h_x\left(\mathcal{A}_t^{\theta,\lambda}\right)\widetilde{x}_t^\theta d\lambda\right]+\mathbb{E}\left[\int_0^1 h_x\left(\mathcal{A}_t^{\theta,\lambda}\right)X_t d\lambda\right]\\&+\mathbb{E}\left[\int_0^1 h_y\left(\mathcal{A}_t^{\theta,\lambda}\right)\widetilde{y}_t^\theta d\lambda\right]+\mathbb{E}\left[\int_0^1 h_y\left(\mathcal{A}_t^{\theta,\lambda}\right)Y_t d\lambda\right]\\&+\mathbb{E}\left[\int_0^1 h_z\left(\mathcal{A}_t^{\theta,\lambda}\right)\widetilde{z}_t^\theta d\lambda\right]+\mathbb{E}\left[\int_0^1 h_z\left(\mathcal{A}_t^{\theta,\lambda}\right)Z_t d\lambda\right]\\&+\mathbb{E}\left[\int_0^1 h_u\left(\mathcal{A}_t^{\theta,\lambda}\right)v_t d\lambda\right].\end{aligned}$$

Letting θ tend to 0, and according the assumption on the function h in assumption 2.1, we apply Lesbesgue's bounded convergence theorem, we get

$$\frac{1}{\theta}\mathbb{E}\left[h^\theta(t)-h(t)\right]\underset{\theta\to 0}{\longrightarrow}\mathbb{E}\left[\int_0^T\left(h_x(t)X_t+h_y(t)Y_t+h_z(t)Z_t+h_u(t)v_t\right)dt\right].$$

which is the result shown in 2.26. ∎

2.3 Stochastic maximum principle

In this section, we introduce our primary finding: the Pontryagin stochastic maximum principle for a fractional Brownian motion driven FBSDE, where the Hurst parameter $H\in\,]0,1[$. We will establish this result in terms of both necessary and sufficient optimality conditions, addressing an optimal control in the stochastic optimization problem (2.2), (2.3) and (2.4). Our approach begins by defining the Hamiltonian functional related to the problem, as outlined in the subsequent definition.

Definition 2.3 *The Hamiltonian functional $\mathcal{H}$, associated to the control problem* (2.2), (2.3), *and* (2.4), *is defined by*

$$\begin{aligned}\mathcal{H}\left(t,x_t,y_t,z_t,u_t,p_t,P_t,q_t\right) &= h\left(t,x_t,y_t,z_t,u_t\right)+p_t f\left(t,x_t,y_t,z_t,u_t\right)+q_t b\left(t,x_t,u_t\right)\\ &\quad+P_t\sigma\left(t,x_t,u_t\right).\end{aligned} \tag{2.27}$$

where $\mathcal{H}\left(t,x_t,y_t,z_t,u_t,p_t,P_t,q_t\right):[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathcal{U}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

For the sake of simplicity, we introduce the following notations.

Notation 2.2 *For* $\tau\in\{u,v\}$, $\kappa\in\{x,y,z,u\}$, $\phi\in\{f,h\}$, $\Pi\in\{b,\sigma\}$, *we note*

$$\begin{aligned}\mathcal{H}^{\tau}\left(t,\iota\right) &= \mathcal{H}\left(t,x_t^u,y_t^u,z_t^u,\tau_t,p_t^u,P_t^u,q_t^u\right),\ \mathcal{H}_\kappa\left(t,\iota\right)=\frac{\partial\mathcal{H}}{\partial\kappa}\left(t,\iota\right),\\ \mathcal{H}\left(t,\iota\right) &= \mathcal{H}\left(t,x_t^u,y_t^u,z_t^u,u_t,p_t^u,P_t^u,q_t^u\right),\ \mathcal{H}_\kappa\left(t\right)=\frac{\partial\mathcal{H}}{\partial\kappa}\left(t\right),\\ \phi_\kappa\left(t,\varsigma\right) &= \phi_\kappa\left(t,x_t^u,y_t^u,z_t^u,u_t\right),\ \Pi_\kappa\left(t,\varrho\right)=\Pi_\kappa\left(t,x_t^u,u_t\right).\end{aligned}$$

Combining the definition of the Hamiltonian and the previous notations 2.2, we introduce the adjoint dynamics.

$$\begin{cases} dp_t^u &= \mathcal{H}_y\left(t,\iota\right)dt+\dfrac{1}{Ht^{2H-1}}\mathcal{H}_z\left(t,\iota\right)dB_t^H\\ &= \left\{h_y\left(t,\varsigma\right)+p_t^u f_y\left(t,\varsigma\right)\right\}dt+\dfrac{1}{Ht^{2H-1}}\left\{h_z\left(t,\varsigma\right)+p_t^u f_z\left(t,\varsigma\right)\right\}dB_t^H,\\ dq_t^u &= -\mathcal{H}_x\left(t\right)dt+\dfrac{1}{Ht^{2H-1}}P_t^u dB_t^H\\ &= -\left\{h_x\left(t,\varsigma\right)+p_t^u f_x\left(t,\varsigma\right)+q_t^u b_x\left(t,\varrho\right)+P_t^u\sigma_x\left(t,\varrho\right)\right\}dt\\ &+ \dfrac{1}{Ht^{2H-1}}P_t^u dB_t^H\\ p_0^u &= g_y\left(y_0^u\right),\quad \text{and } q_T^u=l_x\left(x_T^u\right).\end{cases} \tag{2.28}$$

Remark 2.2 *From lemma 2.4, by virtue of the convergences* (2.24), (2.25) *and* (2.26), *the variational inequality* (2.7) *becomes*

$$0 \leq \mathbb{E}\left[l_x\left(x_T\right)X_T\right]+\mathbb{E}\left[g_y\left(y_0\right)Y_0\right]+\mathbb{E}\left[\int_0^T\left\{h_x\left(t\right)X_t+h_y\left(t\right)Y_t+h_z\left(t\right)Z_t+h_u\left(t\right)v_t\right\}dt\right]. \tag{2.29}$$

2.3.1 A necessary maximum principle

We are now able to express the necessary optimality conditions for the stochastic control problem (2.2), (2.3), and (2.4).

Theorem 2.1 *(Necessary optimality conditions) Let* $(x_t^u, y_t^u, z_t^u, u_t)$ *be the optimal solution to our problem, then there exists a unique adapted process* (p, P, q) *solution to the differential equation* (2.28) *and the variational inequality*

$$\mathbb{E}\left[\int_0^T \mathcal{H}_u\left(t, x_t^u, y_t^u, z_t^u, u_t, p_t^u, P_t^u, q_t^u\right)\left(v_t-u_t\right)dt\right] \geq 0, \tag{2.30}$$

holds for all $v \in \mathcal{U}, \mathbb{P}.a.s.$

Proof. Let u be optimal for $\{(2.2),(2.3),(2.4)\}$ and v some random control from $\mathcal{U}$. We apply the integration by parts formula to p_tY_t, then take the expression of dY_t and dp_t from (2.16) and (2.28), respectively, we obtain

$$\mathbb{E}\left[p_0^uY_0^u\right]=\mathbb{E}\left[\int_0^T\left\{-Y_t^{u}h_y\left(t,\varsigma\right)-Z_t^uh_z\left(t,\varsigma\right)+p_t^uf_x\left(t,\varsigma\right)X_t^u+p_t^uf_u\left(t,\varsigma\right)v_t\right\}dt\right]. \tag{2.31}$$

Similarly, applying the integration by parts formula to $q_t X_t$, and recalling dX_t and dq_t from (2.15) and (2.28), respectively, it comes

$$\begin{aligned}\mathbb{E}\left[q_T^u X_T^u\right] &= \mathbb{E}\left[\int_0^T \{-X_t^u \mathcal{H}_x(t,\iota) + q_t^u b_x(t,\varrho) X_t^u + q_t^u b_u(t,\varrho) v_t + P_t^u \sigma_x(t,\varrho) X_t^u \right. \\ &\quad \left. + P_t^u \sigma_u(t,\varrho) v_t\}\right]\end{aligned} . \tag{2.32}$$

Since we have $p_0^u = g_y(y_0^u)$ and $q_T^u = l_x(x_T^u)$, by substituting equations (2.31) and (2.32) into the variational inequality (2.29) introduced in remark 2.2, we get directly $\mathbb{E}\left[\int_0^T \mathcal{H}_u(t) v_t dt\right] \geq 0$. Due to the convexity of the set $\mathcal{U}$, we may choose some perturbed control $u_t^\theta = u_t + \theta(v_t - u_t) \in \mathcal{U}$, as u is optimal, we get

$$\mathbb{E}\left[\int_0^T \mathcal{H}_u(t)(v_t - u_t) dt\right] \geq 0.$$

which leads to the result. ■

2.3.2 A sufficient maximum principle

After establishing the necessary optimality condition in Theorem 2.1, this paragraph delves in which the condition (2.30) transitions from being necessary to sufficient. The principal result is presented in the following theorem. Prior to that, we provide the next notation.

Notation 2.3 *For* $\tau \in \{u, v\}$, $\kappa \in \{x, y, z, u\}$, $\phi \in \{f, h\}$, $\Pi \in \{b, \sigma\}$, *we propose*

$$\Pi^\tau(t,\varrho) = \Pi(t, x_t^u, \tau_t), \ \ \phi^\tau(t,\varsigma) = \phi(t, x_t^u, y_t^u, z_t^u, \tau_t).$$

Assumption 2.2 *To drive the sufficient optimality condition, first, we suppose the following:*

1. *The function l and g are convex with respect to x and y, respectively.*

2. *The Hamiltonian $\mathcal{H}$ is convex with respect to (x, y, z, u).*

Theorem 2.2 (Sufficient optimality condition) *Assume the assumptions* 2.2 *are satisfied. Then u is the optimal control for the problem* (2.2), (2.3), *and* (2.4), *if the variational inequality* (2.30) *holds for all $v \in \mathcal{U}$ $\mathbb{P}$.a.s.*

Proof. Let $v \in \mathcal{U}$ and $u \in \mathcal{U}$ candidate to be optimal, by using the fact that the functions g and l are convex in y and x, respectively, taking into consideration that $p_0^u = g_y(y_0^u)$ and $q_T^u = l_x(x_T^u)$, get

$$\begin{aligned} g(y_0^v) - g(y_0^u) &\geq g_y(y_0^v)(y_0^v - y_0^u) = p_0(y_0^v - y_0^u), \\ \text{and } l(x_T^v) - l(x_T^u) &\geq l_x(x_T^v)(x_T^v - x_T^u) = q_T(x_T^v - x_T^u). \end{aligned} \tag{2.33}$$

Recall notations 2.1 and 2.2, from (2.3), the difference between $\mathcal{J}(v)$ and $\mathcal{J}(u)$ becomes

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \mathbb{E}\left[\int_0^T (h^v(t,\varsigma) - h^u(t,\varsigma))\,dt + q_T^u(x_T^v - x_T^u) + p_0^u(y_0^v - y_0^u)\right]. \tag{2.34}$$

Using integration by parts formula to $p_t^u(y_t^v - y_t^u)$

$$\begin{aligned} d(p_t^u(y_t^v - y_t^u)) &= (y_t^v - y_t^u)\,dp_t^u + p_t^u d(y_t^v - y_t^u) + d\langle p_t^u, (y_t^v - y_t^u)\rangle \\ &= \mathcal{H}_y(t,\iota)(y_t^v - y_t^u)\,dt + (y_t^v - y_t^u)\frac{1}{Ht^{2H-1}}\mathcal{H}_z(t,\iota)\,dB_t^H \\ &\quad - p_t^u f^v(t,\varsigma)\,dt + p_t^u f^u(t,\varsigma)\,dt + p_t^u(z_t^v - z_t^u)\,dB_t^H \\ &\quad + \mathcal{H}_z(t,\iota)(z_t^v - z_t^u)\,dt. \end{aligned}$$

That implies

$$\begin{aligned} \mathbb{E}[p_0^u(y_0^v - y_0^u)] &= -\mathbb{E}\left[\int_0^T \mathcal{H}_y(t,\iota)(y_t^v - y_t^u)\,dt\right] - \mathbb{E}\left[\int_0^T \mathcal{H}_z(t,\iota)(z_t^v - z_t^u)\,dt\right] \\ &\quad + \mathbb{E}\left[\int_0^T p_t^u f^v(t,\varsigma)\,dt\right] - \mathbb{E}\left[\int_0^T p_t^u f^u(t,)\,dt\right], \end{aligned} \tag{2.35}$$

also, using integration by parts formula to $q_t^u\left(x_t^v - x_t^u\right)$

$$\begin{aligned} d\left(q_t^u\left(x_t^v - x_t^u\right)\right) &= \left(x_t^v - x_t^u\right) dq_t^u + q_t^u d\left(x_t^v - x_t^u\right) + d\left\langle q_t^u, \left(x_t^v - x_t^u\right)\right\rangle \\ &= -\mathcal{H}_x\left(t,\iota\right)\left(x_t^v - x_t^u\right) dt + \left(x_t^v - x_t^u\right)\frac{1}{Ht^{2H-1}} P_t^u dB_t^H \\ &+ q_t^u\left(b^v\left(t,\varrho\right) - b^u\left(t,\varrho\right)\right) dt + q_t^u\left(\sigma^v\left(t,\varrho\right) - \sigma^u\left(t,\varrho\right)\right) dB_t^H \\ &+ P_t^u\left(\sigma^v\left(t,\varrho\right) - \sigma^u\left(t,\varrho\right)\right) dt. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}\left[q_T^u\left(x_T^v - x_T^u\right)\right] &= -\mathbb{E}\left[\int_0^T \mathcal{H}_x\left(t,\iota\right)\left(x_t^v - x_t^u\right) dt\right] + \mathbb{E}\left[\int_0^T q_t^u\left(b^v\left(t,\varrho\right) - b^u\left(t,\varrho\right)\right) dt\right] \\ &+ \mathbb{E}\left[\int_0^T P_t^u\left(\sigma^v\left(t,\varrho\right) - \sigma^u\left(t,\varrho\right)\right) dt\right]. \end{aligned} \tag{2.36}$$

By injecting (2.35) and (2.36) in (2.34) , we obtain

$$\begin{aligned} \mathcal{J}\left(v\right) - \mathcal{J}\left(u\right) \geq \mathbb{E}&\left[\int_0^T \left\{h^v\left(t,\varsigma\right) + p_t f^v\left(t,\varsigma\right) + q_t b^v\left(t,\varrho\right) + P_t\sigma^v\left(t,\varrho\right)\right\} dt\right] \\ - \mathbb{E}&\left[\int_0^T \left\{h^u\left(t,\varsigma\right) + p_t f^u\left(t,\varsigma\right) + q_t b^u\left(t,\varrho\right) + P_t\sigma^u\left(t,\varrho\right)\right\} dt\right] \\ - \mathbb{E}&\left[\int_0^T \mathcal{H}_y\left(t,\iota\right)\left(y_t^v - y_t^u\right) dt\right] - \mathbb{E}\left[\int_0^T \mathcal{H}_z\left(t,\iota\right)\left(z_t^v - z_t^u\right) dt\right] \\ - \mathbb{E}&\left[\int_0^T \mathcal{H}_x\left(t,\iota\right)\left(x_t^v - x_t^u\right) dt\right] \end{aligned}$$

recalling the definition of the Hamiltonian from 2.3, the above inequality becomes

$$\begin{aligned} \mathcal{J}\left(v\right) - \mathcal{J}\left(u\right) \geq \mathbb{E}&\left[\int_0^T \mathcal{H}^v\left(t,\iota\right) dt\right] - \mathbb{E}\left[\int_0^T \mathcal{H}^u\left(t,\iota\right) dt\right] \\ - \mathbb{E}&\left[\int_0^T \mathcal{H}_y\left(t,\iota\right)\left(y_t^v - y_t^u\right) dt\right] - \mathbb{E}\left[\int_0^T \mathcal{H}_z\left(t,\iota\right)\left(z_t^v - z_t^u\right) dt\right] \\ - \mathbb{E}&\left[\int_0^T \mathcal{H}_x\left(t,\iota\right)\left(x_t^v - x_t^u\right) dt\right]. \end{aligned}$$

More than we have the Hamiltonian function is convex, we use this property in the last inequality, and according to the simplification and the assumption of the theorem (the condition 2.30), we get which assures

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \mathbb{E}\left[\int_0^T \mathcal{H}_u(t, \iota)(v_t - u_t)\, dt\right] \geq 0.$$

Hence $\mathcal{J}(v) - \mathcal{J}(u) \geq 0$. Then u is an optimal control. ■

2.4 LQ problem

In this section, as an application of our result, we consider a one-dimensional linear quadratic (LQ) control problem for Stochastic Maximum principle, We consider the linear forward backward stochastic differential dynamics

$$\begin{cases} dx_t &= xdt + xdB_t^H, \\ -dy_t &= (y_t + z_t + \upsilon_t)\, dt - z_t dB_t^H, \\ x_0 &= \varkappa \text{ and } y_T^v = \xi,\ t \in [0, T]. \end{cases} \tag{2.37}$$

We associate this system (2.37) with the following linear quadratic functional cost

$$\mathcal{J}(\upsilon) = \mathbb{E}\left[\int_0^T h(t, x_t^v, y_t^v, z_t^v, v_t)\, dt + l(x_T^v)\right], \tag{2.38}$$

where

$$h(t, x_t^v, y_t^v, z_t^v, v_t) = -\frac{1}{2}\left(x^2 + y_t^2 + z_t^2 + v_t^2\right) \text{ and } l(x_T^v) = \frac{1}{2}x_T^2.$$

that we want to minimize (2.38) over the set $\mathcal{U}$ and find some optimal control u, that satisfies

$$\mathcal{J}(u) = \inf_{v \in \mathcal{U}} \mathcal{J}(v). \tag{2.39}$$

The triplet $\{(2.37), (2.38), (2.39)\}$ forms our linear quadratic control problem. From the definition 2.3, the stochastic Hamiltonian takes the form

$$\begin{aligned}\mathcal{H}(t, x_t, y_t, z_t, u_t, p_t, P_t, q_t) &= -\frac{1}{2}\left(x^2 + y_t^2 + z_t^2 + u_t^2\right) + p_t\left(y_t + z_t + u_t\right) \\ &+ q_t x_t + P_t x_t.\end{aligned}$$

As (2.28), the triplet (p^u, P^u, q^u) is the adjoint process, solution of the system

$$\begin{cases} dp_t^u = -\{y_t^u - p_t^u\}\, dt + \dfrac{1}{Ht^{2H-1}}\{-z^u + p_t^u\}\, dB_t^H, \\ dq_t^u = -\{x_t^u - q_t^u - P_t^u\}\, dt + \dfrac{1}{Ht^{2H-1}} P_t^u dB_t^H, \\ p_0^u = 0 \text{ and } q_T^u = x_T^u. \end{cases} \tag{2.40}$$

We minimize the Hamiltonian functional with respect to u over $\mathcal{U}$. We have

$$\mathcal{H}_u(t) = -\widehat{u}_t + \widehat{p}_t = 0, \tag{2.41}$$

then

$$\widehat{u}_t = \widehat{p}_t. \tag{2.42}$$

In view of theorem 2.1 and the convexity of $l(x_T) = \dfrac{1}{2}x_T^2$ and $\mathcal{H}(t)$ in x_T and (x, y, z, u), respectively, $\widehat{u}_t$ is optimal. In what follows, we derive a feedback state of such control (2.41). We proceed as Young [82], and introduce the adjoint process p_t, solution of (2.40), as a linear combination

$$\widehat{p}_t = \alpha_t \widehat{y}_t + \beta_t, \tag{2.43}$$

with α_t and β_t are two deterministic differentiable functions.

Differentiating (2.43) then using (2.37), (2.42) and (2.43) yield to

$$
\begin{aligned}
d\widehat{p}_t &= \dot{\alpha}_t \widehat{y}_t dt + \alpha_t d\widehat{y}_t + \dot{\beta}_t dt \\
&= \dot{\alpha}_t \widehat{y}_t dt - \alpha_t \left(\widehat{y}_t + \widehat{z}_t + \widehat{u}_t\right) dt + \alpha_t \widehat{z}_t dB_t^H + \dot{\beta}_t dt \\
&= \dot{\alpha}_t \widehat{y}_t dt - \alpha_t \left(\widehat{y}_t + \widehat{z}_t + \alpha_t \widehat{y}_t + \beta_t\right) dt + \alpha_t \widehat{z}_t dB_t^H + \dot{\beta}_t dt \\
&= \left\{\widehat{y}_t \left(\dot{\alpha}_t - \alpha_t - \alpha_t^2\right) - \alpha_t \widehat{z}_t - \alpha_t \beta_t + \dot{\beta}_t\right\} dt + \alpha_t \widehat{z}_t dB_t^H. \qquad (2.44)
\end{aligned}
$$

On the other hand, injecting (2.37) and (2.43) in (2.40) , then $d\widehat{p}_t$ becomes

$$
d\widehat{p}_t = \left\{-\widehat{y}_t + \alpha_t \widehat{y}_t + \beta_t\right\} dt + \frac{1}{Ht^{2H-1}} \left\{-\widehat{z} + \alpha_t \widehat{y}_t + \beta_t\right\} dB_t^H. \qquad (2.45)
$$

Identifying diffusion terms of (2.44) and(2.45) yields to $\widehat{z}_t = \frac{\alpha_t \widehat{y}_t + \beta_t}{\alpha_t Ht^{2H-1}+1}$.

Identifying the drift terms, we get

$$
\left\{ \dot{\alpha}_t = \alpha_t^2 + 2\alpha_t - 1, \quad \alpha_0 = 0, \right. \qquad (2.46)
$$

and

$$
\left\{ \dot{\beta}_t - (1 + \alpha_t)\, \beta_t = \alpha_t \widehat{z}_t, \quad \beta_0 = 0. \right. \qquad (2.47)
$$

The equation (2.46) is a Riccati equation and (2.47) is an ordinary differential equation (ODE in short).

2.4.1 Integrating Riccati and ordinary differential equations

This part is devoted to finding the explicit solution of the Riccati differential equation (2.46), then the ordinary differential equation 2.47. At first, applying some simple algebra to (2.46) , we obtain

$$
\frac{dt}{d\alpha_t} = \frac{1}{2\sqrt{2}} \left[\frac{1}{\alpha_t - \alpha_1} - \frac{1}{\alpha_t - \alpha_2}\right], \qquad (2.48)
$$

where $\alpha_1 = \sqrt{2} - 1$ and $\alpha_2 = -\sqrt{2} - 1$. Integrating (2.48) from 0 to t, it comes

$$2t\sqrt{2} = \left(\ln|(\alpha_s - \alpha_1)| - \ln|(\alpha_s - \alpha_2)|\right)\Big|_0^t.$$

Since $\alpha_0 = 0$, and by simple simplifications, we obtain $\frac{\alpha_t - \alpha_1}{\alpha_t - \alpha_2} = \frac{\alpha_1}{\alpha_2} \exp\left\{2t\sqrt{2}\right\}$.

Finally, the solution of the Riccati equation (2.46) is

$$\alpha_t = \frac{\alpha_1\left(1 - \exp\left\{2t\sqrt{2}\right\}\right)}{1 - \frac{\alpha_1}{\alpha_2}\exp\left\{2t\sqrt{2}\right\}}. \tag{2.49}$$

On the other hand, the explicit solution of the ODE (2.47) is

$$\beta_t = \eta \exp\{-\varrho(t)\} + \int_0^t \alpha_s \widehat{z}_s \exp\{\varrho(s) - \varrho(t)\}\, ds, \tag{2.50}$$

where α_t is introduced in (2.49), $\varrho(t)$ is the derivative of $-(1 + \alpha_t)$ and η is some real constant. Consequently, replacing (2.49) and (2.50) in (2.42), we have

$$\widehat{u}_t = \frac{\alpha_1\left(1 - \exp\left\{2\sqrt{2}t\right\}\right)}{1 - \frac{\alpha_1}{\alpha_2}\exp\left\{2\sqrt{2}t\right\}}\widehat{y}_t + \eta \exp\{-\varrho(t)\} + \int_0^t \alpha_s \widehat{z}_s \exp\{\varrho(s) - \varrho(t)\}\, ds. \tag{2.51}$$

Corollary 2.1 *The solution to the Riccati equation* (2.46) *is explicitly given by* (2.49)*. Moreover, The equation* (2.47) *has an explicit solution represented by* (2.50)*.*

Corollary 2.2 *If equations* (2.46) *and* (2.47) *each have solutions denoted as* $\alpha(.)$ *and* $\beta(.)$ *respectively, then the feedback control (2.42) in our linear quadratic stochastic optimal control problem* $\{(2.37), (2.38)\}$ *yields optimality.*

Theorem 2.3 *If the equations* (2.46) *and* (2.47) *admits the solution* $\alpha(.)$ *and* $\beta(.)$ *given by the expressions* (2.49) *and* (2.50) *, respectively, then the optimal control of the problem of linear quadratic stochastic has the state feedback form* (2.51)*.*

Chapter 3

A Risk-Sensitive Stochastic Maximum Principle for Backward Stochastic Differential Equation with Application

In this chapter, we present our second result of this thesis: Pontryagin's Control Problem of risk-sensitive for fractional backward stochastic differential equation with application. We aim to establish both necessary and sufficient optimality conditions for a system under risk-sensitive control problem. We achieve this goal by employing Pontryagin's maximum principle and the system's dynamics are described by a backward stochastic differential equation (BSDE) driven by a fractional Brownian motion (fBm) with a Hurst parameter $H \in \left]0,1\right[$.

Our strategy to solving the problem is inspired from the approach outlined by Djehiche et al. in their publication [23]. Their methodology considers not only the state and control variables but also incorporates the means of the state process within the drift, diffusion, and initial cost functions. In the context of risk-sensitive control, the system is influenced by a nonlinear BSDE driven by an fBm $dy_t = -f(t, y_t, z_t, \upsilon_t)\,dt + z_t dB_t^H$, where $y_T = \xi$. The criterion to be minimized, with initial risk-sensitive cost is: $\mathcal{J}^{\varepsilon}(\upsilon) = \mathbb{E}\left[\exp\varepsilon\left(\int_0^T h(t, y_t, z_t, \upsilon_t)\,dt + g(y_0)\right)\right]$. A control u is called optimal if it solves $\mathcal{J}^{\varepsilon}(u) = \inf_{\upsilon\in\mathcal{U}} \mathcal{J}^{\varepsilon}(\upsilon)$.

We have been effectively addresses the existence of an optimal solution to achieve the objectives outlined, and it establishes both necessary and sufficient optimality conditions for these models. The methodology employed entails a systematic transformation of the risk-sensitive control problem. Initially, it is reframed in the context of an augmented state process and a terminal payoff problem. This transformation considers the convexity of the control set. Following this, the intermediate first-order adjoint processes are streamlined into a simpler form. Necessary and sufficient optimality conditions are subsequently derived, utilizing the logarithmic transform as illustrated in Lemma 3.2. To be more accurate, the approach, as depicted in the study by Lim and Zhou [57], indicates that employing a square-integral martingale suffices for converting the pair (p_1, q_1) into the adjoint process $(\widetilde{p}_1(t), 0)$, where $\widetilde{p}_1(t)$ remains a square integrable martingale. This suggests that $\widetilde{p}_1(t)$ equals $\widetilde{p}_1(T)$, which also equals the constant $\mathbb{E}[\widetilde{p}_1(T)]$. It's crucial to emphasize that while this generic martingale is not directly associated with the adjoint process $p(t)$ as detailed in Lim and Zhou, it plays a significant role in the adjoint equation linked with the risk-sensitive SMP, as indicated in Theorem 3.2 bellow.

This chapter is extracted from our second results [40] and is structured as follows: we start by present a comprehensive problem formulation, describe the risk-sensitive model, and define the assumptions that govern our approach in Section 3.1. Section 3.2 is devoted to examining our system of BSDE driven by an fBm and explains the rational behind our decision to convert the system into forward-backward stochastic differential equation. In Section 3.3, we introduce our initial main result, which consists of the necessary optimality conditions for the risk-sensitive control problem. Before that, we prove the relationship between the expected exponential utility and the quadratic backward stochastic differential equation. Moving on to Section 3.4, we present the second principal result, which pertains to sufficient optimality conditions for risk-sensitive performance. The last section we conclude this chapter with an application to the linear quadratic control problem.

3.1 Statement of risk-sensitive problem

Consider $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^H\right)_{t\varepsilon[0,T]}, \mathbb{P}\right)$ as a filtered probability space that satisfies the usual conditions. Let T be a strictly positive real number, we define B^H as one dimensional fBm with Hurst parameter H belonging to the interval 0 to 1, and U is a nonempty convex subset of $\mathbb{R}$. Subsequently, a controller influences the system through an $\mathcal{F}_t^H$-adapted stochastic process and satisfies $\mathbb{E}\left[\int_0^T |v_t|^2 \, dt\right] < \infty$. The collection of such controls is called admissible and is represented by $\mathcal{U}$, and we suppose it convex

Let $\mathcal{S}^2\left([0,T], \mathbb{R}\right)$ the set of one-dimensional continuous random $\mathcal{F}_t^H$-adapted processes $\{\phi_t, t \in [0,T]\}$ which satisfy $\|\phi\|_{\mathcal{S}^2}^2 = \mathbb{E}\left[\sup_{t\in[0,T]} |\phi_t|^2\right] < \infty$.

In the same way we denote by $\mathcal{M}^2\left([0,T], \mathbb{R}\right)$ the set of one dimensional jointly $\mathcal{F}_t^H$-adapted random processes which satisfy $\|\phi\|_{\mathcal{M}^2}^2 = \mathbb{E}\left[\int_0^T \left(Hs^{2H-1}\right)^2 |\phi_s|^2 \, ds\right] < \infty$.

Having introduced the primary tools and definitions of our framework in the previous chapters, now we examine the following controlled BSDE

$$\begin{cases} dy_t^v &= -f\left(t, y_t^v, z_t^v, v_t\right) dt + z_t^v dB_t^H, \\ y_T^v &= \xi, \; t \in [0,T], \end{cases} \tag{3.1}$$

where $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \rightarrow \mathbb{R}$, and the terminal condition ξ is an $\mathcal{F}_t^H-$measurable random variable such that $\mathbb{E}|\xi|^2 < \infty$, and v is an admissible control.

We define the criterion to be minimized, with initial risk-sensitive cost, as follows

$$\mathcal{J}^{\varepsilon}(v) = \mathbb{E}\left[\exp\varepsilon\left(g\left(y_0^v\right) + \int_0^T h\left(t, y_t^v, z_t^v, v_t\right) dt\right)\right], \tag{3.2}$$

where ε is the risk-sensitive index, $g : \mathbb{R} \to \mathbb{R}$, $h : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$.

The following assumptions concerning the driver f of the system (3.1) and the cost functional (3.2) are necessaire for ensuring the well-posedness of our problem.

Assumption 3.1 *-The functions f and h are continuously differentiable with respect to (y, z, u).*

-The function g is continuously differentiable, and it is bounded by $|g(y)| \leq c(1 + |y|)$ for $c > 0$.

-There exists $C > 0$, such that all the derivatives of f and h are bounded by $C(1 + |y| + |z| + |v|)$.

Assumption 3.2 *The second derivatives of the Hamiltonian $\mathcal{H}^{\varepsilon}$ with respect to (y, z) are bounded.*

Theorem 3.1 *The system (3.1) admits a unique solution .*

Proof. See [6]. ■

In the control problem, our objective is to minimize the cost functional $\mathcal{J}^{\varepsilon}$ over the set of admissible controls $\mathcal{U}$. If $u \in \mathcal{U}$ is an optimal control (solution), that means

$$\mathcal{J}^{\varepsilon}(u) = \inf_{v \in \mathcal{U}} \mathcal{J}^{\varepsilon}(v). \tag{3.3}$$

An admissible control that solves the problem $\{(3.1), (3.2), (3.3)\}$ is referred to as optimal. Our purpose is to derive both necessary and sufficient optimality conditions, met by such a control, in the context of the stochastic maximum principle with risk-sensitive performance.

Remark 3.1 *Solving the problem* $\{(3.1),(3.2),(3.3)\}$ *and determining the necessary condition through classical methods is challenging due to the presence of the exponential function in the expression of the cost functional* $\mathcal{J}^{\varepsilon}$*. Therefore, as we present the following section, we will follow a new approach.*

3.2 Risk-neutral SMP for fractional backward stochastic differential equation

To address our problem and surmount this obstacle, we must first take an intermediate step, wherein we introduce an auxiliary state process x_t^{υ} that satisfies to the following stochastic differential equation $dx_t^{\upsilon} = h(t, y_t^{\upsilon}, z_t^{\upsilon}, \upsilon_t)\, dt, \quad x^{\upsilon}(0) = 0$ ·

Then, the control problem $\{(3.1),(3.2),(3.3)\}$ is equivalent to

$$\begin{cases} \inf\limits_{\upsilon\in\mathcal{U}} \mathbb{E}\left[\exp\varepsilon\left(x_T^{\upsilon} + g\left(y_0^{\upsilon}\right)\right)\right] = \inf\limits_{\upsilon\in\mathcal{U}} \mathbb{E}\left[\Delta(x_T^{\upsilon}, y_0^{\upsilon})\right], \\ \text{subject to} \\ dx_t^{\upsilon} = h\left(t, y_t^{\upsilon}, z_t^{\upsilon}, \upsilon_t\right) dt, \\ dy_t^{\upsilon} = -f\left(t, y_t^{\upsilon}, z_t^{\upsilon}, \upsilon_t\right) dt + z_t^{\upsilon} dB_t^H, \\ x_0^{\upsilon} = 0,\ y_T^{\upsilon} = \xi. \end{cases} \tag{3.4}$$

We define

$$\begin{aligned} A_T^{\varepsilon} = &\ \exp\left[\varepsilon\left(\int_0^T h\left(t, y_t^{\upsilon}, z_t^{\upsilon}, \upsilon_t\right) dt + g\left(y_0^{\upsilon}\right)\right)\right], \\ \text{and} \quad \Theta_T = &\ \int_0^T h\left(t, y_t^{\upsilon}, z_t^{\upsilon}, \upsilon_t\right) dt + g\left(y_0^{\upsilon}\right), \end{aligned} \tag{3.5}$$

the risk-sensitive loss functional is given by

$$\Theta_{\varepsilon} := \frac{1}{\varepsilon} \log \mathbb{E}\left[\exp\left\{\varepsilon\left(\int_0^T h\left(t, y_t^{\upsilon}, z_t^{\upsilon}, \upsilon_t\right) dt + g\left(y_0^{\upsilon}\right)\right)\right\}\right] = \frac{1}{\varepsilon} \log \mathbb{E}\left[\exp\left\{\varepsilon\Theta_T\right\}\right]. \tag{3.6}$$

When the risk-sensitive index ε is small, the loss functional Θ_{ε} can be expanded as

$\mathbb{E}\left(\Theta_T\right)+\frac{\varepsilon}{2}Var\left(\Theta_T\right)+O\left(\varepsilon^2\right)$, where, $Var(\Theta_T)$ denotes the variance of Θ_T. If ε is less than 0, the variance of Θ_T, as a measure of risk, enhances the performance Θ_ε, in which case the optimizer is referred to as a risk seeker. Whereas if $\varepsilon > 0$, the variance of Θ_T worsens the performance Θ_ε, and the optimizer is labeled as risk averse. For more detailed information, please refer to the paper [77].

Notation 3.1 *Throughout this paper, we will use the following notations.*

For $\varrho \in \{f,\ h\}$, $\tau \in \{\upsilon, u\}$ *and* $\zeta \in \{x, y, z, \upsilon, u\}$, *we put*

$$\begin{cases} \varrho\left(t\right) &=& \varrho\left(t, y_t, z_t, \upsilon_t\right), \\ \varrho^{\tau}\left(t\right) &=& \varrho\left(t, y_t, z_t, \tau_t\right), \\ \varrho_{\zeta}\left(t\right) &=& \frac{\partial \varrho}{\partial \zeta}\left(t, y_t, z_t, \upsilon_t\right). \end{cases}, \begin{cases} \widetilde{\mathcal{H}}^{\varepsilon}\left(t\right) &=& \widetilde{\mathcal{H}}^{\varepsilon}\left(t, x_t, y_t, z_t, \upsilon_t, \overrightarrow{p}\left(t\right), \overrightarrow{q}\left(t\right)\right), \\ \widetilde{\mathcal{H}}^{\varepsilon}_{\zeta}\left(t\right) &=& \frac{\partial \widetilde{\mathcal{H}}^{\varepsilon}}{\partial \zeta}\left(t, x_t, y_t, z_t, \upsilon_t, \overrightarrow{p}\left(t\right), \overrightarrow{q}\left(t\right)\right). \end{cases}$$

and

$$\begin{cases} \mathcal{H}^{\varepsilon}\left(t\right) &=& \mathcal{H}^{\varepsilon}\left(t, y_t, z_t, \upsilon_t, \widetilde{p}_2\left(t\right), \widetilde{q}_2(t), V^{\varepsilon}\left(t\right), l\left(t\right)\right), \\ \mathcal{H}^{\varepsilon}_{\zeta}\left(t\right) &=& \frac{\partial \mathcal{H}^{\varepsilon}}{\partial \zeta}\left(t, y_t, z_t, \upsilon_t, \widetilde{p}_2\left(t\right), \widetilde{q}_2(t), V^{\varepsilon}\left(t\right), l\left(t\right)\right). \end{cases}$$

If we assume that assumption 3.1 holds true, then by applying the stochastic maximum principle for risk-neutral performance of forward-backward type control to the augmented state dynamics (x, y, z), we can determine the adjoint equation satisfied by a unique $\mathcal{F}^H$-adapted pair of processes $((p_1, q_1), (p_2, q_2))$, which solves the following system of forward-backward stochastic differential equations.

Lemma 3.1 *A unique pair of* $\mathcal{F}^H$*-adapted processes* $((p_1, q_1), (p_2, q_2))$ *exists, which serves as a solution to the following system of forward-backward stochastic differen-*

tial equation

$$\begin{cases} d\overrightarrow{p}(t) = \begin{pmatrix} dp_1(t) \\ dp_2(t) \end{pmatrix} & = \begin{pmatrix} 0 & 0 \\ h_y(t) & f_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} dt \\ & + \begin{pmatrix} q_1(t) \\ \frac{\widetilde{\mathcal{H}}_z^{\varepsilon}(t)}{Ht^{2H-1}} \end{pmatrix} dB_t^H, \\ p_1(T) = \varepsilon A_T^{\varepsilon}, & p_2(0) = \varepsilon A_T^{\varepsilon} g_y(y_0^u), \end{cases} \tag{3.7}$$

with $\mathbb{E}\left[\sum\limits_{i=1}^{2} \sup_{t\in[0,T]} |p_i(t)|^2 + \int_0^T \left(Ht^{2H-1}\right)^2 |q_1(t)|^2 dt\right] < \infty.$

Proof. In this context, we assume that $\widetilde{\mathcal{H}}^{\varepsilon}$ represents the Hamiltonian associated with the optimal state dynamics (x, y, z) and the pair of adjoint processes $(\overrightarrow{p}(t), \overrightarrow{q}(t))$, where

$$\widetilde{\mathcal{H}}^{\varepsilon}(t) = h(t) p_1(t) + f(t) p_2(t). \tag{3.8}$$

We have

$$\begin{cases} dp_1(t) &= \widetilde{\mathcal{H}}_x^{\varepsilon}(t) dt + q_1(t) dB_t^H, \ p_1(T) = \Delta_x(x_T^v, y_0^v), \\ dp_2(t) &= \widetilde{\mathcal{H}}_y^{\varepsilon}(t) dt + \frac{1}{Ht^{2H-1}} \widetilde{\mathcal{H}}_z^{\varepsilon}(t) dB_t^H, \ p_2(0) = \Delta_y(x_T^v, y_0^v), \end{cases}$$

$$\begin{aligned} \text{where} \quad p_1(T) &= \varepsilon \exp \varepsilon (x_T^v + g(y_0^v)) = \varepsilon A_T^{\varepsilon}, \\ \text{and} \quad p_2(0) &= \varepsilon g_y(y_0^v) \exp \varepsilon (x_T^v + g(y_0^v)) = \varepsilon g_y(y_0^v) A_T^{\varepsilon}. \end{aligned}$$

That implies

$$\begin{cases} dp_1(t) &= 0dt + q_1(t) dB_t^H, \ p_1(T) = \varepsilon A_T^{\varepsilon}, \\ dp_2(t) &= (h_y(t) p_1(t) + f_y(t) p_2(t)) dt \\ &+ \frac{1}{Ht^{2H-1}} (h_z(t) p_1(t) + f_z(t) p_2(t)) dB_t^H, \ p_2(0) = \varepsilon g_y(y_0^v) A_T^{\varepsilon}. \end{cases}$$

Which is equivalent the system (3.7). ■

The following theorem is referred to as the stochastic maximum principle for risk-neutral control in the forward-backward type form.

Theorem 3.2 *Under the assumption that 3.1 holds, when* (x, y, z) *represents an optimal solution to the risk-neutral control problem* (3.4)*, there exist pairs of* $\mathcal{F}_t^H-$*adapted processes* (p_1, q_1) *and* (p_2, q_2) *that fulfill* (3.7)*, as follows:*

$$\widetilde{\mathcal{H}}_{\upsilon}^{\varepsilon}(t)(\upsilon_t - u_t) \geq 0, \tag{3.9}$$

for all $\upsilon \in \mathcal{U}$*, almost every* $t \in [0, T]$ *and* $\mathbb{P}$*-almost surely.*

Proof. See [39]. ■

3.3 Results statment

In this section, we unveil the main result of our study. However, before delving into that, we introduce several lemmas that will aid us in our analysis. To start, our objective is to establish the relationship between exponential utility and the backward quadratic stochastic equation with a Hurst parameter H. It is essential to articulate the expected exponential utility.

3.3.1 Expected exponential utility and backward quadratic stochastic equation

First and foremost, it is crucial to express the expected exponential utility in the following format

$$e^{\varepsilon Y_t^{\varepsilon}} = \widetilde{\mathbb{E}}\left[A_{t,T}^{\varepsilon} \mid \mathcal{F}_t^H\right] = \widetilde{\mathbb{E}}\left[\exp\varepsilon\left(\int_t^T h(s)\,ds + g(y_0^{\upsilon})\right) \mid \mathcal{F}_t^H\right], \tag{3.10}$$

Here, $\widetilde{\mathbb{E}}$ denotes the quasi-conditional expectation introduced by [42].

The process Y^{ε} represents the initial component of the $\mathcal{F}_t^H$-adapted paire of processes (Y^{ε}, l), satisfies the following quadratic fractional BSDE

$$\begin{cases} dY_t^{\varepsilon} = -\left(h(t) + \frac{\varepsilon H t^{2H-1}}{2} |l(t)|^2\right) dt + l(t)\, dB_t^H, \\ Y_T^{\varepsilon} = g\left(y_0^{\upsilon}\right), \end{cases} \tag{3.11}$$

where $\mathbb{E}\left[\int_0^T \left(Ht^{2H-1}\right)^2 |l(t)|^2 dt\right] < \infty$.

For further insights into the optimization of expected exponential utility, readers can refer to the papers [21] and [43].

Lemma 3.2 *We have equivalence between assertions* (3.10) *and* (3.11)

Proof. we will divide the proof into two steps.

Step 1: From (3.10) we can write

$$e^{\varepsilon\left(Y_t^{\varepsilon} + \int_0^t h(s)ds\right)} = \widetilde{\mathbb{E}}\left[\exp \varepsilon \left(\int_0^T h(s)\, ds + g\left(y_0^{\upsilon}\right)\right) \mid \mathcal{F}_t^H\right] = \widetilde{\mathbb{E}}\left[A_T^{\varepsilon} \mid \mathcal{F}_t^H\right],$$

by the fractional Clark-Ocone formula [41], there exists a unique $\mathcal{F}_t^H$-adapted square integrable process M such that

$$e^{\varepsilon\left(Y_t^{\varepsilon} + \int_0^t h(s)ds\right)} = \mathbb{E}\left[A_T^{\varepsilon}\right] + \int_0^t M_s dB_s^H. \tag{3.12}$$

We applying Itô formula to (3.12) , get

$$\begin{aligned} d\left(e^{\varepsilon\left(Y_t^{\varepsilon} + \int_0^t h(s)ds\right)}\right) &= \left[\varepsilon dY_t^{\varepsilon} + \varepsilon h(t)\, dt + \frac{1}{2}\varepsilon^2 \left\langle dY_t^{\varepsilon}, dY_t^{\varepsilon}\right\rangle\right] e^{\varepsilon\left(Y_t^{\varepsilon} + \int_0^t h(s)ds\right)} \\ &= M_t dB_t^H, \end{aligned}$$

which implies

$$dY_t^{\varepsilon} + h(t)\, dt + \frac{1}{2}\varepsilon \left\langle dY_t^{\varepsilon}, dY_t^{\varepsilon}\right\rangle = \frac{M_t}{\varepsilon} e^{-\varepsilon\left(Y_t^{\varepsilon} + \int_0^t h(s)ds\right)} dB_t^H. \tag{3.13}$$

Hence the second term is an $\mathcal{F}_t^H$-quasi-martingale, then

$$\langle dY_t^\varepsilon, dY_t^\varepsilon \rangle = \left(\frac{M_t}{\varepsilon} e^{-\varepsilon\left(Y_t^\varepsilon + \int_0^t h(s)ds\right)} dB_t^H \right)^2 = Ht^{2H-1} \left| l\left(t\right) \right|^2 dt,$$

where $l\left(t\right) := \frac{M_t}{\varepsilon} e^{-\varepsilon\left(Y_t^\varepsilon + \int_0^t h(s)ds\right)}$. Then (3.13) becomes

$$dY_t^\varepsilon + h\left(t\right) dt + \frac{\varepsilon}{2} Ht^{2H-1} \left| l\left(t\right) \right|^2 dt = l\left(t\right) dB_t^H.$$

Finally, we arrive at.

$$dY_t^\varepsilon = -\left(h\left(t\right) + \frac{\varepsilon H t^{2H-1}}{2} \left| l\left(t\right) \right|^2 \right) dt + l\left(t\right) dB_t^H,\ Y_T^\varepsilon = g\left(\overset{v}{y_0}\right).$$

Step 2: We will prove the opposite, by applying Itô to $e^{\varepsilon Y_t^\varepsilon}$,and from (3.11) we get

$$d\left(e^{\varepsilon Y_t^\varepsilon}\right) = -\varepsilon e^{\varepsilon Y_t^\varepsilon} h\left(t\right) dt + \varepsilon e^{\varepsilon Y_t^\varepsilon} l\left(t\right) dB_t^H. \tag{3.14}$$

On the other side we have

$$\begin{aligned} d\left(e^{\varepsilon\left(Y_t^\varepsilon + \int_0^t h(s)ds\right)}\right) &= e^{\varepsilon Y_t^\varepsilon} d\left(e^{\varepsilon \int_0^t h(s)ds}\right) + e^{\varepsilon \int_0^t h(s)ds} d\left(e^{\varepsilon Y_t^\varepsilon}\right) \\ &= e^{\varepsilon\left(Y_t^\varepsilon + \int_0^t h(s)ds\right)} \varepsilon h\left(t\right) dt + e^{\varepsilon \int_0^t h(s)ds} d\left(e^{\varepsilon Y_t^\varepsilon}\right), \end{aligned}$$

we replacing (3.14) into above equality we get

$$d\left(e^{\varepsilon\left(Y_t^\varepsilon + \int_0^t h(s)ds\right)}\right) = e^{\varepsilon\left(Y_t^\varepsilon + \int_0^t h(s)ds\right)} \varepsilon l\left(t\right) dB_t^H,$$

but we have $\int_t^T d\left(e^{\varepsilon\left(Y_s^\varepsilon + \int_0^s h(u)du\right)}\right) = e^{\varepsilon\left(Y_T^\varepsilon + \int_0^T h(s)ds\right)} - e^{\varepsilon\left(Y_t^\varepsilon + \int_0^t h(s)ds\right)}$, that implies

$$e^{\varepsilon\left(Y_T^\varepsilon+\int_0^T h(s)ds\right)}=e^{\varepsilon\left(Y_t^\varepsilon+\int_0^t h(s)ds\right)}+\int_t^T e^{\varepsilon\left(Y_s^\varepsilon+\int_0^s h(u)du\right)}\varepsilon l\left(s\right)dB_s^H,$$

by passing mathematical quasi-conditional expectation

$$\widetilde{\mathbb{E}}\left[e^{\varepsilon\left(Y_T^\varepsilon+\int_0^T h(s)ds\right)}\mid\mathcal{F}_t^H\right]=\widetilde{\mathbb{E}}\left[e^{\varepsilon\left(Y_t^\varepsilon+\int_0^t h(s)ds\right)}\mid\mathcal{F}_t^H\right]=e^{\varepsilon\left(Y_t^\varepsilon+\int_0^t h(s)ds\right)}.$$

Since $Y_T^\varepsilon=g\left(\overset{v}{y_0}\right)$, finally we get

$$\begin{aligned}e^{\varepsilon Y_t^\varepsilon}&=e^{-\varepsilon\int_0^t h(s)ds}\widetilde{\mathbb{E}}\left[e^{\varepsilon\left(Y_T^\varepsilon+\int_0^T h(s)ds\right)}\mid\mathcal{F}_t^H\right]=\widetilde{\mathbb{E}}\left[e^{\varepsilon\left(g\left(\overset{v}{y_0}\right)+\int_t^T h(s)ds\right)}\mid\mathcal{F}_t^H\right]\\&=\widetilde{\mathbb{E}}\left[A_{t,T}^\varepsilon\mid\mathcal{F}_t^H\right].\end{aligned}$$

Which is equal to the given form (3.10). ■

3.3.2 New adjoint equations and risk-sensitive necessary optimality conditions

To derive our result, we adopt the methodology introduced by Djehiche et al. in [23], and introduce a transformation of the adjoint processes (p_1,q_1) and (p_2,q_2) into a new pair by removing the first component (p_1,q_1). This enables us to express the stochastic maximum principle exclusively in terms of the new process, which consists solely of the last two adjoint processes.

At first, we remark that $dp_1\left(t\right)=q_1\left(t\right)dB_t^H,\;\; p_1\left(T\right)=\;\varepsilon A_T^\varepsilon$.

The explicit solution of this fractional BSDE is

$$p_1\left(t\right)=\varepsilon\widetilde{\mathbb{E}}\left[A_T^\varepsilon\mid\mathcal{F}_t^H\right]=\varepsilon V^\varepsilon\left(t\right),\tag{3.15}$$

and $V_t^\varepsilon := \widetilde{\mathbb{E}}\left[A_T^\varepsilon \mid \mathcal{F}_t^H\right],\ 0 \leq t \leq T$. If we take a close-up look of (3.15), we can determine a transformation of $(\overrightarrow{p}, \overrightarrow{q})$ into an adjoint process $(\widetilde{p}, \widetilde{q})$, where $\widetilde{p_1}(t) = \frac{1}{\varepsilon V_t^\varepsilon} p_1(t) = 1$; that mean

$$\widetilde{p}(t) = \begin{pmatrix} \widetilde{p_1}(t) \\ \widetilde{p_2}(t) \end{pmatrix} := \frac{1}{\varepsilon V_t^\varepsilon} \overrightarrow{p}(t),\ 0 \leq t \leq T. \tag{3.16}$$

By using (3.7) and (3.16), we get $\widetilde{p_1}(T) = 1$ and $\widetilde{p_2}(0) = g_y(y_0^v)$.

To explore the characteristics of this novel process $(\widetilde{p}(t), \widetilde{q}(t))$, it is imperative to demonstrate the following crucial properties of the quasi-martingale V^ε. As demonstrated in Lemma 3.2, the process Y^ε serves as the initial component of the $\mathcal{F}_t^H$-adapted pair of processes (Y^ε, l), representing the unique solution to quadratic backward stochastic differential equation (3.11). The next lemma serves as a supporting result within this chapter, aiding us in achieving our primary objective in the following paragraphs.

Lemma 3.3 *Suppose that assumption 3.1 holds, then*

$$\mathbb{E}\left[\sup |Y_t^\varepsilon|\right] \leq C_T. \tag{3.17}$$

In particular, V^ε *solves the following linear backward SDE*

$$\begin{cases} dV_t^\varepsilon = \varepsilon l(t) V_t^\varepsilon dB_t^H, \\ V_T^\varepsilon = A_T^\varepsilon. \end{cases} \tag{3.18}$$

Then, the process defined on $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^H\right)_{t\in[0,T]}, \mathbb{P}\right)$ *for every* $0 \leq t \leq T$, *by*

$$\frac{V_t^\varepsilon}{V_0^\varepsilon} = \exp\left(\varepsilon \int_0^t l(s)\, dB_s^H - \frac{\varepsilon^2}{2} \int_0^t H s^{2H-1} |l(s)|^2 ds\right), \tag{3.19}$$

is a uniformly bounded $\mathcal{F}_t^H$*-quasi-martingale.*

Proof. First, we prove (3.17) , we have h and g are bounded by constant $C > 0$, we get

$$\begin{cases} -\varepsilon C \leq \varepsilon g\left(y_0^v\right) \leq \varepsilon C, \\ -\varepsilon CT \leq \varepsilon \int_0^T h\left(t\right) dt \leq \varepsilon CT, \end{cases}$$

by addition the two above inequalities and since the exponential function is positive, then from (3.5) we get

$$0 \leq e^{-(1+T)\varepsilon C} \leq A_T^\varepsilon \leq e^{(1+T)\varepsilon C}. \tag{3.20}$$

Therefore V^ε is uniformly bounded $\mathcal{F}_t^H$-quasi-martingale satisfying

$$0 \leq e^{-(1+T)\varepsilon C} \leq V_t^\varepsilon \leq e^{(1+T)\varepsilon C},\ 0 \leq t \leq T. \tag{3.21}$$

The sufficient conditions of the logarithmic transform established in [24], can be applied in the quasi-martingale V^ε as follows

$$V_t^\varepsilon = \exp \varepsilon \left(\int_0^t h\left(s\right) ds + Y_t^\varepsilon \right), \tag{3.22}$$

for every $0 \leq t \leq T$, and $V_0^\varepsilon = \exp\left(\varepsilon Y_0^\varepsilon\right) = \mathbb{E}\left[A_T^\varepsilon\right]$. It is very easy to see from (3.21) and the boundedness of h that $\mathbb{E}\left[\sup\left|Y_t^\varepsilon\right|\right] \leq C_T$, where, C_T is a positive constant that depends only on T and the boundedness of l and g.

Second, we move to prove (3.18) , by applying Itô to (3.22) , then from (3.11) we obtain

$$\begin{aligned} dV_t^\varepsilon &= d\left[\exp \varepsilon \left(\int_0^t h\left(s\right) ds + Y_t^\varepsilon \right)\right] \\ &= \varepsilon h\left(t\right) V_t^\varepsilon dt + \varepsilon V_t^\varepsilon \left[-\left(h\left(t\right) + \tfrac{\varepsilon}{2} H t^{2H-1} \left|l\left(t\right)\right|^2\right) dt + l\left(t\right) dB_t^H\right] \\ &+ \tfrac{\varepsilon^2}{2} V_t^\varepsilon H t^{2H-1} \left|l\left(t\right)\right|^2 dt \\ &= \varepsilon V_t^\varepsilon l\left(t\right) dB_t^H. \end{aligned}$$

Finally, we can prove (3.19), by replacing formula of Y_t^{ε} in (3.22), we get

$$
\begin{aligned}
V_t^{\varepsilon} &= \exp\left\{\varepsilon \int_0^t h(s)\,ds - \int_0^t \varepsilon\left(h(s) + \frac{\varepsilon}{2} H s^{2H-1} |l(s)|^2\right) ds \right. \\
&\left. +\varepsilon \int_0^t l(s)\,dB_s^H \right\} \\
&= \exp\left\{-\frac{\varepsilon^2}{2}\int_0^t H s^{2H-1} |l(s)|^2 ds + \varepsilon \int_0^t l(s)\,dB_s^H\right\}.
\end{aligned}
$$

Then

$$
d\left(\ln V_t^{\varepsilon}\right) = d\left(-\frac{\varepsilon^2}{2}\int_0^t H s^{2H-1} |l(s)|^2 ds + \varepsilon \int_0^t l(s)\,dB_s^H\right).
$$

By introducing the integral and from properties Logarithmic function, we obtain

$$
\ln \frac{V_t^{\varepsilon}}{V_0^{\varepsilon}} = -\frac{\varepsilon^2}{2}\int_0^t H s^{2H-1} |l(s)|^2 ds + \varepsilon \int_0^t l(s)\,dB_s^H.
$$

At last, we have $\frac{V_t^{\varepsilon}}{V_0^{\varepsilon}} = \exp\left(\varepsilon \int_0^t l(s)\,dB_s^H - \frac{\varepsilon^2}{2}\int_0^t H s^{2H-1} |l(s)|^2 ds\right)$.

In view of (3.18), the last expression is a uniformly bounded $\mathcal{F}_t^H$-quasi-martingale. ■

In Coming up, we assert and demonstrate the necessary optimality condition for the system governed by SDE with a risk-sensitive performance objective. thereafter, we introduce the Risk-Sensitive SMP theorem.

Proposition 3.1 *The risk-sensitive for the adjoint equation satisfied by* $(\widetilde{p}_2, \widetilde{q}_2)$ *and* (V^{ε}, l) *becomes*

$$
\begin{cases}
d\widetilde{p}_2(t) &= \mathcal{H}_y^{\varepsilon}(t)\,dt + \frac{1}{Ht^{2H-1}} \mathcal{H}_z^{\varepsilon}(t)\,dB_t^{H,\varepsilon},\ \widetilde{p}_2(0) = g_y\left(y_0^u\right), \\
dV_t^{\varepsilon} &= \varepsilon l(t)\, V_t^{\varepsilon} dB_t^H,\ V_T^{\varepsilon} = A_T^{\varepsilon}.
\end{cases}
\tag{3.23}
$$

The solution $(\widetilde{p}, \widetilde{q}, V^{\varepsilon}, l)$ *of the system* (3.23) *is unique, such that*

$$
\mathbb{E}\left[\sup_{t\in[0,T]} |\widetilde{p}(t)|^2 + \sup_{t\in[0,T]} |V^{\varepsilon}(t)|^2 + \int_0^T \left(Ht^{2H-1}\right)^2 \left(|\widetilde{q}(t)|^2 + |l(t)|^2\right) dt\right] < \infty, \tag{3.24}
$$

where

$$\begin{aligned}\mathcal{H}^{\varepsilon}(t) &:= \mathcal{H}^{\varepsilon}\left(t, y_t, z_t, \begin{pmatrix} \widetilde{p_2}(t) \\ \widetilde{q_2}(t) \end{pmatrix}, V^{\varepsilon}(t), l(t)\right) \\ &= \left(f(t) - z_t \varepsilon l(t) H t^{2H-1}\right) \widetilde{p_2}(t) + h(t).\end{aligned} \tag{3.25}$$

Proof. Under the assumption 3.2**,** the system (3.23) has a unique solution. Now we identify the processes $\widetilde{\alpha}$ and $\widetilde{\beta}$ such that

$$d\widetilde{p}(t) = \widetilde{\alpha}(t)\, dt + \widetilde{\beta}(t)\, dB_t^H. \tag{3.26}$$

By applying integration by parts formula to the process $\overrightarrow{p}(t) = \varepsilon V^{\varepsilon}(t) \widetilde{p}(t)$, and using the expression of V_t^{ε} in (3.18), we obtain

$$\begin{aligned} d\overrightarrow{p}(t) &= d\left(\varepsilon V^{\varepsilon}(t) \widetilde{p}(t)\right) \\ &= \varepsilon^2 \widetilde{p}(t)\, l(t)\, V_t^{\varepsilon} dB_t^H + \varepsilon V^{\varepsilon}(t)\, d\widetilde{p}(t) + \varepsilon^2 l(t)\, V_t^{\varepsilon} \widetilde{\beta}(t)\, H t^{2H-1} dt. \end{aligned}$$

That implies

$$d\widetilde{p}(t) = \frac{1}{\varepsilon V_t^{\varepsilon}} d\overrightarrow{p}(t) - \varepsilon \widetilde{p}(t)\, l(t)\, dB_t^H - \varepsilon l(t)\, \widetilde{\beta}(t)\, H t^{2H-1} dt.$$

According to the system (3.7), we have

$$\begin{aligned} d\widetilde{p}(t) &= \left\{ \frac{1}{\varepsilon V_t^{\varepsilon}} \left[\begin{pmatrix} 0 & 0 \\ h_y(t) & f_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} \right] - \varepsilon l(t)\, \widetilde{\beta}(t)\, H t^{2H-1} \right\} dt \\ &+ \left[\frac{1}{\varepsilon V_t^{\varepsilon}} \begin{pmatrix} q_1(t) \\ \frac{\widetilde{\mathcal{H}}_z^{\varepsilon}(t)}{H t^{2H-1}} \end{pmatrix} - \varepsilon \widetilde{p}(t)\, l(t) \right] dB_t^H. \end{aligned}$$

By using the relation (3.16) , we get

$$\begin{aligned} d\widetilde{p}(t) &= \left[\begin{pmatrix} 0 & 0 \\ h_y(t) & f_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p_1}(t) \\ \widetilde{p_2}(t) \end{pmatrix} - \varepsilon l(t)\widetilde{\beta}(t) Ht^{2H-1}\right] dt \\ &+ \left[\begin{pmatrix} \widetilde{q_1}(t) \\ \frac{1}{Ht^{2H-1}}\left(h_z(t)\widetilde{p_1}(t) + f_z(t)\widetilde{p_2}(t)\right) \end{pmatrix} - \varepsilon\widetilde{p}(t) l(t)\right] dB_t^H. \end{aligned} \tag{3.27}$$

By identifying the coefficients between (3.26) and (3.27), we get the drift term

$$\widetilde{\alpha}(t) = \begin{pmatrix} 0 & 0 \\ h_y(t) & f_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p_1}(t) \\ \widetilde{p_2}(t) \end{pmatrix} - \varepsilon l(t)\widetilde{\beta}(t) Ht^{2H-1},$$

and the diffusion term of the process $\widetilde{p}(t)$

$$\widetilde{\beta}(t) = \begin{pmatrix} \widetilde{q_1}(t) \\ \frac{1}{Ht^{2H-1}}\left(h_z(t)\widetilde{p_1}(t) + f_z(t)\widetilde{p_2}(t)\right) \end{pmatrix} - \varepsilon\widetilde{p}(t) l(t). \tag{3.28}$$

Finally, we obtain

$$d\widetilde{p}(t) = \left\{\begin{pmatrix} 0 & 0 \\ h_y(t) & f_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p_1}(t) \\ \widetilde{p_2}(t) \end{pmatrix} - \varepsilon l(t)\widetilde{\beta}(t) Ht^{2H-1}\right\} dt + \widetilde{\beta}(t) dB_t^H.$$

It easily confirmed that

$$\begin{aligned} d\widetilde{p_1}(t) &= -\varepsilon l(t)\widetilde{\beta_1}(t) Ht^{2H-1}dt + \widetilde{\beta_1}(t) dB_t^H \\ &= \widetilde{\beta_1}(t)\left[dB_t^H - \varepsilon l(t) Ht^{2H-1}dt\right],\ \widetilde{p_1}(T) = 1. \end{aligned}$$

Bearing in mind (3.19) , the fractional Girsanov's theorem permits us to write

$d\widetilde{p_1}(t) = \widetilde{\beta}_1(t)\, dB_t^{H,\varepsilon}$, $\widetilde{p_1}(T) = 1$ $\mathbb{P}^\varepsilon - a.s$, where

$$dB_t^{H,\varepsilon} = dB_t^H - \varepsilon l(t)\, Ht^{2H-1}dt, \tag{3.29}$$

is a $\mathbb{P}^\varepsilon$ fBm, and

$$\frac{d\mathbb{P}^\varepsilon}{d\mathbb{P}} := \exp\left(\varepsilon \int_0^t l(s)\, Hs^{2H-1}dB_s^H - \frac{\varepsilon^2}{2}\int_0^t \left(Hs^{2H-1}\right)^2 |l(s)|^2\, ds\right).$$

As stated by in (3.19) and (3.20), the probability measures $\mathbb{P}^\varepsilon$ and $\mathbb{P}$ are equivalent. Hence, noting that $\widetilde{p_1}(t) = \frac{1}{\varepsilon V_t^\varepsilon} p_1(t)$, is square integrable, we get that $\widetilde{p_1}(t) = \widetilde{\mathbb{E}}^{\mathbb{P}^\varepsilon}\left[\widetilde{p_1}(T) \mid \mathcal{F}_t^H\right] = 1$, and so on, we can find that the process $\widetilde{q_1}(t)$ is a finite quadratic variation, such that $\mathbb{E}\left[\int_0^T |\widetilde{q_1}(t)|^2\, dt\right] = 0$.This implies that, for almost every $0 \le t \le T$, $\widetilde{q_1}(t) = 0$, $\mathbb{P}^\varepsilon$ and $\mathbb{P} - a.s.$ we have

$$d\widetilde{p}(t) = \begin{pmatrix} 0 & 0 \\ h_y(t) & f_y(t) \end{pmatrix} \begin{pmatrix} \widetilde{p_1}(t) \\ \widetilde{p_2}(t) \end{pmatrix} dt + \widetilde{\beta}(t)\, dB_t^{H,\varepsilon}. \tag{3.30}$$

By replacing the relation(3.28) into (3.30), we obtain

$$\begin{aligned} d\widetilde{p}(t) &= \begin{pmatrix} 0 & 0 \\ h_y(t) & f_y(t) \end{pmatrix} \begin{pmatrix} \widetilde{p_1}(t) \\ \widetilde{p_2}(t) \end{pmatrix} dt \\ &+ \left\{ \begin{pmatrix} \widetilde{q_1}(t) \\ \frac{1}{Ht^{2H-1}}\left(h_z(t)\, \widetilde{p_1}(t) + f_z(t)\, \widetilde{p_2}(t)\right) \end{pmatrix} - \varepsilon \widetilde{p}(t)\, l(t) \right\} dB_t^{H,\varepsilon}. \end{aligned} \tag{3.31}$$

Therefore, the second component $\widetilde{p_2}(t)$ given in (3.31) has the form

$$\begin{aligned} d\widetilde{p_2}(t) &= \left[h_y(t)\, \widetilde{p_1}(t) + f_y(t)\, \widetilde{p_2}(t)\right] dt \\ &+ \frac{1}{Ht^{2H-1}}\left[h_z(t)\, \widetilde{p_1}(t) + f_z(t)\, \widetilde{p_2}(t) - Ht^{2H-1}\varepsilon \widetilde{p_2}(t)\, l(t)\right] dB_t^{H,\varepsilon}. \end{aligned}$$

The main risk-sensitive for the second adjoint equation satisfied by $(\widetilde{p_2}, \widetilde{q_2})$ and (V^ε, l)

becomes

$$\begin{cases} d\widetilde{p}_2(t) & = & \mathcal{H}_y^{\varepsilon}(t)\,dt + \frac{1}{Ht^{2H-1}}\mathcal{H}_z^{\varepsilon}(t)\,dB_t^{H,\varepsilon},\ \widetilde{p}_2(0) = g_y(y_0^u). \\ dV_t^{\varepsilon} & = & \varepsilon l(t)\,V_t^{\varepsilon}dB_t^H,\ V_T^{\varepsilon} = A_T^{\varepsilon}, \end{cases}$$

The solution $(\widetilde{p}, \widetilde{q}, V^{\varepsilon}, l)$ of the system (3.23) is unique, such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\widetilde{p}(t)|^2 + \sup_{t\in[0,T]}\left|V^{\theta}(t)\right|^2 + \int_0^T \left(Ht^{2H-1}\right)^2\left(|\widetilde{q}(t)|^2 + |l(t)|^2\right)dt\right] < \infty,$$

where

$$\begin{aligned} \mathcal{H}^{\varepsilon}(t) &:= \mathcal{H}^{\varepsilon}(t, y_t, z_t, \widetilde{p}_2(t), \widetilde{q}_2(t), V^{\varepsilon}(t), l(t)) \\ &= \left(f(t) - z_t\varepsilon l(t)\,Ht^{2H-1}\right)\widetilde{p}_2(t) + h(t). \end{aligned}$$

The proof is completed. ■

Theorem 3.3 *(Risk-sensitive SMP) Assume that assumption* 3.1 *holds, if* (y, z, u) *is an optimal solution to the risk-sensitive control problem* $\{(3.1), (3.2), (3.3)\}$*, then there exist pairs of* $\mathcal{F}_t^H-$*adapted processes* $(\widetilde{p}_2, \widetilde{q}_2)$ *and* (V^{ε}, l) *that satisfy* (3.23) *and* (3.24)*, such that*

$$\mathcal{H}_v^{\varepsilon}(t)(v_t - u_t) \geq 0, \tag{3.32}$$

for all $v \in \mathcal{U}$*, almost every* $0 \leq t \leq T$ *and* $\mathbb{P}$*-a.s, where the Hamiltonian* $\widetilde{\mathcal{H}}^{\varepsilon}$ *associated in* (3.4) *is given by*

$$\widetilde{\mathcal{H}}^{\varepsilon}(t, x_t, y_t, z_t, v_t, \overrightarrow{p}(t), \overrightarrow{q}(t)) = \varepsilon V_t^{\varepsilon}\mathcal{H}^{\varepsilon}(t, y_t, z_t, v_t, \widetilde{p}_2(t), \widetilde{q}_2(t), V_t^{\varepsilon}, l(t)), \tag{3.33}$$

where $\mathcal{H}^{\varepsilon}$ *is given by* (3.25).

Proof. To arrive at a risk-sensitive stochastic maximum principle expressed in terms of the adjoint processes $(\widetilde{p}_2, \widetilde{q}_2)$ and (V^{ε}, l), which solve (3.23), where the Hamiltonian

$\widetilde{\mathcal{H}}^{\varepsilon}$ associated with (3.9) given by (3.8) satisfies

$$\widetilde{\mathcal{H}}^{\varepsilon}\left(t, x_t, y_t, z_t, \upsilon_t, \overrightarrow{p}\left(t\right), \overrightarrow{q}\left(t\right)\right) = \varepsilon V_t^{\varepsilon}\mathcal{H}^{\varepsilon}\left(t, y_t, z_t, \upsilon_t, \widetilde{p}_2\left(t\right), \widetilde{q}_2(t), V_t^{\varepsilon}, l\left(t\right)\right),$$

and $\mathcal{H}^{\varepsilon}$ is the risk-sensitive Hamiltonian given by (3.25).

Hence, since $V^{\varepsilon} > 0$,the variational inequality (3.9) translates into $\mathcal{H}_{\upsilon}^{\varepsilon}\left(t\right) \geq 0$, for all $\upsilon \in \mathcal{U}$, almost every $0 \leq t \leq T$ and $\mathbb{P}$-almost surely. ■

3.4 Risk-sensitive sufficient optimality conditions

In this section, we aim to explore the conditions that lead to the transformation of the necessary optimality condition (3.9) into a sufficient condition for optimality.

Assumption 3.3 *We suppose:*

1. *The Hamiltonian function $\widetilde{\mathcal{H}}^{\varepsilon}$ is convex for all (y, z, u).*
2. *The finction g is convex.*

Theorem 3.4 *If the above assumption 3.3 hold, and for any $\upsilon \in \mathcal{U}$, the process $y_T^{\upsilon} = \xi$ is a one-dimensional $\mathcal{F}_T^H-$measurable random variable with $\mathbb{E}|\xi|^2 < \infty$, then u constitutes an optimal solution to the control problem $\{(3.1), (3.2), (3.3)\}$ if and only if it complies with the necessary optimality condition (3.9).*

Proof. Let $\upsilon, u \in \mathcal{U}$ (u be the optimal), we have

$$\mathcal{J}^{\varepsilon}\left(\upsilon\right) - \mathcal{J}^{\varepsilon}\left(u\right) = \mathbb{E}\left[\exp\varepsilon\left(x_T^{\upsilon} + g\left(y_0^{\upsilon}\right)\right) - \exp\varepsilon\left(x_T^{u} + g\left(y_0^{u}\right)\right)\right],$$

by a Taylor's expansion of exponential function, then by (3.5), and according to (3.7), we get

$$\begin{aligned}\mathcal{J}^{\varepsilon}(v)-\mathcal{J}^{\varepsilon}(u) &= \mathbb{E}\left[\varepsilon g_y\left(y_0^u\right)\left(y_0^v-y_0^u\right)\exp\varepsilon\left(x_T^u+g\left(y_0^u\right)\right)\right] \\ &+ \mathbb{E}\left[\varepsilon\left(x_T^v-x_T^u\right)\exp\varepsilon\left(x_T^u+g\left(y_0^u\right)\right)\right] \\ &= \mathbb{E}\left[p_2(0)\left(y_0^v-y_0^u\right)\right]+\mathbb{E}\left[p_1(T)\left(x_T^v-x_T^u\right)\right].\end{aligned} \tag{3.34}$$

By applying integration by parts formula to $p_1(t)\left(x_t^v-x_t^u\right)$, we obtain

$$d\left(p_1(t)\left(x_t^v-x_t^u\right)\right) = \left(x_t^v-x_t^u\right)q_1(t)\,dB_t^H+p_1(t)\left(h^v(t)-h^u(t)\right)dt.$$

By introducing integral then passing expectation, we get

$$\mathbb{E}\left[p_1(T)\left(x_T^v-x_T^u\right)\right]=\mathbb{E}\left[\int_0^T p_1(t)\left(h^v(t)-h^u(t)\right)dt\right], \tag{3.35}$$

and applying also integration by parts formula to $p_2(t)\left(y_t^v-y_t^u\right)$

$$\begin{aligned}d\left(p_2(t)\left(y_t^v-y_t^u\right)\right) &= \left(y_t^v-y_t^u\right)\widetilde{\mathcal{H}}_y^{\varepsilon}(t)\,dt+\tfrac{1}{Ht^{2H-1}}\left(y_t^v-y_t^u\right)\widetilde{\mathcal{H}}_z^{\varepsilon}(t)\,dB_t^H \\ &- p_2(t)\left(f^v(t)-f^u(t)\right)dt+p_2(t)\left(z_t^v-z_t^u\right)dB_t^H \\ &+ \tfrac{1}{Ht^{2H-1}}\widetilde{\mathcal{H}}_z^{\varepsilon}(t)\left(z_t^v-z_t^u\right)Ht^{2H-1}dt.\end{aligned}$$

By introducing integral then passing matematical expectation

$$\begin{aligned}\mathbb{E}\left[p_2(0)\left(y_0^v-y_0^u\right)\right] &= \mathbb{E}\left[-\int_0^T\left(y_t^v-y_t^u\right)\widetilde{\mathcal{H}}_y^{\varepsilon}(t)\,dt\right] \\ &+ \mathbb{E}\left[\int_0^T p_2(t)\left(f^v(t)-f^u(t)\right)dt\right] \\ &- \mathbb{E}\left[\int_0^T\widetilde{\mathcal{H}}_z^{\varepsilon}(t)\left(z_t^v-z_t^u\right)dt\right].\end{aligned} \tag{3.36}$$

By replacing (3.35) and (3.36) into (3.34), we get

$$\begin{aligned} \mathcal{J}^{\varepsilon}(\upsilon)-\mathcal{J}^{\varepsilon}(u) &= \mathbb{E}\Big[\int_0^T \widetilde{\mathcal{H}}^{\varepsilon}(t,x_t^{\upsilon},y_t^{\upsilon},z_t^{\upsilon},\upsilon_t,\overrightarrow{p}(t),\overrightarrow{q}(t)) \\ &- \widetilde{\mathcal{H}}^{\varepsilon}(t,x_t^{u},y_t^{u},z_t^{u},,u_t\overrightarrow{p}(t),\overrightarrow{q}(t))\,dt\Big] \\ &- \mathbb{E}\Big[\int_0^T (y_t^{\upsilon}-y_t^{u})\widetilde{\mathcal{H}}_y^{\varepsilon}(t)\,dt\Big]-\mathbb{E}\Big[\int_0^T \widetilde{\mathcal{H}}_z^{\varepsilon}(t)(z_t^{\upsilon}-z_t^{u})\,dt\Big]. \end{aligned} \tag{3.37}$$

With using the Hamiltonian convexity at (y,z,υ) in the above inequality and as a direct result from (3.9), we arrive at $\mathcal{J}^{\varepsilon}(\upsilon)-\mathcal{J}^{\varepsilon}(u)\geq 0$. ■

Remark 3.2 *Due to equation* (3.33)*, a relationship between the Hamiltonian concerning risk-neutral and the Hamiltonian concerning risk-sensitive. Specifically, we have*

$$\begin{aligned} \mathcal{J}^{\varepsilon}(\upsilon)-\mathcal{J}^{\varepsilon}(u) &\geq \mathbb{E}\Big[\int_0^T \varepsilon V_t^{\varepsilon}\mathcal{H}_{\upsilon}^{\varepsilon}(t,y_t^{u},z_t^{u},u_t,\widetilde{p}_2(t),\widetilde{q}_2(t),V_t^{\varepsilon},l(t))(\upsilon_t-u_t)\,dt\Big] \\ &\geq 0. \end{aligned}$$

Given that $\varepsilon V_t^{\varepsilon}$ *is greater than* 0*, the previous inequality can be restated as*

$$\mathcal{J}^{\varepsilon}(\upsilon)-\mathcal{J}^{\varepsilon}(u)\geq \mathbb{E}\Big[\int_0^T \mathcal{H}_{\upsilon}^{\varepsilon}(t,y_t^{u},z_t^{u},u_t,\widetilde{p}_2(t),\widetilde{q}_2(t),V_t^{\varepsilon},l(t))(\upsilon_t-u_t)\,dt\Big]\geq 0$$

Referring to the necessary optimality conditions (3.32)*, the latest inequality indicates that* $\mathcal{J}^{\varepsilon}(\upsilon)-\mathcal{J}^{\varepsilon}(u)\geq 0$.

3.5 Application

3.5.1 A control problem with linear quadratic risk sensitivity

We present an example of risk-sensitive backward stochastic linear quadratic problem, and give the explicit optimal control in the feedback forme, and illustrate our main results (Risk-sensitive SMP) in Theorem 3.3.

We consider the following state dynamics

$$\begin{cases} dy_t &= -(\alpha y_t + \upsilon_t)\, dt + z_t dB_t^H, \\ y_T^{\upsilon} &= \xi,\ t \in [0,T], \end{cases} \tag{3.38}$$

and our risk-sensitive cost functional has the form

$$\mathcal{J}^{\varepsilon}(\upsilon) = \mathbb{E}\left[\exp \varepsilon \left(-\frac{1}{2}\int_0^T \left(\upsilon_t^2 + y_t^2\right) dt + \frac{1}{2} y_0^2\right)\right], \tag{3.39}$$

where: α is real constant and $\varepsilon > 0$.

Our aim is to minimize over $\mathcal{U}$, (3.39) subject to (3.38) by choosing υ which satisfies the following equality

$$\mathcal{J}^{\varepsilon}(u) = \inf_{\upsilon \in \mathcal{U}} \mathcal{J}^{\varepsilon}(\upsilon). \tag{3.40}$$

Hence, we can apply Theorem 3.3 to solve our linear-quadratic risk-sensitive stochastic optimal control problem $\{(3.38), (3.39), (3.40)\}$. The Hamiltonian function (3.25) is defined by

$$\begin{aligned} \mathcal{H}^{\varepsilon}\left(t, y_t, z_t, \upsilon_t, \widetilde{p}_2(t), \widetilde{q}_2(t), V^{\varepsilon}(t), l(t)\right) &:= \left(\alpha y_t + \upsilon_t - z_t \varepsilon l(t) H t^{2H-1}\right) \widetilde{p}_2(t) \\ &- \tfrac{1}{2}\left(\upsilon_t^2 + y_t^2\right), \end{aligned}$$

minimizing the Hamiltonian over $\mathcal{U}$ we obtain

$$u_t = \widetilde{p}_2(t). \tag{3.41}$$

Then, (3.38) becomes

$$dy_t^u = -\left(\alpha y_t^u + \widetilde{p}_2(t)\right) dt + z_t^u dB_t^H, \quad y_T^u = \xi. \tag{3.42}$$

By taking into consideration (3.29) , the optimal adjoint equation (3.23) can be written as follow

$$\begin{aligned} d\widetilde{p}_2(t) &= \mathcal{H}_y^{\varepsilon}(t)\,dt + \frac{1}{Ht^{2H-1}}\mathcal{H}_z^{\varepsilon}(t)\,dB_t^{H,\varepsilon} \\ &= \left\{-y_t^u + \left(\alpha + \varepsilon^2 l^2(t)\,Ht^{2H-1}\right)\widetilde{p}_2(t)\right\}dt - \varepsilon l(t)\,\widetilde{p}_2(t)\,dB_t^H. \end{aligned} \tag{3.43}$$

Through (3.42) and (3.43) we have produced a system called fully coupled forward backward system

$$\begin{cases} dy_t^u &= -\left(\alpha y_t^u + \widetilde{p}_2(t)\right)dt + z_t^u dB_t^H, \\ d\widetilde{p}_2(t) &= \left\{-y_t^u + \left(\alpha + \varepsilon^2 l^2(t)\,Ht^{2H-1}\right)\widetilde{p}_2(t)\right\}dt \\ & - \varepsilon l(t)\,\widetilde{p}_2(t)\,dB_t^H, \\ y_T^u &= \xi, \quad \widetilde{p}_2(0) = y_0^u. \end{cases} \tag{3.44}$$

Which his solution is difficult to find explicitly. To solve this system, we consider the following linear form

$$\widetilde{p}_2(t) = \varphi(t)\,y_t^u + \chi(t), \tag{3.45}$$

with φ and χ are deterministic functions. Then applying Itô's formula on (3.45) , we obtain

$$\begin{aligned} d\widetilde{p}_2(t) &= \dot{\varphi}_t y_t^u dt + \varphi_t dy_t + \dot{\chi}_t dt \\ &= \left(\dot{\varphi}_t y_t^u - \alpha\varphi_t y_t^u - \varphi_t\widetilde{p}_2(t) + \dot{\chi}_t\right)dt + \varphi_t z_t^u dB_t^H, \end{aligned} \tag{3.46}$$

where $\dot{\varphi}$ and $\dot{\chi}_t$ are the derivative of φ and χ with respect to t.

Putting (3.45) into (3.46)

$$
\begin{aligned}
d\widetilde{p_2}(t) &= \left(\dot{\varphi}_t y_t^u - \alpha\varphi_t y_t^u - \varphi_t\left(\varphi_t y_t^u + \chi_t\right) + \dot{\chi}_t\right) dt + z_t^u dB_t^H \\
&= \left(\left(\dot{\varphi}_t - \alpha\varphi_t - \varphi_t^2\right) y_t^u - \varphi_t\chi_t + \dot{\chi}_t\right) dt + \varphi_t z_t^u dB_t^H .
\end{aligned} \tag{3.47}
$$

On the other hand, after substituting (3.45) into (3.43), we arrive at

$$
\begin{aligned}
d\widetilde{p_2}(t) &= \left\{\left(\left(\alpha + \varepsilon^2 l^2(t) H t^{2H-1}\right)\varphi_t - 1\right) y_t^u \right. \\
&+ \left.\left(\alpha + \varepsilon^2 l^2(t) H t^{2H-1}\right)\chi_t\right\} dt - \varepsilon l(t)\left(\varphi_t y_t + \chi_t\right) dB_t^H .
\end{aligned} \tag{3.48}
$$

By identification drift terms between (3.47) and (3.48), we obtain the following Riccati and ordinary differential equations, respectively.

$$
\begin{cases}
\dot{\varphi}_t - \varphi_t^2 - 2\varphi_t\left(\alpha + \frac{1}{2}\varepsilon^2 l^2(t) H t^{2H-1}\right) + 1 = 0, \\
\varphi_0 = 1.
\end{cases} \tag{3.49}
$$

$$
\begin{cases}
\dot{\chi}_t - \left(\varphi_t + \alpha + \varepsilon^2 l^2(t) H t^{2H-1}\right)\chi_t = 0, \\
\chi_0 = 0.
\end{cases} \tag{3.50}
$$

By using the same identification, we get $\varphi_t z_t = -\varepsilon l(t)\left(\varphi_t y_t + \chi_t\right)$, which implies

$$
\chi_t = -\frac{1}{\varepsilon l(t)}\varphi_t z_t - \varphi_t y_t . \tag{3.51}
$$

Finally, from (3.41) and by substituting (3.51) into (3.45) we have

$$
u_t(z_t) = \widetilde{p_2}(t) = -\frac{1}{\varepsilon l(t)}\varphi_t z_t , \tag{3.52}
$$

where $\varphi(t)$ is determined by (3.49).

Theorem 3.5 *We assume that* (φ_t, χ_t) *are the solutions of system* (3.49) *and* (3.50), *then the optimal control of our linear-quadratic risk-sensitive stochastic optimal control problem* $\{(3.38),(3.39),(3.40)\}$ *has the state feed-back form* (3.52).

3.5.2 Explicit solution of the Riccati equation

In this part we will find the explicit solution of the Riccati differential equation (3.49). For simplicity we set

$$N = -\left(\alpha + \frac{1}{2}\varepsilon^2 l^2(t) H t^{2H-1}\right). \tag{3.53}$$

Therefore we have $\dot{\varphi_t} = \varphi_t^2 - 2\varphi_t N - 1$.

It is easily to find the discriminant $\Delta = 4(N^2+1)$, And through it we can find two solutions $\varphi_1 = N - \sqrt{N^2+1}$ and $\varphi_2 = N + \sqrt{N^2+1}$.

Then we get $dt = \frac{1}{\varphi_t^2 - 2N\varphi_t - 1} d\varphi_t$.Integrating from 0 to t, taking into consideration $\varphi_0 = 1$, then after some simplification, we arrive at last

$$\varphi_t = \frac{\left(N+\sqrt{N^2+1}\right) - \left(N-\sqrt{N^2+1}\right)\left|\frac{1-\left(N+\sqrt{N^2+1}\right)}{1-\left(N-\sqrt{N^2+1}\right)}\right| \exp\left(2t\sqrt{N^2+1}\right)}{1-\left|\frac{1-\left(N+\sqrt{N^2+1}\right)}{1-\left(N-\sqrt{N^2+1}\right)}\right| \exp\left(2t\sqrt{N^2+1}\right)},$$

if we putting

$$A = \left|\frac{1-\left(N+\sqrt{N^2+1}\right)}{1-\left(N-\sqrt{N^2+1}\right)}\right| \exp\left(2t\sqrt{N^2+1}\right), \tag{3.54}$$

we obtain

$$\varphi_t = \frac{\left(N+\sqrt{N^2+1}\right) - \left(N-\sqrt{N^2+1}\right)A}{1-A}. \tag{3.55}$$

Corollary 3.1 *The explicit solution of the Riccati equation* (3.49) *is given by* (3.55), *where the coefficients* N *and* A *are given by* (3.53) *and* (3.54), *respectively.*

Conclusion and Perspectives

In this thesis, as a first result, we provide a solution to a stochastic optimization problem, employing the Pontryagin stochastic maximum principle. Our problem involves a forward-backward stochastic dynamics governed by a fractional Brownian motion, where H lies within the range of 0 to 1. Leveraging the convex nature of the control domain, we deduce the optimal trajectory of this system by introducing perturbed control in lemmas 2.1, 2.2, 2.3 and 2.4. Utilizing these findings alongside variational calculus, we establish necessary optimality conditions in theorem 2.1. Additionally, by imposing certain concavity constraints on the system's drivers, we derive sufficient optimality conditions in theorem 2.2.

The second one concerns the necessary and sufficient optimality conditions for a fractional backward stochastic differential equation within a risk-sensitive framework. We adopt a methodology akin to Djehich et al. [23], employing various advanced mathematical techniques. These include the logarithmic transformation, serving as a generalization of the method introduced by Elkaroui and Hamadène in [24], transitioning from risk-neutral to risk-sensitive logarithmic quasi-martingales.

A key disparity between our risk-sensitive optimal control problem $\{(3.1), (3.2), (3.3)\}$ and conventional risk-neutral problems lies in the exponential of integral type cost functional (3.2). To tackle this, we exploit the convexity of the Hamiltonian and introduce an exponential utility for the cost function, echoing the approach in [20]. Furthermore, we establish a connection between expected exponential utility and a backward stochastic differential quadratic equation, as detailed in Lemma 3.2.

After conducting this study, we have identified several avenues for future research. One such area of interest involves addressing optimal control problems where the state equation is influenced by fractional Brownian motion. Specifically, we intend to seek the following scenarios:

- We plan to explore the stochastic maximum principle for control systems with controlled jump diffusions.
- Additionally, we intend to investigate the singular risk-sensitive stochastic maximum principle.
- Furthermore, we aim to examine the risk-sensitive stochastic maximum principle of Mean-Field type, particularly in the context of a relaxed control problem, to provide insights into its behavior and potential optimizations.

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