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Stability and Bifurcations and Control in Fractional Order Chaotic Systems

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Dedication

I dedicate this effort to,

My Parents "Abd Elmadjid and Salima" who motivate and encourage me.

and never stop giving of themselves in countless ways,

May God bless them all with good health and full lifespans...

Those that I love a lot. Who have been a

staunch supporter of me throughout all of my endeavors.

My beloved daughter " Ines ", my husband "Younes", my sister "Leila" and brothers "Yacine, Sami and Anis" especially for you. . .

My husband's family "kabache", My entire family, friends, colleagues and for everyone who has helped me in any way,

no matter how small or large their contribution...

I say, Thank you

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Abstract

The inclusion of fractional-order dynamics in the study of nonlinear systems has broadened our understanding of complex behaviors, such as stability, chaos and bifurcations, and has opened up new possibilities in control theory. These systems involve derivatives and integrals of non-integer order, introducing a new level of flexibility and versatility in modeling real-world phenomena. This thesis aims to study the stability and bifurcations in a fractional order chaotic systems and the control of chaos. To achieve our goal we introduced in the first tow chapters the necessary basic notions such as: fractional derivation, chaos theory, stability of fractional systems and bifurcation theory. The main results of this thesis are presented in the last tow chapters where we gave the necessary and sufficient conditions for stability, we showed the existence of Hopf bifurcations in both cases: integer and fractional also we proved the effect of fractional order in the critical point location of Hopf's bifurcation points, stability and chaos control.

Key words: Fractional order, Dynamic system, Stability, Bifurcations, Hopf bifurcation, Chaos, Control, Effect of fractional order, Jerk system, Localisation.

Resumé

Les systèmes dynamiques d'ordre fractionnaire ont gagné en popularité dans divers domaines en raison de leur capacité à modéliser des comportements complexes avec plus de précision que les systèmes d'ordre entiers traditionnels. Ces systèmes impliquent des dérivées et des intégrales d'ordre non entier, introduisant un nouveau niveau de flexibilité et de polyvalence dans la modélisation des phénomènes du monde réel. Cette thèse a pour but d'étudier la stabilité et la bifurcation dans un système chaotique d'ordre fractionnaire ainsi le contrôle du chaos. Pour avoir notre but on a introduit dans les deux premiers chapitres les notions de base nécessaires tels que: la dérivation fractionnaire, la théorie du chaos et la stabilité des systèmes fractionnaires ainsi la théorie des bifurcations. Les principaux résultats de cette thèse se présentent dans le troisième et le quatrième chapitre où on a donné les conditions nécessaires et suffisantes pour la stabilité, on a montré l'existence des bifurcations de Hopf dans les deux cas : entier et fractionnaire ainsi on a preuvé l'effet de l'ordre fractionnaire dans l'emplacement de point critique des points de bifurcations de Hopf ainsi sur la stabilité et le contrôle du chaos.

Mots clés : Ordre fractionnaire, Système dynamique, Stabilité, Bifurcations, Bifurcation de Hopf, Chaos, Contrôle, Effet de l'ordre fractionnaire, Système Jerk, Localisation.

ملخص

لقد اكتسبت الأنظمة الديناميكية ذات الرتب الكسرية شعبية في مختلف المجالات، نظرا لقدرتها على نمذجة السلوكيات المعقدة بشكل أكثر دقة من الأنظمة التقليدية ذات الرتب الصحيحة. تتضمن هذه الأنظمة مشتقات وتكاملات ذات رتب غير صحيحة، مما يوفر مستوى جديدا من المرونة والتنوع في نمذجة ظواهر العالم الحقيقي.

تهدف هذه الأطروحة إلى دراسة الاستقرار والتشعب في النظام الفوضوي الكسري والسيطرة على الفوضى. ولتحقيق هدفنا قدمنا في الفصل الأول والثاني المفاهيم الأساسية اللازمة مثل: الاشتقاق الكسري، ونظرية الفوضى واستقرار الأنظمة الكسرية، ونظرية التشعب. أهم نتائج هذا العمل تم عرضها في الفصلين الثالث والرابع حيث قدمنا الشروط اللازمة والكافية للاستقرار، وبينا وجود تشعبات هوف في الحالتين: الأنظمة ذات الرتب الصحيحة والكسرية كما أثبتنا تأثير الرتب الكسرية في موقع النقطة الحرجة لنقاط تشعب هوف، الاستقرار والسيطرة على الفوضى.

الكلمات المفتاحية: الرتب الكسرية، النظام الديناميكي، استقرار، التشعبات، تشعبات هوبف، الفوضى، التحكم، تأثير الرتب الكسرية، نظام **jerk**؛ موقع.

Contents

List of Figures

Introduction

In recent years, several researchers have focused their attention on the study of fractional calculus, it has been found that many systems in interdisciplinary fields can be described by fractional differential equations. Indeed, there are many applications such as:

Physics and engineering: Fractional calculus provides a more accurate way for explaining complex behaviours and phenomena numerous branches of physics and engineering, such as viscoelasticity, diffusion and heat conduction, as well as electromagnetic, complex circuit analysis and transmission line phenomena [45, 49, 66] .

Geological, atmospheric and oceanic investigations, the capacity of fractional calculus to capture memory effects and non-local phenomena is extremely useful. These applications help us comprehend Earth's processes and events better, which helps us make more accurate forecasts and wiser decisions [61].

Communication Fractional calculus offers techniques to account for intricate and memory-reliant processes in communication that traditional calculus could miss. This may result in communication systems that are more effective and dependable.

Biology, epidemiology and healthcare Fractional calculus is useful in biology, epidemiology and healthcare because these programs help people understand diseases better, make more precise forecasts and make better healthcare decisions [65, 82] .

Transportation Fractional calculus offers an invaluable foundation for comprehending and improving transportation systems, resulting in more efficient traffic flow, safer vehicle dynamics and effective logistical operations [33]. So the efficiency of these equations in the modelling of many real-world problems motivated a lot of researchers to investigate their quantitative and qualitative aspects.

A wide range of phenomena in both living and nonliving systems that exhibit nonlinear behavioral changes over time are referred to as "Dynamic Systems" behavioral change can occur in clouds formations in the sky or in a chemical reaction; it can reflect a sudden transition of gait pattern in a biological system, or a shift in flying pattern formation in a flock of birds. Dynamic systems aim to study the complex processes driving those changes. They are complex because whether occurring in a single system/organism, or a group of individuals, change occurs as the product of multileveled interactions between the various elements constituting these systems.

The stability analysis of fractional-order systems is a complex and intriguing area of study that has gained significant attention due to its unique characteristics compared to integer-order systems. Fractional-order systems exhibit behaviors that are not present in integer-order systems, such as non-local memory effects and the ability to model complex physical phenomena more accurately. Indeed, in the theory of stability linear systems, of integer order, we know well that a system is stable, if and only if the roots of the characteristic polynomial have negative real parts, that is to say located on the left half of the complex plan. Furthermore, the notion of the stability of linear fractional systems is a little different from that of classical systems. In fact, we have clearly observed that stable fractional systems may well have roots of the characteristic polynomial in the right half of the complex plane, which shows that fractional systems are memory systems which are more stable (when the fractional order is less than 1) compared to integer-order systems, and therefore, they exhibit dynamic behavior much more sophisticated. Recently, stability analysis of fractional differential equations has been studied and some basic analytical results are obtained, we can refer to [20, 21, 46, 54, 55, 60] for the recent history of stability analysis of fractional systems.

With in the field of dynamical systems theory, bifurcation theory examines how a system's behavior changes qualitatively when one or more parameters are varied. These changes often involve the emergence of new solutions, alterations in stability and the creation of complex dynamical patterns. Bifurcations are critical points where the qualitative character of the system's behavior undergoes a significant transformation.

The most active topics of interest under current study, investigated in the field of chaotic fractional-order dynamical systems, are the study of Hopf bifurcation and chaos control, therefore, when a fractional-order system undergoes a Hopf bifurcation, loses its stability and becomes finally a chaotic fractional-order system, and how to control and synchronize this chaos have been very important problems. Due to this fact, the Hopf bifurcation of integer order has been thoroughly investigated during the past long time [1, 9, 30, 57, 85] . However, there are a few results about fractional Hopf bifurcation. In [2, 4, 13, 16, 21, 25, 49] the authors proposed some works about fractional Hopf bifurcation.

Currently, it has been discovered that a number of fractional-order differential systems, including the Rossler system [47], Chua circuit [31], Duffing system [8], jerk model [6], Chen system [51], the fractional-order Lü system [83] and Newton-Leipnik system [70], exhibit chaotic behaviors.

These systems' control has been viewed as a difficult task because of their delicate and complicated dynamics. However, due to its ability to improve control performance, enable correct modeling, manage complex dynamics, provide robustness, regulate multi-physics systems, account for time delays and accommodate future technologies, fractional order system control is crucial. Several methods have been proposed to control these systems and stabilize their behavior. More information is available at [3, 14, 25, 26, 35, 62, 68, 81] .

Numerous emerging industries and technologies use fractional order control such as robotics, mechatronics, biomedical systems, renewable energy systems, chemical processes and others are included . Through the use of fractional order control techniques, these systems may be better managed and optimized, improving performance, energy efficiency and safety.

Motivated by the above considerations, the work presented in this thesis based on the study of stability, bifurcations and control in fractional chaotic system, it is divided into two parts. We start with preliminary chapters: 1 and 2, and the second part presents our main results. The content of each chapter is outlined as follows:

The first chapter, is the introductory chapter which comprises basic notions concerning the calculus fractional and general notions of the theory of dynamical systems, along with some findings about the stability of non-integer order systems.

In the second chapter, we briefly recall some characteristics of chaos and its applications, also we introduce the preliminary notions and properties of bifurcations and their types with basic examples.

In the third chapter, we explore the stability of a fractional-order chaotic "Jerk system " by applying the Routh-Hurwitz criteria. The analysis involves conducting a Hopf bifurcation study with respect to the fractional order and a specific parameter. Conditions for ensuring the occurrence of Hopf bifurcation are proposed under the bifurcation parameter, both for the fractional order and its corresponding integer order. We also carry out numerical investigations using both commensurate and incommensurate fractional chaotic systems. The findings reveal that the critical Hopf bifurcation value undergoes a shift in location under the influence of the fractional order.

Finally, the fourth chapter is mainly devoted to the principles of control of chaotic systems, tow control methods are summarized , as well as their applications . We propose another version of "Jerk System" to apply the theories of stability and control using the generalized Routh-Hurwitz criterion to fractional order.

The thesis is concluded with a conclusion and perspectives.

Part I

Preliminaries

Chapter 1

Stability of Fractional Dynamical Systems

The general concepts of fractional dynamical system are presented in this chapter, along with some findings about the stability of non-integer order systems.

1.1 Fractional Differentiation

The fractional derivative is a generalization of the concept of differentiation to non-integer order. It is often used in various fields of science and engineering, especially in the study of systems exhibiting complex and anomalous behaviors. The construction of fractional derivatives involves several mathematical approaches, among which the most common methods are the Grünwald-Letnikov definition, the Riemann-Liouville definition and the Caputo definition.[23, 41, 67].

1.1.1 Grunwald-Letnikov Fractional Derivative

The Grunwald-Letnikov definition of the fractional derivative is based on the idea of using a discrete approximation to define the fractional derivative.

For a function $f(t) \in C\ [a, b]$, the first derivative of the function $f(t)$ is defined by :

$$f'(t) = D^{(1)}f(t) = \frac{df(t)}{dt} = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h}. \tag{1.1}$$

If we apply this formula again, we get the second-order derivative:

$$\begin{aligned} f''(t) = D^{(2)}f(t) = \frac{d^{(2)}f(t)}{dt} &= \lim_{h\to 0} \frac{f'(t) - f'(t-h)}{h} \\ &= \lim_{h\to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}\ , \end{aligned} \tag{1.2}$$

we can generalize this formula for the $n^{th}-$ derivative

$$D^{(n)}f(t) = \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(t - mh), \tag{1.3}$$

where $\binom{n}{m} = \frac{n(n-1)......(n-m+1)}{m!}$.

We pose $h = \frac{t-a}{n}$, where a is a real number, the generalization in the sense of Grünwald-Letnikov of this formula for the non-integer order α $(n-1 < \alpha < n)$ is défined by:

$${}_{a}^{GL}D_t^{\alpha} = \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} \frac{\Gamma(m-\alpha)}{\Gamma(m+1)\Gamma(-\alpha)} f(t - mh)\ , \tag{1.4}$$

such as:

$$(-1)^m \binom{\alpha}{m} = \frac{-\alpha(1-\alpha)(2-\alpha)....(m-\alpha-1)}{m!} = \frac{\Gamma\,(m-\alpha)}{\Gamma(m+1)\Gamma(-\alpha)}\ . \tag{1.5}$$

Riemann-Liouville Integral

Cauchy's formula for repeated integration

$$I^n f(t) = \int_a^t ds_1 \int_a^{s_1} ds_2 .... \int_a^{s_{n-1}} f(s_n) ds_n = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s) ds, \; n \in \mathbb{N}^* \;, \quad (1.6)$$

holds for $n \in \mathbb{N}, a, t \in \mathbb{R}, t > a$. Replacing $(n-1)!$ by $\Gamma(\alpha)$ and the power n in the integrand with some $\alpha \in \mathbb{R}_+$, we have Riemann-Liouville fractional integral:

Definition 1.1.1 *The Riemann-Liouville integral is defined by*

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds \quad , \; \alpha > 0, t > a, \quad (1.7)$$

where $\Gamma(.)$ is the gamma function.

Proposition 1.1.1 *for* $f \in C([a;b))$ *:*

1. $I_{a+}^{\alpha}[I_{a+}^{\beta} f(t)] = I_{a+}^{\alpha+\beta} f(t) \qquad ,\alpha, \beta > 0.$
2. $I_{a+}^{\alpha}[I_{a+}^{\beta} f(t) = I_{a+}^{\beta} I_{a+}^{\alpha} f(t) \qquad , \alpha, \beta > 0.$
3. $\frac{d}{dx}\left[I_{a+}^{\alpha} f(t)\right] = I_{a+}^{\alpha-1} f(t) \qquad ,\alpha > 1.$

1.1.2 Riemann-Liouville Fractional Derivative

Definition 1.1.2 *For* $m \in \mathbb{N}^*$*; and* $a \in \mathbb{R}$*; the Riemann-Liouville derivative with fractional-order* α *of function* $f \in C([a;+\infty);\mathbb{R})$ *is given by*

$$^{RL}D_a^{\alpha} f(t) = \begin{cases} D^m I_a^{m-\alpha} f(t) = \frac{d^m}{dt^m}\left(\frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau\right) & , m-1 < \alpha < m. \\ \frac{d^m}{dt^m} f(t) & , \alpha = m. \end{cases} \quad . \quad (1.8)$$

(1) The Riemann -Liouville fractional derivative is a linear operator.

(2) The Riemann-Liouville fractional differential operator is the left inverse operator of the fractional integral I_a^α e.i $\left({}^{RL}D_a^\alpha\right)\left(I_a^\alpha f\right)(t) = f(t)$.

Example 1.1.1

$f(t)$	$I_{a+}^\alpha f(t)$	$D_{a+}^\alpha f(t)$	Specifications
$(t-a)^\beta$	$\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta}$	$\frac{\Gamma(\beta+1)}{\Gamma(-\alpha+\beta+1)}(t-a)^{\beta-\alpha}$	$a \in \mathbb{R}$ and $\alpha > 0, \beta > -1$
C	$\frac{C}{\Gamma(\alpha+1)}(t-a)^\alpha$	$\frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}$	$a \in \mathbb{R}$ and $\alpha \in \mathbb{R}, C \in \mathbb{R}$
$e^{\lambda t}$	$\lambda^{-\alpha} e^{\lambda t}$	$\lambda^{+\alpha} e^{\lambda t}$	$a = -\infty, \alpha > 0, \lambda > 0$
$e^{-\lambda t}$	$\lambda^{-\alpha} e^{-\lambda t}$	$\lambda^{\alpha} e^{-\lambda t}$	$a = +\infty, \alpha > 0, \lambda > 0$

1.1.3 Caputo Fractional Derivative

Definition 1.1.3 *For $m-1 < \alpha < m$; $m \in \mathbb{N}^*$; and $f \in C^m([a;+\infty))$, the fractional operator:*

$$
{}^C D_a^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} \frac{d^m}{ds^m} f(s) ds = I_a^{m-\alpha} D^m f(t) & , m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t) & , \alpha = m \end{cases} , \quad (1.9)
$$

is called the Caputo fractional derivative.

Proposition 1.1.2 *For $m-1 < \alpha < m$; $m \in \mathbb{N}^*$; et $f \in C^m([a;+\infty))$, We have the following properties:*

1. $({}^C D_a^\alpha)$ is a linear operator.

2. $({}^C D_a^\alpha)(I_a^\alpha f)(t) = f(t)$.

3. If ${}^C D_a^\alpha f(t) = 0$, so $f(t) = \sum_{j=0}^{m-1} c_j (t-a)^j$, $(c)_{j=0,\dots,m-1} \in \mathbb{R}$.

4. $I_a^\alpha({}^C D_a^\alpha f(t)) = f(t) + \sum_{j=0}^{m-1} c_j (t-a)^j, (c)_{j=0,\dots,m-1} \epsilon \mathbb{R}$

Example 1.1.2 *We consider the following function :*$f : t \to t^\beta . \beta \geq 0$

For $0 < m-1 < \alpha < m$*, we have :*

$$ {}^C D_a^\alpha f(t) = I^{m-\alpha} (D^m) t^\beta, $$

where:

$$ {}^C D^m t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)} t^{\beta-m}, $$

so,

$$ I^{m-\alpha} = \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)} t^{\beta-m} \right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha} s^{\beta-m} ds, $$

assume that :$s = yt \Rightarrow ds = tdy$*, we find ,*

$$ \begin{aligned} \int_0^t (t-s)^{m-\alpha} s^{\beta-m} ds &= \int_0^1 (t-ty)^{m-\alpha-1} (ty)^{\beta-m} tdy \\ &= \int_0^1 t^{m-\alpha-1}(1-y)^{m-\alpha-1} y^{\beta-m} t^{\beta-m+1} \\ &= t^{\beta-\alpha} \int_0^1 (1-y)^{m-\alpha-1} y^{\beta-m} dy \\ &= t^{\beta-\alpha} \beta(m-\alpha, \beta-m+1) \\ &= t^{\beta-\alpha} \frac{\Gamma(m-\alpha)\Gamma(\beta-m+1)}{\Gamma(\beta-\alpha+1)}, \end{aligned} $$

therefore,

$$ I^{m-\alpha} \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)} t^{\beta-m} \right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m)\Gamma(m-\alpha)} \frac{\Gamma(m-\alpha)\Gamma(\beta-m+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. $$

We have :

$$^{C}D^{m}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha},$$

for $\beta = 0$; one obtian :

$$^{C}D^{m}t^{0} = D^{m}1 = 0.$$

We can say that the fractional derivative of any constant function using Caputo definition is consistent, since it is equal to zero.

1.1.4 Laplace Transforms of Fractional Derivatives

Basic Laplace Transform Tools

Definition 1.1.4 *The Laplace transform of a function $f(t)$ is define as follow :[78]*

$$F(s) = L\{f(t); s\} = \int_{0}^{\infty} e^{-st} f(t)dt, s \in \mathbb{C} \tag{1.10}$$

For this integral to exist we must have

$$e^{-\alpha t}|f(t)| \leq M \qquad \textit{for all } T < t,$$

where M and T are positive constants. The original function $f(t)$ can be recovered from the Laplace transform.

Definition 1.1.5 *The inverse Laplace transform $f(t)$ for , $s \in \mathbb{C}$ and $F(s)$ is the Laplace transform is defined as*

$$f(t) = L^{-1}(F(s))(t) = \int_{c-i\infty}^{c+i\infty} e^{st} f(s) ds. \qquad c = \mathrm{Re}(s) > c_0, \tag{1.11}$$

where c_0 is the convergence index of the integral (1.11).

Definition 1.1.6 *The convolution of two functions is given by :*

$$f(t) * g(t) = \int_0^t f(t-\tau) g(\tau) d\tau = \int_0^t g(t-\tau) f(\tau) d\tau. \tag{1.12}$$

The Laplace transform of the convolution of two functions $f(t)$ and $g(t)$ is defined as:

$$L\left\{f(t) * g(t); s\right\} = F(s)G(s). \tag{1.13}$$

The Laplace transform of the derivative of order n of the function f can be written:

$$L\left\{f^{(n)}(t); s\right\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^{(k)} f^{(n-k-1)}(0). \tag{1.14}$$

Laplace Transform of the Riemann-Liouville Fractional Derivative

The Laplace transform of Riemann-Liouville fractional derivative is defined by:

$$L\left\{{}_0^{RL}D^\alpha f(t); s\right\} = s^\alpha F(s) - \sum_{k=0}^{m-1} s^k \left[{}_0D^{(\alpha-k-1)} f(t)\right]_{t=0}, m-1 \le \alpha < m. \tag{1.15}$$

Laplace Transform of Caputo Fractional Derivative

The Laplace transform of Caputo's fractional derivative is defined by:

$$L\left\{{}_0^{C}D^\alpha f(t); s\right\} = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0), m-1 \le \alpha < m.$$

Laplace Transform of the Grunwald-Letnikov Fractional Derivative

- First we shall consider the case $0 < \alpha < 1$, the Laplace transform of Grunwald-Letnikov fractional derivative is defined by:

$$L\left\{{}_{GL}D^{\alpha}_{0+}f(t);s\right\} = \frac{f(0)}{s^{1-\alpha}} + \frac{1}{s^{1-\alpha}}\left(sF(s) - f(0)\right) = s^{\alpha}F(s). \tag{1.16}$$

- If $\alpha > 1$ the Laplace transform of the Grünwald-Letnikov fractional derivative does not exist in the classical sense, because in such a case we have non-integrable functions in the sum of the formula (1.14).

1.1.5 Comparison Between Caputo and Riemann-Liouville Derivative's

- The relation linking the derivative in the sense of Riemann-Liouville to that of Caputo is given by:

$$^{C}D^{\alpha}_{a}f(t) =^{RL} D^{\alpha}_{a}f(t) - \sum_{k=0}^{m-1}\frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma\left(k-\alpha+1\right)}, \tag{1.17}$$

where $f \in C([a;b))$; and $m-1 < \alpha < m$, $m \in \mathbb{N}^{*}$,

We note that if $f^{(k)}(a) = 0$ for $k = 0, 1, 2, \ldots.., n-1$, we will have

$$^{C}D^{\alpha}_{a}f(t) =^{RL} D^{\alpha}_{a}f(t).$$

- The derivative of a constant function in the sense of Caputo is zero, on the other hand by Riemann-Lioville it is $\frac{C}{\Gamma(1-\alpha)}\left(x-\alpha\right)^{-\alpha}$.

- The Laplace transform formula of the fractional derivative in the Riemann-Liouville sense is given by:

$$L\left\{{}^{RL}_{0}D^{\alpha}f(t);s\right\} = s^{\alpha}F(s) - \sum_{k=0}^{m-1}s^{k}\left[{}_{0}D^{(\alpha-k-1)}f(t)\right]_{t=0}, m-1 \leq \alpha < m.$$

The Laplace transform formula of the fractional derivative in the Caputo sense is given by:

$$L\left\{{}_0^C D^\alpha f(t); s\right\} = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0), m-1 \leq \alpha < m.$$

- Caputo's definition is well used since its Laplace transformation allows initial conditions taking the same forms as those of integer-order derivatives, which have clear physical interpretations and applications behavior in the actual modeling process.
- We have the derivative of Caputo is ${}^C D_a^\alpha f(x) = I_a^{n-\alpha}\left[\frac{d^n}{dx^n} f(x)\right]$ on the other hand the Riemann-Lioville derivative is ${}^{Rl} D_a^\alpha f(x) = \frac{d^m}{dx^m}\left[I_a^{m-\alpha} f(x)\right].$

1.2 Dynamical Systems

Dynamical systems theory is a broad and interdisciplinary field that studies the behavior of systems over time. It explores concepts like stability, bifurcations, attractors and chaos, which are crucial in understanding the long-term behavior of dynamic systems. A dynamical system described by a mathematical function presents two types of variables : dynamic and static. Dynamic variables are fundamental quantities that change over time; static variables, also called parameters of the system, are fixed.

We will consider two types of dynamical systems: those with continuous (real) time $T = \mathbb{R}$ and those with discrete (integer) time $T = \mathbb{Z}$.

1.2.1 Continuous Dynamical Systems

In the case where time is continuous, the fractional dynamical system, corresponding to a vector field f; is defined as :

$${}^C D^\alpha x = f(t, x, \mu) \quad , \tag{1.18}$$

where $x \in \Omega \subseteq \mathbb{R}^n$, $\mu \in D \subseteq \mathbb{R}^p$,$\alpha = [\alpha_{1,}\alpha_2, ..., \alpha_n]^T$, $f : \mathbb{R} \times \Omega \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$ and ${}^cD^\alpha$ designates the derivation operator of Caputo.

1.2.2 Discrete Dynamical Systems

In the case where time is discrete, the dynamical system is presented by an application (iterative function),

$$x_{n+1} = f(n, x_n, \mu), \; n \; \in \; \mathbb{N}. \tag{1.19}$$

For example discrete-time systems appear naturally in ecology and economics when the state of a system at a certain moment of time t completely determines its state after a year, say at $t + 1$.

1.2.3 Autonomous and Non-autonomous Systems

When f depends explicitly on time, the system (1.18) is said **non-autonomous** system. Otherwise, we say the system (1.18) is **autonomous**.

1.2.4 Poincaré Section

A Poincaré section, named after the French mathematician Henri Poincaré, is a way to analyze the long-term behavior of a dynamic system, particularly in the context of continuous-time dynamical systems described by differential equations.

Making a Poincaré section amounts to cutting the trajectory in phase space, in order to study the intersections of this trajectory with, for example in dimension three, a plan. We then move from a continuous time dynamic system to a system discrete-time dynamics. It is demonstrated that the properties of the system are preserved after completing a judiciously chosen section of Poincare.

Definition 1.2.1 *Consider a dynamical system defined by :*

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0 \end{cases}, \tag{1.20}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n, n > 2$, and $\Omega \subset \mathbb{R}^{n-1}$. The intersection of the plane Ω and the trajectory of the system (1.20) allows us to define a function H as follows:

$$\begin{cases} H : U \subset \Omega \to \Omega \\ x(t) = \varphi\left(x(t), \delta\right) \end{cases},$$

where δ designates the time, it takes for the trajectories $x(t)$ and to start from U to arrive Ω . The function H is called the first return application.

1.3 Fractional Differential Equations (FDE)

1.3.1 Cauchy Problem

Generally, a FDE admits an infinite number of solutions, to choose between the different solutions the one which describes the problem, it is necessary to consider others data and other conditions which depend on the value of the solution in one initial instant t_0 denoted $y(t_0)$: This condition is called the initial condition.

We start by providing a general definition of a differential equation of fractional order before discussing the existence and uniqueness of a Cauchy problem for a fractional differential equation.

Definition 1.3.1 *A differential equation of fractional order caputo type is given by the equation:*

$$^{C}D^{\alpha}x = f(x),$$

where $\alpha > 0, \alpha \notin \mathbb{N}$, $n = [\alpha] + 1$, $f : A \subset \mathbb{R} \to \mathbb{R}$, and $^{C}D^{\alpha}$ designates the derivation operator of Caputo.

We consider the Cauchy problem for a fractional differential equation Caputo type :

$$\begin{cases} {}^{C}D^{\alpha}x = f(x), t \in [0,T] \\ {}^{C}D^{\alpha}x^{k}(0) = b_k, k = 0, 1, ...n-1 \end{cases}, \tag{1.21}$$

where $T > 0, b_k \in \mathbb{R}$.

1.3.2 Existence and Uniqueness Theorem

The following theorem allows us to affirm the existence and uniqueness of the solution of problem with initial values (1.21)

Theorem 1.3.1 *Let $K > 0, h^* > 0$, $x_0^{(i)} \in \mathbb{R}, i = 0, 1, ..., n-1$, and $f : G = [0, h^*] \times \mathbb{R} \to \mathbb{R}$, a continuous function, satisfying the Lipschitz condition by contribution to x:*

$$|f(t, x_1) - f(t, x_2)| < K |x_1 - x_2|,$$

where in the case $\alpha \in (0, 1)$, the parameter h is given by the relation

$$h = \min \left\{ h^*, (K\Gamma(\alpha+1)/M)^{\frac{1}{\alpha}} \right\},$$

and

$$M = \sup_{t,z\epsilon\mathbb{R}} |f(t,z)|,$$

then, the problem (1.21), admits a single solution $x \in C[0, h]$.

Theorem 1.3.2 *Under the assumptions of theorem, the problem with initial conditions (1.21) is equivalent to the Volterra integral equation:*

$$x(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau. \tag{1.22}$$

1.3.3 Numerical Solving Fractional Equations

In general, to solve nonlinear differential equations, we use numerical methods, so that analytical resolution in this case is generally impossible. There are several methods for the numerical resolution of diferential equations of fractional order, namely: the method of diferences Grünwald-Letnikov fractional methods, the Adomian decomposition method, the variational iteration method and the Adams-Basheforth-Moulton method. In this part, we focus on the numerical method : the Adams-Basheforth-Moulton method.

The Adams-Basheforth-Moulton Method

The Adams-Basheforth-Moulton method is a numerical method introduced by Diethelm and Freed [22], based on the Volterra equation (1.22)

We assume that x_k is the approximation of $x(t_j)$ for all $j = 1, ..., k$ in the interval $[0, T]$

In order to obtain x_{k+1}, we use the quadrature product formula of trapezoids, where the nodes t_j for $j = 0, ...k+1$, and interpret the function $(x_{k+1} - .)^{\alpha-1}$. We obtain the approximation:

$$\int_0^{k+1} (t_{k+1} - \tau)^{\alpha-1} g(\tau) d\tau \simeq \sum_{j=0}^{k+1} a_{j,k+1} g(tj),$$

where

$$a_{j,k+1} = \int_0^{k+1} (t_{k+1} - \tau)^{\alpha-1} \omega_{j,k+1} d\tau,$$

and

$$\omega_{j,k+1} = \begin{cases} \frac{\tau - t_{j-1}}{t_j - t_{j-1}}, & \text{if } t_{j-1} < \tau < t_j \\ \frac{t_{j+1} - \tau}{t_{j+1} - t_j}, & \text{if } \quad t_j < \tau < t_{j+1} \\ 0, & \text{else} \end{cases},$$

and like $t = jh$, for all $j = 0, ...k+1$,we have

$$a_{j,k+1} = \begin{cases} \frac{h^{\alpha}}{\alpha(\alpha+1)} \left(k^{\alpha+1} - (k-\alpha)(k+1)^{\alpha}\right), \text{ if } j = 0 \\ \frac{h^{\alpha}}{\alpha(\alpha+1)}((k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}), \text{if } 1 \leq j \leq k \\ \frac{h^{\alpha}}{\alpha(\alpha+1)}, \text{if } j = k+1 \end{cases}, \qquad (1.23)$$

then, we find the implicit equation of the Adams-Moulton one-step method:

$$x_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^{j}}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}) \right). \qquad (1.24)$$

For the preacher's formula we adopt the same method explained above but this time we replace the integral by a formula for the rectangle :

$$\int_{0}^{k+1} (t_{k+1} - \tau)^{\alpha-1} g(\tau) \, d\tau \simeq \sum_{j=0}^{k} b_{j,k+1} g(t_j),$$

where

$$b_{j,k+1} = \frac{h^{\alpha}}{\alpha} \left((k+1-j)^{\alpha} - (k-j)^{\alpha}\right), \qquad (1.25)$$

then we have:

$$x_{k+1} = \sum_{k=0}^{n-1} \frac{t_{k+1}^{j}}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j). \qquad (1.26)$$

The algorithm of the Adams-Bashforth-Moulton method is well determined by the equations (1.24) and (1.26) with the weights $a_{j,k+1}$ and $b_{j,k+1}$ being defined respectively according to the equations (1.23) and (1.25).

1.4 Stability of Fractional Dynamical System

The problem of stability consists of studying the behavior of a given system after it has suffered a disturbance from the move away from its equilibrium position. In all that follows, we consider the following fractional differential system:

$$^{C}D^{\alpha}x = f(x), \tag{1.27}$$

where $\alpha = [\alpha_1,\alpha_2, ..., \alpha_n]^T$, $i = 1, 2, ..n$, $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ a continuous function and $^{C}D^{\alpha}x$ denotes the Caputo fractional derivative .

Remark 1.4.1 *If all derivation orders $\alpha_i, i = 1, 2, ..., n$ of the system (1.27) are equal, we say that the system is commensurable. Otherwise, the system says unocommensurable*

1.4.1 Equilibrium Point

Definition 1.4.1 *A point $x_e \in \mathbb{R}^n$ is called as an equilibrium point of Eq (1.27), if $f(x_e) = 0$.*

Definition 1.4.2 *In the sense of Lyapunov the equilibrium point x_e of the system (1.27) is:*

1. ***Stable** if*

$$\forall\varepsilon > 0, \exists\ \delta > 0 : \ \|x(t_0) - x_e\| < \delta \Rightarrow \|x(t, x(t_0)) - x_e\| < \varepsilon, \forall\ t \geq t_0. \tag{1.28}$$

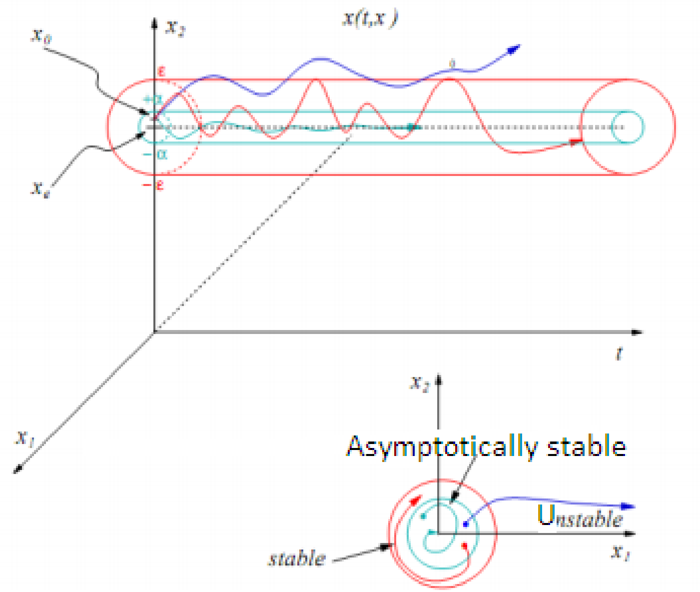
2. ***Asymptotically stable** if **stable** and :*

$$\exists\ \delta > 0 : \ \|x(t_0) - x_e\| < \delta \Rightarrow \lim_{t\to\infty} \|x(t, x(t_0)) - x_e\| = 0. \tag{1.29}$$

3. ***Exponentially stable** if:*

$$\forall\varepsilon > 0, \exists\ \delta > 0 : \ \|x(t_0) - x_e\| < \delta \Rightarrow \|x(t, x(t_0)) - x_e\| < a\,\|x(t_0) - x_e\| \exp(-bt), \ \forall\ t > t_0. \tag{1.30}$$

4. ***Unstable** if equation (1.28) is not satisfied.*



1.4.2 Stability of Autonomous Linear Systems

In the theory of the stability of linear systems with invariant time, we know well that a system is stable if the roots of the characteristic polynomial are negative or with negative real parts if they are complex conjugates and therefore located on the left half of the complex plane. Furthermore, in the case of linear fractional systems with invariant time, the definition of stability is different from integer order systems. Indeed, the interesting notion is that fractional systems can indeed have roots in the right half of the complex plane ([55] [64][54]).

Theorem 1.4.1 *Consider the fractional order autonomous system*

$$\begin{cases} {}^{C}D^{\alpha}x(t) = Ax(t) \\ x(t_0) = x_{0,} \end{cases}, \tag{1.31}$$

where ${}^{C}D^{\alpha}$ *is the Caputo fractional derivative operator,* $x \in \mathbb{R}^n$, $0 < \alpha < 1$ *and* $A \in \mathbb{R}^n \times \mathbb{R}^n$.

1. The system (1.31) is asymptotically stable, if and if only if, $|\arg(\lambda)| > \alpha\frac{\pi}{2}$, for all λ: eigenvalues of the matrix A, furthermore the state vector $x(t)$ tends towards 0 and verifies the following condition :$\|x(t)\| < Nt^{-\alpha}$, $t > 0, \alpha > 0$.

2. The system (1.31) is stable, if and only if, the condition :$|\arg(\lambda)| \geq \alpha\frac{\pi}{2}$ is verified for any eigenvalue of the matrix A, and critical eigenvalues satisfy $|\arg(\lambda)| = \alpha\frac{\pi}{2}$ have a geometric multiplicity which coincides with their algebraic multiplicity.

3. The system (1.31) is unstable, if there exists an eigenvalue of A verifying $|\arg(\lambda)| < \alpha\frac{\pi}{2}$.

Remark 1.4.2 *if* $1 < \alpha < 2$*, the system (1.31) is asymptotically stable, if and if only if,* $|\arg(\lambda_i)| > \alpha\frac{\pi}{2}$*, for all* $i = 1, 2, ..., n$

The following figure shows stable and unstable regions

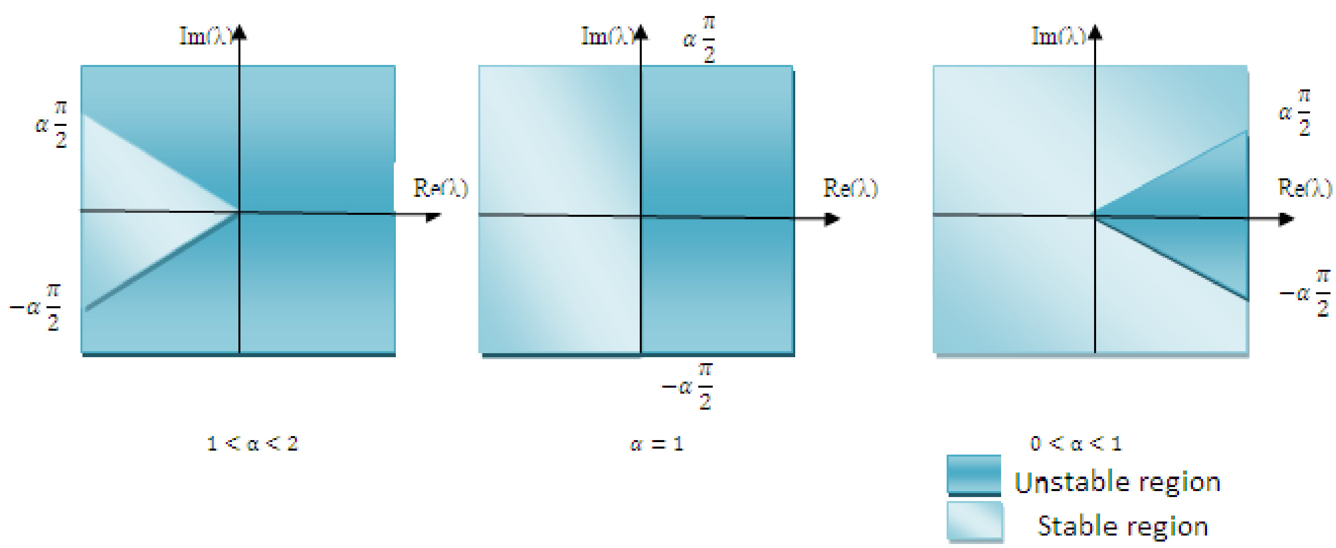


Figure 1.1: Stability Regions

Corollary 1.4.1 *If* $\alpha_1 \neq \alpha_2 \neq ...... \neq \alpha_n$ *and all* α_i *are rational numbers between* 0 *and* 1*, such that* $\alpha_i = \frac{v_i}{u_i}; u_i, v_i \in \mathbb{Z}^+$*, let* m *the least common multiple of denomenators* u_i *where* $i = \overline{1, n}$ *and putting* $\rho = \frac{1}{m}$*, then the system (1.31) is asymptotically stable if all the roots of the equation*

$$\det\left[diag\left([\lambda^{m\alpha_1}, ...\lambda^{m\alpha_n}]\right) - A\right] = 0$$

Satisfy $|\arg(\lambda)| \geq \rho\frac{\pi}{2}$.

1.4.3 Stability of Nonlinear Systems (Linearization)

Now, consider a nonlinear fractional system given by:([78], [76])

$$^{C}D^{\alpha}x(t) = f(x(t)),\ 0 < \alpha < 1, x \ \in \ \mathbb{R}^{n}. \tag{1.32}$$

Suppose that x_e is a equilibrium point of the system (1.32), $f(x_e) = 0$.

To analyze the stability of this point, we linearize the system (1.32) (around the equilibrium point),

We define $\epsilon = x - x_e$, so

$$^{C}D^{\alpha}x(t) = f(x_e + \epsilon(t)), \tag{1.33}$$

by expansion in Taylor series of the function in of the point x_e we find:

$$f(x_e + \epsilon(t)) \approx f(x_e) + \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}\right|_{x=x_e} \epsilon(t),$$

where $\epsilon = [\epsilon_1, \epsilon_2, ..., \epsilon_n]$, we know that $f(x_e) = 0$, then:

$$f(x_e + \epsilon(t)) \approx \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}\right|_{x=x_e} \epsilon(t),$$

and Eq (1.33) becomes,

$$\begin{aligned} ^{C}D^{\alpha}x(t) &=^{C} D^{\alpha}(x_e + \epsilon\,(t)) =^{C} D^{\alpha}x_e +^{C} D^{\alpha}\epsilon\,(t) \\ &=^{C} D^{\alpha}\epsilon\,(t)\,, \quad \left(\text{because } ^{C}D^{\alpha}x_e = 0\right) \end{aligned}$$

we obtain;

$$^{C}D^{\alpha}\epsilon\left(t\right)=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}\end{pmatrix}_{x=x_e}\epsilon(t),$$

so,

$$^{C}D^{\alpha}\epsilon\left(t\right)=J_f(x_e)\epsilon(t), \tag{1.34}$$

where $J_f(x_e)$ is the Jacobian matrix associated with f at the point x_e.

We can now apply the previous theorem to study the local stability of the equilibrium solutions of the system of non-linear autonomous fractional equations (1.32).

1.4.4 Hartman-Grobman Theorem

In the study of dynamic systems, the Hartman-Grobman theorem often known as the linearization theorem is an important theorem concerning the local behavior of dynamic systems around a equilibrium point.

Consider the dynamical system (1.32), let x_e be an equilibrium point of the system (1.32) and $J_f(x_e)$ is the Jacobian matrix associated with f at the point x_e.

Definition 1.4.3 *Tow flows φ_t and ψ_t are said to be topologically equivalent in neighborhoods of equilibrium points, if there exists a homeomorphism h which sends the equilibrium point of the first flow at the equilibrium point of the second flow cand which combines the points $(h\circ\varphi_t=\psi_t\circ h)$.*

Theorem 1.4.2 *(Hartman – Grobman)*

If $J_f(x_e)$ admits pure non-zero or imaginary eigenvalues, then there exists a homeomorphism which transforms the orbits of the nonlinear flow into those of the flow linear in some neighborhood of x_e.

This theorem will allow us to link the dynamics of the nonlinear (1.32) system to the dynamics of the linearized system (1.34) .

1.4.5 Fractional Order Routh-Hurwitz Criterion

The Routh–Hurwitz stability criterion is a useful tool for investigating the stability property of linear and nonlinear dynamical systems by analyzing the coefficients of the corresponding characteristic polynomial without calculating the eigenvalues of its Jacobian matrix. Recently some of these conditions have been generalized to fractional systems of order $\alpha \in [0, 1)$.[5]

Consider the fractional system:

$$^{C}D^{\alpha}x(t) = f(x,t),$$

where $0 < \alpha \leq 1$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$.

We have seen in the previous sections the necessary and sufficient condition for the system is asymptotically stable (local) for all eigenvalues λ_i of the Jacobian matrix of f

$$|\arg(\lambda_i)|_{i=1,n} > \alpha\frac{\pi}{2}.$$

This condition poses an interesting question namely what are the conditions that all the roots of the polynomial equation

$$p(\lambda) = 0, \qquad p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ...+ a_n,$$

satisfy

$$|\arg(\lambda_i)|_{i=1,n} > \alpha\frac{\pi}{2}, \tag{1.35}$$

where all the coefficients in (1.35) are real.

For $\alpha = 1$ the solution is Routh–Hurwitz conditions, so we need a new version of this criterion.

$$a_1 > 0, \qquad \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0, ...$$

For $\alpha \in [0, 1)$ these conditions are sufficient but not necessary,so we need a new version of this criterion.

The discriminant $D(p)$ of a polynomial

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n,$$

is defined by

$$D(p) = (-1)^{n(n-1)/2} R(p, \grave{p}),$$

where $\grave{p}$ is the derivative of p and $R(p, \grave{p})$ is the result of $(2n-l)\,(2n-l)$ of $p(\lambda)$ and its derivative $\grave{p}(\lambda)$ is given by

$$R(P, \grave{P}) = \begin{vmatrix} 1 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & 1 & a_1 & \cdots & a_n & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & \cdots & 0 & 1 & a_1 & \cdots & a_n \\ n & (n-1)a_1 & .... & a_{n-1} & 0 & \cdots & 0 \\ 0 & n & (n-1)a_1 & \cdots & a_{n-1} & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & .. & 0 & n & (n-1)a_1 & .. & a_{n-1} \end{vmatrix}.$$

For $n = 3$ we have

$$D(p) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3\,(a_1)^3 - 4\,(a_2)^3 - 27\,(a_3)^2\,.$$

Remark 1.4.3 *Notice that if $D(p) > 0(< 0)$ there is an even (odd) number of pairs of complex roots for the equation $P(\lambda) = 0$,*

For $n = 3$, $D(p) > 0$ implies that all the roots are real and $D(p) < 0$ implies that there is only one real root and one complex root and its complex conjugate.

1. For $n = 1$ the condition for (1.35) is $a_1 > 0$.

2. For $n = 2$ the conditions for (1.35) are either Routh-Hurwitz or

$$a_1 < 0, \quad 4a_2 > (a_1)^2, \quad \left|\tan^{-1}\left(\sqrt{4a_2 - (a_1)^2 / a_1}\right)\right| > \alpha\pi/2.$$

3. For $n = 3$, Using the results of [5], we have the following fractional-order Routh-Hurwitz conditions:

- If $D(p) > 0$, then the necessary and sufficient condition for the equilibrium point x_e to be locally asymptotically stable is $a_1 > 0, a_3 > 0, a_1 a_2 > a_3$.
- If $D(p) < 0, a_1 \geq 0, a_2 \geq 0, a_3 > 0$, then x_e is locally asymptotically stable for $\alpha < 2/3$. However, if $D(p) < 0, a_1 < 0, a_2 < 0, \alpha > 2/3$, then x_e is unstable.
- If $D(p) < 0, a_1 > 0, a_2 > 0, a_1 a_2 - a_3 = 0$, then x_e is locally asymptotically stable for all $\alpha \in (0.1)$.
- The necessary condition for the equilibrium point x_e to be locally asymptotically stable is $a_3 > 0$.

1.4.6 Fractional-Order Extension of Lyapunov Direct Method

Lyapunov stability provides an important tool for the analysis of stability in nonlinear systems; the method consists of finding a candidate Lyapunov function for a given nonlinear system. If such a function exists, the system is stable, this method is difficult to implement, but it is much more general in scope. Note that the direct Lyapunov method gives us a sufficient condition of stability, that is to say that the system can be stable even in the face of the impossibility of finding a Lyapunov function because there is no general rule for finding such a function, however, in mechanics problems, energy is often a good candidate. Let's start by defining stability in the Mittag-Leffle sense :

Definition 1.4.4 *[50]The solution of the system (1.27) is said to be Mittag-Leffler stable* *if there exist,* $\lambda \geq 0$ *and* $b > 0$ *such that*

$$\|x(t)\| \leq \{m[x(t_0)]\, E_\alpha(-\lambda(t-t_0)^\alpha)\}^b, \tag{1.36}$$

where $0 < \alpha < 1, m(0) = 0, m(x) \geq 0$ *and* $m(x)$ *locally the Lipschitz on* $x \in B \subset \mathbb{R}^n$ *with the Lipschitz constant* m_0 *.*

Definition 1.4.5 *The solution of the system (1.27) is said to be stable in the generalized sense of Mittag-Leffler if*

$$\|x(t)\| \leq \left\{m[x(t_0)]\,(t-t_0)^{-\gamma}\, E_{\alpha,1-\gamma}(-\lambda(t-t_0)^\alpha)\right\}^b, \tag{1.37}$$

where $0 < \alpha < 1, -\alpha < \gamma < 1-\gamma, \lambda \geq 0, b > 0, m(0) = 0, m(x) \geq 0$ and $m(x)$ locally the Lipschitz on $x \in B \subset \mathbb{R}^n$ with the Lipschitz constant m_0 .

- Mittag-Leffler stability implies asymptotic stability.
- Let $\lambda = 0$, we find from (1.37),

$$\|x(t)\| \leq \left[\frac{m[x(t_0)]}{\Gamma(1-\gamma)}\right]^b (t-t_0)^{-\gamma b},$$

which implies that the asymptotically stable is a special case of the Mittag-Leffler stability.

In the following, we extend the Lyapunov direct method to the case of fractional order systems, which leads to the Mittag-Leffler stability.

Theorem 1.4.3 *[84] Let* $x_e = 0$ *be an equilibrium point for the system (1.27) and* $D \subset \mathbb{R}^n$ *be a domain containing the origin. Let* $V(t, x(t))$ $:[0,\infty) \times D \to \mathbb{R}$ *be a continuously differentiable function and locally Lipschitz with respect to* x*, such that*

$$\alpha_1 \|x\|^a \leq V(t, x(t)) \leq -\alpha_2 \|x\|^{ab},$$

$$^{C}D^{\beta}V(t,x(t)) \leq -\alpha_3 \left\| x \right\|^{ab} \ ,$$

where $t \geq 0, x \in D, \beta \in (0,1), \alpha_i$ *(*$i = 1.2.3$*),* a *and* b *are positive constants .Then* $x_e = 0$ *is Mittag-Leffler stable. If the assumptions hold globally on* $\mathbb{R}^n$*, Then* $x_e = 0$ *is globally Mittag-Leffler stable.*

Example 1.4.1 *Consider the following fractional system:*

$${}_{0}^{C}D^{\alpha} \left| x(t) \right| = - \left| x(t) \right|, \alpha \ \in \ (0,1),$$

let the Lipschitzienn function $V(t,x) = |x|$*, we have,*

$${}_{0}^{C}D^{\alpha}V ={}_{0}^{C} D^{\alpha} \left| x \right| \leq_{0} D^{\alpha} \left| x \right| \leq - \left| x \right|,$$

so it is enough to take $\alpha_1 = \alpha_2 = 1, \alpha_3 = -1,$ *and the application of the theorem gives us*

$$\left| x(t) \right| \leq \left| x(0) \right| E_{\alpha} \left(-t^{\alpha} \right).$$

Chapter 2

Chaos and Bifurcations Theory

Chaos and bifurcations theory are two intertwined concepts within the field of nonlinear dynamics, offering insights into the behavior of complex systems. In this chapter we present basic tools of chaos, its characteristics and some well-known types of bifurcations of codimension one.

2.1 Basic tools

2.1.1 Lyapunov Exponents

The idea of lyapunov exponents is utilized in the study of dynamic systems, especially in the context of chaos theory. In a dynamical system, they offer a numerical representation of the exponential divergence or convergence of neighboring trajectories.

The Lyapunov exponents are the average rates of exponential divergence or convergence of neighboring trajectories in a system given a set of differential equations describing the system.We note the exponent Lyapunov by λ_i. We present below the principle of Lyapounov exponents to evaluate the behavior of dynamic systems with discrete, continuous and fractional [27].

Discreet case:

Let's consider a discreet dynamic system of dimension 1 defined in the interval $[0, 1]$

$$x_{n+1} = f(x_n) \qquad n \in \mathbb{N}.$$

Let x_0 an initial condition, we add a very small error ε to disrupt the condition initial x_0

Let W_1 and W_2 two orbits initialized by x_0 and $x_0 + \varepsilon$ respectively.

We need to evaluate the exponential distance between the two orbits W_1 and W_2 after n iterations such that the distance is defined by

$$D(x_n) = \left| f^n(x_0 + \varepsilon) - f^n(x_0) \right|,$$

for very large n, we have

$$D(x_n) \simeq \varepsilon \exp(n\lambda), \tag{2.1}$$

we have $x_0, x_1 = f(x_0), x_2 = f(x_1) = f(f(x_0))........x_n = f(x_{n-1}) = f^n(x_0)$,

When $\epsilon \to 0$, we obtain

$$\varepsilon \exp(n\lambda) \simeq \frac{d^n f(x)}{dx},$$

and

$$\begin{aligned} \ln \frac{d(x_n)}{d(x_0)} &\simeq \ln \frac{d^n f(x)}{dx} \\ &= \ln \prod_{k=0}^{n-1} \left| \grave{f}(x_k) \right| \\ &= \sum_{k=0}^{n-1} \ln \left| \grave{f}(x_k) \right|. \end{aligned}$$

According to equation (2.1), we can define the Lyapounov exponent as follows

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \grave{f}(x_k) \right|. \tag{2.2}$$

Continuous Case

Consider a continuous dynamic system of dimension n defined by the following differential equations

$$\frac{dx_i(t)}{dt} = f_i(x(t)) \qquad i = (1, ..., n), \tag{2.3}$$

where x_i are the coordinates of the system.

Using the Taylor expansion of equation (2.3) to evaluate a small disturbance $Dx(t)$ around a trajectory $x(t)$

$$\begin{aligned} \frac{d}{dt}(x_i(t) + Dx_i(t)) &= f_i(x(t) + Dx(t)) \\ &= f_i(x_i(t)) + \sum_j \frac{\partial f_i x(t)}{\partial x_j} Dx_j + 0(Dx(t)) \\ &\simeq f_i(x_i(t)) + \sum_j \frac{\partial f_i x(t)}{\partial x_j} Dx_j, \end{aligned}$$

according to (2.3), we have

$$\frac{dDx_i(t)}{dt} \simeq \sum_j \frac{\partial f_i x(t)}{\partial x_j} Dx_j. \tag{2.4}$$

If we write equation (2.4) in matrix form, we find

$$\dot{Dx_i(t)} = -VDx_i(t), \tag{2.5}$$

where $V_{i,j} = -\frac{\partial f_i x(t)}{\partial x_j}$.

Suppose V solution of the matrix equation:

$$\dot{V} = -VDV_i,$$

then, equation (2.5) integrates:

$$Dx(t) = V(t)Dx(0).$$

The norm of $Dx(t)$ is given by

$$|Dx(t)|^2 = Dx^T(t)V^T(t)V(t)Dx(0).$$

We define the finite-time Lyapounov exponents as the eigenvalues of $\frac{1}{2t}\log\left(V^T(t)V(t)\right)$

$$\lambda_i(t) = \left\{spect(\log(V^T(t)V(t)^{\frac{1}{2t}})\right\}, \lambda_i < ... < \lambda_n.$$

If the matrix check the general conditions of article [63]. then it converges when $t \to \infty$. Which allows us to define the Lyapounov exponents as follows:

$$\lambda_i = \lim_{t\to\infty} \lambda_i(t)$$

Fractional case:

To calculate the Lyapounov exponents in fractional -order system case, We propose algorithm developed by T. Rosenstein et al.[69] and the Benettin–Wolf algorithm [19].

First of all, to define the Lyapunov exponents we need the following results

Theorem 2.1.1 *Consider fractional differential equations*

$$\begin{cases} {}^{C}D^{\alpha}x = f(x) & 0 < \alpha < 1 \\ x(0) = 0 \end{cases}, \tag{2.6}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and ${}^{C}D^{\alpha}$is the Caputo fractional derivative.

The variation equation for the system (2.6) is as follows:

$$\begin{cases} {}^{C}D^{\alpha}\Psi(t) = D_x f(x)\Psi(t) \\ \Psi(0) = I \end{cases}, \tag{2.7}$$

where Ψ is the matrix solution of the system (2.7), D_x is the Jacobian of f and I is the matrix identity.

Let $\lambda_i(t)$ $i = 1, 2, ..., n$ the eigenvalues of $\Psi(t)$ of the system (2.7), which satisfy,

$$|\lambda_1(t)| < |\lambda_2(t)| < .... < |\lambda_n(t)| .$$

Then, the exponents of Lyapunov λ_i of trajectories $x(t)$ solving equation (2.7) are defined by

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \sup \ln |\lambda_i(t)| \qquad i = 1, 2, ..., n$$

2.1.2 Attractors and Basin of Attraction

Attractors

In mathematics, particularly in the field of dynamical systems and chaos theory, attractors refer to sets of values or states toward which a system evolves over time. These attractors play a crucial role in understanding the long-term behavior of dynamic systems.

Definition 2.1.1 *A is an attractor if:*

1. For every neighborhood U of A, there exists a neighborhood V of A such that every solution $x(x_0, t) = \varphi_t(x_0)$ will stay in U if $x_0 \in V$.
2. $\cap \varphi_t(V) = A, t \geq 0$.
3. There exists a dense orbit in A.

There are different types of attractors, and they are often associated with the solutions of differential equations or iterative processes.

Regular Attractor

A regular attractor is a specific type of attractor that exhibits some form of regularity or pattern in its behavior. There are several types of regular attractors, including:

The fixed point: A fixed point attractor is a single point in the state space toward which a system tends to evolve. Once the system reaches this point, it stays there.

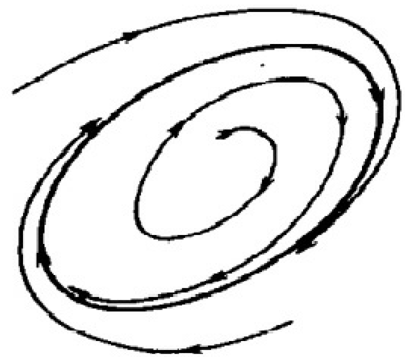
Limit Cycle: A limit cycle is a closed trajectory in the state space to which the system repeatedly converges. The system oscillates around this cycle indefinitely.

A torus: which corresponds to the attractor obtained by the movements resulting from two independent oscillations, for example: electric oscillators

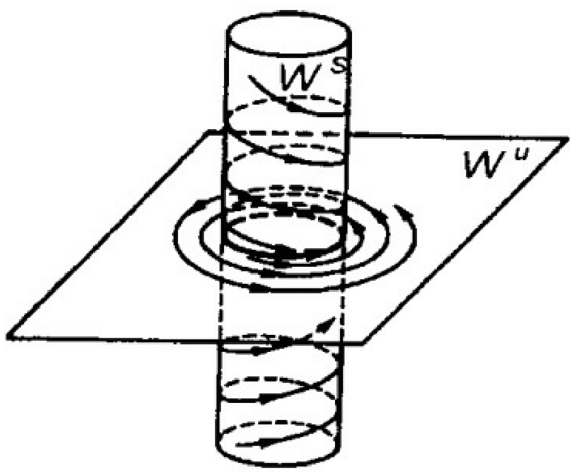
Strange Attractors

A strange attractor is a more complex and chaotic form of attractor. Unlike regular attractors, strange attractors have a fractal structure and exhibit sensitive dependence on initial conditions.

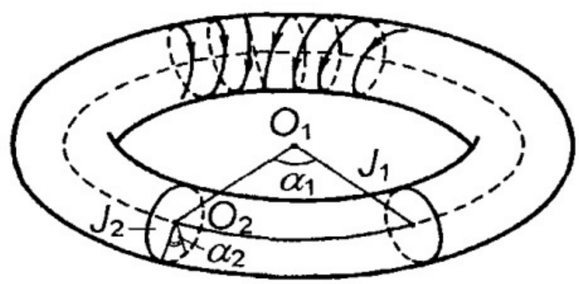
Fixed point



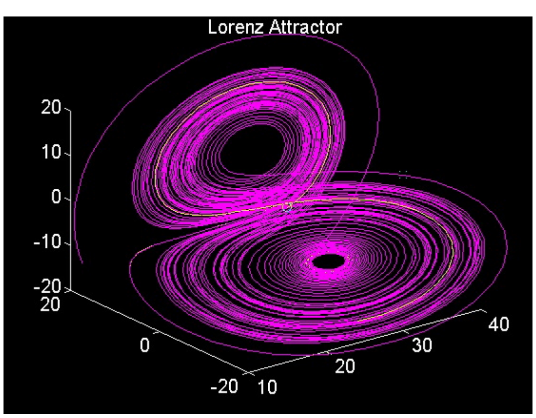
limit cycle



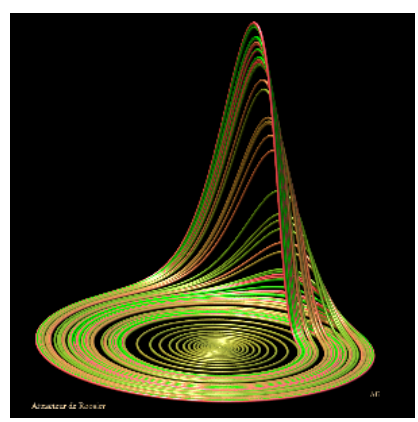
A limit cycle in natural space



Invariant tori



Lorenz Strange Attractor



Rosler Strange Attractor

Definition 2.1.2 (Basin of attraction) *The basin of attraction $B(A)$ of an attractor A is the set of initial conditions (the set of all initial states of the orbits) is for a long time a behavior approach towards. Different attractors may have distinct basins of attraction and the boundaries between these basins can be important in understanding the global behavior of a system.*

2.2 Chaos Theory

Chaos theory is a branch of mathematics and science that deals with complex systems characterized by sensitive dependence on initial conditions. It emerged as a field of study in the late 20th century. As it is widely known, the chaotic attractor has been found throughout the world in many research and has powerful applications in various fields including physics, biology, economics, engineering and even the social sciences. it has been used to study such as weather forecasting, the study of turbulent fluid dynamics, the behavior of financial markets, and the modeling of biological systems. It has also inspired a philosophical and scientific exploration of the limits of predictability in complex systems. Researchers use mathematical techniques, computer simulations and experimentation to study chaotic systems and gain insights into their behavior.

2.2.1 Characteristics of Chaos

Chaos: Mathematically, this term is used to describe dynamical systems in which small changes in initial conditions lead to large changes in the solution after some period of time. It can be characterized by the following point:

Nonlinear Dynamics

The focus of chaos theory is on systems that exhibit nonlinear dynamics, meaning that their behavior is not easily predictable from linear relationships between their components. It is possible for nonlinear systems to exhibit intricate and frequently unexpected behavior.

Sensitive Dependence on Initial Conditions

Sensitivity to initial conditions is a phenomenon discovered for the first time, from the end of the 19th century by Poincarè, then was rediscovered in 1963 by Lorenz during his work in meteorology. which suggests that a small change in the initial conditions of a system can lead to vastly different outcomes. In other words, tiny variations in the starting state of a system can result in significant and unpredictable differences in its future behavior.Figure (2.1) illustrates the temporal evolution of a trajectory of the system of Lu with three conditions diferent initials close .

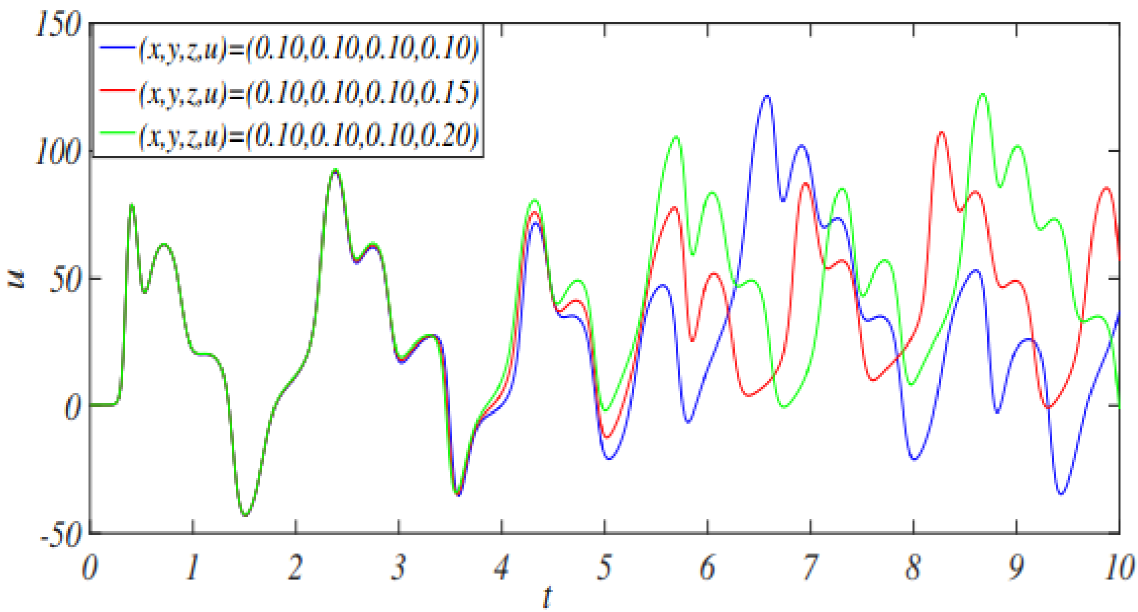


Figure 2.1: The temporal evolution of a trajectory u of the system of Lu with three conditions diferent initials .

Non-periodicity

Typically, chaotic systems don't display regular, recurring cycles or patterns. Instead, they exhibit an aperiodic, seemingly irregular activity that may be characterized as a complicated, nuanced, and unexpected series of occurrences.

Strange Attractors

Chaotic systems tend to exhibit strange attractors, which are geometric patterns in phase space that the system's trajectory approaches over time. These attractors can have a fractal structure and represent the underlying order within the seemingly random behavior of the system.

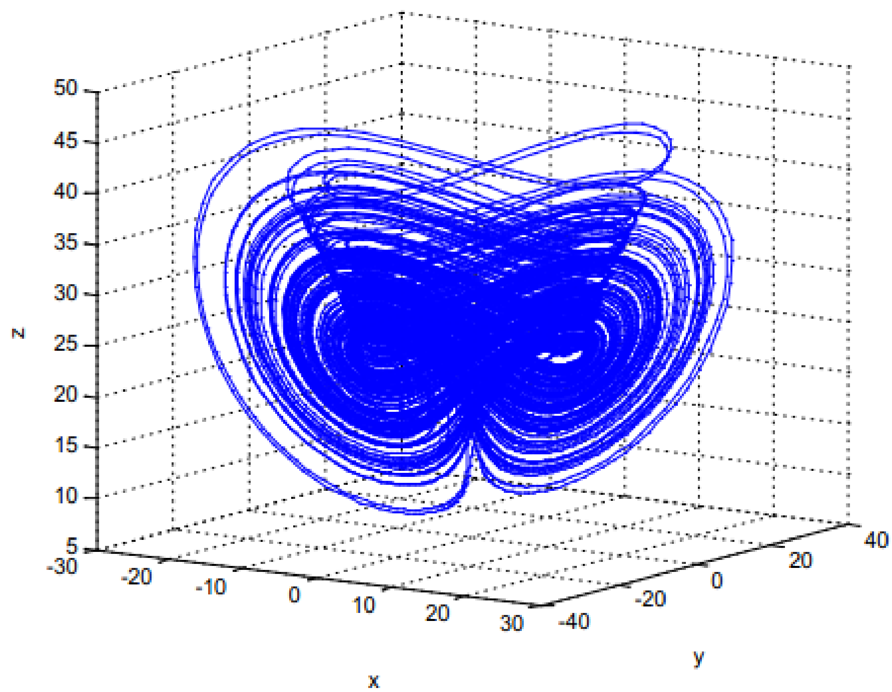


Figure 2.2: Chen's Chaotic Attractor

Determinism

Determinism means that the system is non-random and has no parameters or stochastic entry. This property is specific to all systems whose evolution is defined by a set of differential equations or difference equations. In the random phenomena, it is impossible to predict the trajectory of any particle. On the contrary and although they appear, at first glance, random, dynamic systems chaotic are governed by certain equations accounting for the phenomenon, but whose solutions sensitive to initial conditions.

2.2.2 Some Applications of Chaos

Weather forecasting

By taking into consideration how sensitive atmospheric conditions are to preliminary observations, chaos theory has been applied to enhance weather prediction models. Meteorologists have found it useful in comprehending the boundaries of predictability in intricate systems such as the Earth's atmosphere.

Physics

Complex physical systems including fluid dynamics, turbulence, and nonlinear optics have been studied using chaos theory. It has aided in the comprehension of the behavior of chaotic complex systems, offering new perspectives on events that were previously hard to explain.

Engineering

Chaos theory has been used to a number of engineering fields, such as signal processing, telecommunications and control theory. It has aided in the analysis of complicated engineering systems' behavior and stability as well as the creation of reliable control systems.

Biology and Medicine

Chaos theory has been used to examine biological systems, including brain networks, genetic regulatory networks, and the human pulse. This may result in an enhanced comprehension of disease dynamics and physiological systems.

For exemple, in biology, makes it possible to explain cerebral oscillations (electroencephalogram, that is to say a graphic recording of the electrical activity of the brain by means of Electro placed on the scalp of a Strange Attractor. Thus, the arrhythmias typical of many heart diseases can be explained Also by chaos theory.

Art and Creative Expression

A number of musicians and artists have used chaos theory as a source of inspiration to produce original and dynamic literary, musical and artistic creations.

In Computer Science

Chaos theory concepts and methods can be applied in computer science in various ways. For secure communication, chaos-based encryption approaches take use of chaotic systems' unexpected characteristics. To improve the security of data transmission, keys for encryption techniques can be generated by chaotic systems.

2.3 Bifurcation Theory

Definition of Bifurcation

A bifurcation is a change in the topological type of the system (qualitative and quantitative change) when modifying the control parameter, that is to say the disappearance or change of stability (from stable to unstable or the reverse) and the emergence of new solutions.

Bifurcation Diagram

A bifurcation diagram is a graphical representation used in the study of dynamical systems, particularly in the field of chaos theory and nonlinear dynamics. It provides insights into how the behavior of a dynamic system changes as a control parameter is varied. Bifurcation diagrams are commonly used to analyze and visualize the emergence of complex and chaotic behavior in such systems. Note that the bifurcation diagram dependsin general on the region of phase space considered.

Types of Bifurcations

There are various types of bifurcations, each associated with different changes in system behavior. We are only talking here about the bifurcation of codimension one ($k = 1$), Some well-known types of bifurcations of codimension one include [43], [44]:

2.3.1 Saddle-Node Bifurcation

In this bifurcation, a pair of equilibrium points, one stable and one unstable, collide and annihilate each other as a parameter is varied. This leads to the creation or destruction of an equilibrium point. This bifurcation has a lot of other names, including limit point, fold bifurcation and turning point.

Consider the following equation

$$\dot{x} = f(x, r).$$

Where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a sufficiently regular function with $f(0,0) = 0$, $\frac{\partial f}{\partial x}(0,0) = 0$ (that is to say (0.0)is a non-hyperbolic equilibrium point) et $\frac{\partial f}{\partial r}(0,0) \neq 0$

By a development limited to the vicinity of the equilibrium point we find:

$$\begin{aligned}
\dot{x} &= f(0,0) + x\frac{\partial f}{\partial x}(0,0) + r\frac{\partial f}{\partial r}(0,0) + \frac{x^2}{2}\frac{\partial^2 f}{\partial x^2}(0,0) + ... \\
&= Ar + Bx^2,
\end{aligned}$$

where

$$A = \frac{\partial f}{\partial r}(0,0) \text{ et } B = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0).$$

Assume that,

$$\begin{aligned}
&U = \frac{B}{A}x, \mu = \frac{B}{A}r, T = At \\
&\dot{U} = \frac{B}{A}\dot{x} \\
&\dot{U} \simeq \frac{B}{A}(Ar + Bx^2) \\
&\dot{U} = Br + \frac{B^2}{A}x^2 \\
&\dot{U} = A(\frac{B}{A}r + \frac{B^2}{A^2}x^2) \\
&\dot{U} = A\left(\mu + U^2\right),
\end{aligned}$$

we have $\frac{\partial U}{\partial t} = \frac{\partial U}{\partial T}\frac{\partial T}{\partial t} \simeq A(\mu + U)$

$$\Rightarrow \frac{\partial U}{\partial T} = \mu + U^2.$$

So, the last equation called the normal form of the saddle-node bifurcation.

Example 2.3.1 *Consider the following equation:*

$$\dot{x}(t) = \mu - \alpha x^2,$$

where μ is the control parameters.

For $\alpha > 0$ we are speaking about subcritical bifurcation. Lets $\alpha = 1$, the equilibrium points are easy to determine and they are immediately obtained : $x_e = \pm\sqrt{\mu}$, we can summarize the result in the following table :

Equilibrium point	$\mu < 0$	$\mu > 0$
$x_e = \sqrt{\mu}$	doesn't exist	stable
$x_e = -\sqrt{\mu}$	doesn't exist	unstable

The visualization can be done in the bifurcation diagram:

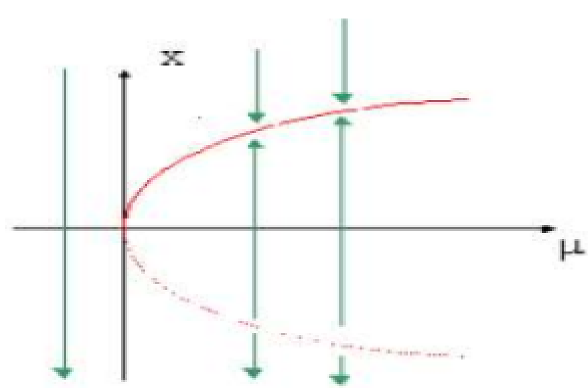


Figure 2.3: Saddle-Node Bifurcation

2.3.2 Transcritical Bifurcation

Here, two equilibrium points (one stable and one unstable) exchange stability as a parameter is changed. This results in the creation or destruction of an equilibrium point.

Let us still consider the equation:

$$\dot{x}(t) = f(x, r).$$

Where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(0,0) = 0$, $\frac{\partial f}{\partial x}(0,0) = 0$ and we add a third condition $\frac{\partial f}{\partial r}(0,0) = 0$, with $\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$ and $\frac{\partial^2 f}{\partial x \partial r}(0,0) \neq 0$. By a development limited to the vicinity of the equilibrium point (0.0), we find:

$$f(x,r) = f(0,0) + x\frac{\partial f}{\partial x}(0,0) + r\frac{\partial f}{\partial r}(0,0) + \frac{x^2}{2}\frac{\partial^2 f}{\partial x^2}(0,0) + xr\frac{\partial f}{\partial x \partial r}(0,0) + ...$$

We suppose that :

$$A = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0,$$

and

$$B = \frac{\partial f}{\partial x \partial r}(0,0) \neq 0.$$

Then, the equation becomes

$$f(x,r) \simeq Ax^2 + Bxr,$$

let's make the changes

$$U = \frac{x}{A}, T = A^2 t \text{ and } \mu = \frac{Br}{A^2},$$

so,

$$\begin{aligned}
\dot{U} &= \frac{\dot{x}}{A} \\
&\simeq x^2 + \frac{Br}{A}x \\
&= A^2U^2 + BUr \\
&\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial U}{\partial T}\frac{\partial T}{\partial t} \simeq A^2U^2 + BUr \\
&\Rightarrow \frac{\partial T}{\partial t} = U^2 + \mu U.
\end{aligned}$$

The last equation called the normal form of the Transcritical bifurcation.

Example 2.3.2 *Consider the following equation:*

$$f(x, \mu) = \mu x - x^2. \tag{2.8}$$

The usual analysis gives :

$$f(x, \mu) = 0 \Longleftrightarrow \mu x - x^2 = 0 \Longleftrightarrow x(\mu - x) = 0$$

$$\begin{cases} x_1 = 0 \\ x_2 = \mu \end{cases}.$$

The equation $f(x, \mu) = 0$ admits two equilibrium points

$$\frac{df(x, \mu)}{dx} = \mu - 2x \text{ so } \frac{df(x, \mu)}{dx} \mid_{x_1} = \mu \text{ and } \frac{df(x, \mu)}{dx} \mid_{x_2} = -\mu,$$

so: the equilibrium point $x_1 = 0$ stable for $\mu < 0$, unstable for$\mu > 0$, $x_2 = \mu$, stable for $\mu > 0$ and unstable for $\mu < 0$.

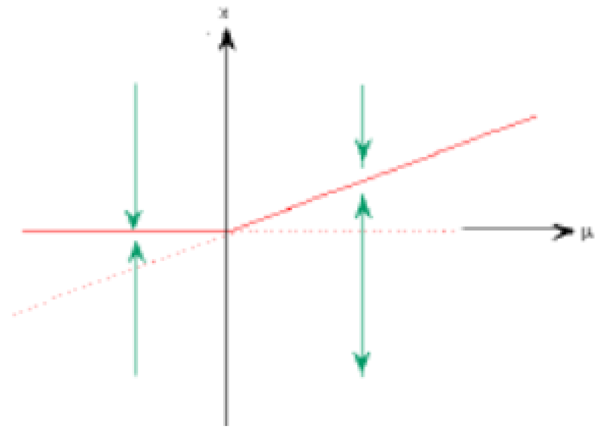


Figure 2.4: Transcritical bifurcation:

2.3.3 Pitchfork Bifurcation

In this case, a single stable equilibrium point splits into three equilibrium points, where one is stable and the other two are unstable, or vice versa. This occurs as a parameter is varied. It is the bifurcation associated with the differential equation

$$\dot{x} = f\left(x, \mu\right), x \ \in \ \mathbb{R} \text{ and } \mu \ \in \ \mathbb{R},$$

taking in this part $f(0,0) = 0$, $\frac{\partial f}{\partial r}(0,0) = 0$, $\frac{\partial^2 f}{\partial x^2}(0,0) = 0$,$\frac{\partial f}{\partial c^2}(0,0) = 0$, $\frac{\partial^2 f}{\partial x \partial r}(0,0) \neq 0$, $\frac{\partial^3 f}{\partial x^3}(0,0) \neq 0$.

By a development limited to the vicinity of the equilibrium point we find:

$$\begin{aligned} f(x,r) &= f(0,0) + x\frac{\partial f}{\partial x}(0,0) + r\frac{\partial f}{\partial r}(0,0) + \frac{x^2}{2}\frac{\partial^2 f}{\partial x^2}(0,0) + xr\frac{\partial f}{\partial x \partial r}(0,0) + \frac{r^2}{2}\frac{\partial^2 f}{\partial r^2}(0,0) \\ &+ \frac{x^3}{6}\frac{\partial^3 f}{\partial x^3}(0,0) + ... \\ &= xr\frac{\partial f}{\partial x \partial r}(0,0) + \frac{x^3}{6}\frac{\partial^3 f}{\partial x^3}(0,0) + ... \end{aligned}$$

Assume that :

$$A = \frac{1}{6}\frac{\partial^3 f}{\partial x^3}(0,0) \neq 0,$$

and

$$B = \frac{\partial^2 f}{\partial x \partial r}(0,0) \neq 0.$$

Then, the equation becomes

$$f(x,r) \simeq Ax^3 + Bxr$$

Let's make the changes

$$U = \frac{x}{A}, T = A^3 t \text{ and } \mu = \frac{Br}{A^3},$$

so,

$$\begin{aligned} \dot{U} &= \frac{\dot{x}}{A} \\ &\simeq x^3 + \frac{Br}{A}x \\ &= A^3U^3 + BUr \end{aligned}$$

By following:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial U}{\partial T}\frac{\partial T}{\partial t} \simeq A^3U^3 + BUr \\ \Rightarrow \frac{\partial T}{\partial t} &= U^3 + \mu U. \end{aligned}$$

The last equation called the normal form of the Pitchfork bifurcation.

Example 2.3.3 *There are two kinds of this bifurcation :*

Supercritical, having a normal form:

$$f(x,\mu) = \mu x - x^3, \tag{2.9}$$

and subcritical, having a normal form :

$$f(x,\mu) = \mu x + x^3.$$

Let's start with supercritical pitchfork bifurcation, we calculate the equilibrium points.

$$f(x,\mu) = 0$$

$$\mu x - x^3 = 0 \Leftrightarrow x(\mu - x^2) = 0$$

$$\Longleftrightarrow \begin{cases} x = 0 \\ \text{or} \\ \mu - x^2 = 0 \end{cases} \Longleftrightarrow \begin{cases} x = 0, \\ \text{or} \\ x^2 = \mu. \end{cases}$$

So, if $\mu < 0$, we have a single point of equilibrium at $x = 0$.

If $\mu > 0$, we have three equilibrium points

$$\begin{cases} x_1 = 0 \\ x_{2,3} = \pm\sqrt{\mu} \end{cases}.$$

We study the stability of these equilibrium points:

$$\frac{df(x,\mu)}{dx} = \mu - 3x^2 \text{ so } \begin{cases} \frac{df(x,\mu)}{dx} \mid_{x_1} = \mu, \\ \frac{df(x,\mu)}{dx} \mid_{x_{2,3}} = -2\mu. \end{cases},$$

as a result :

- If $\mu < 0$ we have the only equilibrium point where $x = 0$ is stable.
- If $\mu > 0$ we have the equilibrium point:

$$\begin{cases} x = 0 \text{ is unstable,} \\ x = \pm\sqrt{\mu} \text{ is stable.} \end{cases}.$$

- if $\mu = 0$ we have a single point of equilibrium where $x = 0$ is stable.

In the case of a subcritical pitchfork bifurcation, the same calculation yields

$$f(x,\mu) = 0,$$

$$\mu x + x^3 = 0 \Leftrightarrow x(\mu + x^2) = 0$$

$$\Longleftrightarrow \begin{cases} x = 0 \\ \text{or} \\ \mu + x^2 = 0 \end{cases} \Longleftrightarrow \begin{cases} x = 0, \\ \text{or} \\ x^2 = -\mu. \end{cases},$$

so, if $\mu > 0$, we have a single point of equilibrium $x = 0$.

If $\mu < 0$, we have three equilibrium points

$$\begin{cases} x_1 = 0 \\ x_{2,3} = \pm\sqrt{-\mu} \end{cases},$$

we study the stability of these equilibrium points :

$$\frac{df(x,\mu)}{dx} = \mu + 3x^2 \text{ so } \begin{cases} \frac{df(x,\mu)}{dx} \mid_{x_1} = \mu, \\ \frac{df(x,\mu)}{dx} \mid_{x_{2,3}} = -2\mu. \end{cases},$$

as a result :

- If $\mu > 0$ we have the only equilibrium point where $x = 0$ is unstable.
- If $\mu < 0$ we have the equilibrium point:

$$\begin{cases} x = 0 \text{ is stable,} \\ x = \pm\sqrt{\mu} \text{ is unstable.} \end{cases}.$$

Remark 2.3.1 *In the fractional order case, the conditions of the saddle-node bifurcation, transcritical bifurcation and pitchfork bifurcation do not exchange.*

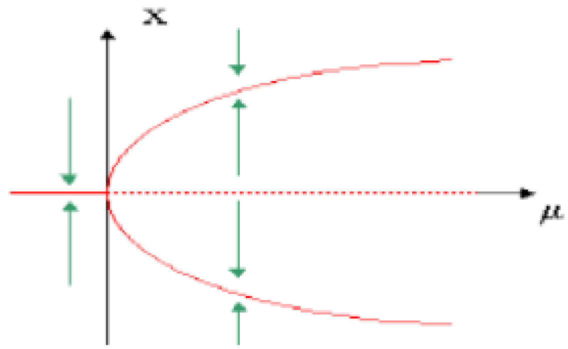


Figure 2.5: Pitchfork bifurcation:

2.3.4 Hopf Bifurcation

The term Hopf bifurcation (also sometimes called Poincare-Andronov-Hopf bifurcation) refers to the local birth or death of a periodic solution (self-excited oscillation) from an equilibrium as a parameter crosses a critical value, This occurs when a stable periodic solution of a system changes its stability as a parameter is varied, giving rise to the creation of a limit cycle. The Hopf bifurcation theorem makes the above precise.

We can also distinguish two types of Hopf bifurcation:

- Super-critical Hopf bifurcation where the equilibrium undergoes a change in stability towards instability.
- Sub-critical Hopf bifurcation or the equilibrium undergoes a change in instability towards stability.

Figure (2.6) gives a representation:

Example 2.3.4 *Consider the following system of two differential equations depending on one parameter:*

$$\begin{cases} \dot{x}_1 = \mu x_1 - x_2 - x_1\left(x_1^2 + x_2^2\right). \\ \dot{x}_2 = x_1 + \mu x_2 - x_2(x_1^2 + x_2^2) \end{cases}, \tag{2.10}$$

for all μ, the system has the one equilibrium $x_1 = x_2 = 0$ and its eigenvalues $\lambda_{1,2} = \mu \pm i$.

By asking the complex variable $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, $|z|^2 = z\bar{z} - x_1^2 + x_2^2$.

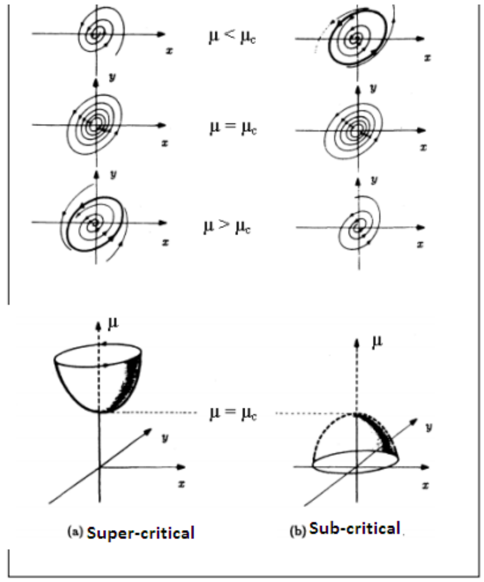


Figure 2.6: Hopf bifurcation in a phase plane

And

$$\dot{z} = \dot{x}_1 + i\dot{x}_2 = \mu\left(x_1 + ix_2\right) + i\left(x_1 + ix_2\right) - \left(x_1 + ix_2\right)\left(x_1^2 + x_2^2\right),$$

therefore, we can rewrite system (2.10) in the following complex form:

$$\dot{z} = (\mu + i)z - z\left|z\right|^2.$$

To study this equation, using the representation $z = re^{i\theta}$ we get

$$\dot{z} = \dot{r}e^{i\theta} + ri\dot{\theta}e^{i\theta},$$

the polar form of the system (2.10) is provided by

$$\begin{cases} \dot{r} = r\left(\mu - r^2\right) \\ \dot{\theta} = 1 \end{cases},$$

the first equation is nothing more than a pitchfork bifurcation of control parameter μ

If $\mu < 0$, The system has a point of stable equilibrium which corresponds here to a focus point: the trajectories spiral towards the origin. when $\mu = 0$, it remains stable . And when

$\mu > 0$ a stable periodic trajectory is then formed or limited cycle

The hopf bifurcation corresponds to an oscillatory instability.

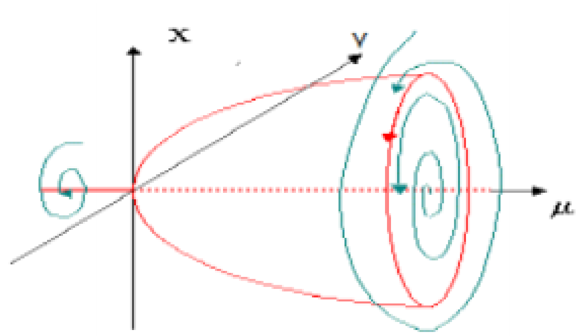


Figure 2.7: Hopf Bifurcation

Consider the Fractional order system

$$D^{\alpha}x = f(x,\beta),\ 0 < \alpha \leq 1, x \in \mathbb{R}^3, \beta \in \mathbb{R}. \tag{2.11}$$

In order to obtain the Hopf bifurcation conditions in fractional order commensurate system (3.2), we first recall the integer order case:.

Integer Order Case

Theorem 2.3.1 *[1], [2]The conditions of system (2.11), with $\alpha = 1$, to undergo a Hopf bifurcation at the equilibrium point x_e when $\beta = \beta^*$, are:*

- The Jacobian matrix has two complex-conjugate eigenvalues $\lambda_{1.2} = \theta(\beta) \pm iw(\beta)$ and one real $\lambda_3(\beta)$.
- $\theta(\beta^*) = 0, w(\beta^*) > 0$ and $\lambda_3(\beta^*) \neq \mathbf{0}$.
- $\frac{d\theta}{d\beta}|_{\beta=\beta^*} \neq 0$.

Fractional Order Case

In the fractional case, the stability of x_e is related to the sign of $m_i(\alpha,\beta) = \alpha\frac{\pi}{2} - \min_{1\leq i\leq 3}|\arg(\lambda_i(\beta))|, i = 1, 2, 3$. If there is i such that $m_i(\alpha,\beta) > 0$, then x_e is unstable. If $m_i(\alpha,\beta) < 0$ for all

$i = 1, 2, 3$, then x_e is locally asymptotically stable. So, the function $m_i(\alpha, \beta)$ has the same effect as the real part of eigenvalues in integer-order systems. That's why, one extends the Hopf bifurcation conditions to the fractional systems by replacing $Re(\lambda_i)$ with $m_i(\alpha, \beta)$ as follows :

Theorem 2.3.2 *[49]The system (2.11) undergoes fractional Hopf bifurcation at the equilibrium point x_e, if there exists a Hopf critical value $\beta = \beta^*$ such that the following conditions are satisfy:*

(i) The Jacobian matrix has two complex-conjugate eigenvalues $\lambda_{1.2} = \theta(\beta) \pm iw(\beta)$ and one real $\lambda_3(\beta)$.

(ii) $m(\beta^*) = \alpha\frac{\pi}{2} - \min_{1\leq i\leq 3} |\arg(\lambda_i(\beta))| = 0$.

(iii) $\frac{dm(\beta)}{d\beta} |_{\beta=\beta^*} \neq 0$.

Part II

Main Results

Chapter 3

Fractional order effect on the localisation of Hopf bifurcation point

Among the most important chaotic systems that have been studied, are the jerk systems. Jerk in physics, refers to the rate of change of acceleration. It is the third derivative of position with respect to time. It can be written in the form ODE as the third-order dynamics [73, 74, 75].

$$\frac{d^3x}{dt^3} = \varphi(x, \frac{dx}{dt}, \frac{d^2x}{dt}), \tag{3.1}$$

in (3.1), x, $\frac{dx}{dt}$, $\frac{d^2x}{dt}$ and $\frac{d^3x}{dt^3}$ stands for the displacement, the velocity, the acceleration and the jerk, respectively. Therefore, we identify Eq (3.1) of the third order as the jerk differential equation.

The applications of jerk have many instances and one of the examples one can shoot is simply that the jerk is all about the rate at which any object's acceleration changes with time or with respect to time, here are a few examples: Transportation, vehicle dynamics, robotics, automation medical imaging, radiology, virtual reality and video games [33]. In technical applications the jerk information can be used to control bodies in motion for accurate and precise control, to improve the control response and to avoid excessive input of acceleration.

However, in the field, some flaws appeared when applying the Jerk systems, beyond the direct influence of the forces themselves, rapid shifts in the forces can also modify acceleration, jerk, and higher-order derivatives, which can have undesirable impacts. In addition to inducing fatigue cracks in metals and other materials, jerk can inflict harm on humans and racing animals.

Through this study, we believe that to overcome some of these obstacles and solve some of these problems can be used the fractional derivatives. The changes in acceleration jerk can be quieter when the order of the derivatives is reduced, which allows expanding the region of stability and the narrowing of the chaotic region. For this, we proposed the fractional version of Jerk systems.

3.1 Description and Stability Analysis of the Model

In mathematics, Eq (3.1) can be recast into a system form as follow:

$$\begin{cases} \dot{x}(t) = y \\ \dot{y}(t) = z \\ \dot{z}(t) = \varphi(x, y, z) \end{cases},$$

when we assume that $x(t) = x(t), y(t) = \dot{x}(t)$ and $z(t) = \ddot{x}(t)$.

The fractional jerk system is defined as follow:

$$\begin{cases} \frac{d^{\alpha_1} x}{dt^{\alpha_1}} = & y \\ \frac{d^{\alpha_2} y}{dt^{\alpha_2}} = & z \\ \frac{d^{\alpha_3} z}{dt^{\alpha_3}} = & -\beta z - y + \Psi(x) \end{cases}, \tag{3.2}$$

where $\Psi(x)$ is nonlinear function, in this study is given as $\Psi(x) = (|x| - 1)$, β is the parameter $(\beta > 0)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the fractional order of system (3.2), it is also supposed that α_i lies in $(0, 1], i = 1, 2, 3$. The proposed system has two equilibrium points : $E_1(1, 0, 0)$ and

$E_2(-1, 0, 0)$.

Stability of the Equilibrium Points

In this subsection we proceed with commensurate order $\alpha = \alpha_1 = \alpha_2 = \alpha_3$.

The Jacobian matrix $J(x^*, y^*, z^*)$ associated with the equilibrium point (x^*, y^*, z^*) is given by :

$$J(x^*, y^*, z^*) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{x^*}{|x^*|} & -1 & -\beta \end{pmatrix}.$$

To study the equilibrium point stability conditions, we apply the Routh-Hurwitz criteria **??**.

Stability Conditions of $E_1(1, 0, 0)$

The characteristic polynomial associated with the equilibrium point E_1 is given by :

$$p(\lambda) = \lambda^3 + \beta\lambda^2 + \lambda - 1,$$

according to the fourth condition of Routh-Hurwitz criteria, we find that E_1 is unstable ($a_3 = -1 < 0$).

Stability Conditions of $E_2(-1, 0, 0)$

The characteristic polynomial associated with the equilibrium point E_2 is given by :

$$p(\lambda) = \lambda^3 + \beta\lambda^2 + \lambda + 1, \tag{3.3}$$

its discriminant is given by $D(p) = -4\beta^3 + \beta^2 + 18\beta - 31$, we note that $D(p)$ is negative for all β positive (see Fig. 3.1).

According the second condition of Routh-Hurwitz criteria, E_2 is locally asymptotically stable when $\alpha < 2/3$. E_2 is a saddle point of index 2, thus the necessary condition for the fractional order system (3.2) to remain chaotic is $\alpha > \frac{2}{\pi}\arctan(\frac{|\lambda_{1.2}|}{R(\lambda_{1.2})})$; Consequently, for $\beta = 0.6$, the lowest fractional order α, for which the fractional-order system (3.2) demonstrates chaos at the above-mentioned parameters is given by the inequality $\alpha > 0.931$.

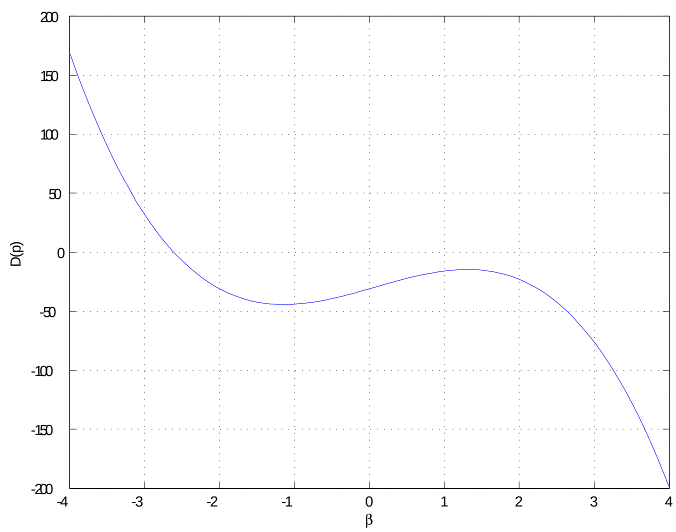


Figure 3.1: Representation of $D(p_{E_2})$ as function of β.

The following figures show that the system (3.2) is stable for $\alpha = 0.85$ (Fig.3.2), and chaotic for $\alpha = 0.95$ (Fig.3.3).

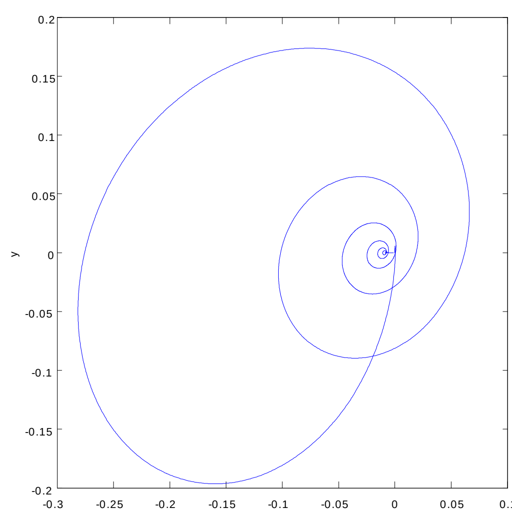


Figure 3.2: Phase portrait of system (3.2) for $\alpha = 0.85$.

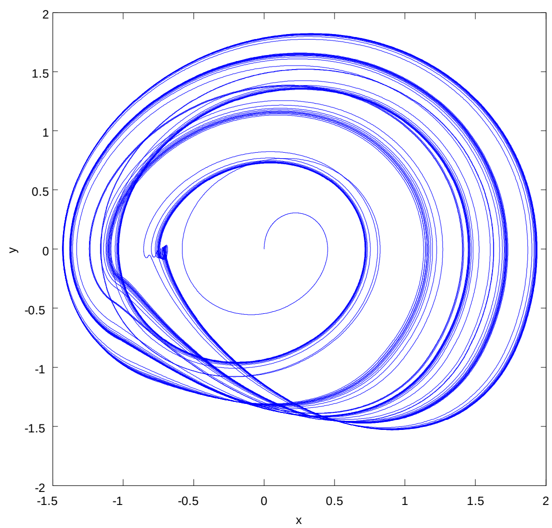


Figure 3.3: Phase portrait of system (3.2) for $\alpha = 0.98$

3.2 Hopf Bifurcation

Hopf Bifurcation Versus the Parameter β

The system is said to undergo a Hopf bifurcation when an equilibrium point switches the stability along- with creation or destruction of certain periodic orbits. In order to obtain the Hopf bifurcation conditions in fractional order commensurate system (3.2), we first recall the integer order case:

3.2.1 Integer Order Case

Theorem 3.2.1 *The conditions of system (3.2), with $\alpha = 1$, to undergo a Hopf bifurcation at the equilibrium point E_2 when $\beta = \beta^*$, are:*

- The Jacobian matrix has two complex-conjugate eigenvalues $\lambda_{1.2} = \theta(\beta) \pm iw(\beta)$ and one real $\lambda_3(\beta)$.
- $\theta(\beta^*) = 0, w(\beta^*) > 0$ and $\lambda_3(\beta^*) \neq \mathbf{0}$.
- $\frac{d\theta}{d\beta}|_{\beta=\beta^*} \neq 0$.

Proof. The Jacobian matrix at equilibrium E_2 of system (3.2) is :

$$J_{E_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -\beta \end{pmatrix},$$

we want to determine the sufficient conditions for J_{E_2} has real eigenvalue negative $\lambda_0(\beta)$ and two complex-conjugate eigenvalues

$\lambda_{\pm}(\beta) = \theta(\beta) \pm iw(\beta)$, with the real part $\theta(\beta)$ satisfies the conditions :

$$\theta(\beta^*) = 0 \quad \text{and} \quad \frac{d\theta}{d\beta}|_{\beta=\beta^*} \neq 0 \, ,$$

we can write the characteristic polynomial (3.3) as follows:

$$(\lambda - \lambda_0)(\lambda - \lambda_+)(\lambda - \lambda_-) = 0, \tag{3.4}$$

then,

$$\lambda^3 - (2\theta + \lambda_0)\lambda^2 + \left(|\lambda_+|^2 + 2\theta\lambda_0\right)\lambda - |\lambda_+|^2 \lambda_0 = 0, \tag{3.5}$$

from (3.4) and (3.5) , one obtains :

$$\begin{cases} 2\theta + \lambda_0 = -\beta \\ |\lambda+|^2 + 2\theta\lambda_0 = 1 \\ |\lambda_+|^2 \lambda_0 = -1 \end{cases}, \tag{3.6}$$

then, $\lambda_0 = -(2\theta + \beta) < 0, \beta > 0$, the Jacobian matrix J_{E_2} has two pure imaginary eigenvalues, if and only if, $a_1 a_2 = a_3$, So, $\beta = 1$, (in this case, we have $\lambda_0 = -1$, $\lambda_+ = i$, and $\lambda_- = -i$).

Hence, the critical Hopf bifurcation value is $\beta^* = 1$ and from (3.6), thus, one can get the following :

$$-2\theta-\beta-2\theta\left(4\theta^2+\beta^2+4\theta\beta\right)+1=0, \tag{3.7}$$

the differentiation of (3.7) with respect to β, we get :

$$\frac{d\theta}{d\beta}=\frac{8\theta^2+20\theta\beta+1}{-24\theta^2-2\beta^2-16\theta\beta-2},$$

so,

$$\left.\frac{d\theta}{d\beta}\right|_{\beta=\beta^*}=-\frac{1}{4}\neq 0.$$

It follows that the system (3.2) satisfies the Hopf bifurcation conditions at E_2. The corresponding bifurcation diagram and phase portrait are presented in (Figs. 3.4 and 3.5)

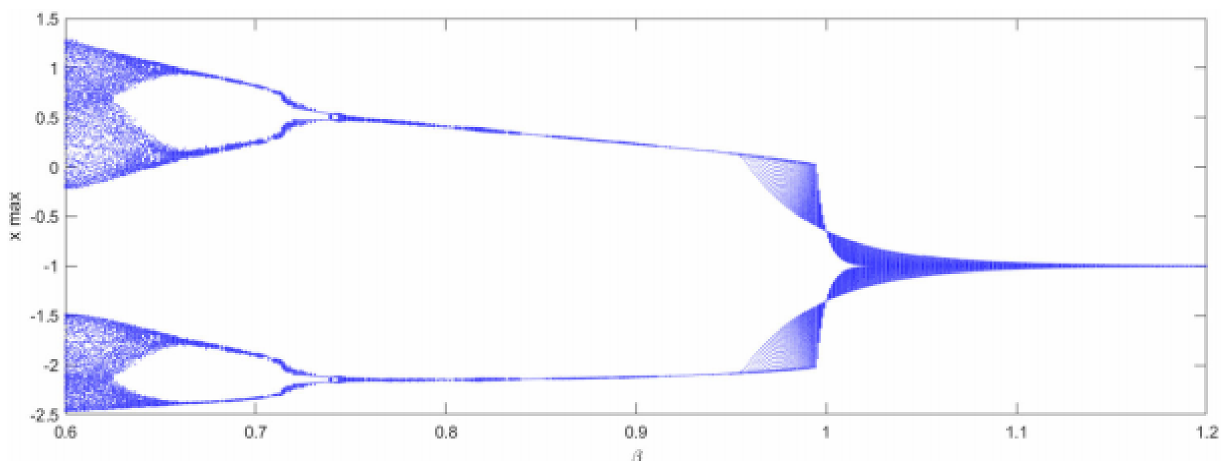


Figure 3.4: Bifurcation diagram of system (3.2) versus β, when $\alpha=1$.

■

3.2.2 Fractional Order Case

In the fractional case, the stability of E is related to the sign of $m_i(\alpha,\beta)=\alpha\frac{\pi}{2}-\left|\arg(\lambda_i(\beta))\right|, i=1,2,3$. If there is i such that $m_i(\alpha,\beta)>0$, then E is unstable. If $m_i(\alpha,\beta)<0$ for all $i=1,2,3$, then E_2 is locally asymptotically stable. So, the function $m_i(\alpha,\beta)$ has the same effect as the real part of eigenvalues in integer-order systems. That's why, one extends the

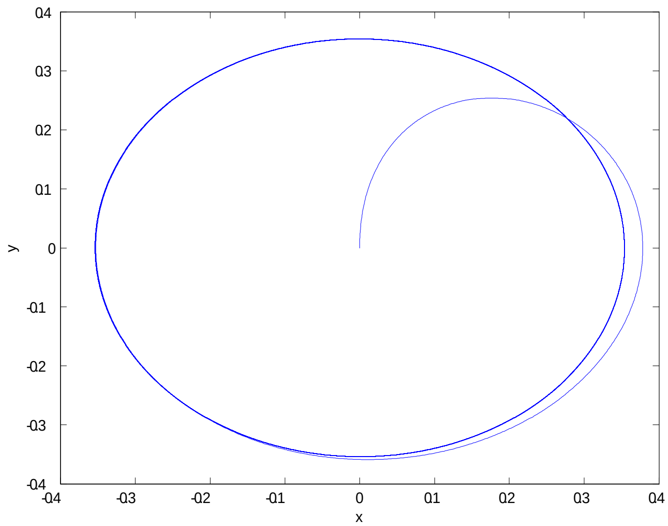


Figure 3.5: Phase portrait of system (3.2) for $\beta = 1$ and $\alpha = 1$.

Hopf bifurcation conditions to the fractional systems by replacing $Re(\lambda_i)$ with $m_i(\alpha, \beta)$ as follows :

Hopf Bifurcation Versus the Fractional Order α

It is clear that the fractional-order α can also act as a bifurcation parameter in fractional-order systems. So the parameter β is fixed and the fractional order α is considered as a control parameter for the critical Hopf bifurcation, using the proposed conditions in this theorem :

Theorem 3.2.2 *The system (3.2) undergoes fractional Hopf bifurcation at the equilibrium point E_2 , if there exists a Hopf critical value $\alpha = \alpha^*$ such that the following conditions are satisfy:*

(i) The Jacobian matrix has two complex-conjugate eigenvalues $\lambda_{1.2} = \theta \pm iw$ and one real negative root $\lambda_3(\beta)$.

(ii) $m(\alpha^*) = \alpha^* \frac{\pi}{2} - \min_{1 \leq i \leq 3} |\arg(\lambda_i)| = 0$.

(iii) $\frac{dm(\alpha)}{d\alpha} |_{\alpha=\alpha^*} \neq 0$.

The first and second conditions are sometimes called singularity conditions while the third is transversality condition.

Proof. For proving (i) :

Firstly, we Suppose that the characteristic polynomial (3.3) has a pair purely imaginary conjugate roots, so, $\lambda_{1.2} = \pm iw$.

By remplacing $\lambda_{1.2}$ in Eq (3.3), one get

$$(iw)^3 + \beta\,(iw)^2 + iw + 1 = 0.$$

thus,

$$\begin{cases} 1 - \beta w^2 = 0 \\ w - w^3 = 0 \end{cases} \Rightarrow \begin{cases} 1 - \beta w^2 = 0 \\ w(1 - w^2) = 0 \end{cases} . \tag{3.8}$$

From (3.8), $w = 0$ or $w = \pm 1$.

Hence, $\beta = 1$.

Secondly, we suppose that $\beta = 1$, therefore, in this case, we have, $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_0 = -1$;So, Eq (3.3) has a pair purely imaginary conjugate roots.

On other side, we have

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda 3) = \lambda^3 + \beta\lambda^2 + \lambda + 1,$$

then,

$$(-1)\lambda_1\lambda_2\lambda_3 = a_3 = 1,$$

we have $a_3 = 1 > 0$; and $\lambda_1\lambda_2$ are a pair complex-conjugate eigenvalues $(\lambda_1\lambda_2 > 0)$.Hence, $\lambda_3 < 0$.

In addition, according to Routh-Hurwitz theorem, the roots of (3.3) have negative real parts if and only if $D(p) > 0$, $a_1 = \beta > 0, a_3 = 1 > 0$, $a_1 a_2 - a_3 = \beta - 1 > 0$.

In our study $D(p) < 0$ Eq(3.2)

Therefore, under conditions given below, condition (i) will be guaranteed if:

$$\begin{cases} \beta \neq 1 \\ \beta - 1 \leq 0 \end{cases} . \tag{3.9}$$

To prove (ii) notice that $\min_{1\leq i\leq 3\leq} |\arg(\lambda_i)| = \arctan\left|\frac{w}{\theta}\right|$, then $m(\alpha^*) = \alpha^*\frac{\pi}{2} - \min_{1\leq i\leq 3} |\arg(\lambda_i)| = m(\alpha^*) = \alpha^*\frac{\pi}{2} - \arctan\left|\frac{w}{\theta}\right| = 0$. So $\alpha^* = \frac{2}{\pi}\arctan\left|\frac{w}{\theta}\right|$. It is clear that $\alpha^* \in (0.1)$. Moreover, condition (iii) that the sign of $m(\alpha)$ can change when the bifurcation parameter α passes through the critical value α^*. ■

Hopf Bifurcation Versus the Parameter β

In this subsection, the parameter α is fixed and the parameter β is considered a control parameter, to analyse the occurrence of Hopf bifurcation in the system (3.2), one follows the same method of therom (3.2.2) with

$$m(\beta) = \alpha\frac{\pi}{2} - \min_{1\leq i\leq 3} |\arg(\lambda_i(\beta))| .$$

Theorem 3.2.3 *The system (3.2) undergoes fractional Hopf bifurcation at the equilibrium point E_2, if there exists a Hopf critical value $\beta = \beta^*$ such that the following conditions are satisfy:*

- The Jacobian matrix has two complex-conjugate eigenvalues $\lambda_{1.2} = \theta(\beta) \pm iw(\beta)$ and one real $\lambda_3(\beta)$.
- $\theta(\beta^*) = 0, w(\beta^*) > 0$ and $\lambda_3(\beta^*) \neq \mathbf{0}$.
- $\frac{d\theta}{d\beta}\mid_{\beta=\beta^*} \neq 0$.

Proof.

- Observing that condition (i) can be verified using conditions (3.9) in theorem (3.2.2).
- We can also find the critical value β^* according to the conditions in Theorem (3.2.3), it is given by the constraints bellow :

$$\begin{cases} \alpha\pi/2 - \arctan\left|\frac{w(\beta^*)}{\theta(\beta^*)}\right| = 0 \\ \frac{\dot{w}(\beta^*)\theta(\beta^*) - w(\beta^*)\dot{\theta}(\beta^*)}{\theta^2(\beta^*) + w^2(\beta^*)} \neq 0 \end{cases}.$$

■

3.3 Numerical Results

3.3.1 Numerical explorations Versus the Parameter α

In this case, the parameter β is fixed at $\beta = 0.6$, and the fractional order α is considered as a bifurcation parameter, using the proposed conditions in theorem 3.2.2, one finds :

The discriminant : $D(p) = -20.704 < 0, \beta = 0.6 \neq 1$ and $\beta - 1 = 0.6 - 1 = -0.4 < 0$, therefore the condition (3.9) is checked.

The eigenvalues of the characteristic equation (3.3) of system (3.2) are given by :$\lambda_{1.2} = 0.117\,78 \pm 1.087\,6i, \lambda_3 = -0.835\,55$ $(\theta = 0.117\,78 > 0)$

Now one can use the condition (ii) to find the critical value of bifurcation parameter:

$$\alpha^* = \frac{2}{\pi}\arctan\left|\frac{1.087\,6}{0.117\,78}\right| = 0.931\,31.$$

Finally, one can get $\frac{dm(\alpha)}{d\alpha}|_{\alpha=\alpha^*} = \frac{\pi}{2} \neq 0$. So the proposed fractional-order Hopf bifurcation conditions are verified.

When $\alpha < 0.931\,31$ the equilibrium points E_2 is stable , when $\alpha = \alpha^* = 0.931\,31$ system (3.2) under-goes a Hopf bifurcation as mentioned above, and the equilibrium point E_2 becomes unstable.The resulting bifurcation diagram is shown in (Fig.3.6).

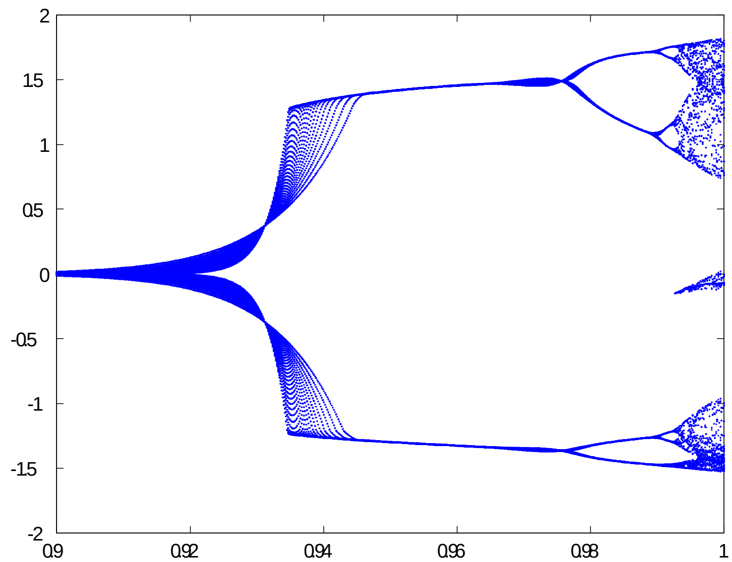


Figure 3.6: Bifurcation diagram of system (3.2) versus α, when $\beta = 0.6$.

3.3.2 Numerical explorations Versus the Parameter β

In this case, the fractional order α is fixed and the parameter β is considered as a bifurcation parameter, using the proposed conditions in previous theorem and through (Figs.3.7-3.8-3.9-3.10)

Case 1: For $\alpha = 0.98$, we find that the critical Hopf bifurcation value is localized at $\beta^* = 0.88$, $\frac{dm(\beta)}{d\beta}|_{\beta=\beta^*} = 0.35458 \neq 0$.

When $\beta < 0.88$ the equilibrium point E_2 is chaotic , when $\beta = \beta^* = 0.88$, system (3.2) undergoes a Hopf bifurcation as mentioned above and the fixed points E_2 becomes stable. The bifurcation diagram and phase portrait is shown in (Figs. 3.7 and 3.8)

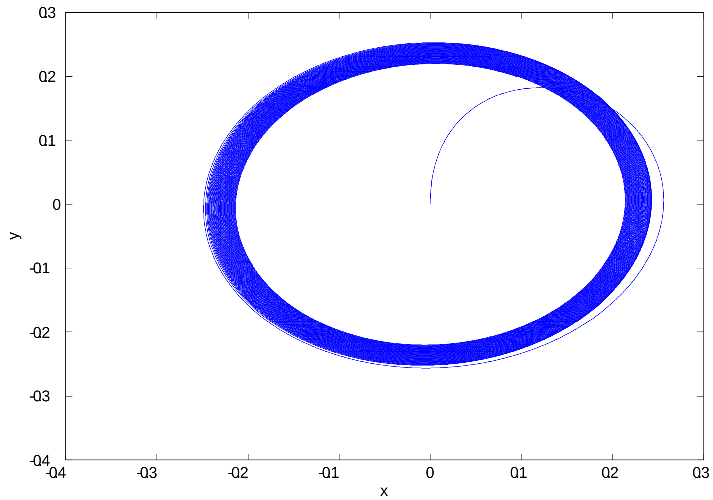


Figure 3.7: Phase portrait with $\alpha = 0.98$ and $\beta^* = 0.88$.

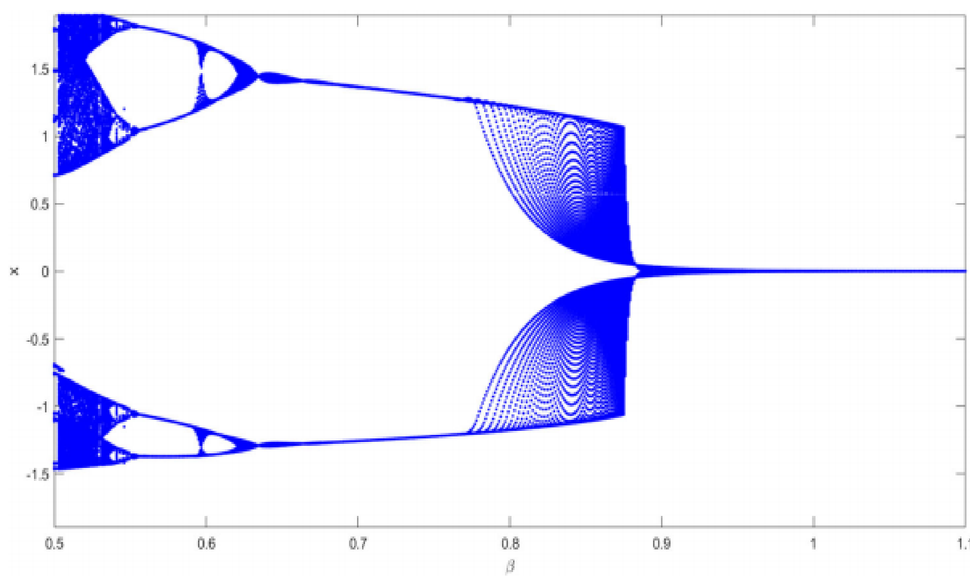


Figure 3.8: Bifurcation diagram of system (3.2) versus β,when $\alpha = 0.98$.

Case 2 For $\alpha = 0.95$, we find that the critical Hopf bifurcation value is localized at $\beta^* = 0.72$, $\frac{dm(\beta)}{d\beta}|_{\beta=\beta^*} = 0.24734 \neq 0$.

When $\beta < 0.72$ the equilibrium points E_2 is chaotic, when $\beta = \beta^* = 0.72$ system (3.2) under-goes a Hopf bifurcation as mentioned above, and the fixed points E_2 becomes stable. The bifurcation diagram is shown in (Figs. 3.9and 3.10).

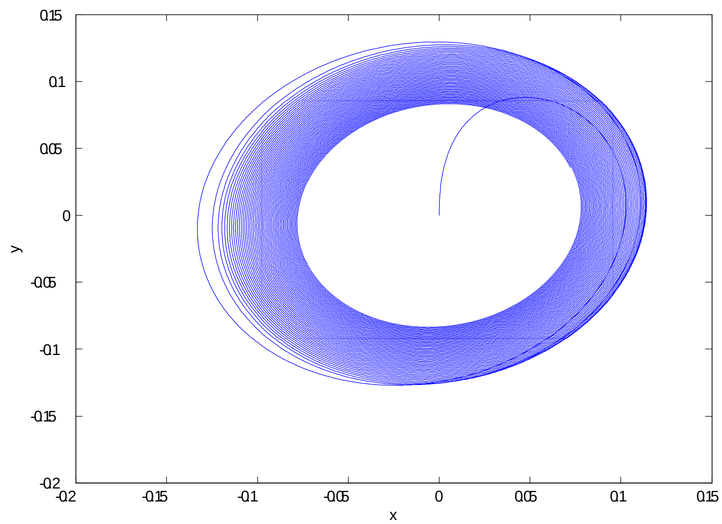


Figure 3.9: Phase portrait with $\alpha = 0.95$ and $\beta^* = 0.72$.

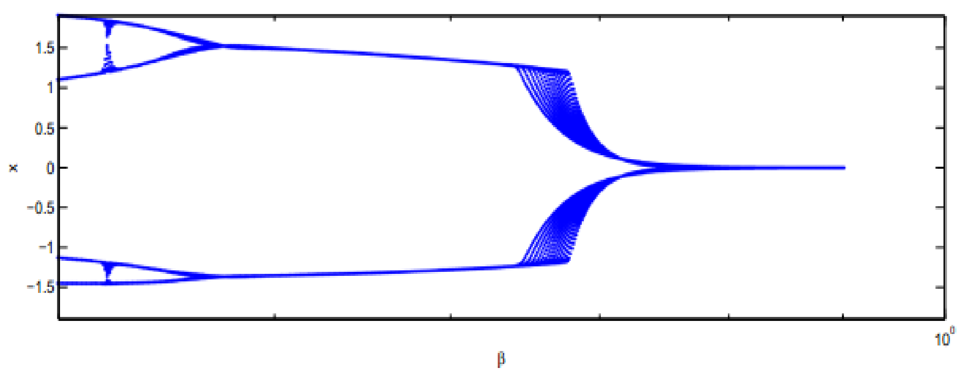


Figure 3.10: Bifurcation diagram of system (3.2) virsus β,when $\alpha = 0.95$.

3.3.3 Fractional order effect on the localisation of Hopf bifurcation point

In this part, to present the effectiveness of the fractional order derivative α on the localization of the Hopf bifurcation point β^*. We give some numerical results by varying the fractional order α in two cases: commensurate and incomerttmensurate.

The first table shows the change in the location of the bifurcation with the change of the critical Hopf bifurcation point β^* according to the variation fractional order α in the case commensurate system (i.e $\alpha = \alpha_1 = \alpha_2 = \alpha_3$).

Tables 2, 3, and 4 present the change of location of β^* according to the variation of one of the orders α_1, α_2 or α_3 in the incommensurate system. It observes that by decreasing the fractional order α, β^* decreases, the location of the critical Hopf bifurcation point in a dynamical system can be influenced by the fractional order of the system. consequently, the stability region can be enlarged and the instability region reduces. Therefore, the appropriate fractional order can be chosen in order to maximise the stability region and minimise the instability region, and vice versa, as needed. In general, we can say that the flexibility that characterizes fractional order derivatives, makes it possible to control the region of stability, the latter can be widened or narrowed according to the needs. For example, it is useful to expand the stability region when we are dealing with cancerous diseases, and vice versa, it is useful to exploit it in the case of encryption when the instability field is the most so that the encryption is more secure. The effect of fractional order on the location of the critical Hopf bifurcation point can be summarized as follows:

- Changing the fractional order of the system can lead to a shift in the location of the critical Hopf bifurcation point. However, in integer-order systems, the Hopf bifurcation point is typically associated with a specific set of system parameters. in fractional order systems.

- Fractional order systems can exhibit increased sensitivity to changes in the fractional order near the critical Hopf bifurcation point. Small variations in the fractional order can have a significant impact on the system's behavior, leading to changes in the bifurcation characteristics.

  - The introduction of non-integer order dynamics in the system equations can alter the stability properties near the critical Hopf bifurcation point.

  - The fractional order terms introduce memory effects and additional degrees of freedom, which can affect the stability and emergence of limit cycles.

- Fractional order systems can exhibit more complex dynamics near the critical Hopf bifurcation point compared to integer-order systems.

Overall, the fractional order of a system can influence the location and characteristics of the critical Hopf bifurcation point, introducing new dynamical behaviors and increasing or decreasing the complexity of the system's dynamics.

Table 3.1: Location of critical Hopf bifurcation points of β^* according to the variation of fractional order α ($\alpha = \alpha_1 = \alpha_2 = \alpha_3$)

Fractional derivative order α	0.93	0.94	0.95	0.96	0.97	0.98	0.99	1
Hopf bifurcation critical value β^*	0.61	0.66	0.72	0.77	0.83	0.88	0.94	1

Table 3.2: Location of critical Hopf bifurcation points of β^* according to the variation of fractional order $\alpha_1 (\alpha_2 = \alpha_3 = 1)$

fractional derivative order α_1	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99	1
Hopf bifurcation critical value β^*	0.88	0.9	0.91	0.92	0.93	0.94	0.96	0.97	0.99	1

Table 3.3: Location of critical Hopf bifurcation points of β^* according to the variation of fractional order $\alpha_2 (\alpha_1 = \alpha_3 = 1)$

Fractional derivative order α_2	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99	1
Hopf bifurcation critical value β^*	0.76	0.78	0.81	0.83	0.86	0.89	0.91	0.94	0.97	1

Table 3.4: Location of critical Hopf bifurcation points of β^* according to the variation of fractional order $\alpha_3(\alpha_1 = \alpha_2 = 1)$

Fractional derivative order α_3	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99	1
Hopf bifurcation critical value β^*	0.86	0.87	0.89	0.91	0.92	0.94	0.96	0.97	0.99	1

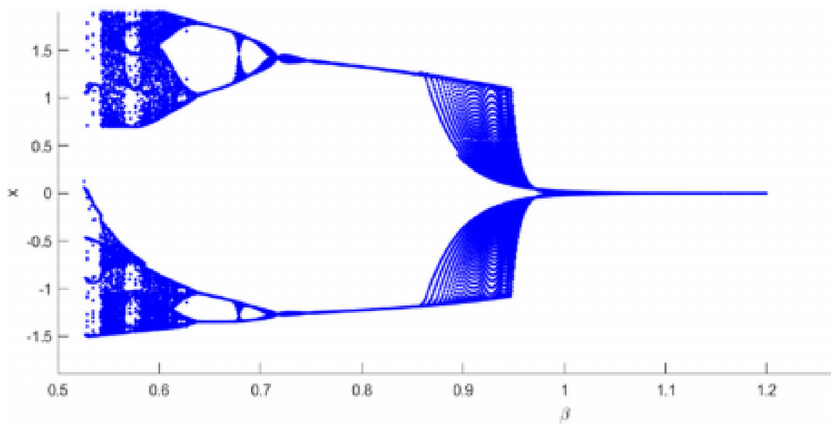


Figure 3.11: Bifurcation diagram of incommensurate system (3.2) versus β with $\alpha_1 = 0.98$ and $\alpha_2 = \alpha_3 = 1$.

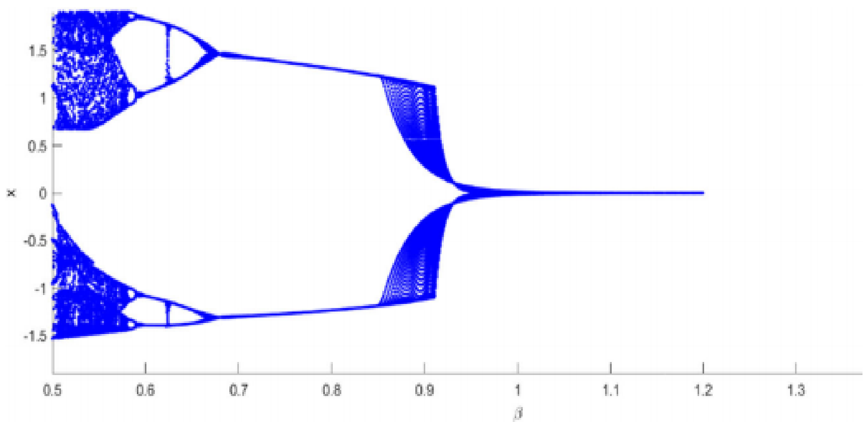


Figure 3.12: Bifurcation diagram of incommensurate system (3.2) versus β with $\alpha_2 = 0.98$ and $\alpha_1 = \alpha_3 = 1$.

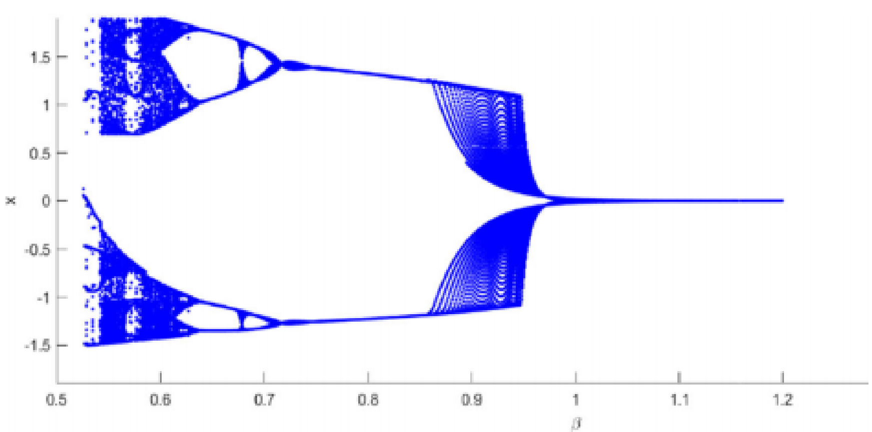


Figure 3.13: Bifurcation diagram of incommensurate system (3.2) versus β with $\alpha_3 = 0.98$ and $\alpha_1 = \alpha_2 = 1$.

Chapter 4

Chaos Control of the fractional order systems

With the advent of the concept of chaos in scientific literature, the behavior chaotic was seen as a phenomenon that interests many researchers and it can be useful or dangerous in nature. With this fact, the question of control has become a central problem, therefore is several work concerning this aspect began to emerge in the early 1990s; so according to the control class chaos chosen, several techniques were adapted and developed according to the need.

4.1 Control Strategies

In this section, we proposed tow control methods, as well as their applications.

4.1.1 Ott-Grebogi-Yorke (OGY) Method

This method was proposed by **Ott**, **Grebogi** and **Yorke** in the early 90s, relies on the fact that a Chaos often contains embedded unstable periodic orbits. The OGY method begins by:

- Identifying these periodic orbits within the chaotic system, examine them, then choose one which represents the performance of the system chaotic .
- We adjust the parameters of small disturbances in order to stabilize the unstable orbit.

The principle: Consider the following discrete dynamic system :

$$X_{n+1} = F\left(X_n, r\right),$$

where $X_n \ \epsilon \ \mathbb{R}^n$and $r \ \epsilon \ \mathbb{R}$ represents the control parameter.

Suppose that for a some values r^* from r the system admits a chaotic attractor.

Let $\bar{x}(r^*)$ a fixed point in the attractor, for r sufficiently close to r^* and in a neighborhood of $\bar{x}(r^*)$, we have the following approximation:

$$X_{n+1} - \bar{x}(r^*) = A\left[X_{n+1} - \bar{x}(r^*)\right] + B\left(r - r^*\right),$$

where $A = \frac{\partial F}{\partial x}$ and $B = \frac{\partial F}{\partial r}$.

The matrix A is broken down as follows:

$$A = \lambda_u e_u f_u^T + \lambda_s e_s f_s^T,$$

where λ_s and λ_u are the stable and unstable eigenvalues ($|\lambda_s| < 1, |\lambda_u| > 1$) in the own directions e_s and e_u respectively. f_u^T and $f_s^T$are the covariance vectors such that:

$$f_u^T e_u = f_s^T e_s = 1$$

$$f_u^T e_s = f_s^T e_u = 0$$

The strategy of the OGY method consists of adjusting the control parameter r, in order to stabilize the system on point $\bar{x}(r^*)$. It is therefore necessary that

$$f_u^T(X_{n+1} - \bar{x}(r^*)) = 0,$$

by using linearization around the fixed point and the decomposition of A we will have

$$r - r^* = -K^T(X_{n+1} - \bar{x}(r^*)),$$

where $K^T = \frac{\lambda_u}{f_u^T B}$, $f_u^T B \neq 0$.

The aim of the control is to satisfy the following condition:

$$|r - r^*| < \delta,$$

we can also write

$$\left|K^T(X_{n+1} - \bar{x}(r^*))\right| < \delta,$$

therefore, the control is determined by:

$$\partial r = \begin{cases} -K^T(X_{n+1} - \bar{x}(r^*)), & \text{if } \left|K^T(X_{n+1} - \bar{x}(r^*))\right| < \delta \\ 0 & \text{else} \end{cases}.$$

Remark 4.1.1 *If the system is continuous then we discretize it using the Poincare section.*

Application of the method to the Hénon system Consider the following Hénon system:

$$\begin{cases} x_{n+1} = a - x_n^2 + by_n \\ y_n = x_n \end{cases}, \tag{4.1}$$

where a and b represents the control parameters.

The system has two fixed points :

$$(x_1, y_1) = (\frac{1}{2}(b + \sqrt{4a - 2b + b^2 + 1} - 1), \frac{1}{2}(b + \sqrt{4a - 2b + b^2 + 1} - 1)).$$
$$(x_2, y_2) = (\frac{1}{2}(b - \sqrt{4a - 2b + b^2 + 1} - 1), \frac{1}{2}(b - \sqrt{4a - 2b + b^2 + 1} - 1)).$$

We pose $b = 0.3$, the system (4.1) represents a chaotic attractor for the value $a^* = 1.4$ of the parameter a.

The figure (4.1) represents the attractor with the evolution of time coordinates.

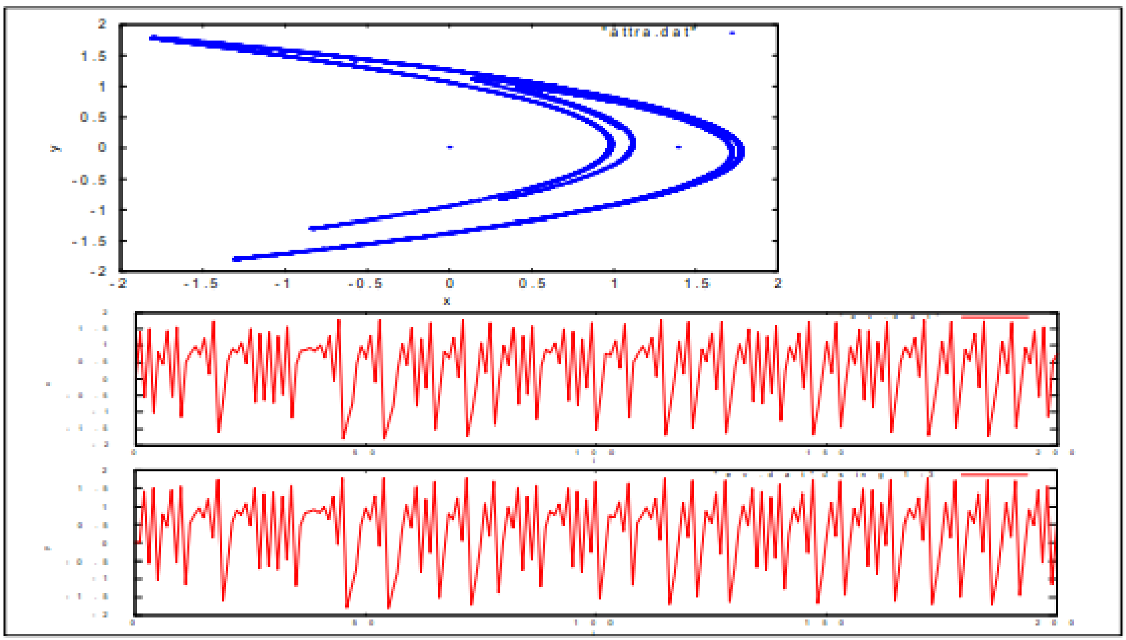


Figure 4.1: The attractor of Hénon and the evolution of time coordinates for$a = 1.4$, $b = 0.3$ and $(x_0, y_0) = (0.01, 0.01)$.

The fixed points for these parameter values are:

$$(x_1, y_1) = (0.88390, 0.88390),$$

$$(x_2, y_2) = (-1.5839, -1.5839),$$

in our case we choose the point (x_1, y_1)

We have

$$A = \begin{bmatrix} -1.7678 & 0.3 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The eigenvalues are: $\lambda_u = -1.9237$ and $\lambda_s = 0.15595$

The eigenvectors are given by:

$$[e_u, e_s] = \begin{bmatrix} 0.88728 & 0.15408 \\ -0.46123 & 0.98806 \end{bmatrix}.$$

We know that

$$f_u^T e_u = f_s^T e_s = 1 \text{ and } f_u^T e_s = f_s^T e_u = 0,$$

we obtain:

$$\begin{bmatrix} f_u \\ f_s \end{bmatrix} = \begin{bmatrix} 1.0425 & -0.16257 \\ 0.48666 & 0.93619 \end{bmatrix}$$

Calculation of k:

$$K = \frac{-1.9237}{1.0425} [1.0425 \quad -0.16257] =$$

$$K = [-1.9237 \quad 0.29999],$$

We choose $\delta = 0.01$

then,

$$\delta a = [1.9237 \quad -0.29999] \, \delta X_n \text{ with } \delta X_n = \begin{bmatrix} x_n - x^* \\ y_n - y^* \end{bmatrix}.$$

The control region is defined by:

$$(x_n - x^*)^2 + (y_n - y^*)^2 < 0.01$$

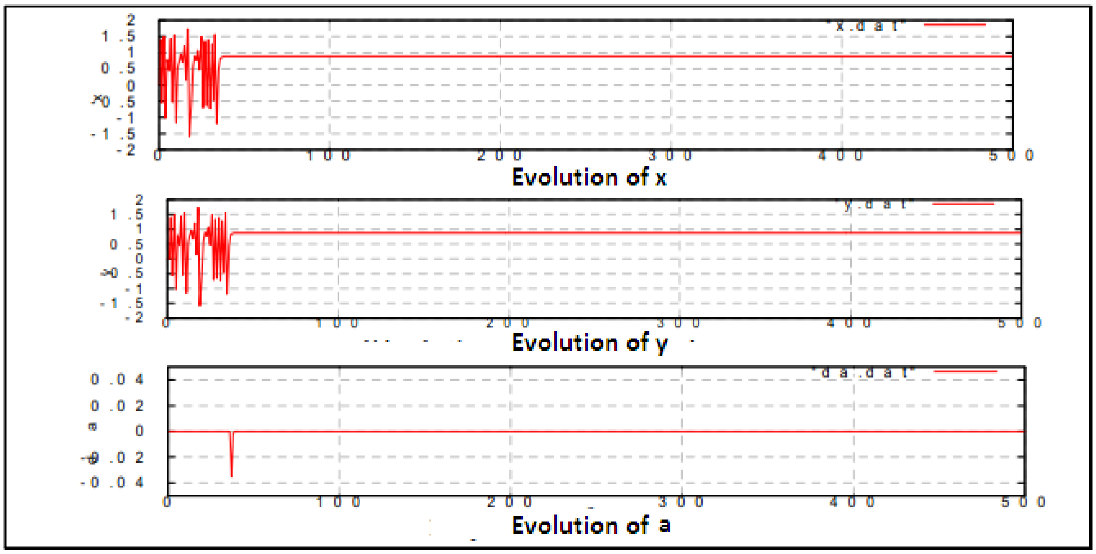


Figure 4.2: Control resut for the application of Heno by applying the method OGY

4.1.2 Feedback control

Feedback control is a process used in various fields to regulate or manage systems by continuously monitoring their performance and adjusting the system's behavior based on the observed output. It is a crucial concept in engineering, electronics, biology, economics and other disciplines. This method consists of disturbing the system state variables to reach the target orbit, it has the advantage of guaranteeing robust stability.

This method is very simple, it was applied successfully to different systems [7], [15].

The principle

Consider a continuous dynamic system :

$$^{C}D^{\alpha}x(t) = f(x, u, t),$$

where x state vector, $F : \mathbb{R}^n \to \mathbb{R}^n$ continuous function, $u(t)$ the control vector and $^{C}D^{\alpha}$ Caputo fractional order $\alpha \in]0, 1]$.

The principle is to find a control law $u(t) = w(x,t)$,w is a nonlinear vector (including the linear case) in such a way that the controlled system

$$^{C}D^{\alpha}x(t) = f(x, w(x,t), t).$$

Can be driven by feedback control $w(x,t)$ to reach the target orbit $x^{*}(t)$

$$\lim_{t \to t_f} \|x(t) - x^{*}(t)\| = 0,$$

generally we determine the control $u(t)$ which guides the state vector$x(t)$ corresponding to the nonlinear system:

$$^{C}D^{\alpha}x(t) = f(x,t) + u(t),$$

towards the target orbit $x^{*}(t)$ as follows

$$u(t) =^{C} D^{\alpha}x^{*}(t) - f(x(t),t) + k(x(t) - x^{*}(t)),$$

where k the return gain.

4.2 Feadbak Control of fractional Jerk System

4.2.1 Description of the Model

Many dynamic systems are better characterized by a dynamic fractional order model, generally based on the notion of differentiation or integration of non integer order. In this work we choose "**The Jerk System**" to apply the theories of stability and control using the generalized Routh-Hurwitz criterion to fractional order.

The Jerk system defined as follows :

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = y \\ \frac{d^\alpha y}{dt^\alpha} = z \\ \frac{d^\alpha z}{dt^\alpha} = -2y + (|x| - 1) \end{cases}, \tag{4.2}$$

where α is the fractional-order of system (**4.2**) which has two equilibrium points : $E_1(1, 0, 0)$ and $E_2(-1, 0, 0)$.

4.2.2 Stability of the equilibrium points

The Jacobian matrix $J_f(x^*, y^*, z^*)$ associated with the point of equilibrium is given by:

$$J_f(x^*, y^*, z^*) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{x^*}{|x^*|} & -2 & 0 \end{pmatrix}.$$

To study the equilibrium point stability conditions, we apply the **Routh-Hurwitz** criterion (**4.3**) in which all real parts of the eigenvalues are negative if and only if the following condition is true:

$$a_1 > 0, a_3 > 0, a_1 a_2 > a_3, \tag{4.3}$$

where $a_{1,}a_2$ and a_3 are defined as :

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3.$$

Stability Condition of $\mathbf{E_1(1, 0, 0)}$

The Jacobian matrix at equilibrium E_1 is :

$$J_f(x_1, y_1, z_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix}.$$

The characteristic polynomial associated with the equilibrium point E_1 is given by:

$$p(\lambda) = \lambda^3 + 2\lambda - 1,$$

applying the **Routh-Hurwitz** criterion (4.3), we find that E_1 is unstable .

Stability condition of E$_2$(-1,0,0)

The Jacobian matrix at equilibrium E_2 is :

$$J_f(x_2, y_2, z_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & 0 \end{pmatrix}.$$

The characteristic polynomial associated with the equilibrium point E_2 is given by: $p(\lambda) = \lambda^3 + 2\lambda + 1$.

Their eigenvalues are given as: $\lambda_0 = 0.45340, \lambda_{1.2} = -0.22670 \pm 1.4677i$.

Applying the **Routh-Hurwitz** criterion (4.3), we find that E_2 is locally asymptotically stable when $\alpha < 2/3$. E_2 is a saddle point of index 2, thus the necessary condition for the fractional-order system (4.2) to remain chaotic is $\alpha > \frac{2}{\pi} \arctan\left(\frac{|\lambda_{1.2}|}{\mathrm{Re}(\lambda_{1.2})}\right)$; Consequently, the lowest fractional order α, for which the fractional-order system (4.2) demonstrates chaos at the above-mentioned parameters, is given by the inequality $\alpha > 0.903\,56$, (see Figs.4.3 and 4.4).

The following fugures show that the system (4.2) is chaotic for α=0.95 and stable for α=0.85.

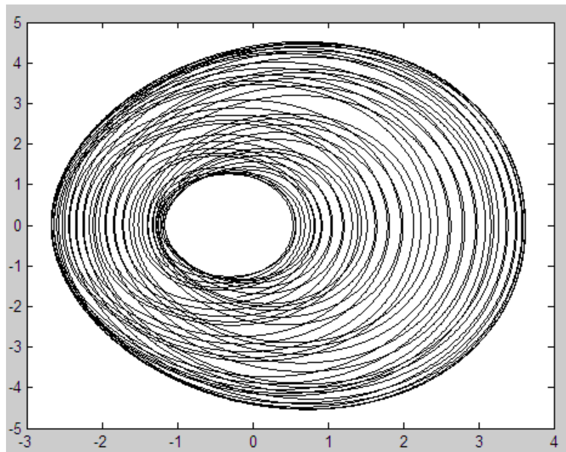


Figure 4.3: The system (4.2) is chaotic for $\alpha = 0.95$

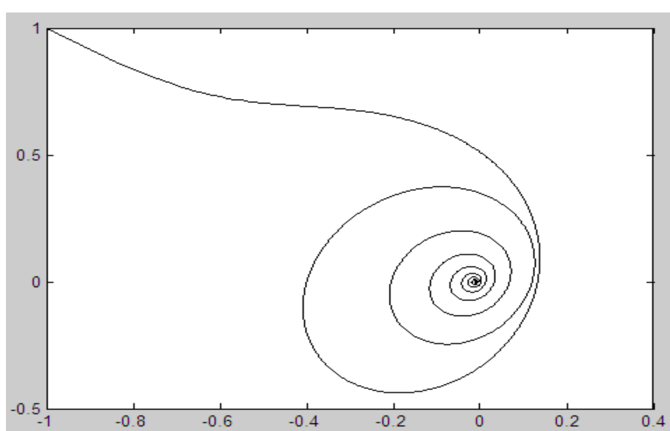


Figure 4.4: The system (4.2) is stable for $\alpha = 0.85$

4.2.3 Chaos control

The controlled fractional-order system associated with the system (4.2) is given by :

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = y - k_1 (x - x^*) \\ \frac{d^\alpha y}{dt^\alpha} = z - k_2 (y - y^*) \\ \frac{d^\alpha z}{dt^\alpha} = -2y + (|x| - 1) - k_3 (z - z^*) \end{cases} , \tag{4.4}$$

where (x^*, y^*, z^*) represents an arbitrary equilibrium point of system (4.2) . The goal is to drive the system trajectories to any of the two unstable equilibrium point E. For simplicity, we are going to choose the feedback gains $K = diag(0, k_2, 0)$.

The sufficient condition for the stabilization of controlled systems (4.4) is given by the following proposition:

Proposition 4.2.1 *If* $k_2 = 1/2$ *, then the trajectories of the controlled system (4.4) are driven to the stable equilibrium point* E_2*.*

Proof. The characteristic equation of the controlled system (4.4) at E_2 is given as:

$$\lambda^3 + k_2 \lambda^2 + 2\lambda + 1 = 0.$$

Its discriminant is given by $D(p) = -4k_2^3 + 4k_2^2 + 36k_2 - 59$, then following the graph $D(P) < 0$ if $k_2 \in [-3.4, 2]$.

If $k_2 = 1/2$, then the stability condition (3) holds and the trajectories of the controlled system (4.4) are driven to the stable equilibrium points E_2 for all $\alpha \in]0; 1[$. ■

4.2.4 Numerical Results

For $k_2 = 1/2$ and $k_1 = k_3 = 0$, It follows that $D(P) = -40.25 < 0, a_1 > 0, a_3 > 0$ and $a_1 a_2 = a_3$

Therefore, the stability conditions (3) is checked. This implies that the trajectories of the controlled fractional-order system (4.4) converge to the equilibrium E_2 as shown in Fig (4.5) . But in the integer-order case, there are two pure imaginary eigenvalues of the characteristic equation. This means that the integer-order form of the controlled system (4.4) is not stabilized to the same equilibrium point E_2, see Fig (4.6) .

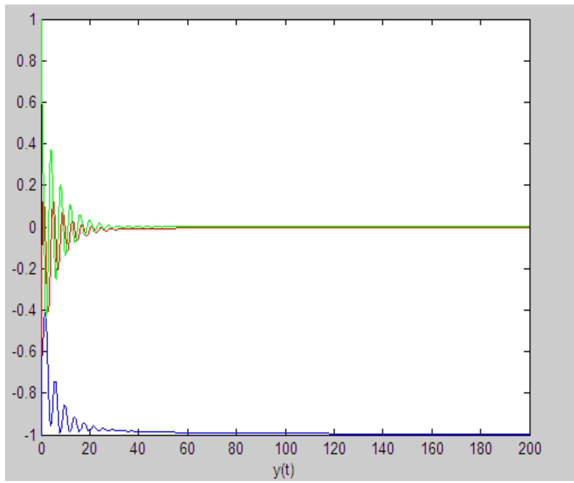


Figure 4.5: The trajectories of the controlled system (4.4) stabilized to the equilibrium point E_2 for $\alpha = 0.95$

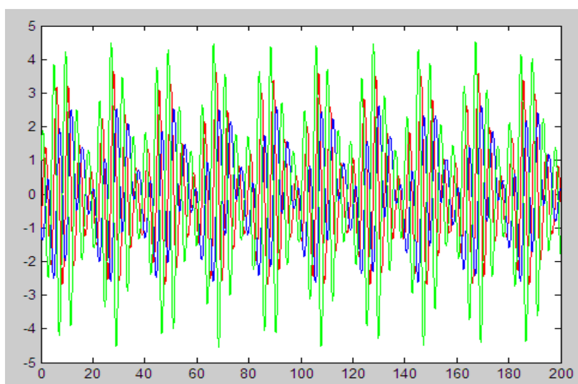


Figure 4.6: The trajectories of the controlled system (4.4) not stabilized to the equilibrium point E_2 for $\alpha = 1$

The results obtained in this section show the effect of the fractional order on the control, which proves the effectiveness of the method applied to distinguish the fractional case and that of the integer order case and to underline the importance of the control of the fractional systems, those systems that have proven to be more accurate than its integer order counterparts.

Conclusion

The objective of this thesis falls within the framework of the study of stability, bifurcation and control for a fractional order systems, it is organized as follows:

Chapter 1: We cited the approaches of the fractional integration and the fractional derivation by presenting three famous approaches (Grünwald-Letnikov, Riemann-Liouville, Caputo), their Laplace transforms, and their properties, then we cited the notions of stability of dynamical systems theory

Chapter 2: is reserved for introducing chaos and bifurcation theory in fractional order dynamical systems.

Chapter 3: we choose "The Jerk system" to apply the stability theory and control based on the Routh-Hurwitz criterion generalized to the fractional order. Thus we have demonstrated the influence of fractional order on the location of Hoph's bifurcation point and on stability.

Chapter 4: we finish our work by presented two methods for controlling chaos (the OGY method and the Feedback method). We also studied the problem of stabilization of equilibrium points, chaos control of another " Jerk System (2)" in fractional case. We showed that the fractional-order systems are controlled to their equilibrium points, however, their integer-order counterparts are not.

This thesis shows the importance to fractional order systems in applications. Fractional order systems can exhibit more complex dynamics near the critical Hopf bifurcation point compared to integer-order systems.

Our next project will be the study of stability, bifurcations and control in other dynamical systems, with other methods and we try to apply it in various fields.

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