People 's Democratic Republic
of Algeria Ministry of Higher
Education and Scientific
Research

THIRD CYCLE LMD FORMATION

A Thesis submitted in partial execution of the requirements for the degree of

DOCTOR IN MATHEMATICS

Suggested by

Mohamed Khidher University Biskra

Presented by

Salima DOUBBAKH

Titled

Malliavin Smoothness of Solutions of BSDE and Applications

Supervisor: **Pr. Nabil KHELFALLAH**

Examination Committee :

Mr. Abdelhakim NECIR	Professor	University of Biskra	President
Mr. Nabil KHELFALLAH	Professor	University of Biskra	Supervisor
Mr. Salah Eddine REBIAI	Professor	University of Batna 2	Examiner
Mr. Farid CHIGHOUB	Professor	University of Biskra	Examiner

College year 2024

Dedication

To the one who encouraged me to persevere and be patient throughout
my life, to the most prominent man in my life
(My dear father),
To the one through whom I exalt and upon whom I rely, the generous heart
(My beloved mother),
To the highest symbol of sincerity, loyalty, and companionship on the path.
(My dear husband and all his family),
To those who contributed and helped me in my academic life
(My dear uncle),
To those who were my best support
(My brothers and sisters and all their little families),
To my family, to my friends and colleagues . . .

Salima Doubbakh

Acknowledgment

First, I thank God Almighty with abundant, kind praise that fills the heavens and the earth, and I thank him before anything for what he has honoured me with in completing this thesis, and I hope from God that it will benefit me in my religion and my world and that I will obtain God's satisfaction with it. I express my gratitude to my thesis supervisor, Pr. KHELFALLAH Nabil, for the exceptional guidance provided. A heartfelt appreciation for his continuous encouragement, unwavering support, and valuable advice throughout these past years. Special thanks are extended to Pr. NECIR Abdelhakim, from the University of Biskra who has honored me by agreeing to chair the thesis jury. Sincere appreciation also goes to Pr. CHIGHOUB Farid from the University of Biskra and Pr. REBIAI Salah Eddine from the University of Batna 2, for accepting to serve as examiners for my thesis. I would also like to extend my thanks to Pr. EDDAHBI M'hamed from King Saud University for the efforts and assistance he has generously offered me. I am grateful to my family and friends for their unwavering support throughout the preparation of my significant thesis. Lastly, I extend my thanks to all the members of the LMA Applied Mathematics Laboratory at the University of Biskra.

ملخص

نهتم في الموضوع الأول من هذه الأطروحة بدراسة تنظيم ماليافين و مخططات التقريب العددية لفئة من المعادلات التفاضلية العشوائية التراجعية ذات المعامل التربيعي.

أما الموضوع الثاني ،فيتعلق بدراسة الشروط اللازمة و الكافية للتحكم الأمثل لنوع من المعادلات التفاضلية العشوائية.

تتألف هذه الأطروحة من ثلاثة فصول، يتناول الفصل الأول وجود و وحدانية الحل و إثبات استمرارية هولدر ،و ذلك باستخدام الارتباط بين هذا النوع من المعادلات مع معادلات تفاضلية عشوائية تراجعية ذات مولدات تحقق شرط ليبشيتز. الأداة المستخدمة في ذلك هي حساب ماليافين و تحويل زفينكن.

في الفصل الثاني ،يتم التركيز على مخططات التقريب العددية سواء كانت صريحة أو ضمنية، كما نقترح مخططا متقطعا كاملا تحت شروط معينة ، ثم نستنتج معدلات تقاربها.

يستعرض الفصل الثالث ،الشروط اللازمة و الكافية للتحكم الأمثل لنوع من المعادلات التفاضلية العشوائية ذات معاملات ليبشيزية، لكن غير قابلة للتفاضل و ذلك باستخدام حساب ماليافين و نظرية رادماشر.

الكلمات المفتاحية: المعادلات التفاضلية العشوائية التراجعية ذات المعامل التربيعي ؛وجود و وحدانية الحل ؛استمرارية هولدر ؛تحويل زفينكن ؛مخططات التقريب العددية ؛معدل التقارب ؛ الشرط اللازم و الكافي ؛حساب ماليافين ؛نظرية رادمشر.

Résumé

Cette thèse explore deux domaines distincts dans le champ des systèmes stochastiques : la régularité au sens de Malliavin des solutions et la théorie du contrôle. Le premier domaine se concentre sur la régularité de Malliavin des solutions d'un type spécifique d'équations différentielles stochastiques rétrogrades quadratiques (EDSR-Q), ainsi que sur la convergence de leurs schémas d'approximation numérique. Le deuxième domaine aborde les problèmes de contrôle optimal pour les systèmes stochastiques dont les coefficients ne sont pas réguliers.

Dans le premier chapitre, nous examinons l'existence et l'unicité dans $\mathbb{L}^q(q \geq 2)$ des solutions des EDSRs quadratiques unidimensionnelles , ainsi que leurs propriétés. Nous établissons la continuité Hölderienne des solutions dans $\mathbb{L}^p$ pour $(q > 4)$ et $(2 \leq p < \frac{1}{2})$, et présentons des résultats importants concernant la r égularité des solutions des EDSR-Q. Ces résultats sont obtenus en exploitant la connexion entre les EDSR-Q et les EDSR Lipschitz (L-EDSR), en utilisant notamment le calcul de Malliavin et la transformation de Zvonkin.

Le deuxième chapitre utilise les résultats existants dans la littérature sur les L-EDSRs pour construire et étudier les taux de convergence de différents types de schémas numériques pour la solution des EDSR-Q dans des cas explicites et implicites. Bien que ces schémas ne soient pas complètement discrets par rapport à la variable z, nous introduisons et examinons un "schéma complètement discret" sous certaines conditions restrictives.

Enfin, le dernier chapitre se concentre sur les conditions nécessaires et suffisantes d'optimalité pour une classe d'équations différentielles stochastiques contrôlées, où les coefficients sont globallement Lipschitz par rapport à la variable d'état mais pas nécessairement partou différentiables. Dans cette analyse, nous faisons usage du calcul de Malliavin et le Théorème de Radmecher comme principaux outils. Cela élargit le champ d'application de la théorie du contrôle stochastique et permet de traiter des problèmes plus complexes rencontrés dans divers domaines d'application.

Mots clés. Équations différentielles stochastiques rétrogrades quadratiques; Calcul de Malliavin ; Schéma explicite ; Schéma implicite ; Taux de convergence ; Continuité Hölderienne; Équations différentielles stochastiques ; Principe du maximum stochastique ; Théorème de Rademacher; Principe variationnel d'Ekeland ; Inégalité de Krylov.

Abstract

This thesis studies two different topics in the stochastic systems fields: The solutions' Malliavin regularity and control theory. The first is related to the Malliavin smoothness of the solutions of a specific type of quadratic backward stochastic differential equation (chapter 1) and the convergence of their numerical approximating schemes (chapter 2). The second topic refers to optimal control problems for stochastic systems with non-smooth coefficients (chapter 3).

Chapter one focuses on the $\mathbb{L}^q(q \geq 2)$-existence and uniqueness of the solutions of the one-dimensional quadratic backward stochastic differential equation (Q-BSDEs for short) and their properties. The $\mathbb{L}^p$-Hölder continuity of the solutions for any ($q > 4$ and $2 \leq p < \frac{q}{2}$) are established and some important results concerning the smoothness of the solution of Q-BSDEs are presented. These findings are obtained based on the connection between the underlying Q-BSDEs and the related Lipschitz BSDE (L-BSDEs for short). The natural tools are the Malliavin calculus and the so-called Zvonkin's transformation.

Chapter two uses some existing results on L-BSDEs literature to construct and study the convergence rates of different types of numerical schemes for the solution of Q-BSDE in different cases: explicit and implicit. Those schemes are not completely discrete with respect to the z-variable. However, under some restrictive conditions, a completely discrete scheme" is introduced and studied.

The last chapter investigates the necessary and sufficient optimality conditions for a class of controlled stochastic differential equations where the coefficients are merely Lipschitz continuous in the state variable but not necessarily differentiable everywhere.

The Malliavin calculus and Radmecher's theorem are the main tools in this analysis.

Key-Words: Quadratic backward stochastic differential equations; Malliavin calculus; Explicit scheme; Implicit scheme; Rate of convergence; Hölder continuity; Stochastic differential equations; Stochastic maximum principle; Rademacher's Theorem; Ekelands variational principle; Krylov's inequality.

Contents

List of Symbols and Abbreviations

The different symbols and abbreviations used in this thesis.

Symbols

- $(\Omega, \mathcal{F}, \mathrm{P})$: probability space.
- $W = (W_t)_{t\in[0,T]}$: Brownian motion.
- $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$: is the natural filtration generated by the Brownian motion W.
- $\sigma(A)$: σ-algebre generated by A.
- $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P})$: filtered probability space.
- $\mathcal{X}_A$: the indicator function of the set A.
- $\mathbb{R}$: the set of all real numbers.
- $\mathbb{R}^n$: n-dimensional real Euclidean space.
- ∂_y: partial derivative with respect to the variable "y".
- $\mathbb{E}[\cdot]$: the mathematical expectation.
- $\mathbb{E}[\cdot|\mathcal{F}]$: conditional expectation.
- $u(\cdot)$: optimal strict control.
- $\mathcal{J}(u)$: The expected cost corresponding to the control u.
- $\mathcal{H}$: The Hamiltonian.
- $\mathcal{P}$: the progressive σ-field defined on the product space $[0,\ T] \times \Omega$.
- $\mathrm{P}\otimes dt$: the product measure of P with the Lebesgue measure dt.

- $\langle .,. \rangle_H$: the scalar product of the separable Hilbert space H.

- $\mathbb{L}^p(\Omega)$: denotes the space of all $\mathcal{F}_T$-measurable random variables X satisfying $\mathbb{E}\,|X|^p < +\infty$, for any $p \geq 2$.

- D: the Malliavin derivative

- $\mathbb{D}^{1,2}$: the set of all random variables which are Malliavin differentiable in $\mathbb{L}^2(\Omega)$.

- $\mathbb{D}^{1,p}$: the set of all random variables which are Malliavin differentiable in $\mathbb{L}^p(\Omega)$.

- $H := \mathbb{L}^2([0,T])$: Hilbert space.

- $H^{\otimes n}$: the n-fold tensor product of a Hilbert space H.

- $\mathbb{L}^p\left(\Omega, H^{\otimes k}\right)$: Lebesgue space of p-integrable functions defined on a measurable space Ω with values in the tensor product $H^{\otimes k}$.

- $\mathcal{H}^p_{\mathcal{F}}([0,\ T])$ denotes the Banach space of all progressively measurable processes $\varphi : ([0,\ T] \times \Omega, \mathcal{P}) \to (\mathbb{R}, \mathcal{B})$ with norm

$$\|\varphi\|_{\mathcal{H}^p} = \left(\mathbb{E}\left(\int_0^T |\varphi_t|^2 \,\mathrm{d}t\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} < +\infty.$$

- $\mathcal{S}^p_{\mathcal{F}}([0,\ T])$: denotes the Banach space of all the RCLL (right continuous with left limits) adapted processes $\varphi : ([0,\ T] \times \Omega, \mathcal{P}) \to (\mathbb{R}, \mathcal{B})$ with norm

$$\|\varphi\|_{\mathcal{S}^p} = \left(\mathbb{E} \sup_{0 \leq t \leq T} |\varphi_t|^p\right)^{\frac{1}{p}} < +\infty.$$

- $\mathcal{M}^{2,p}$: for any $p \geq 2$, denotes the class of square-integrable random variables F with a stochastic integral representation of the form

$$F = \mathbb{E}\,[F] + \int_0^T u_t \mathrm{d}W_t,$$

where u is a progressively measurable process satisfying $\sup_{0 \leq t \leq T} \mathbb{E}|u_t|^p$ is finite.

- $\mathbb{L}_a^{1,p}$: stand for the set of all H-valued processes $\{u_t\}_{0\leq t\leq T}$, which are progressively measurable and have real-valued versions, such that:

  (a) For almost all $t \in [0,\ T]$, $u_t \in \mathbb{D}^{1,p}$.

  (b) $\mathbb{E}[(\int_0^T |u_t|^2 \,\mathrm{d}t)^{\frac{p}{2}} + (\int_0^T \int_0^T |D_\theta u_t|^2 \,\mathrm{d}\theta \mathrm{d}t)^{\frac{p}{2}}] < +\infty$.

  - $\mathcal{W}_1^2(\mathbb{R})$: the space of continuous functions g from $\mathbb{R}$ to $\mathbb{R}$ such that g' is continuous and g'' is integrable on $\mathbb{R}$.

Abbreviations:

- a.e: almost everywhere.
- a.s: almost surely.
- i.e: that is to say.
- w.r.t: with respect to
- RCLL: right continuous with left limits.
- SDE: stochastic differential equations.
- Q-SDE: quadratic stochastic differential equations.
- BSDE: backward stochastic differential equations.
- F-BSDE: forward-backward stochastic differential equations.
- L-BSDE: Lipschitz backward stochastic differential equations.
- Q-BSDE: quadratic backward stochastic differential equations.
- NCO : necessary conditions of optimality.
- SCO : sufficient conditions of optimality.

General Introduction

The Malliavin calculus is essential for analyzing the smoothness of solutions to backward stochastic differential equations (BSDEs). The methods provided are highly effective in establishing solutions' existence, uniqueness, and regularity and examining their sensitivity to parameter changes.

One of the significant advantages of this stochastic calculus of variations is its capacity to calculate derivatives of solutions of BSDEs with respect to parameters and initial conditions. This enables a comprehensive examination of the regular characteristics of solutions, such as their level of smoothness and continuity, providing vital insights into the dynamics of stochastic processes.

Furthermore, Malliavin calculus is a highly effective technique for analyzing stochastic differential equations (SDEs) and their applications in diverse domains such as economics, physics, and biology. Examining stochastic optimal control, filtering theory, and analyzing stochastic partial differential equations is essential. The applications of this concept span from pricing options and managing risk in finance to analyzing stochastic processes in physics and biology.

Recall that backward stochastic differential equations (BSDEs) were first studied by Bismut [10] in the linear case; Bismut introduced a linear backward stochastic differential equation to represent the conjugate variable or adjoint process in the stochastic version of the Pontryagin maximum principle. Pardoux and Peng [48] published a seminal paper in which they studied the general nonlinear backward stochastic differential equations (BSDEs) where the generator is global Lipschitz and the terminal condition is square integrable. Since then, there has been extensive research on BSDEs; we refer to [13, 27, 40] for a more complete presentation of the theory. This type of equation has proven to be a powerful tool for studying stochastic processes and has applications in many fields, including finance, economics, engineering, and mathematical biology.

This thesis focuses on two issues. The initial topic concerns the examination of the Malliavin regularity of solutions to a certain class of backward stochastic differential equations that demonstrate quadratic growth. In addition, we devise diverse numerical techniques to solve these equations and ascertain the rate at which their numerical schemes converge. These equations are characterized by the following form:

$$Y_t = \xi + \int_t^T \Big(h(r, Y_r) + h_1(r) Z_r + f(Y_r) \, |Z_r|^2 \Big) \, \mathrm{d}r - \int_t^T Z_r \mathrm{d}W_r, \ 0 \le t \le T, \tag{0.1}$$

where the process $h : [0,\ T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a given bounded and global Lipschitz function in y uniformly in r, $h_1 : [0,\ T] \to \mathbb{R}$ is a bounded function, $f : \mathbb{R} \to \mathbb{R}$ is an integrable function and ξ is a given terminal datum such that ξ need not be a function of a forward diffusion; this means that the random variable ξ can be taken arbitrarily. Since finding explicit solutions for BSDEs is difficult, several researchers have focused on numerical methods. Generally speaking, numerical approximation schemes are a very important research topic in recent years that many studies have focused on. One of the challenges for solving BSDEs numerically is that the equations are nonlinear and high-dimensional, which makes it difficult to obtain exact solutions. Therefore, researchers have developed a variety of numerical approximation schemes to solve BSDEs, including Monte Carlo methods, finite difference methods, and numerical methods based on partial differential equation theory. In particular, in the Markovian case, Douglas *et al.* [23] established numerical methods for a class of forward-backward SDEs based on the four-step scheme developed by Ma *et al.* [39] to solve general F-BSDEs requiring the numerical resolution of quasi-linear parabolic PDE. Bally [8] proposed a time discretization scheme and obtained its convergence rate. Zhang [52] established some $\mathbb{L}^2$-regularity on Z and found that their scheme converges and also derived its convergence rate. It is worth mentioning the work of Briand *et al.* [16], where a scaled random walk replaces Brownian motion. Good references for this are [18, 29, 30, 36, 45]. The research paper we relied on is Hu *et al.* [34] in the non-Markovian case, which is considered the first generalized result because they investigate BSDE with a general terminal value and random generator. Notably, they do not impose the condition that the terminal value should stem from a forward diffusion equation. Similarly, the generator's randomness need not arise from a forward equation and verify a Lipschitz condition. They initially establish the $\mathbb{L}^p$-Hölder continuity of the solution. Subsequently, they devise some numerical approximation schemes for backward stochastic differential equations, determining the convergence rate based on the attained $\mathbb{L}^p$-Hölder continuity results. The Malliavin calculus serves as the main tool in their analysis. Many researchers have turned their interest toward Q-BSDE theory in the last few years. The very well-known result of the existence of the solution was proved by Kobylanski in [37] when the terminal condition is bounded, the generator coefficient ($h(r,y)+h_1(r)z+f(y)\,|z|^2$ in our case) is continuous and has a quadratic growth in z. Later, Bahlali *et al.* studied in [4] one-dimensional Q-BSDE with a measurable generator in cases where $h(r,y)+h_1(r)z=0$ and the terminal condition is merely square integrable. Q-BSDE-theory has been developed very remarkably in different perspectives; in particular, in Bahlali *et al.* [5], the authors studied a BSDE whose generator shows logarithmic growth and provided a relation between this latter and one type of Q-BSDE. Subsequently, in [41], Madoui *et al.* focused on solving a class of quadratic BSDEs with Jumps.

As opposed to ordinary BSDEs, only a few studies are devoted to the numerical study of Q-BSDEs. Indeed, Imkeller and Dos Reis in [35] gave explicit convergence rates for the difference between the solution of a Q-BSDE and its truncation; in the same context, Richou in [50] provided a new time discretization scheme with a non-uniform time step for such BSDEs and also obtained an explicit convergence rate for this scheme. Recently, Chassagneux and Richou in [17] introduced a fully implementable algorithm for a Q-BSDE based on quantization and illustrated their convergence results with numerical examples. The question arises at this stage: Can we extend some existing findings in the global Lipschitz BSDE framework into the quadratic BSDE? These findings concern the $\mathbb{L}^p$-Hölder continuity of the solution, numerical approximation schemes, and their rate of convergence.

During our journey to answer this question, we have faced several drawbacks and difficulties. Below, we list the important four among them. The first one is how to choose the form of the generator itself to transform the initial Q-BSDE to the Lipschitz BSDE (L-BSDE in short). After several attempts, we ended up with the following generator's form $h(r,y)+h_1(r)z+f(y)|z|^2$, where the function h, h_1 are bounded, and h is Lipschitz in y. The second drawback concerns the function h_1, which makes it difficult to ensure that the generator of the transformed L-BSDE is Hölder continuous. We were forced to assume that h_1 is a constant function to overcome this difficulty.

The third difficulty concerns the problem of the explicit scheme's rate convergence. More precisely, we can not prove the convergence of the approximating Z-component of the underlying Q-BSDE. Defer from Hu *et al.* [34], because of pure technical reasons, we have chosen to work on $\mathbb{L}^p$ $(p<2)$ rather than $\mathbb{L}^2$.

The fourth drawback is due to the presence of the integral of Z over some intervals in both explicit and implicit numerical schemes, making it difficult to provide fully discrete schemes. To get around this obstacle, we restricted ourselves to the cases where the generator has the following two forms,

$$\beta(s)z+f(y)|z|^2 \quad \text{and} \quad \alpha(s)+\beta(s)z+\frac{1}{2}|z|^2,$$

the functions α and β are assumed to be deterministic, bounded and Hölder continuous.

The answer to the above mentioned question is the contents of the first and second chapters.

In the first chapter, we study the existence and uniqueness of solutions to Q-BSDE (0.1) by using the relationship between some types of Q-BSDEs and Lipschitz BSDE (L-BSDE in short). Indeed, the following space transformation

$$F(x)=\int_0^x \exp(2\int_0^y f(t)\mathrm{d}t)\mathrm{d}y, \tag{0.2}$$

allows the elimination of the Q-BSDE's generator or the quadratic part of it to obtain L-BSDE. Because this transformation function is a bijection, one can transfer the properties interchangeably between the Q-BSDE and L-BSDE under consideration. Applying Itô-Krylov's formula to $F(Y_t)$, one can obtain

$$F(Y_t)=F(\xi)+\int_t^T (h(s,Y_s)F'(Y_s)+h_1(s)F'(Y_s)Z_s)\,\mathrm{d}s-\int_t^T F'(Y_s)Z_s\mathrm{d}W_s. \tag{0.3}$$

To simplify notations, we put

$$F(Y_t) = \bar{Y}_t,\ \bar{Z}_t = F'(Y_t)\, Z_t \text{ and } F(\xi) = \bar{\xi}. \tag{0.4}$$

Thus, Equation (0.3) reads

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{h}(s, \bar{Y}_s, \bar{Z}_s)\mathrm{d}s - \int_t^T \bar{Z}_s \mathrm{d}W_s, \tag{0.5}$$

where $\bar{h}(s,y,z) = h(s, F^{-1}(y))F'(F^{-1}(y)) + h_1(s)z$, which is a Lipschitz generator.

If (Y, Z) is a solution to Q-BSDE (0.1), then Itô-Krylov's formula applied to $F(Y_t)$ shows that $(\bar{Y}, \bar{Z})$ is a solution to (0.5). Conversely, if $(\bar{Y}, \bar{Z})$ is a solution to (0.5), then by applying Itô-Krylov's formula to $F^{-1}\left(\bar{Y}_t\right)$, we show that $(Y = F^{-1}\left(\bar{Y}\right),\ Z = \frac{\bar{Z}}{F'(F^{-1}(\bar{Y}))})$ is a solution to $\mathrm{Q}-\mathrm{BSDE}$ (0.1). Our second aim is to prove, under some extra conditions on f, and the $\mathbb{L}^p$-Hölder continuity of the solution $(\bar{Y}, \bar{Z})$ proven in Hu *et al.* [34], in the case where the terminal value is twice differentiable in the sense of Malliavin calculus, and the first and second derivatives satisfy some integrability conditions; also the generator satisfied similar assumption (see Assumption 2 in Chapter 1, for more details). Several statements concerning the path regularity property of the solutions of Q-BSDE (0.1) in the sense of Malliavin calculus and thus establish the following estimates, for any $q > 4$, $2 \le p < \frac{q}{2}$ and $s, t \in [0,\ T]$

$$\mathbb{E}|Y_t - Y_s|^p \ \le\ K|t-s|^{\frac{p}{2}} \quad \text{and} \quad \mathbb{E}|Z_t - Z_s|^p \ \le\ K|t-s|^{\frac{p}{2}}, \tag{0.6}$$

where K is a constant independent of s and t. Moreover, we shall prove that the Q-BSDE's solution (Y, Z) is Malliavin differentiable, and the process Z can be determined as the trace of the Malliavin derivative of Y. Our results are illustrated by three examples.

In the second chapter, depending on the results of Hu *et al.* in [34], we construct different types of numerical schemes for the solution of Q-BSDE (0.1) and give the rate of their convergence. Our starting point is to define both explicit and implicit numerical schemes of Q-BSDE (0.1) as

$$(Y^\pi, Z^\pi) = \left(F^{-1}\left(\bar{Y}^\pi\right), \frac{\bar{Z}^\pi}{F'(F^{-1}\left(\bar{Y}^\pi\right))}\right).$$

where the approximating pairs $\left(\bar{Y}^\pi, \bar{Z}^\pi\right)$ associated global Lipschitz BSDE (0.5).

Firstly, we prove that the rate of convergence of the explicit schemes is given by

$$\mathbb{E}\sup_{0\leq t\leq T}|Y_t - Y_t^{\pi}|^2 \;\leq\; K(|\pi| + \mathbb{E}|\xi - \xi^{\pi}|^2). \tag{0.7}$$

while the one of Z takes the following form

$$\int_0^T \mathbb{E}\,|Z_t - Z_t^{\pi}|^p\,\mathrm{d}t \;\leq\; K\left(|\pi| + \mathbb{E}\,|\xi - \xi^{\pi}|^2\right)^{\frac{p}{2}} \quad \forall\, 1\leq p<2. \tag{0.8}$$

Secondly, we focus on the rate of convergence of the implicit numerical scheme for Q-BSDE (0.1), which is given as follows: for any $p\geq 2$

$$\mathbb{E}\sup_{0\leq t\leq T}|Y_t - Y_t^{\pi}|^p \;\leq\; K\left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\xi - \xi^{\pi}|^p\right),$$

and

$$\mathbb{E}\left(\int_0^T |Z_t - Z_t^{\pi}|^2\,\mathrm{d}t\right)^{\frac{p}{2}} \;\leq\; K\left(|\pi|^{\frac{p}{2}} + \max\left(\mathbb{E}\,|\xi - \xi^{\pi}|^p,\; |\pi|^{-\frac{p}{2}}\,\mathbb{E}\,|\xi - \xi^{\pi}|^{2p}\right)\right).$$

Notice that both explicit and implicit numerical schemes are not completely discrete due to the use of the integral of the process Z in each iteration of the schemes. However, we suggest a "fully discrete scheme" where the Lipschitz part h of the generator is independent of y and is assumed to be linear in z. More precisely, we will deal with the following form of Q-BSDE for $0\leq t\leq T$

$$Y_t = \xi + \int_t^T \left(\alpha\,(r) + \beta\,(r)\,Z_r + f(Y_r)\,|Z_r|^2\right)dr - \int_t^T Z_r dW_r$$

and prove the following rate of convergence in two cases: the first case, when $\alpha\equiv 0$ and the second one when $f\equiv\frac{1}{2}$

$$\mathbb{E}\max_{0\leq i\leq n}\left\{\left|Y_{t_i} - Y_{t_i}^{\pi}\right|^p + \left|Z_{t_i} - Z_{t_i}^{\pi}\right|^p\right\} \;\leq\; C\,|\pi|^{\frac{p}{2} - \frac{p}{2\ln\frac{1}{|\pi|}}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.$$

The second topic is concerned with the optimality conditions of a controlled state process governed by the subsequent stochastic differential equation:

$$\begin{cases} \mathrm{d}X_t &= b(t,X_t,u_t)\mathrm{d}t + \sigma\,(t,X_t)\,\mathrm{d}W_t,\\ X_0 &= x\in\mathbb{R}, \end{cases} \tag{0.9}$$

where: $b : [0,T]\times\mathbb{R}^d\times U \to \mathbb{R}^d$, $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d\otimes\mathbb{R}^d$, are given deterministic functions, T be a fixed strictly positive real number which serves as a finite time horizon,

U be some Borel subset of $\mathbb{R}^d$, $(W_t)_{t\in[0,T]}$ is a d-dimensional Brownian motion, x is the initial state at time 0 and u stands for the control process. The primary goal is to establish the necessary as well as sufficient conditions of optimality in the case where the coefficients b and σ are merely Lipschitz continuous in the state variable but not necessarily differentiable everywhere.

Let $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,\ T]}$ be a sub-filtration of $\mathbb{F}$ and U be some Borel subset of $\mathbb{R}^d$. We denote $\mathcal{U}^{\mathcal{G}}$ as the set comprising all admissible controls, defined as the set of all measurable, $\mathbb{G}$-adapted, and $\mathrm{d}t \otimes \mathrm{d}P$-integrable processes $u : [0,\ T] \times \Omega \to U$.

The cost to be minimized is expressed as the expected value of:

$$\mathcal{J}(u) = \mathbb{E}\Big[\int_0^T \ell(t, X_t, u_t)\mathrm{d}t + g(X_T)\Big], \tag{0.10}$$

where ℓ and g are given real-valued functions defined respectively on $[0,\ T] \times \mathbb{R}^d \times U$ and $\mathbb{R}^d$. Keep in mind that an optimal control is characterized as an admissible control that meets the following criteria:

$$\mathcal{J}(\hat{u}) = \min_{u\in\mathcal{U}^{\mathcal{G}}} \mathcal{J}(u). \tag{0.11}$$

Henceforth, the above control problem will be referred to as "**Problem A**".

Typically, the study of the **Problem A** can be carried out using two distinct approaches. The first method is Bellman's dynamic programming, named after the American mathematician Richard Bellman. The second approach, which is the centre of interest of this work, is named after the Soviet mathematician Lev Semenovich Pontryagin. Different mathematical methodologies are needed to handle these two approaches. Dynamic programming seeks to derive a second-order partial differential equation, commonly called the Hamilton-Jacobi-Bellman equation, that describes the value function.

The second approach explores optimality determination based on Pontryagin's necessary conditions, commonly known as the stochastic maximum principle (SMP for short). Generally, we use two different mathematical tools to study the stochastic maximum principle: Frechet derivatives and Malliavin calculus. With Frechet derivatives, the adjoint processes, which play a vital role in the stochastic maximum principle, are described by stochastic linear systems. Conversely, Malliavin calculus provides an advantage by offering explicit expressions for the adjoint processes, which are given by employing the Malliavin derivatives of the optimal state process.

In the context of the first way, the pioneering work in the SMP field is credited to Kushner [38], who focused on the classical regular case, where the coefficients of the controlled SDE are sufficiently smooth. Following this groundbreaking research, a substantial body of literature has emerged, including significant contributions by Benssoussan [9], Bismut [11], Haussmann [33], and Peng [49]. For a comprehensive overview of this subject and an extensive list of references, you may consult the well-regarded book authored by Yong and Zhou [51]. Orrieri [47] established a version of SMP for a controlled SDE with dissipative drift, noteworthy to highlight the pioneering contribution of Azizi and Khelfallah [1], who explored the initial version of SMP for controlled backward SDEs beyond the global Lipschitz framework. We also refer the reader to the following papers [2, 6, 12, 31], which address the SMP for BSDE and F-BSDE systems.

Then, the researchers turned their attention to relaxing the smoothness of the coefficients of the controlled systems and those of the cost functional. The first attempt in this direction is due to Mezerdi [15], who introduced a pioneering approach to handle SDEs with non-smooth drift. The author developed an SMP similar to Kushner's but with less regularity on the drift of the controlled SDE. This accomplishment was made possible by employing Clarke's generalized gradients and the stable convergence of probability measures. The work highlights treating a broad class of optimization problems for stochastic controlled systems with non-differentiable coefficients. Following this, Bahlali *et al.* [7] introduced a second version to address stochastic differential equations featuring Lipschitz coefficients. Their methodology is based on Krylov's inequality, assuming the diffusion matrix satisfies the condition of uniform ellipticity. In a similar vein, Bahlali *et al.* [3] have established an SMP for the optimal control of SDEs with degenerate diffusion coefficients. Remarkably, this result was derived using techniques similar to those originally initiated by Bouleau and Hirsch in [14]. Expanding upon this groundwork, Chighoub *et al.* [21] broadened the investigation to encompass scenarios where the coefficients of the cost functional are merely Lipschitz (non-differentiable).

The second way to handle the SMP is initiated with Meyer-Brandis *et al.* [43]. This work examines a controlled Itô-Lévy process in which the controller may have access to a subset of the accessible information. The system coefficients and the objective performance functional are permitted to be random, potentially non-Markovian. The application of Malliavin calculus allows for deriving a maximal principle for the optimal control of a system, in which the adjoint process is explicitly described. subsequently, the main focus of the paper [44] is to extend the previous work to mixed regular-singular control issues, where the control variable consists of two components: one is absolutely continuous, and the second is singular. At this stage, the natural question is: Can we relax the smoothness of the controlled system's coefficients? The third chapter of this thesis is devoted to answering this question and exploring some extensions to some non-Lipschitz-controlled stochastic systems. The main objective of this chapter is to examine **Problem A** when the coefficients b and σ are solely Lipschitz continuous, without necessarily being differentiable everywhere. The primary goal of our work is to investigate the **Problem A** when the coefficients b and σ are merely Lipschitz continuous but not necessarily differentiable everywhere. Firstly, we want to prove the following SMP represented by the necessary condition for optimality

$$\mathbb{E}\Big[\partial_u \mathcal{H}(t,\hat{X}_t,\hat{u}_t,\hat{Y}(t))\Big|\mathcal{G}_t\Big] = 0 \text{ for a.e. } (t,\ \omega), \tag{0.12}$$

where $\hat{u}$ refers to the optimal control, $\hat{X}$ to its associated optimal solution, and the usual Hamiltonian is defined by:

$$\mathcal{H}(t,X_t,u_t,Y(t)) = \ell(t,X_t,u_t) + Y(t)b(t,X_t,u_t), \tag{0.13}$$

here Y stands for the adjoint process, which will be determined later. The key tool to achieve these fundamental findings is known as Rademacher's theorem [20], which states that all Lipschitz functions have bounded and measurable derivatives almost everywhere. This enables us to define the resulting adjoint equation. With this in mind, we will use Frankowska's approach [28] to approximate b and σ by sequences b_n and σ_n, respectively, which converge uniformly and belong to the class $\mathcal{C}^1$. This will yield the following set of regularized control problems, indexed by n,

$$\begin{cases} \mathrm{d}X_t^n &= b_n(t,X_t^n,u_t)\mathrm{d}t + \sigma_n\left(t,X_t^n\right)\mathrm{d}W_t, \\ X_0^n &= x \in \mathbb{R}, \end{cases} \tag{0.14}$$

and

$$\mathcal{J}_n(u) = \mathbb{E}\Big[\int_0^T \ell(t,X_t^n,u_t)\mathrm{d}t + g(X_T^n)\Big]. \tag{0.15}$$

Since b_n and σ_n are $\mathcal{C}^1-$functions, Ekeland's variational principle [25], is instrumental in deriving a set of near-optimal conditions for the regularized family of control problems (0.14) and (0.15). Subsequently, by utilizing the result of Mezerdi [44] and Meyer-Brandis *et al.* [43], one can derive the following SMP:

$$\mathbb{E}\Big[\partial_u \mathcal{H}_n(t, \hat{X}_t^n, \hat{u}_t^n, \hat{Y}^n(t))\Big|\mathcal{G}_t\Big] = O\left(\delta_n\right) \text{ for a.e. } (t, \omega), \tag{0.16}$$

where $\hat{u}^n$ is near optimal control for the cost $\mathcal{J}_n$, $\hat{X}^n$ denotes its corresponding near optimal solution, such that the Hamiltonian is defined similarly as in (0.13) for any n. Then, By utilizing Krylov's estimate, we employ limit arguments to demonstrate the SMP for the original problem (0.12). Furthermore, we establish sufficiency conditions, indicating that if there exists $\hat{u} \in \mathcal{U}^{\mathcal{G}}$ satisfying (0.12), then $\hat{u}$ also satisfies (0.11). Finally, we apply our results to study the necessary and sufficient optimality conditions for controlled quadratic SDEs.

Scientific Contributions

Publications Based on this Thesis

- Doubbakh, S., Khelfallah, N., Eddahbi, M., & Almualim, A. (2023). Malliavin Regularity of Non-Markovian Quadratic BSDEs and Their Numerical Schemes. Axioms, 12(4), 366.
- Eddahbi, M., Doubbakh, S. & Khelfallah, N. (2024). Pontraygin's Optimal Variational Inequalities for SDEs with non-Differentiable Coefficients: A Malliavin Calculus Approach. Submitted to Stochastics: An International Journal of Probability and Stochastic Processes.

Conference Papers

- Doubbakh, S. Khelfallah, N. Eddahbi, M. ($\mathbb{L}^q \geq 2$)**-Solutions of Quadratic Backward Stochastic Differential Equations.** *The 6th International Workshop on Applied Mathematics and Modelling « WIMAM'2022 »*, 26 and 27 October, 2022 Guelma, Algeria.
- Doubbakh, S. Khelfallah, N. ($\mathbb{L}^p \geq 2$)**-Solutions of Quadratic Backward Stochastic Differential Equations with jumps.** *The First international Workshop on Applied Mathematics 1st-IWAM'2022*, 6-8 December, 2022 Costantine, Algeria.
- Doubbakh, S. Khelfallah, N. **Smoothness of Solutions of QBSDEs.** *The First International Workshop on Applied Mathematics 1st-IWAM'2022*, 6-8 December, 2022 Costantine, Algeria.
- Doubbakh, S. Khelfallah, N. Eddahbi, M. **Malliavin Regularity for Quadratic**

BSDE. *The first National Applied Mathematics Seminar 1st-NAMS'23*, 14-15 May,2023 Biskra, Algeria.

Chapter 1

$\mathbb{L}^p$-Hölder Continuity of the Solutions of Q-BSDEs $(2 \leq p < \frac{q}{2})$

1.1 Introduction

This chapter focuses on two important results regarding a class of quadratic backward stochastic differential equations (Q-BSDEs), taking the following form

$$Y_t = \xi + \int_t^T \left(h(r, Y_r) + h_1(r) Z_r + f(Y_r) |Z_r|^2 \right) \mathrm{d}r - \int_t^T Z_r \mathrm{d}W_r, \ 0 \leq t \leq T, . \tag{1.1}$$

The first one studies the $\mathbb{L}^q (q \geq 2)$-existence and uniqueness problem for Q-BSDE (1.1). The second one establishes the $\mathbb{L}^p$-Hölder continuity of its solutions for any $(q > 4, \ 2 \leq p < \frac{q}{2})$, under some assumptions on the coefficients. The natural tool that we shall use is the Malliavin calculus and Zvonkin's transformation allow us to eliminate the quadratic term and obtain standard L-BSDE. taking advantage of this transformed L-BSDE's existence and uniqueness propriety, we can construct a unique solution to the initial Q-BSDE and study the regularity of its solutions. Additionally, we shall prove that the Q-BSDE's solutions (Y, Z) are Malliavin differentiable, and the process Z can be determined as the trace of the Malliavin derivative of Y. It is worth mentioning that the content of this and the next Chapter is the subject of our paper Doubbakh et *al.* [22].

The next section is devoted to some basic notions of Malliavin calculus, the main tool in proving our crucial results.

1.2 Malliavin Calculus

Over the past few years, Malliavin calculus has generated great interest. One of the reasons for this is undoubtedly the wide range of applications in mathematical finance, particularly in models based on Brownian motion. Following a symposium on the applications of Malliavin calculus in finance, the journal Mathematical Finance dedicated its entire January 2003 volume to this new reality. At the same time, empirical data highlighted the limitations of Brownian financial models, leading to a renewed interest in jump processes, such as Lévy processes. In risk theory models, which involve an insurance company's portfolio, Lévy processes have also experienced significant growth. This situation and the need for discontinuity have created fertile ground for the development of Malliavin calculus for Lévy processes and the further development of applications for this important family of stochastic processes.

The question asked was, what is Malliavin's calculus? It is a stochastic calculus of variations, or in other words, an infinite-dimensional differential calculus on the Wiener space, which is the canonical space of Brownian motion. It represents a fusion between probability theory and differential calculus. The earliest works on this subject date back to the 1970s, specifically in 1976 when Paul Malliavin published "Stochastic Calculus of Variations and Hypoelliptic Operators" [42], focusing on the existence and regularity of the density function of random vectors. The initial application of this theory was to provide a probabilistic proof of Hörmander's Theorem (Hörmander's "sum of squares" Theorem) on hypoelliptic differential operators.

1.2.1 Notations and Preliminaries

Let $W = \{W_t\}_{0 \leq t \leq T}$ be a real-valued Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathrm{P}, \{\mathcal{F}_t\}_{0 \leq t \leq T})$, such that $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the natural filtration generated by the Brownian motion W and the P-null sets and let $\mathcal{F} = \mathcal{F}_T$.

Now, let us define the core concept of Malliavin calculus: the Malliavin derivative. To begin with we define the following spaces

- $\mathbb{L}^q(\Omega)$ denotes the space of all $\mathcal{F}_T$-measurable random variables X satisfying $\mathbb{E}\,|X|^q < +\infty$.

- $H := \mathbb{L}^2([0,T])$ denotes the separable Hilbert space of square-integrable functions defined on the interval $[0,T]$.

We represent by $\langle \cdot, \cdot \rangle_H$ the scalar product of the separable Hilbert space H and by $\|h\|_H$ the norm of any element h of H. For any $h \in H$, we denote the Wiener integral by

$$\mathcal{W}(h) = \int_0^T h(t)\mathrm{d}W_t,$$

We denote by $C_p^{\infty}(\mathbb{R}^n)$ the set of all infinitely continuously differentiable functions $g : \mathbb{R}^n \to \mathbb{R}$ such that g and all their partial derivatives have polynomial growth. For $(h_1, \ldots, h_n) \in H^{\otimes n}$, set

$$F = g\left(\mathcal{W}(h_1), \cdots, \mathcal{W}(h_n)\right). \tag{1.2}$$

We denote by $\mathcal{S}$ the class of all smooth random variables of the form (1.2). Denote $\partial_i g := \frac{\partial g}{\partial x}$, for $i = 1, \ldots, n$..

Definition 1.1

The derivative of a smooth random variable F of the form (1.2) is the H-valued random variable given by

$$DF = \sum_{i=1}^{i=n} \partial_i g\left(\mathcal{W}(h_1), \cdots, \mathcal{W}(h_n)\right) h_i.$$

The subsequent outcome is an integration-by-parts formula

Lemma 1.1

Suppose that F is a differentiable random variable and h belongs to the set H. In such a case. Then

$$\mathbb{E}\left(\langle DF, h\rangle_H\right) = \mathbb{E}\left(F\mathcal{W}(h)\right).$$

The domain of D in $\mathbb{L}^p(\Omega)$, denoted as $\mathbb{D}^{1,p}$, for any $p \geq 1$, refers to the closure of the set of smooth random variables $\mathcal{S}$ with respect to the norm,

$$\|F\|_{1,p} = \left[\mathbb{E}\,|F|^p + \mathbb{E}\left(\|DF\|_H^p\right)\right]^{\frac{1}{p}},$$

When p is equal to 2, the space $\mathbb{D}^{1,2}$ becomes a Hilbert space with the scalar product given by

$$\langle F, G\rangle_{1,2} = \mathbb{E}\left(FG\right) + \mathbb{E}\left(\langle DF, DG\rangle_H\right),$$

We have the ability to define the iteration of the operator D in a manner that ensures the iterated derivative $D^k F$, for a smooth random variable F, becomes a random variable with values in $H^{\otimes k}$. Consequently, for any natural number $k \geq 1$ and every $p \geq 1$, we introduce a seminorm on $\mathcal{S}$ that is defined as follows:

$$\|F\|_{k,p} = \left[\mathbb{E}\left(|F|^p\right) + \sum_{j=1}^{k} \mathbb{E}\left(\left\| D^j F \right\|_{H^{\otimes j}}^p \right) \right]^{\frac{1}{p}},$$

This family of seminorms verifies the following properties:

(i) Monotonicity: $\|F\|_{k,p} \leq \|F\|_{j,q}$ for any $F \in \mathcal{S}$, given that $p \leq q$ and $k \leq j$.

(ii) Closability: the operator D^k is closable from $\mathcal{S}$ into $\mathbb{L}^p\left(\Omega, H^{\otimes k}\right)$ for all $p \geq 1$.

The following result is the chain rule, which can be easily proved by approximating the random variable F by smooth random variables and the function ϕ by $\phi^* \psi_\varepsilon$, where ψ_ε is an approximation of the identity.

Proposition 1.2

Let $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives, and fix $p \geq 1$. Suppose that $F = (F_1, \ldots, F_m)$ is a random vector whose components belong to the space $\mathbb{D}^{1,p}$. Then $\phi(F) \in \mathbb{D}^{1,p}$, and

$$D(\phi(F)) = \sum_{i=1}^{m} \partial_i \phi(F) \, DF^i.$$

Example 1.1

$D_t W(T) = \mathcal{X}_{[0,T]}(t) = 1$ *(for* $t \in [0,T]$.

Example 1.2

By the chaine rule, we get

$$D_t\left(\exp W(t_0)\right) = \exp W(t_0) \, .\mathcal{X}_{[0,t_0]}(t).$$

Example 1.3

If f is an element of $\mathbb{L}^2([0,T])$, then the Wiener integral

$$\int_0^T f(s) \mathrm{d}W_s,$$

is differentiable in the sense of Malliavin, meaning that it belongs to $\mathbb{D}^{1,2}$ and its *derivative is*

$$D_t\left(\int_0^T f(s)\mathrm{d}W_s\right) = f(t),$$

Let $\mathbb{L}_a^{1,p}$ stand for the set of all H-valued processes $\{u_t\}_{0\leq t\leq T}$, which are progressively measurable and have real-valued versions such that:

(a) For almost all $t\in[0,\ T]$, $u_t\in\mathbb{D}^{1,p}$.

(b) $\mathbb{E}[(\int_0^T |u_t|^2\,\mathrm{d}t)^{\frac{p}{2}} + (\int_0^T\int_0^T |D_\theta u_t|^2\,\mathrm{d}\theta\mathrm{d}t)^{\frac{p}{2}}] < +\infty$.

For more details on Malliavin calculus, we refer the reader to Nualart's seminal book [46].

1.3 Some Properties of L-BSDE

Before addressing the important results of this thesis, which are related to the existence, and uniqueness of the solution for Q-BSDE (0.1) and their Malliavin regularity, we recall the most crucial results established by Hu *et al.* in [34] concerning the Malliavin calculus for L-BSDE.

1.3.1 Estimates on the Solution of BSDEs

Let $\mathcal{P}$ be the progressive σ-field defined on the product space $[0,\ T]\times\Omega$, to begin with defining, for any $q\geq 1$, the following spaces which will be used frequently in the sequel:

- $\mathcal{H}^q_{\mathcal{F}}([0,\ T])$ denotes the Banach space of all progressively measurable processes $\varphi : ([0,\ T]\times\Omega,\mathcal{P})\to(\mathbb{R},\mathcal{B})$ with norm

$$\|\varphi\|_{\mathcal{H}^q} = \left(\mathbb{E}\left(\int_0^T |\varphi_t|^2\,\mathrm{d}t\right)^{\frac{q}{2}}\right)^{\frac{1}{q}} < +\infty.$$

- $\mathcal{S}^q_{\mathcal{F}}([0,\ T])$ denotes the Banach space of all the RCLL (right continuous with left limits) adapted processes $\varphi : ([0,\ T]\times\Omega,\mathcal{P})\to(\mathbb{R},\mathcal{B})$ with norm

$$\|\varphi\|_{\mathcal{S}^q} = \left(\mathbb{E}\sup_{0\leq t\leq T}|\varphi_t|^q\right)^{\frac{1}{q}} < +\infty.$$

- $\mathcal{M}^{2,q}$, for any $q \geq 2$, denote the class of square-integrable random variables F with a stochastic integral representation of the form

$$F = \mathbb{E}F + \int_0^T u_t \mathrm{dW}_t,$$

where u is a progressively measurable process satisfying $\sup_{0\leq t\leq T} \mathbb{E}|u_t|^p$ is finite.

We consider the following BSDE

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{h}(r, \bar{Y}_r, \bar{Z}_r)\mathrm{d}r - \int_t^T \bar{Z}_r \mathrm{d}r, \quad 0 \leq t \leq T \tag{1.3}$$

where: the generator $\bar{h}$ in the BSDE (1.3) is a measurable function.

$\bar{h} : ([0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}, \mathcal{P} \otimes \mathcal{B} \otimes \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$, and the terminal value $\bar{\xi}$ is an $\mathcal{F}_T$-measurable random variable.

Definition 1.2

A solution to the BSDE (1.3) is a pair of progressively measurable processes $(\bar{Y}, \bar{Z})$ such that $\int_0^T |\bar{Z}_t|^2 \mathrm{d}t < \infty$, $\int_0^T |\bar{h}(t, \bar{Y}_t, \bar{Z}_t)| \mathrm{d}t < \infty$, a. s. and

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{h}(r, \bar{Y}_r, \bar{Z}_r)\mathrm{d}r - \int_t^T \bar{Z}_r \mathrm{d}r.$$

The next lemma provides a useful estimate on the solution to BSDE (1.3).

Lemma 1.3

Fix $q \geq 2$. Suppose that $\bar{\xi} \in \mathbb{L}^q(\Omega)$, $\bar{h}(t, 0, 0) \in \mathcal{H}^q_{\mathcal{F}}([0,T])$ and $\bar{h}$ is uniformly Lipschitz in (y, z); namely, there exists a positive number L such that $\mu \times \mathrm{P}$ a.e.

$$\left|\bar{h}(t, y_1, z_1) - \bar{h}(t, y_2, z_2)\right| \leq L\left(|y_1 - y_2| + |z_1 - z_2|\right)$$

for all $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}$. Then there exists a unique solution pair $(\bar{Y}, \bar{Z}) \in \mathcal{S}^q_{\mathcal{F}}([0, T]) \times \mathcal{H}^q_{\mathcal{F}}([0,T])$ to (1.3). Moreover, we have the following estimate for the solution

$$\mathbb{E} \sup_{0\leq t\leq T} \left|\bar{Y}_t\right|^q + \mathbb{E}\left(\int_0^T \left|\bar{Z}_t\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} \leq K\left(\mathbb{E}\left|\bar{\xi}\right|^q + \mathbb{E}\left(\int_0^T \left|\bar{h}(t,0,0)\right|^2 \mathrm{d}t\right)^{\frac{q}{2}}\right). \tag{1.4}$$

Proof: See Hu *et al.* [34]. ■

1.3.2 The Malliavin Regularity for L-BSDE

It is important to remember the most significant results established by Hu *et al.* in [34], regarding the $\mathbb{L}^p$-Hölder continuity of the solution in the case where the generator is global Lipschitz. As we will see later, for a given L-BSDE the process $\bar{Z}$ will be expressed in terms of the Malliavin derivative of the solution $\bar{Y}$, which will satisfy a linear BSDE with random coefficients. To study the properties of $\bar{Z}$ we need to analyze a class of linear BSDEs.

Given two progressively measurable processes $\{\alpha_t\}_{0\leq t\leq T}$ and $\{\beta_t\}_{0\leq t\leq T}$ with $0 \leq t \leq T$. We will make use of the following integrability conditions:

(H1) For any $\lambda > 0$,

$$C_\lambda := \mathbb{E}\exp\left(\lambda \int_0^T \left(|\alpha_t| + \beta_t^2\right) \mathrm{d}t\right) < \infty.$$

(H2) For any $p \geq 1$,

$$K_p := \sup_{0\leq t\leq T} \mathbb{E}\left(|\alpha_t|^p + |\beta_t|^p\right) < \infty.$$

Under condition (H1), we denote by $\{\rho_t\}_{0\leq t\leq T}$ the solution of the linear stochastic differential equation

$$\begin{cases} \mathrm{d}\rho_t &= \alpha_t\rho_t \mathrm{d}t + \beta_t\rho_t \mathrm{d}W_t, \quad 0 \leq t \leq T, \\ \rho_0 &= 1. \end{cases} \tag{1.5}$$

The proof of the main Theorem in this context is crucially dependent on the Theorem presented below.

Theorem 1.4

Let $q > p \geq 2$ and let $\bar{\xi} \in \mathbb{L}^q(\Omega)$ and $\bar{f} \in \mathcal{H}^q_{\mathcal{F}}([0,T])$. Let $\{\alpha_t\}_{0\leq t\leq T}$ and $\{\beta_t\}_{0\leq t\leq T}$ are two progressively measurable processes satisfying conditions (H1) and (H2). Suppose that the random variables $\bar{\xi}\rho_T$ and $\int_0^T \rho_t\bar{f}_t \mathrm{d}t$ belong to $\mathcal{M}^{2,q}$, where $\{\rho_t\}_{0\leq t\leq T}$ is the solution to equation (1.5). Then the following linear BSDE,

$$\bar{Y}_t = \bar{\xi} + \int_t^T \left(\alpha_r\bar{Y}_r + \beta_r\bar{Z}_r + \bar{f}_r\right) \mathrm{d}r - \int_t^T \bar{Z}_r \mathrm{d}W_r, \quad 0 \leq t \leq T,$$

has a unique solution pair $(\bar{Y}, \bar{Z})$, and there is a constant $K > 0$ such that

$$\mathbb{E}\left|\bar{Y}_t - \bar{Y}_s\right|^p \leq K\,|t-s|^{\frac{p}{2}}$$

for all $s, t \in [0, T]$.

Proof: See Hu *et al.* [34]. ■

Throughout this section, we will make use of the following important assumptions on the terminal value $\bar{\xi}$ and the generator $\bar{h}$ of L-BSDE (1.3) :

Assumption 1 Fix $q > 4$ and $2 \leq p < \frac{q}{2}$

(1.i) $\bar{\xi} \in \mathbb{D}^{2,q}$ and satisfies

$$\mathbb{E}|D_\theta \bar{\xi} - D_{\theta'} \bar{\xi}|^p \leq L\,|\theta - \theta'|^{\frac{p}{2}}\,, \tag{1.6}$$

$$\sup_{0 \leq \theta \leq T} \mathbb{E}|D_\theta \bar{\xi}|^q < +\infty \tag{1.7}$$

and

$$\sup_{0 \leq \theta \leq T} \ \sup_{0 \leq u \leq T} \ \mathbb{E}|D_u D_\theta \bar{\xi}|^q \ < \ +\infty. \tag{1.8}$$

where $L > 0$ is a constant and $\theta, \ \theta' \in [0, \ T]$.

(1.ii) $\bar{h}$ has continuous and uniformly bounded first- and second-order partial derivatives with respect to $\bar{y}$ and $\bar{z}$, and $\bar{h}(\cdot, \ 0, 0) \in \mathcal{H}^q_{\mathcal{F}}([0, \ T])$.

(1.iii) $\bar{\xi}$ and $\bar{h}$ satisfy respectively the conditions (1.i) and (1.ii). Let $(\bar{Y}, \bar{Z})$ be the unique solution of (1.3) with terminal value $\bar{\xi}$ and generator $\bar{h}$ such that $\bar{h}(t, \ \bar{Y}_t, \bar{Z}_t)$, $\partial_{\bar{y}} \bar{h}(t, \ \bar{Y}_t, \bar{Z}_t)$ and $\partial_{\bar{z}} \bar{h}(t, \bar{Y}_t, \bar{Z}_t)$ belong to $\mathbb{L}^{1,q}_a$ and $D.\bar{h}(t, \ \bar{Y}_t, \bar{Z}_t)$, $D.\partial_{\bar{y}} \bar{h}(t, \ \bar{Y}_t, \bar{Z}_t)$ and $D.\partial_{\bar{z}} \bar{h}(t, \ \bar{Y}_t, \bar{Z}_t)$ satisfy

$$\sup_{0 \leq \theta \leq T} \mathbb{E}\left(\int_\theta^T \left|D_\theta \bar{h}\left(t, \bar{Y}_t, \bar{Z}_t\right)\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} \ < \ +\infty, \tag{1.9}$$

$$\sup_{0 \leq \theta \leq T} \mathbb{E}\left(\int_\theta^T \left|D_\theta \partial_{\bar{y}} \bar{h}\left(t, \bar{Y}_t, \bar{Z}_t\right)\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} \ < \ +\infty, \tag{1.10}$$

$$\sup_{0 \leq \theta \leq T} \mathbb{E}\left(\int_\theta^T \left|D_\theta \partial_{\bar{z}} \bar{h}\left(t, \bar{Y}_t, \bar{Z}_t\right)\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} \ < \ +\infty. \tag{1.11}$$

There exists $L > 0$ such that for any $t \in (0, T]$ and for any $0 \leq \theta, \ \theta' \leq t \leq T$

$$\mathbb{E}\left[\left(\int_t^T \left|D_\theta \bar{h}\left(t, \bar{Y}_t, \bar{Z}_t\right) - D_{\theta'} \bar{h}\left(t, \bar{Y}_t, \bar{Z}_t\right)\right|^2 \mathrm{d}t\right)^{\frac{p}{2}}\right] \leq L\,|\theta - \theta'|^{\frac{p}{2}}\,. \tag{1.12}$$

For each $\theta \in [0,\ T]$ and each pair of $(\bar{y}, \bar{z})$, $D_\theta \bar{h}(\cdot,\ \bar{Y}_t, \bar{Z}_t) \in \mathbb{L}_a^{1,q}$ and it has continuous partial derivatives with respect to $\bar{y}$, $\bar{z}$, which are denoted by $\partial_{\bar{y}} D_\theta \bar{h}(\cdot,\ \bar{Y}_t, \bar{Z}_t)$ and $\partial_{\bar{z}} D_\theta \bar{h}(\cdot,\ \bar{Y}_t, \bar{Z}_t)$ and the Malliavin derivative $D_u D_\theta \bar{h}(\cdot, \bar{Y}_t, \bar{Z}_t)$ satisfies

$$\sup_{0\leq t\leq T} \sup_{0\leq u\leq T} \mathbb{E}\left(\int_{u\vee\theta}^{T} \left|D_u D_\theta \bar{h}\left(t, \bar{Y}_t, \bar{Z}_t\right)\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} < \infty.$$

Theorem 1.5

Let Assumption 1 be satisfied.

(a) There exists a unique solution pair $\left\{\left(\bar{Y}_t,\ \bar{Z}_t\right)\right\}_{0\leq t\leq T}$ *to BSDE (1.3), and* $\bar{Y}$, $\bar{Z}$ *are in* $\mathbb{L}_a^{1,q}$*. A version of the Malliavin derivatives* $\left\{\left(D_\theta \bar{Y}_t,\ D_\theta \bar{Z}_t\right)\right\}_{0\leq\theta,t\leq T}$ *of the solution pair satisfies the following linear BSDE:*

$$\begin{aligned} D_\theta \bar{Y}_t = D_\theta \bar{\xi} + \textstyle\int_t^T \Big[\partial_{\bar{y}} \bar{h}\left(r, \bar{Y}_r, \bar{Z}_r\right) D_\theta \bar{Y}_r + \partial_{\bar{z}} \bar{h}\left(r, \bar{Y}_r, \bar{Z}_r\right) D_\theta \bar{Z}_r \\ + D_\theta \bar{h}\left(r, \bar{Y}_r, \bar{Z}_r\right)\Big] \mathrm{d}r - \textstyle\int_t^T D_\theta \bar{Z}_r \mathrm{d}W_r, \qquad 0 \leq \theta \leq t \leq T, \end{aligned} \tag{1.13}$$

$$D_\theta \bar{Y}_t = 0, \quad D_\theta \bar{Z}_t = 0, \quad 0 \leq t \leq \theta \leq T \tag{1.14}$$

Moreover, $\left\{D_t \bar{Y}_t\right\}_{0\leq t\leq T}$*, defined by (1.13), gives a version of* $\left\{\bar{Z}_t\right\}_{0\leq t\leq T}$*, namely,* $\mu \times \mathrm{P}$ *a.e.*

$$\bar{Z}_t = D_t \bar{Y}_t. \tag{1.15}$$

(b) There exists a constant $K > 0$*, such that, for all* $s, t \in [0, T]$*:*

$$\mathbb{E}\left|\bar{Z}_t - \bar{Z}_t\right|^p \leq K\left|t - s\right|^{\frac{p}{2}}. \tag{1.16}$$

Proof: **(a)** The proof of the existence and uniqueness of the solution $\left(\bar{Y}, \bar{Z}\right)$, and $\bar{Y}$, $\bar{Z} \in \mathbb{L}_a^{1,2}$ is similar to the Proposition 5.3 in [27], and also the fact $\left(D_\theta \bar{Y},\ D_\theta \bar{Z}\right)$ is given by (1.13) and (1.14). In Proposition 5.3 in [27] the exponent q is equal to 4, and one assume that $\int_0^T \left\|D_\theta \bar{h}(\cdot, \bar{Y}, \bar{Z})\right\|_{H^2}^2 \mathrm{d}\theta < \infty$, which is a consequence of (1.9) and the fact that $\bar{Y}$, $\bar{Z} \in \mathbb{L}_a^{1,2}$. Furthermore, from conditions (1.7) and (1.9) and the estimate (1.4) in Lemma 1.3, we obtain

$$\sup_{0\leq\theta\leq T}\left\{\mathbb{E} \sup_{\theta\leq t\leq T}\left|D_\theta \bar{Y}_t\right|^q + \mathbb{E}\left(\int_\theta^T \left|D_\theta \bar{Z}_t\right|^2 \mathrm{d}t\right)^{\frac{q}{2}}\right\} < \infty.$$

Hence, by Proposition 1.5.5 in [46], $\bar{Y}$ and $\bar{Z}$ belong to $\mathbb{L}_a^{1,q}$.

(b) Let $0 \leq s \leq t \leq T$. In this proof, $C > 0$ will be a constant independent of s and t, and my vary from line to line. By representation (1.15), we have

$$\bar{Z}_t - \bar{Z}_s = D_t\bar{Y}_t - D_s\bar{Y}_s = \left(D_t\bar{Y}_t - D_s\bar{Y}_t\right) + \left(D_s\bar{Y}_t - D_s\bar{Y}_s\right) \tag{1.17}$$

From Lemma 1.3 and equation (1.13) for $\theta = s$ and $\theta' = t$, respectively, we obtain, using conditions (1.7) and (1.12)

$$\begin{aligned} &\mathbb{E}\left|D_t\bar{Y}_t - D_s\bar{Y}_t\right|^p + \mathbb{E}\left(\int_t^T \left|D_t\bar{Z}_r - D_s\bar{Z}_r\right|^2 \mathrm{d}r\right)^{\frac{p}{2}} \\ &\leq C\left[\mathbb{E}\left|D_t\bar{\xi} - D_s\bar{\xi}\right|^p + \mathbb{E}\left(\int_t^T \left|D_t\bar{h}\left(r, \bar{Y}_r, \bar{Z}_r\right) - D_t\bar{h}\left(r, \bar{Y}_r, \bar{Z}_r\right)\right|^2 \mathrm{d}r\right)^{\frac{p}{2}}\right] \\ &\leq C\left|t-s\right|^{\frac{p}{2}}. \end{aligned} \tag{1.18}$$

Denote $\alpha_u = \partial_{\bar{y}}\bar{h}\left(u, \bar{Y}_u, \bar{Z}_u\right)$ and $\beta_u = \partial_{\bar{z}}\bar{h}\left(u, \bar{Y}_u, \bar{Z}_u\right)$ for all $u \in [0, T]$. Then, by 1.ii in Assumption 1, the processes α and β satisfy the conditions (H1) and (H2), and from (1.13) we have for $r \in [s, T]$

$$D_s\bar{Y}_r = D_s\bar{\xi} + \int_r^T \left[\alpha_u D_s\bar{Y}_u + \beta_u D_s\bar{Z}_u + D_s\bar{h}\left(u, \bar{Y}_u, \bar{Z}_u\right)\right] \mathrm{d}u - \int_r^T D_s\bar{Z}_u \mathrm{d}W_u.$$

Then, we use Theorem 1.4 to estimate $\mathbb{E}\left|D_s\bar{Y}_t - D_s\bar{Y}_t\right|^p$ by the following form

$$\mathbb{E}\left|D_s\bar{Y}_t - D_s\bar{Y}_t\right|^p \leq C\left|t-s\right|^{\frac{p}{2}} \tag{1.19}$$

for all s, $t \in [0, T]$. Combining (1.19) with (1.17) and (1.18), we obtain that there is a constant $K > 0$ independent of s and t, such that

$$\mathbb{E}\left|\bar{Z}_t - \bar{Z}_t\right|^p \leq K\left|t-s\right|^{\frac{p}{2}}$$

for all s, $t \in [0, T]$.

For more details, can you see Proposition 5.3 in [27] and Theorem 2.6 in [34]. ■

Remark 1.6

From Theorem 1.5 we know that $\left\{\left(D_\theta\bar{Y}_t, D_\theta\bar{Z}_t\right)\right\}_{0\leq\theta,\, t\leq T}$ satisfies equation (1.13) and $\bar{Z}_t = D_t\bar{Y}_t$, $\mu \times \mathrm{P}$ a.e. Moreover, since (1.7) and (1.9) hold, we can apply the estimate (1.4) in Lemma 1.4 to the linear BSDE (1.13) and deduce $\sup\limits_{0\leq t\leq T} \mathbb{E}\left|\bar{Z}_t\right|^q < \infty$.

Corollary 1.1

Under the assumptions in Lemma 1.3, let $\left(\bar{Y}, \bar{Z}\right) \in \mathcal{S}^q_{\mathcal{F}}([0,T]) \times \mathcal{H}^q_{\mathcal{F}}([0,T])$ *be the unique solution pair to* (1.3)*. If* $\sup\limits_{0\leq t\leq T} \mathbb{E}\left|\bar{Z}_t\right|^q$ *is finite, then there exists a constant* C *such that, for any* $s,t \in [0,T]$*:*

$$\mathbb{E}\left|\bar{Y}_t - \bar{Y}_s\right|^q \leq K\left|t-s\right|^{\frac{q}{2}}.$$

Proof: See Hu *et al.* [34]. ∎

In the following section, we extend these findings to the framework of Q-BSDE.

1.4 $\mathbb{L}^p$-Hölder Continuity of the Solutions of Q-BSDEs $(2 \leq p < \frac{q}{2})$

In this section, we shall study the regularity in the sense of stochastic calculus of variations for solutions of a class of singular BSDEs that show quadratic growth on the variable z.

1.4.1 $\mathbb{L}^q(q \geq 2)$-Solutions of Q-BSDE

Now we establish an existence and uniqueness result to Q-BSDE (0.1) by performing an exponential transformation, known as Zvonkin's transformation as mentioned in [4], we need the following Assumption:

Assumption 2 Fix $q \geq 2$

(2.i) $h(\cdot,0) \in \mathcal{H}^q_{\mathcal{F}}([0,\ T])$ and h is bounded and uniformly Lipschitz in y.

(2.ii) $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a given integrable function.

(2.iii) ξ is q-integrable.

(2.iv) There exists a constant $L > 0$ such that

$$|h(t_2,y) - h(t_1,y)| \leq L\left|t_2 - t_1\right|^{\frac{1}{2}}.$$

We recall the following Lemma which will be used frequently in the sequel.

Lemma 1.7

The function F defined by

$$F(x) = \int_0^x \exp(2\int_0^y f(t)\mathrm{d}t)\mathrm{d}y, \tag{1.20}$$

satisfies,

$$F''(x) - 2f(x)F'(x) = 0, \text{ for a.e. } x \in \mathbb{R}, \tag{1.21}$$

and has the following properties:

(i) *F and F^{-1} are quasi-isometry, that is for any x, $y \in \mathbb{R}$ and $|f|_1 = \int_{\mathbb{R}} |f(x)|\,\mathrm{d}x$*

$$\begin{aligned} e^{-2|f|_1}|x-y| \le |F(x) - F(y)| \le e^{2|f|_1}|x-y| \\ e^{-2|f|_1}|x-y| \le |F^{-1}(x) - F^{-1}(y)| \le e^{2|f|_1}|x-y| \end{aligned} \tag{1.22}$$

(ii) *F is a one-to-one function. Both F and its inverse function F^{-1} belongs to $\mathcal{W}_1^2(\mathbb{R})$.*

Proof: (i) By definition the function F and its inverse F^{-1} are continuous, one to one, strictly increasing function, moreover $F''(x) - 2f(x)F'(x) = 0$ for a. e. $x \in \mathbb{R}$. In addition $F'(x) = \exp(2\int_0^x f(t)\mathrm{d}t)$, hence, for all $x \in \mathbb{R}$

$$m =: e^{-2|f|_1} \le F'(x) \le e^{2|f|_1} := M \text{ and } m =: e^{-2|f|_1} \le (F^{-1})'(x) \le e^{2|f|_1} := M. \tag{1.23}$$

(ii) Using the inequality (1.23), one can show that both F and F^{-1} are $\mathcal{C}^1(\mathbb{R})$. Since the second generalized derivative F'' satisfies (1.21) for almost all x, we get that F'' belongs to $L^1(\mathbb{R})$. therefore, F belongs to the space $\mathcal{W}_1^2(\mathbb{R})$. using again assertion (i), one can check that F^{-1} also belongs to $\mathcal{W}_1^2(\mathbb{R})$. ■

The function F defined for every $x \in \mathbb{R}$ by

$$F(x) = \int_0^x \exp\left(2\int_0^y f(t)\mathrm{d}t\right)\mathrm{d}y \tag{1.24}$$

satisfies

$$F''(x) - 2f(x)F'(x) = 0, \text{ for a.e. } x \in \mathbb{R}.$$

It was shown in [4], that both F and its inverse are global Lipschitz, one to one and $\mathcal{C}^2$ functions from $\mathbb{R}$ onto $\mathbb{R}$.

By applying Itô-Krylov's formula to $F(Y_t)$ shows that

$$\begin{aligned} \mathrm{d}F(Y_t) &= F'(Y_t)\,\mathrm{d}Y_t + \frac{1}{2}F''(Y_t)\,\mathrm{d}\langle Y_\cdot\rangle_t \\ &= -F'(Y_t)\left(h(t,Y_t)+h_1(t)Z_t\right)\mathrm{d}t + Z_tF'(Y_t)\,\mathrm{d}W_t \\ &+ \left(-F'(Y_t)\,f(Y_t) + \frac{1}{2}F''(Y_t)\right)|Z_t|^2\,\mathrm{d}t, \end{aligned}$$

since

$$-F'(x)\,f(x) + \frac{1}{2}F''(x) = 0, \tag{1.25}$$

we obtain

$$F(Y_t) = F(\xi) - \int_t^T Z_sF'(Y_s)\,\mathrm{d}W_s \tag{1.26}$$

$$+ \int_t^T \left(h(s,Y_s)F'(Y_s) + h_1(s)F'(Y_s)\,Z_s\right)\mathrm{d}s. \tag{1.27}$$

If we set

$$F(Y_t) = \bar{Y}_t \text{ then } Y_t = F^{-1}\left(\bar{Y}_t\right),\ \ Z_tF'(Y_t) = \bar{Z}_t \text{ then } Z_t = \frac{\bar{Z}_t}{F'(Y_t)}, \text{ and } F(\xi) = \bar{\xi}. \tag{1.28}$$

BSDE (1.27) becomes:

$$\bar{Y}_t = \bar{\xi} + \int_t^T \left(h(s,F^{-1}\left(\bar{Y}_s\right))F'\left(F^{-1}\left(\bar{Y}_s\right)\right) + h_1(s)\bar{Z}_s\right)\mathrm{d}s - \int_t^T \bar{Z}_s\mathrm{d}W_s. \tag{1.29}$$

To simplify statement (1.29), we use the following notation

$$\bar{h}(s,y,z) = \left(h(s,F^{-1}(y))F'\left(F^{-1}(y)\right) + h_1(s)z\right). \tag{1.30}$$

Then, we have

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{h}(s,\bar{Y}_s,\bar{Z}_s)\mathrm{d}s - \int_t^T \bar{Z}_s\mathrm{d}W_s. \tag{1.31}$$

Proposition 1.8 (*A Priori Estimates*)

Let $\xi \in \mathbb{L}^q(\Omega)$. If $\left(\bar{Y},\bar{Z}\right) \in \mathcal{S}^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$ is a solution of BSDE (1.31) and (Y,Z) satisfies Q-BSDE (0.1), then we have:

(i) $(Z_r)_{0\le r\le T} \in \mathcal{H}^q_{\mathcal{F}}([0,\ T])$,

(ii) $(Y_r)_{0\leq r\leq T} \in \mathcal{S}^q_{\mathcal{F}}([0,\ T])$,

(iii) $\mathbb{E}|\int_0^T (h(r,Y_r) + h_1(r)Z_r + f(Y_r)|Z_r|^2)\,\mathrm{d}r|^q$ *is finite.*

Proof of (i) and (ii): Suppose that $\left(\bar{Y}_t, \bar{Z}_t\right) \in \mathcal{S}^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$ be a solution to BSDE (1.31), then, we have

$$\mathbb{E} \sup_{0\leq t\leq T} \left|\bar{Y}_t\right|^q + \mathbb{E}\left(\int_0^T \left|\bar{Z}_t\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} < \infty.$$

Since $m\,|x-y| \leq |F(x) - F(y)|$ and $m \leq F'(x)$, we have

$$\begin{aligned} &m^q \mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^q + \left(\int_0^T |Z_t|^2\,\mathrm{d}t\right)^{\frac{q}{2}}\right] \\ &\leq \mathbb{E}\sup_{0\leq t\leq T} |F(Y_t) - F(0)|^q + \mathbb{E}\left(\int_0^T \left|Z_t F'(Y_t)\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} \\ &\leq \mathbb{E}\sup_{0\leq t\leq T} \left|\bar{Y}_t\right|^q + \mathbb{E}\left(\int_0^T \left|\bar{Z}_t\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} \\ &< \infty. \end{aligned}$$

■

Proof of (iii).: Since $(Y_., Z_.)$ satisfies Q-BSDE (0.1),

$$\int_0^T \left(h(r,Y_r) + h_1(r)Z_r + f(Y_r)\,|Z_r|^2\right)\mathrm{d}r = \int_0^T Z_r \mathrm{d}W_r + Y_0 - \xi.$$

Now, taking the expectation and using Burkholder–Davis–Gundy inequality, we obtain

$$\begin{aligned} &\mathbb{E}\left|\int_0^T \left(h(r,Y_r) + h_1(r)Z_r + f(Y_r)\,|Z_r|^2\right)\mathrm{d}r\right|^q \\ &\leq C\left(\mathbb{E}\left|\int_0^T Z_r \mathrm{d}W_r\right|^q + \mathbb{E}\left(|Y_0|^q + |\xi|^q\right)\right) \\ &\leq C\left(\mathbb{E}\,|\xi|^q + |Y_0|^q + C_p \mathbb{E}\left(\int_0^T |Z_r|^2\,\mathrm{d}r\right)^{\frac{q}{2}}\right). \end{aligned}$$

Finally,

$$\mathbb{E}\left|\int_0^T (h(r,Y_r) + h_1(r)Z_r + f(Y_r)\,|Z_r|^2)\mathrm{d}r\right|^q$$

is finite thanks to (i) and (ii). ■

Theorem 1.9 ($\mathbb{L}^q$ **($q \ge 2$)-*Solutions of Q-BSDE***)

For any $q \ge 2$, assume that (2.i), (2.ii) and (2.iii) in Assumption 2 are in force. Then, Q-BSDE (0.1) has a unique solution that belongs to $\mathcal{S}^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$.

Proof: If (Y, Z) is a solution to Q-BSDE (0.1), then Itô-Krylov's formula applied to $F(Y_t)$ leads to L-BSDE (1.31) which is described by

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{h}(s, \bar{Y}_s, \bar{Z}_s)\mathrm{d}s - \int_t^T \bar{Z}_s \mathrm{d}W_s.$$

Since $\bar{\xi}$, $\bar{h}$ satisfy the conditions of Lemma 1.3, there exists a unique solution pair $\left(\bar{Y}, \bar{Z}\right) \in \mathcal{S}^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$ to L-BSDE (1.31).

Conversely: By applying Itô-Krylov's formula to $F^{-1}(\bar{Y}_t)$, we obtain

$$\mathrm{d}F^{-1}(\bar{Y}_t) = (F^{-1})'(\bar{Y}_t)\mathrm{d}\bar{Y}_t + \frac{1}{2}(F^{-1})''(\bar{Y}_t)\mathrm{d}\left\langle \bar{Y}_\cdot \right\rangle_t,$$

we know that

$$(F^{-1})'(\bar{Y}_t) = \frac{1}{F'(F^{-1}(\bar{Y}_t))} \quad \text{and} \quad (F^{-1})''(\bar{Y}_t) = \frac{-F''(F^{-1}(\bar{Y}_t))}{(F'(F^{-1}(\bar{Y}_t)))^3}. \tag{1.32}$$

Using notations (1.28), we have

$$Y_t = \xi + \int_t^T \left(h(r, Y_r) + h_1(r) Z_r + f(Y_r)\,|Z_r|^2\right)\mathrm{d}r - \int_t^T Z_r \mathrm{d}W_r,\ \ 0 \le t \le T.$$

Which means that (Y, Z) is the unique solution to Q-BSDE (0.1). Thanks to the priori estimates in Proposition 1.8 one can confirm that $(Y, Z) \in \mathcal{S}^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$. ■

1.4.2 The Malliavin Regularity of non-Markovian Quadratic BSDE

In this subsection, we can obtain some estimates for solutions to Q-BSDE (0.1). To begin with, let $\left(\bar{Y}, \bar{Z}\right)$ be the unique solution of L-BSDE (1.31) associated to Q-BSDE (0.1).

Below, we specify some assumptions on the coefficients.

Assumption 3 Fix $q > 4$ and $2 \le p < \frac{q}{2}$

(3.i) ξ satisfies (1.i) in Assumption 1,

(3.ii) The first- and second-order partial derivatives of h are continuous and uniformly bounded with respect to y and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuously differentiable function such that f and f' are bounded functions.

(3.iii) $h(\cdot, Y_t)$ and $\partial_y h(\cdot, Y_t)$ belong to $\mathbb{L}_a^{1,q}$ and we have

$$\sup_{0 \leq \theta \leq T} \mathbb{E} \left(\int_\theta^T |D_\theta h(t, Y_t|^2 \, \mathrm{d}t \right)^{\frac{q}{2}} < +\infty,$$

$$\sup_{0 \leq \theta \leq T} \mathbb{E} \left(\int_\theta^T |D_\theta \partial_y h(t, Y_t)|^2 \, \mathrm{d}t \right)^{\frac{q}{2}} < +\infty,$$

and there exists $L > 0$ such that for any $t \in (0, T]$ and for any $0 \leq \theta, \theta' \leq t \leq T$

$$\mathbb{E} \left(\int_t^T |D_\theta h(r, Y_r) - D_{\theta'} h(r, Y_r)|^2 \, \mathrm{d}t \right)^{\frac{p}{2}} \leq L \left| \theta - \theta' \right|^{\frac{p}{2}}.$$

(3.iv) For each $\theta \in [0, \; T]$, $D_\theta h(\cdot, Y_t) \in \mathbb{L}_a^{1,q}$ and it has continuous partial derivative with respect to y, which is denoted by $\partial_y D_\theta h(\cdot, Y_t)$ and the Malliavin derivatives $D_u D_\theta h(\cdot, Y_t)$ satisfy

$$\sup_{0 \leq \theta \leq T} \sup_{0 \leq u \leq T} \mathbb{E} \left(\int_{\theta \vee u}^T |D_u D_\theta h(t, Y_t)|^2 \, \mathrm{d}t \right)^{\frac{q}{2}} < +\infty.$$

Remark 1.10

Notice that Assumption 3 and (2.i), (2.iii) in Assumption 2 imply that the terminal value $\bar{\xi}$ and the generator $\bar{h}$ satisfy Assumption 1.

Remark 1.11

From Theorem 1.5, we know that $\left\{ \left(D_\theta \bar{Y}_t, D_\theta \bar{Z}_t \right) \right\}_{0 \leq \theta \leq t \leq T}$ satisfies the following linear BSDE

$$\begin{aligned} D_\theta \bar{Y}_t = D_\theta \bar{\xi} - \int_t^T D_\theta \bar{Z}_r \mathrm{d}W_r \\ + \int_t^T \left[\partial_{\bar{y}} \bar{h} \left(r, \bar{Y}_r, \bar{Z}_r \right) D_\theta \bar{Y}_r + \partial_{\bar{z}} \bar{h} \left(r, \bar{Y}_r, \bar{Z}_r \right) D_\theta \bar{Z}_r + D_\theta \bar{h} \left(r, \bar{Y}_r, \bar{Z}_r \right) \right] \mathrm{d}r \end{aligned} \tag{1.33}$$

and $\bar{Z}_t = D_t \bar{Y}_t$, $\mu \otimes$P-a.e. Moreover, since $\bar{\xi}$ and $\bar{h}$ satisfy (1.7) and (1.9) in Assumption 1, then, from estimate (1.4) in Lemma 1.3, we deduce that

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \bar{Z}_t \right|^q < \infty \tag{1.34}$$

for any $q \geq 2$.

Lemma 1.12

For any $2 \leq p < \frac{q}{2}$, *let Assumption 3, (2.i) and (2.iii) in Assumption 2 be satisfied then, there exists a constant* K *such that, for any* $s,\ t \in [0,\ T]$, *we have:*

(i) $\mathbb{E}\left|Y_t - Y_s\right|^p \leq K\left|t-s\right|^{\frac{p}{2}}$,

(ii) $\mathbb{E}\left|Z_t - Z_s\right|^p \leq K\left|t-s\right|^{\frac{p}{2}}$,

(iii) *For any partition* $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ *of the interval* $[0,\ T]$, *we have*

$$\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\left[\left|Z_t - Z_{t_i}\right|^2 + \left|Z_t - Z_{t_{i+1}}\right|^2\right]\mathrm{d}t \leq K\left|\pi\right|,$$

where $|\pi| = \max_{0\leq i\leq n-1}(t_{i+1}-t_i)$ *and* K *is a constant independent of the partition* π.

Proof: Due to the fact that $\bar{\xi}$ and $\bar{h}$ satisfy Assumption 1, then, thanks to Theorem 1.5 and Corollary 1.1 and with Hölder's inequality, there exists a constant K, that may change from line to line, such that for any $s,\ t \in [0,\ T]$

$$\mathbb{E}\left|\bar{Y}_t - \bar{Y}_s\right|^p \leq K\left|t-s\right|^{\frac{p}{2}}, \tag{1.35}$$

and

$$\mathbb{E}\left|\bar{Z}_t - \bar{Z}_s\right|^p \leq K\left|t-s\right|^{\frac{p}{2}}. \tag{1.36}$$

Now, we proceed to prove the estimates (i) and (ii).

Since F^{-1} is Lipschitz and by using the previous result (1.35), the following estimate holds for all $s,\ t \in [0,\ T]$,

$$\mathbb{E}\left|Y_t - Y_s\right|^p = \mathbb{E}\left|F^{-1}\left(\bar{Y}_t\right) - F^{-1}\left(\bar{Y}_s\right)\right|^p \leq K\left|t-s\right|^{\frac{p}{2}}. \tag{1.37}$$

where K is a positive constant.

Using the fact that both F' and F^{-1} are Lipschitz functions, $m \leq F' \leq M$, we obtain:

$$\begin{aligned}|Z_t - Z_s| &= \left| \frac{\bar{Z}_t}{F'(Y_t)} - \frac{\bar{Z}_s}{F'(Y_t)} + \frac{\bar{Z}_s}{F'(Y_t)} - \frac{\bar{Z}_s}{F'(Y_s)} \right| \\ &\le \frac{1}{m} \left| \bar{Z}_t - \bar{Z}_s \right| + \frac{1}{m^2} \left(\left| \bar{Z}_s \right| |F'(Y_s) - F'(Y_t)| \right) \\ &\le \frac{1}{m} \left| \bar{Z}_t - \bar{Z}_s \right| + \frac{1}{m^2} \left(\left| \bar{Z}_s \right| \left| F'(F^{-1}\left(\bar{Y}_t\right)) - F'(F^{-1}\left(\bar{Y}_s\right)) \right| \right) \\ &\le \frac{1}{m} \left| \bar{Z}_t - \bar{Z}_s \right| + \frac{L}{m^2} \left(\left| \bar{Z}_s \right| \left| \bar{Y}_t - \bar{Y}_s \right| \right),\end{aligned}$$

where L is the Lipschitz constant of $F'(F^{-1}(\cdot))$.

For $\alpha = \frac{q}{p}$, $\beta = \frac{q}{q-p}$, such that, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, using Hölder inequality and taking account of the relations (1.34)–(1.36), we obtain (ii).

Finally, (iii) is a simple consequence of (ii) with $p = 2$. ■

Smoothness of Solutions of Q-BSDEs

In this subsubsection, we will present some important results concerning Q-BSDEs, the natural tool we will use is the Malliavin calculus, and previous results for the regularity of the associated L-BSDE (1.31).

Theorem 1.13

Let $(Y_., Z_.) \in \mathcal{S}^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$ be the unique solution to Q-BSDE (0.1) for any $q \ge 4$, then

(i) $\mathbb{E} \sup_{0 \le t \le T} |Z_t|^q < +\infty$,

(ii) *Y belongs to $\mathbb{L}^{1,q}_a$ and Z belongs to $\mathbb{L}^{1,\frac{q}{2}}_a$,*

(iii) $Z_t = D_t Y_t$, $\mu \otimes$ P-a.e.

(iv) $\sup_{0 \le \theta \le T} \{\mathbb{E} \sup_{0 \le t \le T} |D_\theta Y_t|^q + \mathbb{E}(\int_\theta^T |D_\theta Z_t|^2 \, dt)^{\frac{q}{4}}\} < +\infty$.

Proof of (i) : Keeping in mind that F' is bounded and (1.34), then (i) follows immediately from the fact that $Z_t = \frac{\bar{Z}_t}{F'(F^{-1}(\bar{Y}_t))}$. ■

Proof of (ii) : We will prove that Y, Z belongs to $\mathbb{L}_a^{1,q}$. To do that we should prove the following two assertions.

(a) Due to the Lipschitz continuity of F^{-1} and $\bar{Y} \in \mathbb{D}^{1,q}$, it is obvious that $Y \in \mathbb{D}^{1,q}$. Since $\bar{Z} \in \mathbb{D}^{1,q}$ and $F' \in \mathcal{C}^1(\mathbb{R})$ with bounded derivative and $Y \in \mathbb{D}^{1,q}$, then $Z \in \mathbb{D}^{1,q}$

(b) First, since (Y, Z) is a solution of Q-BSDE (0.1) then we have the following estimates

$$\mathbb{E}\left(\int_0^T |Y_t|^2 \, \mathrm{d}t\right)^{\frac{q}{2}} + \mathbb{E}\left(\int_0^T |Z_t|^2 \, \mathrm{d}t\right)^{\frac{q}{2}} < \infty.$$

We want to prove,

$$\mathbb{E}\left(\int_0^T \int_0^T |D_\theta Y_t|^2 \, \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{2}} + \mathbb{E}\left(\int_0^T \int_0^T |D_\theta Z_t|^2 \, \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{4}} < \infty.$$

Since $D_\theta Y_t = \dfrac{D_\theta \bar{Y}_t}{F'(Y_t)}$, $\bar{Y} \in \mathbb{L}_a^{1,q}$ and the relation $m \leq F'(x) \leq M$, we obtain

$$\mathbb{E}\left(\int_0^T \int_0^T |D_\theta Y_t|^2 \, \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{2}} \leq \frac{1}{m^q} \mathbb{E}\left(\int_0^T \int_0^T \left|D_\theta \bar{Y}_t\right|^2 \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{2}} < \infty.$$

A simple computation shows that

$$D_\theta Z_t = \frac{D_\theta \bar{Z}_t}{F'(Y_t)} - \frac{F''(Y_t)}{F'(Y_t)} Z_t D_\theta Y_t.$$

Thus by the Cauchy–Schwartz inequality and the fact that $\mathbb{E} \sup_{0 \leq t \leq T} |Z_t|^q$ is finite for any $q \geq 4$, we have

$$\mathbb{E}\left(\int_0^T \int_0^T |Z_t D_\theta Y_t|^2 \, \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{4}} \leq \left(\mathbb{E} \sup_{0 \leq t \leq T} |Z_t|^q\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\int_0^T \int_0^T |D_\theta Y_t|^2 \, \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{2}}\right)^{\frac{1}{2}}.$$

and hence,

$$\begin{aligned}\mathbb{E}\left(\int_0^T \int_0^T |D_\theta Z_t|^2 \, \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{4}} &\leq \frac{1}{m^{\frac{q}{2}}} \mathbb{E}\left(\int_0^T \int_0^T \left|D_\theta \bar{Z}_t\right|^2 \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{4}} \\ &+ \frac{M^{\frac{q}{2}}}{m^{\frac{q}{2}}} \left(\mathbb{E} \sup_{0 \leq t \leq T} |Z_t|^q\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\int_0^T \int_0^T |D_\theta Y_t|^2 \, \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{2}}\right)^{\frac{1}{2}}.\end{aligned}$$

■

Proof of (iii) : $\left\{D_t \bar{Y}_t\right\}_{0 \leq t \leq T}$ gives a version of $\left\{\bar{Z}_t\right\}_{0 \leq t \leq T}$, namely, $\bar{Z}_t = D_t \bar{Y}_t$; then

$$Z_t F'(Y_t) = \bar{Z}_t = D_t \bar{Y}_t = D_t(F(Y_t)) = F'(Y_t) D_t Y_t,$$

then, $Z_t = D_t Y_t$. ■

Proof of (iv) : We know that $\bar{\xi}$ and $\bar{h}$ satisfy conditions (1.7) and (1.9) and by invoking the estimate in Lemma 1.3, we obtain

$$\sup_{0\leq\theta\leq T}\left\{\mathbb{E}\sup_{0\leq t\leq T}\left|D_\theta\bar{Y}_t\right|^q+\mathbb{E}\left(\int_\theta^T\left|D_\theta\bar{Z}_t\right|^2\mathrm{d}t\right)^{\frac{q}{4}}\right\} < +\infty. \tag{1.38}$$

Using (1.38) and $m \leq F' \leq M$, we have

$$\begin{aligned}\sup_{0\leq\theta\leq T}\mathbb{E}\sup_{0\leq t\leq T}|D_\theta Y_t|^q &\leq \sup_{0\leq\theta\leq T}\mathbb{E}\sup_{0\leq t\leq T}\left|\frac{1}{F'(Y_t)}D_\theta\bar{Y}_t\right|^q\\ &\leq \frac{1}{m^q}\sup_{0\leq\theta\leq T}\mathbb{E}\sup_{0\leq t\leq T}\left|D_\theta\bar{Y}_t\right|^q < +\infty.\end{aligned}$$

The Cauchy–Schwartz inequality, F' and F'' are bounded functions and the fact that $\mathbb{E}\sup_{0\leq t\leq T}|Z_t|^q$ is finite, leads to

$$\begin{aligned}&\sup_{0\leq\theta\leq T}\mathbb{E}\left(\int_\theta^T|D_\theta Z_t|^2\mathrm{d}t\right)^{\frac{q}{4}}\\ &\leq \sup_{0\leq\theta\leq T}\left\{\mathbb{E}\left(\int_\theta^T\left|\frac{D_\theta\bar{Z}_t}{F'(Y_t)}\right|^2\mathrm{d}t\right)^{\frac{q}{4}}+\mathbb{E}\left(\int_\theta^T\left|\frac{F''(Y_t)}{F'(Y_t)}Z_tD_\theta Y_t\right|^2\mathrm{d}t\right)^{\frac{q}{4}}\right\}\\ &\leq \frac{1}{m^{\frac{q}{2}}}\sup_{0\leq\theta\leq T}\mathbb{E}\left(\int_\theta^T\left|D_\theta\bar{Z}_t\right|^2\mathrm{d}t\right)^{\frac{q}{4}}\\ &+\frac{M^{\frac{q}{2}}}{m^{\frac{q}{2}}}\sup_{0\leq\theta\leq T}\left\{\left(\mathbb{E}\sup_{0\leq t\leq T}|Z_t|^q\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_\theta^T|D_\theta Y_t|^2\mathrm{d}t\right)^{\frac{q}{2}}\right)^{\frac{1}{2}}\right\} < +\infty.\end{aligned}$$

■

Remark 1.14

We would like to point out that the estimations for Y and $D_\theta Y$ in Theorem 1.13 hold also true for $q \geq 2$.

In what follows, we shall adapt the examples studied in [34] to our setting of quadratic BSDEs for which Assumption 3 is satisfied.

Example 1.4

Consider the Q-BSDE (0.1) with generator $h(t,y)+h_1(t)z+f(y)|z|^2$.

(i) Assume that $h : [0,\ T]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a deterministic function twice continuously differentiable with uniformly bounded first- and second-order partial derivatives with respect to y and $\int_0^T|h(t,0)|^2dt < +\infty$.

(ii) We define the terminal value ξ as the multiple stochastic integrals of the form

$$\xi = \int_{[0,\ T]^n} g(t_1, \cdots, t_n) \mathrm{d}W_{t_1} \cdots \mathrm{d}W_{t_n},$$

where $n \ge 2$ is an integer and $g(t_1, \ldots, t_n)$ is a symmetric function in $\mathbb{L}^2([0,\ T]^n)$, such that

$$D_u \xi = n \int_{[0,\ T]^{n-1}} g(t_1, \cdots, t_{n-1}, u) \mathrm{d}W_{t_1} \cdots \mathrm{d}W_{t_{n-1}}$$
$$D_v D_u \xi = n(n-1) \int_{[0,\ T]^{n-2}} g(t_1, \cdots, t_{n-2}, u, v) \mathrm{d}W_{t_1} \cdots \mathrm{d}W_{t_{n-2}}.$$

Then

$$\sup_{0 \le u \le T} \mathbb{E}\left[|D_u \xi|^2\right] \le n \sup_{0 \le u \le T} \int_{[0,\ T]^{n-1}} (g(t_1, \cdots, t_{n-1}, u))^2 \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} \ < \ +\infty,$$

and

$$\sup_{0 \le u \le T} \sup_{0 \le v \le T} \mathbb{E}\left[|D_u D_v \xi|^2\right] \le n(n-1) \sup_{0 \le u \le T} \sup_{0 \le v \le T} \int_{[0,\ T]^{n-2}} (g(t_1, \cdots, t_{n-2}, u, v))^2 \mathrm{d}t_1 \cdots \mathrm{d}t_{n-2} \ < \ +\infty,$$

Moreover, there is a constant $L > 0$ such that for any $u,\ v \in [0,\ T]$

$$\int_{[0,\ T]^{n-1}} |g(t_1, \cdots, t_{n-1}, u) - g(t_1, \cdots, t_{n-1}, v)|^2 \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} \le L\,|u - v|\,.$$

Assumptions (i) and (ii) imply Assumption 3, and thus Z satisfies property (ii) of Lemma 1.12.

Example 1.5

We consider the Q-BSDE (0.1) with generator $h(r, y) + h_1(r)z + f(y)\,|z|^2$.

Let $\Omega = C_0([0,\ T])$ be the classical Wiener space equipped with the Borel σ-field and Wiener measure. Then, Ω is a Banach space with a uniform norm $\|\cdot\|_\infty$ and $W_t = \omega(t)$ is the canonical Wiener process:

(i) Assume that $h : [0,\ T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a twice differentiable deterministic function such that their first- and second-order partial derivatives with respect to y are

uniformly bounded and $\int_0^T h^2(t,0)\,dt < +\infty$.

(ii) *We put* $\xi = \varphi(W)$ *such that* $\varphi : \Omega \to \mathbb{R}$ *is twice Fréchet differentiable, assuming further that the Fréchet derivatives* $\delta\varphi$ *and* $\delta^2\varphi$ *satisfy for all* $\omega \in \Omega$ *and some positive constants* C_1 *and* C_2

$$|\varphi(\omega)| + \|\delta\varphi(\omega)\| + \left\|\delta^2\varphi(\omega)\right\| \leq C_1 \exp\left(C_2 \|\omega\|_\infty^r\right), \quad 0 < r < 2$$

where $\|\cdot\|$ *stands for the operator norm.*

(iii) *We associate with* $\delta\varphi$ *and* $\delta^2\varphi$ *the signed measure* λ *on* $[0,\ 1]$ *and* υ *on* $[0,\ 1] \times [0,\ 1]$, *respectively; there exists a constant* $L > 0$ *such that for all* $0 \leq \theta \leq \theta' \leq 1$, *for some* $p \geq 2$

$$\mathbb{E}\left|\lambda\left((\theta, \theta']\right)\right|^p \leq L\left|\theta - \theta'\right|^{\frac{p}{2}},$$

we know that $D_\theta\xi = \lambda((\theta,1])$ *and* $D_u D_\theta \xi = \upsilon((\theta,1] \times (u,1])$. *From (i), (ii), (iii) and Fernique's Theorem, we can check that Assumption 3 is satisfied and therefore the Hölder continuity property of* Z *(ii) of Lemma 1.12 is established.*

Example 1.6

Consider the following quadratic forward–backward SDE

$$\begin{cases} X_t &= X_0 + \int_0^t b(r, X_r)\mathrm{d}r + \int_0^t \sigma(r, X_r)\mathrm{d}W_r, \\ Y_t &= \varphi\left(\int_0^T X_r^2 \mathrm{d}r\right) - \int_t^T Z_r \mathrm{d}W_r \\ & \quad + \int_t^T \left(h(r, X_r, Y_r) + h_1(r)Z_r + f(Y_r)\left|Z_r\right|^2\right)\mathrm{d}r, \end{cases} \tag{1.39}$$

where b, σ, φ, h, *and* f *are deterministic functions and* $X_0 \in \mathbb{R}$.

We make the following assumptions:

(i) b *and* σ *are twice differentiable and their first- and second-order partial derivatives with respect to* x *are uniformly bounded; in addition, there is a constant* $L > 0$, *such that, for any* $s,\ t \in [0,\ T]$, $x \in \mathbb{R}$

$$|\sigma(t,x) - \sigma(s,x)| \leq L\left|t - s\right|^{\frac{1}{2}}$$

(ii) $\sup_{0\le t\le T}\{|b(t,0)| + |\sigma(t,0)|\} < +\infty$.

(iii) φ *is twice differentiable and there exists a positive constant* C *and integer* n *such that*

$$\left|\varphi\left(\int_0^T x_t^2 \mathrm{d}t\right)\right| + \left|\varphi'\left(\int_0^T x_t^2 \mathrm{d}t\right)\right| + \left|\varphi''\left(\int_0^T x_t^2 \mathrm{d}t\right)\right| \le C\left(1 + \|x\|_\infty\right)^n,$$

where $\|x\|_\infty = \sup_{0\le t\le T}|x_t|$, *for any* $x. \in \mathcal{C}([0,T])$.

(iv) The first- and second-order partial derivatives of $h(t,\cdot,\cdot)$ *with respect to* x *and* y *are continuous and uniformly bounded and* $\int_0^T (h(t,0,0))^2 \mathrm{d}t < +\infty$.

Under assumptions (i) and (iv), equation (1.39) has a unique solution triple (X,Y,Z). *Moreover, the following results hold true; for any real number* $r>0$, *there exists a constant* $C>0$ *such that, for any* $t,s\in[0,\ T]$

$$\mathbb{E}\sup_{0\le t\le T}|X_t|^r < +\infty, \quad \mathbb{E}\left|X_t - X_s\right|^r \le C\left|t-s\right|^{\frac{r}{2}}. \tag{1.40}$$

For any fixed $y\in\mathbb{R}$, *we have* $D_\theta h(t,X_t,y) = \partial_x h(t,X_t,y)D_\theta X_t$. *Then, keeping in mind all the assumptions in this example, by Theorem 2.2.1, Lemma 2.2.2 in [46], and the estimates in (1.40), one can check that the assertions of Assumption 3 are satisfied. Therefore,* Z *enjoys the Hölder continuity property (ii) of Lemma 1.12.*

Chapter 2

The Numerical Schemes for Q-BSDEs and Their Rate of Convergence

2.1 Introduction

Hu *et al.* in [34] construct and develop, by utilizing the Malliavin calculus, multiple numerical approximation schemes for solving backward stochastic differential equations with a global Lipschitz coefficient in the non-Markovian framework. Additionally, they determine their convergence rate using the $\mathbb{L}^p$-Hölder continuity results. The primary objective of this chapter is to generalize the above mentioned results to quadratic backward stochastic differential equations.

2.2 Numerical Schemes for L-BSDEs

In this subsection, we recall the most important results established in [34] concerning the numerical schemes and their rate of convergence in the case where the generator is global Lipschitz. For this purpose, we consider the following L-BSDE

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{h}(r, \bar{Y}_r, \bar{Z}_r)\mathrm{d}r - \int_t^T \bar{Z}_r \mathrm{d}W_r, \quad 0 \leq t \leq T, \tag{2.1}$$

where $\bar{h}$:$[0,\ T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given generator and $\bar{\xi}$ the terminal condition. Note that, if for any $q \geq 2$, $\bar{\xi} \in \mathbb{L}^q(\Omega)$ and $\bar{h}$ is a uniformly Lipshitz function, then Lemma 1.3 shows the existence of a unique solution $\left(\bar{Y}, \bar{Z}\right) \in \mathcal{S}^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$ to BSDE (2.1).

In this subsection and in the rest of the thesis, we let $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ stand for an arbitrary partition of the interval $[0,\ T]$ and $|\pi| = \max_{0\leq i\leq n-1} |t_{i+1} - t_i|$ and we denote $\Delta_i = t_{i+1} - t_i$, $0 \leq i \leq n-1$..

◁ **Explicit scheme:** An explicit scheme for L-BSDE (2.1) has been presented in [34], where the approximate pairs $\left(\bar{Y}^{\pi}, \bar{Z}^{\pi}\right)$ are defined as follows

$$\begin{cases} \bar{Y}^{\pi}_{t_n} = \bar{\xi}^{\pi}, \quad \bar{Z}^{\pi}_{t_n} = 0, \\ \bar{Y}^{\pi}_{t} = \bar{Y}^{\pi}_{t_{i+1}} + \bar{h}\left(t_{i+1}, \bar{Y}^{\pi}_{t_{i+1}}, \mathbb{E}\left[\frac{1}{\Delta_{i+1}}\int_{t_{i+1}}^{t_{i+2}} \bar{Z}^{\pi}_{r}\mathrm{d}r \mid \mathcal{F}_{t_{i+1}}\right]\right)\Delta_i \\ \qquad - \int_{t}^{t_{i+1}} \bar{Z}^{\pi}_{r}\mathrm{d}W_r, \qquad t\in[t_i, t_{i+1}), \end{cases} \tag{2.2}$$

$i = n-1,\ n-2,\ \ldots,\ 0$, where we have by convention,

$$\mathbb{E}\left[\frac{1}{\Delta_{i+1}}\int_{t_{i+1}}^{t_{i+2}} \bar{Z}^{\pi}_{r}\mathrm{d}r \mid \mathcal{F}_{t_{i+1}}\right] = 0 \quad \text{for } i = n-1.$$

Proposition 2.1

Consider the explicit scheme (2.2). Assume that Assumption 1 holds true and the partition π satisfies

$$\max_{0\le i\le n-1}\frac{\Delta_i}{\Delta_{i+1}} \le L_1,$$

where L_1 is a constant. Assume that a constant $L_2 > 0$ exists such that

$$\left|\bar{h}(t_2, \bar{y}, \bar{z}) - \bar{h}(t_1, \bar{y}, \bar{z})\right| \le L_2\left|t_2 - t_1\right|^{\frac{1}{2}} \tag{2.3}$$

for all $t_1,\ t_2 \in [0,\ T]$ and $\bar{y},\ \bar{z}\in\mathbb{R}$. Then, there exist two positive constants δ and K which are independent from π, such that, if $|\pi| < \delta$, then

$$\mathbb{E}\sup_{0\le t\le T}\left|\bar{Y}_t - \bar{Y}^{\pi}_t\right|^2 + \int_0^T \mathbb{E}\left|\bar{Z}_t - \bar{Z}^{\pi}_t\right|^2 \mathrm{d}t \le K\left(|\pi| + \mathbb{E}\left|\bar{\xi} - \bar{\xi}^{\pi}\right|^2\right).$$

Proof: See Theorem 3.1 in [34]. ■

◁ **Implicit scheme:** We also recall the numerical scheme in the implicit case for L-BSDE (2.1), the approximating pair $\left(\bar{Y}^{\pi}, \bar{Z}^{\pi}\right)$ is defined recursively by

$$\begin{cases} \bar{Y}^{\pi}_{t_n} = \bar{\xi}^{\pi} \\ \bar{Y}^{\pi}_{t} = \bar{Y}^{\pi}_{t_{i+1}} + \bar{h}\left(t_{i+1}, \bar{Y}^{\pi}_{t_{i+1}}, \frac{1}{\Delta_i}\int_{t_i}^{t_{i+1}} \bar{Z}^{\pi}_{r}\mathrm{d}r\right)\Delta_i - \int_{t}^{t_{i+1}} \bar{Z}^{\pi}_{r}\mathrm{d}W_r, \end{cases} \tag{2.4}$$

for $t \in [t_i, t_{i+1})$, $i = n-1, n-2, \ldots, 0$, where $\bar{\xi}^\pi$ is an approximation of the terminal value $\bar{\xi}$. In this recursive formula (2.4), on each subinterval $[t_i, t_{i+1})$, $i = n-1, \ldots, 0$, the nonlinear "generator" $\bar{h}$ contains the information of $\bar{Z}^\pi$ on the same interval. In this sense, this formula is different from formula (2.2), and (2.4) is an equation for $\left\{\left(\bar{Y}_t^\pi, \bar{Z}_t^\pi\right)\right\}_{t_i \leq t \leq t_{i+1}}$. When $|\pi|$ is sufficiently small, the existence and uniqueness of the solution to the above equation can be established. In fact, equation (2.4) is of the following form:

$$\bar{Y}_t = \bar{\xi} + g(\int_a^b \bar{Z}_r \mathrm{d}r) - \int_t^b \bar{Z}_r \mathrm{d}W_r, t \in [a, b] \text{ and } 0 \leq a < b \leq T. \tag{2.5}$$

For the BSDE (2.5), we have the following Theorem.

Theorem 2.2

Let $0 \leq a < b \leq T$ and $p \geq 2$. Let $\bar{\xi}$ be $\mathcal{F}_b$-measurable and $\bar{\xi} \in \mathbb{L}^p(\Omega)$. If there exists a constant $L > 0$ such that $g : (\Omega \times \mathbb{R}, \mathcal{F}_b \otimes \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ satisfies

$$|g(z_1) - g(z_2)| \leq L\, |z_1 - z_2|$$

for all z_1, $z_2 \in \mathbb{R}$ and $g(0) \in \mathbb{L}^p(\Omega)$, then there is a constant $\delta(p, L) > 0$, such that, when $b-a < \delta(p, L)$, equation (2.5) has a unique solution $(\bar{Y}, \bar{Z}) \in \mathcal{S}_{\mathcal{F}}^p([a, b]) \times \mathcal{H}_{\mathcal{F}}^p([a, b])$.

Proof: See [34]. ■

Proposition 2.3

Assume that $\bar{h}$ satisfies condition (2.3) in Proposition 2.1. If Assumption 1 holds true and $\bar{\xi}^\pi \in \mathbb{L}^p(\Omega)$, then there exist two positive constants δ and K which are independent from π, such that, whenever $|\pi| < \delta$, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \left|\bar{Y}_t - \bar{Y}_t^\pi\right|^p + \mathbb{E}\left(\int_0^T \left|\bar{Z}_t - \bar{Z}_t^\pi\right|^2 \mathrm{d}t\right)^{\frac{p}{2}} \leq K\left(|\pi|^{\frac{p}{2}} + \mathbb{E}\left|\xi - \bar{\xi}^\pi\right|^p\right).$$

Proof: See [34]. ■

◁ **Totally discrete scheme**: In addition to the two aforementioned types of schemes discussed in [34], the authors propose a totally discrete scheme in the case where the generator $\bar{h}$ takes the following linear form;

$$\bar{h}(t,\bar{y},\bar{z}) = g(t)\bar{y} + h(t)\bar{z} + f_1(t), \tag{2.6}$$

where the functions g, h are bounded and $f_1 \in \mathbb{L}^2([0,T])$ and the following assumptions are in force:

(H1) $\bar{h}$ is deterministic, which implies $D_\theta \bar{h}(t,\bar{y},\bar{z}) = 0$.

(H2) The functions g, h, and f_1 are $\frac{1}{2}$-Hölder continuous in t.

(H3) $\mathbb{E}\sup_{0\le\theta\le T}|D_\theta\bar{\xi}|^r < +\infty$, for all $r \ge 1$.

From (2.1), $\{D_\theta \bar{Y}_t\}_{0\le\theta\le t\le T}$ can be represented as

$$D_\theta \bar{Y}_t = \mathbb{E}\left[\rho_{t,T} D_\theta\bar{\xi} + \int_t^T \rho_{t,r} D_\theta \bar{h}\left(r,\bar{Y}_r,\bar{Z}_r\right)\mathrm{d}r \mid \mathcal{F}_t\right], \tag{2.7}$$

where

$$\rho_{t,r} = \exp\left\{\int_t^r \beta_s \mathrm{d}W_s + \int_t^r \left(h_1(s) - \frac{1}{2}\beta_s^2\right)\mathrm{d}s\right\} \tag{2.8}$$

with $\alpha_s = \partial_{\bar{y}}\bar{h}(s,\bar{Y}_s,\bar{Z}_s)$ and $\beta_s = \partial_{\bar{z}}\bar{h}(s,\bar{Y}_s,\bar{Z}_s)$.

Using $\bar{Z}_t = D_t\bar{Y}_t$, $\mu\otimes\mathrm{P}$ a.e., from (2.1), (2.7) and (2.8), we define recursively

$$\begin{cases} \bar{Y}_t^\pi &= \bar{\xi}^\pi, \qquad \bar{Z}_{t_n}^\pi = D_T\bar{\xi} \\ \bar{Y}_{t_i}^\pi &= \mathbb{E}\left[\bar{Y}_{t_{i+1}}^\pi + \bar{h}\left(t_{i+1},\bar{Y}_{t_{i+1}}^\pi,\bar{Z}_{t_{i+1}}^\pi\right)\Delta_i \mid \mathcal{F}_{t_i}\right] \\ \bar{Z}_{t_i}^\pi &= \mathbb{E}\left[\rho_{t_{i+1},t_n}^\pi D_{t_i}\bar{\xi} + \sum_{k=i}^{n-1}\rho_{t_{i+1},t_{k+1}}^\pi \bar{h}\left(t_{k+1},\bar{Y}_{t_{k+1}}^\pi,\bar{Z}_{t_{k+1}}^\pi\right)\Delta_k \mid \mathcal{F}_{t_i}\right], \end{cases} \tag{2.9}$$

$i = n-1, n-2, \ldots, 0$, such that $\rho_{t_i,t_i}^\pi = 1$, $i = 0,1,\ldots,n$ and for $0 \le i < j \le n$,

$$\rho_{t_i,t_j}^\pi = \exp\left\{\sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}} \partial_{\bar{z}}\bar{h}\left(r,\bar{Y}_{t_k}^\pi,\bar{Z}_{t_k}^\pi\right)\mathrm{d}W_r \right. \tag{2.10}$$
$$\left. + \sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}}\left(\partial_{\bar{y}}\bar{h}\left(r,\bar{Y}_{t_k}^\pi,\bar{Z}_{t_k}^\pi\right) - \frac{1}{2}\left[\partial_{\bar{z}}\bar{h}\left(r,\bar{Y}_{t_k}^\pi,\bar{Z}_{t_k}^\pi\right)\right]^2\right)\mathrm{d}r\right\}.$$

Proposition 2.4

Let $\bar{\xi}$ satisfy (1.i) in Assumption 1 and $\bar{h}$ be a linear function taking the form (2.6). Then, under assumptions (H1)–(H3) there exist two positive constants δ and K which *are independent from π, such that when $|\pi| < \delta$ we have*

$$\mathbb{E} \max_{0 \leq i \leq n} \left\{ \left| \bar{Y}_{t_i} - \bar{Y}^{\pi}_{t_i} \right|^p + \left| \bar{Z}_{t_i} - \bar{Z}^{\pi}_{t_i} \right|^p \right\} \leq K \left| \pi \right|^{\frac{p}{2} - \frac{p}{\left(2 \ln \frac{1}{|\pi|}\right)}} \left(\ln \frac{1}{|\pi|} \right)^{\frac{p}{2}}.$$

Proof: See [34]. ■

2.3 The Rate of Convergence of Q-BSDEs

In this section, with the help of the aforementioned explicit (2.2), implicit (2.4), and fully discrete (2.9) schemes provided for the numerical study of L-BSDE (2.1), we aim to construct numerical schemes for the underlying Q-BSDE (0.1). We also take a further step to study the convergence rates of these schemes. It is important to note that, due to technical concerns, we ought to assume that the function h_1 in Q-BSDE (0.1) is a constant.

2.3.1 An Explicit Scheme for Q-BSDE

From (1.31), we know that, when $t \in [t_i, t_{i+1}]$, the L-BSDE associated to the Q-BSDE (0.1) is given by

$$\bar{Y}_t = \bar{Y}_{t_{i+1}} + \int_t^{t_{i+1}} \bar{h} \left(r, \bar{Y}_r, \bar{Z}_r \right) \mathrm{d}r - \int_t^{t_{i+1}} \bar{Z}_r \mathrm{d}W_r.$$

In the same way as defining the alternative scheme (2.2), the approximating pairs $\left(\bar{Y}^{\pi}, \bar{Z}^{\pi}\right)$ of the above L-BSDE are defined recursively by

$$\begin{cases} \bar{Y}^{\pi}_{t_n} & = \bar{\xi}^{\pi}, \quad \bar{Z}^{\pi}_{t_n} = 0 \\ \bar{Y}^{\pi}_{t} & = \bar{Y}^{\pi}_{t_{i+1}} - \int_t^{t_{i+1}} \bar{Z}^{\pi}_r \mathrm{d}W_r \\ & \quad + h\left(t_{i+1}, F^{-1}\left(\bar{Y}^{\pi}_{t_{i+1}}\right)\right) F'\left(F^{-1}\left(\bar{Y}^{\pi}_{t_{i+1}}\right)\right) \triangle_i \\ & \quad + h_1\left(t_{i+1}\right) \mathbb{E}\left[\frac{1}{\triangle_{i+1}} \int_{t_{i+1}}^{t_{i+2}} \bar{Z}^{\pi}_r \mathrm{d}r \mid \mathcal{F}_{t_{i+1}}\right] \triangle_i \end{cases} \tag{2.11}$$

for $t \in [t_i, t_{i+1})$, $i = n-1,\ n-2, \dots,\ 0$ and $\bar{\xi}^{\pi} \in \mathbb{L}^2(\Omega)$ is an approximation of the final condition $\bar{\xi}$ and by convention

$$\mathbb{E}\left[\frac{1}{\triangle_{i+1}} \int_{t_{i+1}}^{t_{i+2}} \bar{Z}^{\pi}_r \mathrm{d}r \mid \mathcal{F}_{t_{i+1}}\right] = 0 \text{ when } i = n-1.$$

We define the scheme associated with Q-BSDE (0.1) as follows: $Y^{\pi}_t = F^{-1}\left(\bar{Y}^{\pi}_t\right)$ where $Z^{\pi}_t = \frac{\bar{Z}^{\pi}_t}{F'\left(F^{-1}\left(\bar{Y}^{\pi}_t\right)\right)}$. We should point out that (Y^{π}, Z^{π}) does not satisfy a QBSDE; the reason is that the numerical scheme (2.11) is not a BSDE since the non-linear generator $\bar{h}$ contains the information of $\bar{Z}^{\pi}$ on the time interval $[t_{i+1}, t_{i+2}]$ rather than $[t_i, t_{i+1}]$.

Theorem 2.5

Let Assumption 3, (2.i), (2.iii) and (2.iv) in Assumption 2 be satisfied and the partition π satisfies $\max_{0 \le i \le n-1} \frac{\triangle_i}{\triangle_{i+1}} \le L_1$, where L_1 is a positive constant. Then, there exist two positive constants δ and K which are independent of π, such that, for $|\pi| < \delta$, we have the following estimates

$$\mathbb{E} \sup_{0 \le t \le T} |Y_t - Y^{\pi}_t|^2 \le K\left(|\pi| + \mathbb{E}\,|\xi - \xi^{\pi}|^2\right),$$

and for all $1 \le p < 2$,

$$\mathbb{E} \int_0^T |Z_t - Z^{\pi}_t|^p \, \mathrm{d}t \le K\left(|\pi| + \mathbb{E}\,|\xi - \xi^{\pi}|^2\right)^{\frac{p}{2}}.$$

Proof : We consider the approximation scheme (2.11). We have already seen that the inputs $\bar{\xi}$ and $\bar{h}$ of Equation (1.31) satisfy Assumption 1. Using assertion (iv) of Assumption 2,

h_1 is a constant function and the fact that F' is bounded one shows the existence of a constant $L_3 > 0$, such that, for all $t_1,\ t_2 \in [0,\ T]$ and $y,\ z \in \mathbb{R}$

$$\left|\bar{h}(t_2, y, z) - \bar{h}(t_1, y, z)\right| \leq L_3 \left|t_2 - t_1\right|^{\frac{1}{2}}. \tag{2.12}$$

Then, thanks to Proposition 2.1, there exist positive constants K and δ, independent of the partition π, such that, whenever $|\pi| < \delta$, we have

$$\mathbb{E} \sup_{0\leq t\leq T} \left|\bar{Y}_t - \bar{Y}_t^{\pi}\right|^2 + \mathbb{E}\int_0^T \left|\bar{Z}_t - \bar{Z}_t^{\pi}\right|^2 \mathrm{d}t \leq K\left(|\pi| + \mathbb{E}\left|\xi - \xi^{\pi}\right|^2\right). \tag{2.13}$$

Firstly, by using (2.13) and the fact that F and F^{-1} are Lipschitz functions, we have

$$\begin{aligned}\mathbb{E} \sup_{0\leq t\leq T} \left|Y_t - Y_t^{\pi}\right|^2 &= \mathbb{E} \sup_{0\leq t\leq T} \left|F^{-1}\left(\bar{Y}_t\right) - F^{-1}\left(\bar{Y}_t^{\pi}\right)\right|^2 \\ &\leq K\left[|\pi| + \mathbb{E}\left|\xi - \xi^{\pi}\right|^2\right].\end{aligned}$$

Now, we shall show that for $1 \leq p < 2$

$$\mathbb{E}\int_0^T \left|Z_t - Z_t^{\pi}\right|^p \mathrm{d}t \ \leq\ K\left(|\pi| + \mathbb{E}\left|\xi - \xi^{\pi}\right|^2\right)^{\frac{p}{2}}.$$

Applying the Hölder inequality twice and using (2.13), we have

$$\begin{aligned}\mathbb{E}\int_0^T \left|\bar{Z}_t - \bar{Z}_t^{\pi}\right|^p \mathrm{d}t &\leq K\mathbb{E}\left(\int_0^T \left|\bar{Z}_t - \bar{Z}_t^{\pi}\right|^2 \mathrm{d}t\right)^{\frac{p}{2}} \\ &\leq K\left(|\pi| + \mathbb{E}\left|\xi - \xi^{\pi}\right|^2\right)^{\frac{p}{2}},\end{aligned} \tag{2.14}$$

With the Hölder inequality and $\mathbb{E}\sup_{0\leq t\leq T}\left|\bar{Z}_t\right|^q < +\infty$, for any $q \geq 2$, we have

$$\begin{aligned}\mathbb{E}\int_0^T \left|\bar{Z}_t\right|^p \left|\bar{Y}_t - \bar{Y}_t^{\pi}\right|^p \mathrm{d}t &\leq T\left(\mathbb{E}\sup_{0\leq t\leq T}\left|\bar{Y}_t - \bar{Y}_t^{\pi}\right|^2\right)^{\frac{p}{2}} \left(\mathbb{E}\sup_{0\leq t\leq T}\left|\bar{Z}_t\right|^{\frac{2p}{2-p}}\right)^{\frac{2-p}{2}} \\ &\leq K\left(|\pi| + \mathbb{E}\left|\xi - \xi^{\pi}\right|^2\right)^{\frac{p}{2}},\end{aligned} \tag{2.15}$$

and hence, by (2.14) and (2.15), we obtain the desired result. ■

2.3.2 An Implicit Scheme for Q-BSDE

Firstly, we give the numerical scheme for the associated L-BSDE (1.31)

$$\begin{cases}\bar{Y}_{t_n}^{\pi} &= \bar{\xi}^{\pi} \\ \bar{Y}_t^{\pi} &= \bar{Y}_{t_{i+1}}^{\pi} - \displaystyle\int_t^{t_{i+1}} \bar{Z}_r^{\pi}\mathrm{d}W_r \\ & \quad + \left[F'\left(F^{-1}\left(\bar{Y}_{t_{i+1}}^{\pi}\right)\right) h\left(t_{i+1}, F^{-1}\left(\bar{Y}_{t_{i+1}}^{\pi}\right)\right) + \dfrac{h_1\left(t_{i+1}\right)}{\triangle_i}\displaystyle\int_{t_i}^{t_{i+1}} \bar{Z}_s^{\pi}\mathrm{d}s\right]\triangle_i,\end{cases} \tag{2.16}$$

for $t \in [t_i, t_{i+1})$, $i = n-1,\ n-2, \ldots,\ 0$, where the partition π and Δ_i, $i = n-1, \ldots, 0$ are defined as in the previous section and $\bar{\xi}^\pi$ is an approximation (if necessary) of the terminal condition $\bar{\xi}$. Then, we define $Y_t^\pi = F^{-1}(\bar{Y}_t^\pi)$ and $Z_t^\pi = \frac{\bar{Z}_t^\pi}{F'(F^{-1}(\bar{Y}_t^\pi))}$. The pair (Y^π, Z^π) is an approximation of the unique solution (Y, Z) of Q-BSDE (0.1). Then, (Y^π, Z^π) satisfies the following recursive quadratic BSDEs

$$\begin{cases} Y_{t_n}^\pi &= \xi^\pi, \\ Y_t^\pi &= \varphi\left(\int_{t_i}^{t_{i+1}} F'\left(Y_r^\pi\right) Z_r^\pi \mathrm{d}r\right) + \int_t^{t_{i+1}} f(Y_r^\pi)\left|Z_r^\pi\right|^2 \mathrm{d}r - \int_t^{t_{i+1}} Z_r^\pi \mathrm{d}W_r, \end{cases} \tag{2.17}$$

where $t \in [t_i, t_{i+1})$, $i = n-1,\ n-2, \ldots,\ 0$ and

$$\begin{aligned} &\varphi\left(\int_{t_i}^{t_{i+1}} F'\left(Y_r^\pi\right) Z_r^\pi \mathrm{d}r\right) \\ &= F^{-1}\left(F\left(Y_{t_{i+1}}^\pi\right) + F'(Y_{t_{i+1}}^\pi) h\left(t_{i+1}, Y_{t_{i+1}}^\pi\right) \triangle_i + h_1\left(t_{i+1}\right) \int_{t_i}^{t_{i+1}} Z_r^\pi F'\left(Y_r^\pi\right) \mathrm{d}r\right). \end{aligned}$$

In fact, Equation (2.17) can be written in the form:

$$Y_t = F^{-1}\left(\bar{\xi} + g\left(\int_u^v F'\left(Y_r\right) Z_r \mathrm{d}r\right)\right) + \int_t^v f(Y_r)\left|Z_r\right|^2 \mathrm{d}r - \int_t^v Z_r \mathrm{d}W_r. \tag{2.18}$$

for $t \in [u,\ v]$ and $0 \leq u < v \leq T$, $\bar{\xi}$ is $\mathcal{F}_v$-measurable and $g : (\Omega \times \mathbb{R},\ \mathcal{F}_v \otimes \mathcal{B}) \longrightarrow (\mathbb{R},\ \mathcal{B})$ is a given function. In the following Theorem we will give an existence and uniqueness result for this new type of Q-BSDE.

Theorem 2.6

Let $0 \leq u < v \leq T$ and $p \geq 2$. Let g be a Lipschitz function such that $g(0) \in \mathbb{L}^p(\Omega)$ and $\bar{\xi} \in \mathbb{L}^p(\Omega)$, then equation (2.18) admits a unique solution

$$(Y, Z) \in \mathcal{S}_{\mathcal{F}}^p([u,\ v]) \times \mathcal{H}_{\mathcal{F}}^p([u,\ v]).$$

Proof: Itô's formula applied to $\bar{Y}_t = F(Y_t)$, taking into account that $\bar{Z}_t = F'(Y_t) Z_t$, yields

$$\begin{cases} \mathrm{d}\bar{Y}_t &= F'(Y_t)\left(-f(Y_t)\left|Z_t\right|^2 \mathrm{d}t + Z_t \mathrm{d}W_t\right) + \frac{1}{2} F''(Y_t)\left|Z_t\right|^2 \mathrm{d}t = \bar{Z}_t \mathrm{d}W_t \\ \bar{Y}_v &= \bar{\xi} + g\left(\int_u^v \bar{Z}_r \mathrm{d}r\right). \end{cases}$$

Or equivalently in its integral form

$$\bar{Y}_t = \bar{\xi} + g\left(\int_u^v \bar{Z}_r \mathrm{d}r\right) - \int_t^v \bar{Z}_r \mathrm{d}W_r, \quad t \in [u,\ v]. \tag{2.19}$$

Notice that g and $\bar{\xi}$ satisfy all the conditions of Theorem 2.2 and therefore the BSDE (2.19) has a unique solution $(\bar{Y}, \bar{Z}) \in \mathcal{S}^p_{\mathcal{F}}([u,\ v]) \times \mathcal{H}^p_{\mathcal{F}}([u,\ v])$.
Now, Itô's formula applied to $F^{-1}(\bar{Y}_t)$ shows that

$$\mathrm{d}F^{-1}(\bar{Y}_t) = (F^{-1})'(\bar{Y}_t)\mathrm{d}\bar{Y}_t + \frac{1}{2}(F^{-1})''(\bar{Y}_t)\mathrm{d}\langle \bar{Y}_\cdot\rangle_t.$$

By invoking notations (1.28) and (1.32), we have

$$\mathrm{d}F^{-1}(\bar{Y}_t) = \frac{\bar{Z}_t}{F'(F^{-1}\left(\bar{Y}_t\right))}\mathrm{d}W_t - \frac{1}{2}\frac{F''(F^{-1}(\bar{Y}_t))}{\left(F'(F^{-1}\left(\bar{Y}_t\right))\right)^3}\left|\bar{Z}_t\right|^2\mathrm{d}t,$$

which will be written in its integral form as

$$Y_t = Y_v + \int_t^v f(Y_r)\left|Z_r\right|^2\mathrm{d}r - \int_t^v Z_r\mathrm{d}W_r. \tag{2.20}$$

■

Theorem 2.7

For any $p \geq 2$, let $(Y^\pi, Z^\pi) \in \mathcal{S}^p_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^p_{\mathcal{F}}([0,\ T])$ be a solution of equation (2.17). Let Assumption 3 and (2.i), (2.iii) and (2.iv) in Assumption 2 be satisfied. Then, there exist two positive constants δ and K which are independent from π, such that, if $|\pi| < \delta$, the rate of convergence of the implicit scheme (2.17) of Q-BSDE (0.1) is of this type:

$$\mathbb{E}\sup_{0\leq t\leq T}|Y_t - Y_t^\pi|^p \leq K\left(|\pi|^{\frac{p}{2}} + \mathbb{E}|\xi - \xi^\pi|^p\right),$$

and

$$\mathbb{E}\left(\int_0^T |Z_t - Z_t^\pi|^2\,\mathrm{d}t\right)^{\frac{p}{2}} \leq K\left(|\pi|^{\frac{p}{2}} + \max\left(\mathbb{E}\left|\xi - \xi^\pi\right|^p,\ |\pi|^{-\frac{p}{2}}\,\mathbb{E}\left|\xi - \xi^\pi\right|^{2p}\right)\right).$$

Proof: Thanks to assertion (2.iv) in Assumption 2, and considering that h_1 is a constant function and the boundedness of F', we can easily check that $\bar{h}$ satisfies condition (2.12). Assuming that ξ^π is p-integrable, it can be easily verified that $\bar{\xi}^\pi \in \mathbb{L}^p(\Omega)$, for any $p \geq 2$. Keeping in mind that $\bar{\xi}$ and $\bar{h}$ satisfy Assumption 1, Proposition 2.3 shows that there are two positive constants K and δ, independent of the partition π, such that, for $|\pi| < \delta$, we have

$$\mathbb{E}\sup_{0\leq t\leq T}\left|\bar{Y}_t - \bar{Y}_t^\pi\right|^p + \mathbb{E}\left(\int_0^T\left|\bar{Z}_t - \bar{Z}_t^\pi\right|^2\mathrm{d}t\right)^{\frac{p}{2}} \leq K\left(|\pi|^{\frac{p}{2}} + \mathbb{E}\left|\xi - \xi^\pi\right|^p\right). \tag{2.21}$$

Since F and F^{-1} are Lipschitz functions and by using (2.21) we have

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}|Y_t - Y_t^\pi|^p &\le K\mathbb{E}\sup_{0\le t\le T}\left|\bar{Y}_t - \bar{Y}_t^\pi\right|^p \\ &\le K\left(|\pi|^{\frac{p}{2}} + \mathbb{E}\left|\xi - \xi^\pi\right|^p\right).\end{aligned}$$

Now, let us show that:

$$\mathbb{E}\left(\int_0^T |Z_t - Z_t^\pi|^2\,\mathrm{d}t\right)^{\frac{p}{2}} \le K\left(|\pi|^{\frac{p}{2}} + \max\left(\mathbb{E}\left|\xi - \xi^\pi\right|^p,\ |\pi|^{-\frac{p}{2}}\,\mathbb{E}\left|\xi - \xi^\pi\right|^{2p}\right)\right).$$

We have for all $q \ge 1$, $(|x| + |y|)^q \le 2^{q-1}(|x|^q + |y|^q)$,

$$\mathbb{E}\left(\int_0^T |Z_t - Z_t^\pi|^2\,\mathrm{d}t\right)^{\frac{p}{2}} \le K\left(\mathbb{E}\left(\int_0^T \left|\bar{Z}_t - \bar{Z}_t^\pi\right|^2\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \left|\bar{Z}_t\right|^2\left|\bar{Y}_t - \bar{Y}_t^\pi\right|^2\mathrm{d}t\right)^{\frac{p}{2}}\right)$$

so, by Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}\mathbb{E}\left(\int_0^T \left|\bar{Z}_t\right|^2\left|\bar{Y}_t - \bar{Y}_t^\pi\right|^2\mathrm{d}t\right)^{\frac{p}{2}} &\le \mathbb{E}\sup_{0\le t\le T}\left|\bar{Y}_t - \bar{Y}_t^\pi\right|^p\left(\int_0^T \left|\bar{Z}_t\right|^2\mathrm{d}t\right)^{\frac{p}{2}} \\ &\le \left(\mathbb{E}\sup_{0\le t\le T}\left|\bar{Y}_t - \bar{Y}_t^\pi\right|^{2p}\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_0^T \left|\bar{Z}_t\right|^2\mathrm{d}t\right)^p\right)^{\frac{1}{2}}.\end{aligned}$$

Now, for all $x > 0$, $y \ge 0$ and $0 < q < 1$, we have $(x+y)^q \le x^q + qx^{q-1}y$ and by using (2.21), we obtain

$$\begin{aligned}\left(\mathbb{E}\sup_{0\le t\le T}\left|\bar{Y}_t - \bar{Y}_t^\pi\right|^{2p}\right)^{\frac{1}{2}} &\le K\left(|\pi|^p + \mathbb{E}\left|\xi - \xi^\pi\right|^{2p}\right)^{\frac{1}{2}} \\ &\le K\left(|\pi|^{\frac{p}{2}} + |\pi|^{-\frac{p}{2}}\,\mathbb{E}\left|\xi - \xi^\pi\right|^{2p}\right).\end{aligned}$$

Moreover, one has

$$\mathbb{E}\left(\int_0^T \left|\bar{Z}_t\right|^2\mathrm{d}t\right)^p \ \le\ T^p\mathbb{E}\sup_{0\le t\le T}\left|\bar{Z}_t\right|^{2p}\ <\ +\infty.$$

and therefore

$$\mathbb{E}\left(\int_0^T \left|\bar{Z}_t\right|^2\left|\bar{Y}_t - \bar{Y}_t^\pi\right|^2\mathrm{d}t\right)^{\frac{p}{2}}\ \le\ K\left(|\pi|^{\frac{p}{2}} + |\pi|^{-\frac{p}{2}}\,\mathbb{E}\left|\xi - \xi^\pi\right|^{2p}\right).$$

Finally,

$$\begin{aligned}\mathbb{E}\left(\int_0^T |Z_t - Z_t^\pi|^2\,\mathrm{d}t\right)^{\frac{p}{2}} &\le K\left(|\pi|^{\frac{p}{2}} + \mathbb{E}\left|\xi - \xi^\pi\right|^p\right) \\ &\quad + K\left(|\pi|^{\frac{p}{2}} + |\pi|^{-\frac{p}{2}}\,\mathbb{E}\left|\xi - \xi^\pi\right|^{2p}\right) \\ &\le K\left(|\pi|^{\frac{p}{2}} + \max\left(\mathbb{E}\left|\xi - \xi^\pi\right|^p,\ |\pi|^{-\frac{p}{2}}\,\mathbb{E}\left|\xi - \xi^\pi\right|^{2p}\right)\right).\end{aligned}$$

This ends the proof of theorem. ■

Remark 2.8

(i) Implicit and explicit schemes give the same results if $\bar{h}$ does not depend on $\bar{Z}$.

(ii) For both explicit and implicit numerical schemes considered in this section, the problem is how to evaluate the processes $\{Z_t^\pi\}_{0\le t\le T}$ and $\{\bar{Z}_t^\pi\}_{0\le t\le T}$, in order to implement the scheme on computers.

2.3.3 A Fully Discrete Scheme for Q-BSDE

In this part, we consider the following Q-BSDE

$$Y_t = \xi + \int_t^T \left(\alpha(s) + \beta(s)Z_s + f(Y_s)\left|Z_s\right|^2\right) \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s \tag{2.22}$$

under the following Assumptions:

(A1) Assume that α and β are deterministic and bounded functions, moreover, there exists a constant $L > 0$, such that for all $t_1,\ t_2 \in [0,\ T]$

$$|\alpha(t_2) - \alpha(t_1)| + |\beta(t_2) - \beta(t_1)| \le L\left|t_2 - t_1\right|^{\frac{1}{2}}.$$

(A2) $\mathbb{E}[\sup_{0\le\theta\le T}|D_\theta\xi|^r]$ is finite for all $r \ge 1$.

(A3) There exists a constant $M > 0$ such that, $\xi \le M$ P-a.s.

QBSDE (ξ, $\beta(s)z + f(y)\left|y\right|^2$) This paragraph is devoted to the study of a particular case of Q-BSDE (2.22) when $\alpha \equiv 0$. By using Zvonkin's transformation and the notations (1.28), the equation QBSDE (2.22) will be transformed formally a.s. to the following equation

$$\bar{Y}_t = \bar{\xi} + \int_t^T \beta(s)\bar{Z}_s \mathrm{d}s - \int_t^T \bar{Z}_s \mathrm{d}W_s. \tag{2.23}$$

The fully discrete numerical scheme of (2.23) will be defined similarly to the one of (2.9) with $\bar{h}(t_{i+1}, \bar{Y}_{t_{i+1}}^\pi, \bar{Z}_{t_{i+1}}^\pi) = \beta(t_{i+1})\bar{Z}_{t_{i+1}}^\pi$,

$$\begin{cases} \bar{Y}_{t_n}^\pi &= \bar{\xi}, \qquad \bar{Z}_{t_n}^\pi = D_T\bar{\xi}, \\ \bar{Y}_{t_i}^\pi &= \mathbb{E}\left[\bar{Y}_{t_{i+1}}^\pi + \beta(t_{i+1})\bar{Z}_{t_{i+1}}^\pi\Delta_i \mid \mathcal{F}_{t_i}\right], \\ \bar{Z}_{t_i}^\pi &= \mathbb{E}\left[\rho_{t_{i+1},t_n}^\pi\ D_{t_i}\bar{\xi} \mid \mathcal{F}_{t_i}\right], \end{cases} \tag{2.24}$$

where $\rho^{\pi}_{t_i,t_i} = 1$, $i = 0, 1, \ldots, n$ and for $0 \leq i < j \leq n$,

$$\rho^{\pi}_{t_i,t_j} = \exp\left\{\sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}} \beta(s)\mathrm{d}W_s - \frac{1}{2}\sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}} \beta^2(s)\mathrm{d}s\right\}.$$

Theorem 2.9

For any $p \geq 2$, let Assumptions (A1) and (A2) be satisfied and ξ satisfy assertion (1.i) in Assumption 1. Then, the rate of convergence of the fully discrete scheme associated with Q-BSDE (2.22) is given by

$$\mathbb{E}\max_{0\leq i\leq n}\left\{\left|Y_{t_i} - Y^{\pi}_{t_i}\right|^p + \left|Z_{t_i} - Z^{\pi}_{t_i}\right|^p\right\} \;\leq\; C\,|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.$$

Proof: Considering that $\bar{\xi}$ satisfies (1.i) in Assumption 1, and Assumptions (A1) and (A2) are satisfied, then thanks to Proposition 2.4, the following inequality gives the rate of convergence of the fully discrete scheme (2.24) related to BSDE (2.23),

$$\mathbb{E}\max_{0\leq i\leq n}\left\{\left|\bar{Y}_{t_i} - \bar{Y}^{\pi}_{t_i}\right|^p + \left|\bar{Z}_{t_i} - \bar{Z}^{\pi}_{t_i}\right|^p\right\} \;\leq\; C\,|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}. \tag{2.25}$$

Let (Y^{π}, Z^{π}) such that $Y^{\pi}_t = F^{-1}(\bar{Y}^{\pi}_t)$ and $Z^{\pi}_t = \frac{\bar{Z}^{\pi}_t}{F'\left(F^{-1}(\bar{Y}^{\pi}_t)\right)}$ be an approximation of the unique solution pair (Y, Z) of Q-BSDE (2.22). Firstly, since F and F^{-1} are Lipschitz functions, we deduce from (2.25) that

$$\begin{aligned}\mathbb{E}\max_{0\leq i\leq n}\left|Y_{t_i} - Y^{\pi}_{t_i}\right|^p &\leq M\mathbb{E}\max_{0\leq i\leq n}\left|\bar{Y}_{t_i} - \bar{Y}^{\pi}_{t_i}\right|^p\\ &\leq C\,|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.\end{aligned}$$

where M is the Lipschitz constant of F.

Now, by (2.25), Cauchy–Schwartz inequality and $\sup_{0\leq t\leq T}\mathbb{E}[|\bar{Z}_t|^{2p}] < +\infty$, we have

$$\begin{aligned}\mathbb{E}\max_{0\leq i\leq n}\left|\bar{Z}_{t_i}\right|^p\left|\bar{Y}_{t_i} - \bar{Y}^{\pi}_{t_i}\right|^p &\leq \left(\mathbb{E}\max_{0\leq i\leq n}\left|\bar{Z}_{t_i}\right|^{2p}\right)^{\frac{1}{2}}\left(\mathbb{E}\max_{0\leq i\leq n}\left|\bar{Y}_{t_i} - \bar{Y}^{\pi}_{t_i}\right|^{2p}\right)^{\frac{1}{2}}\\ &\leq C\,|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.\end{aligned} \tag{2.26}$$

Then, we have

$$\begin{aligned}\mathbb{E}\max_{0\leq i\leq n}\left|Z_{t_i} - Z^{\pi}_{t_i}\right|^p &\leq C\left(\mathbb{E}\max_{0\leq i\leq n}\left|\bar{Z}_{t_i} - \bar{Z}^{\pi}_{t_i}\right|^p + \mathbb{E}\max_{0\leq i\leq n}\left|\bar{Z}_{t_i}\right|^p\left|\bar{Y}_{t_i} - \bar{Y}^{\pi}_{t_i}\right|^p\right)\\ &\leq C\,|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.\end{aligned}$$

■

QBSDE $(\xi,\ \alpha(s)+\beta(s)z+\frac{1}{2}|z|^2)$ In this paragraph, we consider the case where $f\equiv\frac{1}{2}$, so that the generator of Q-BSDE (2.22) takes the following form $\alpha(s)+\beta(s)z+\frac{1}{2}|z|^2$. The exponential change of variable $\bar{Y}_t=\exp(Y_t)$ transforms formally equation (2.22) a.s.: $\forall\ t\in[0,\ T]$,

$$\begin{aligned}\bar{Y}_t &= \exp(\xi)-\int_t^T Z_s\exp(Y_s)\,\mathrm{d}W_s\\ &\quad+\int_t^T(\alpha(s)\exp(Y_s)+\beta(s)\exp(Y_s)Z_s)\,\mathrm{d}s\\ &= \exp(\xi)+\int_t^T\left(\alpha(s)\bar{Y}_s+\beta(s)\bar{Z}_s\right)\mathrm{d}s-\int_t^T\bar{Z}_s\mathrm{d}W_s. \end{aligned}\tag{2.27}$$

The latter BSDE being linear with α is a positive function, and one can define

$$\forall\ t\in[0,\ T],\quad Y_t=\ln\left(\bar{Y}_t\right),\qquad Z_t=\frac{\bar{Z}_t}{\bar{Y}_t}.$$

We define the fully discrete numerical scheme of (2.27) similarly to (2.9) with

$$\bar{h}(t_{i+1},\bar{Y}^\pi_{t_{i+1}},\bar{Z}^\pi_{t_{i+1}})=\alpha(t_{i+1})\bar{Y}^\pi_{t_{i+1}}+\beta(t_{i+1})\bar{Z}^\pi_{t_{i+1}},$$

$$\begin{cases}\bar{Y}^\pi_{t_n} &= \exp(\xi),\qquad \bar{Z}^\pi_{t_n}=\exp(\xi)\,D_T\xi,\\ \bar{Y}^\pi_{t_i} &= \mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}}+\left(\alpha(t_{i+1})\bar{Y}^\pi_{t_{i+1}}+\beta(t_{i+1})\bar{Z}^\pi_{t_{i+1}}\right)\Delta_i\mid\mathcal{F}_{t_i}\right],\\ \bar{Z}^\pi_{t_i} &= \mathbb{E}\left[\rho^\pi_{t_{i+1},t_n}\ \exp(\xi)\,D_{t_i}\xi\mid\mathcal{F}_{t_i}\right],\end{cases}\tag{2.28}$$

where $\rho^\pi_{t_i,t_i}=1$, $i=0,1,\ldots,n$ and for $0\le i<j\le n$,

$$\rho^\pi_{t_i,t_j}=\exp\left\{\sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}}\beta(s)\mathrm{d}W_s-\sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}}\left(\alpha(s)-\frac{1}{2}\beta(s)^2\right)\mathrm{d}s\right\}.$$

Let ξ satisfy condition (1.i) in Assumption 1, under Assumptions (A1), (A2) and (A3) and thanks to Proposition 2.4, there are positive constants C and δ independent of the partition π, such that, when $|\pi|<\delta$ we have, for any $p\ge 2$,

$$\mathbb{E}\max_{0\le i\le n}\left\{\left|\bar{Y}_{t_i}-\bar{Y}^\pi_{t_i}\right|^p+\left|\bar{Z}_{t_i}-\bar{Z}^\pi_{t_i}\right|^p\right\}\ \le\ C\,|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.\tag{2.29}$$

Theorem 2.10

If ξ satisfies (1.i) in Assumption 1, and Assumptions (A1), (A2) and (A3) hold true, then there are positive constants C and δ independent of the partition π, such that, *when* $|\pi| < \delta$, *we have the following rate of convergence*

$$\mathbb{E}\left[\max_{0\le i\le n}\left\{\left|Y_{t_i} - Y_{t_i}^{\pi}\right|^p + \left|Z_{t_i} - Z_{t_i}^{\pi}\right|^p\right\}\right] \le C\,|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.$$

Proof: We define the approximation of the couple (Y, Z) solution of Q-BSDE (2.22), as follows $Y_t^{\pi} = \ln\left(\bar{Y}_t^{\pi}\right)$ and $Z_t^{\pi} = \frac{\bar{Z}_t^{\pi}}{\bar{Y}_t^{\pi}}$. Noting that the linear BSDE (2.27) has a unique solution $\left(\bar{Y}, \bar{Z}\right)$ where $\bar{Y}$ is given explicitly by $\bar{Y}_t = \mathbb{E}[\exp(\xi)\Gamma_{t,T}|\mathcal{F}_t]$, such that $(\Gamma_{t,s})_{s\ge t}$ is the solution of the following SDE

$$\begin{cases} d\Gamma_{t,s} &= \Gamma_{t,s}\left(\alpha(s)ds + \beta(s)dW_s\right), \\ \Gamma_{t,t} &= 1. \end{cases}$$

Define $P^* = L_T P$, where

$$L_t = \exp\left(\int_0^t \beta(s)dW_s - \frac{1}{2}\int_0^t |\beta(s)|^2\,ds\right),$$

then

$$\bar{Y}_t = \mathbb{E}^*\left[\exp(\xi)\exp\left(\int_t^T \alpha(s)ds\right) \mid \mathcal{F}_t\right],$$

where $\mathbb{E}^*$ stands for the mathematical expectation under P^*. Obviously, if $\xi \ge 0$, $\alpha(s) \ge 0$, we have

$$\bar{Y}_t \ge \exp\left(\int_t^T \alpha(s)ds\right) \ge 1.$$

Due the fact that $\ln(\cdot)$ is a Lipschitz function in the interval $[1, +\infty[$ and by using (2.29), one can obtain

$$\begin{aligned} \mathbb{E}\max_{0\le i\le n}\left|Y_{t_i} - Y_{t_i}^{\pi}\right|^p &= \mathbb{E}\max_{0\le i\le n}\left|\ln(\bar{Y}_{t_i}) - \ln\left(\bar{Y}_{t_i}^{\pi}\right)\right|^p \\ &\le C\mathbb{E}\max_{0\le i\le n}\left|\bar{Y}_{t_i} - \bar{Y}_{t_i}^{\pi}\right|^p \\ &\le C\,|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}. \end{aligned} \tag{2.30}$$

Let $\left(\bar{Y}, \bar{Z}\right)$ be the unique solution of BSDE (2.27). Since $\exp(\xi)$ and the functions α, β are bounded, it was shown by Proposition 2. d (i) in [32] that $\bar{Y}$ and $\bar{Z}$ are bounded. Moreover, thanks to scheme (2.28), it is easy to verify that $\bar{Y}^{\pi}$ is also bounded.

Elementary calculation shows that

$$\begin{aligned}\mathbb{E}\max_{0\leq i\leq n}\left|Z_{t_i}-Z^{\pi}_{t_i}\right|^p &= \mathbb{E}\max_{0\leq i\leq n}\left|\frac{\bar{Z}_{t_i}}{\bar{Y}_{t_i}}-\frac{\bar{Z}^{\pi}_{t_i}}{\bar{Y}^{\pi}_{t_i}}\right|^p \\ &\leq \mathbb{E}\left[\max_{0\leq i\leq n}\left(\left|\frac{1}{\bar{Y}^{\pi}_{t_i}}\left(\bar{Z}_{t_i}-\bar{Z}^{\pi}_{t_i}\right)\right|^p+\left|\frac{\bar{Z}_{t_i}}{\bar{Y}_{t_i}\bar{Y}^{\pi}_{t_i}}\right|^p\left|\bar{Y}_{t_i}-\bar{Y}^{\pi}_{t_i}\right|^p\right)\right] \\ &\leq C\left(\mathbb{E}\max_{0\leq i\leq n}\left|\bar{Z}_{t_i}-\bar{Z}^{\pi}_{t_i}\right|^p+\mathbb{E}\max_{0\leq i\leq n}\left|\bar{Y}_{t_i}-\bar{Y}^{\pi}_{t_i}\right|^p\right).\end{aligned}$$

Finally, by invoking relation (2.29), we obtain

$$\mathbb{E}\max_{0\leq i\leq n}\left|Z_{t_i}-Z^{\pi}_{t_i}\right|^p \leq C\left|\pi\right|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.$$

■

2.4 Simulation results for Q-BSDE

Let $(\Omega,\mathcal{F},P)$ be complete probability space, $(W_t)_{t\leq T}$ be a 1-dimensional Brownian motion defined on fixed interval $[0,\ T]$. We consider for a fixed $n\in\mathbb{N}$,

$$W^n_t := \sqrt{\delta}\sum_{j=1}^{[t/\delta]}X_j,\quad \text{for all}\quad 0\leq t\leq T,\ \delta=\frac{T}{n},$$

where $\{X_j\}_{j=1}^n$ is a $\{1,-1\}$-valued i.i.d. sequence with $P(X_j=1)=P(X_j=-1)=0.5$, i.e., a Bernoulli sequence. By Donsker's theorem, we have $\sup_{0\leq t\leq T}|W^n_t-W_t|\to 0$, as $n\to\infty$.

2.4.1 Examples

In what follows, we will consider the terminal time $T=1$.

Example 2.1

We consider the Q-BSDE as follows

$$Y_t = ln\left|W_1\right|+\int_t^1\left(1+Z_r+\frac{1}{2}\left|Z_r\right|^2\right)\mathrm{d}r-\int_t^1 Z_r\mathrm{d}W_r,\quad 0\leq t\leq 1, \tag{2.31}$$

The exponential change of variable $\bar{Y}=\exp(Y)$ transforms formally this equation a.s.:

$$\bar{Y}_t=\left|W_1\right|+\int_t^1\left(\bar{Y}_r+\bar{Z}_r\right)\mathrm{d}r-\int_t^1\bar{Z}_r\mathrm{d}W_r,\quad 0\leq t\leq 1,$$

such that $\bar{Z}_t = Z_t \exp(Y)$.

The numerical results obtained for Q-BSDE (2.31) are:

n	100	500	1000	2000	5000
Y^n	*1.1571*	*1.1547*	*1.1544*	*1.1543*	*1.1540*

Applying the Monte-Carlo method with 10 000 000 samples to the exact solution of the Q-BSDE (2.31) $Y_0 = \ln\left(\bar{Y}_0\right) = \ln\left(\exp\left(\frac{1}{2}\right)\mathbb{E}\left(|W_1|\exp(W_1)\right)\right)$, *then,* $Y_0 = 1.1540$.

The following figure shows the Y and Z process trajectories for Example 1.

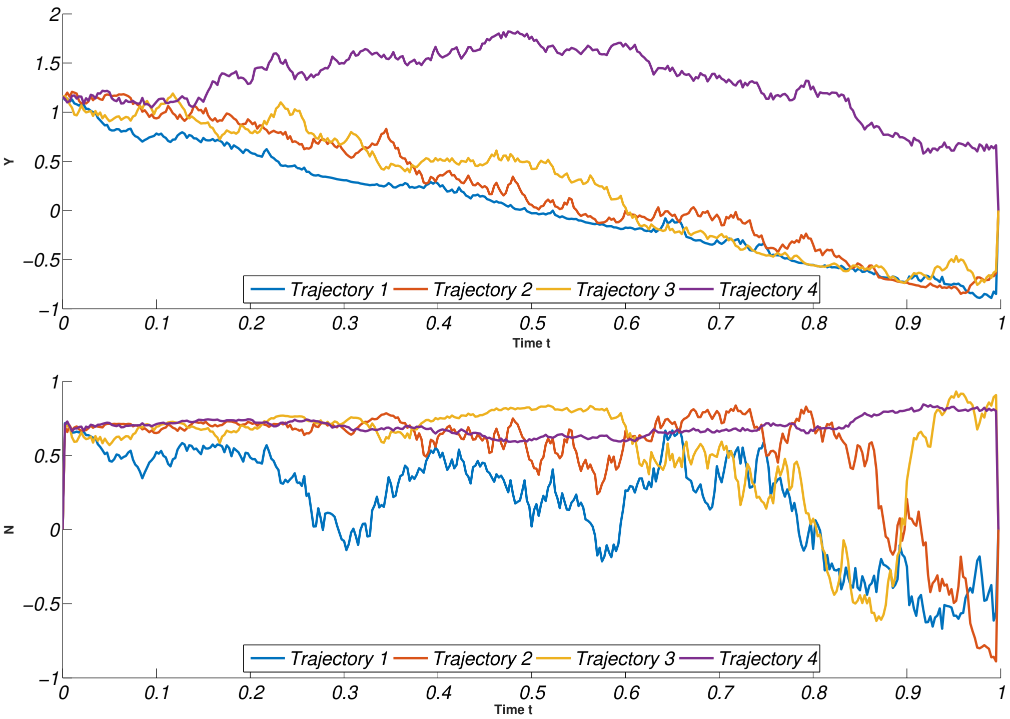

Figure 2.1: The trajectories of the solution of Q-BSDE

Example 2.2

We consider the Q-BSDE as follows

$$Y_t = |W_1| + \int_t^1 \left(Z_r + \frac{1}{2}|Z_r|^2\right) \mathrm{d}r - \int_t^1 Z_r \mathrm{d}W_r, \quad 0 \le t \le 1, \tag{2.32}$$

By the exponential change of variable $\bar{Y} = \exp(Y)$, *we obtain*

$$\bar{Y}_t = \exp|W_1| + \int_t^1 \bar{Z}_r \mathrm{d}r - \int_t^1 \bar{Z}_r \mathrm{d}W_r, \quad 0 \le t \le 1, \tag{2.33}$$

The numerical results obtained for Q-BSDE (2.32) are

n	200	400	600	800	1000
Y^n	*1.5532*	*1.5526*	*1.5446*	*1.5389*	*1.5439*

We apply the Monte-Carlo method, with 10 000 000 samples, to calculate the exact solution of Q-BSDE (2.32) $Y_0 = \ln\left(\bar{Y}_0\right) = \ln\left(\exp\left(-\frac{1}{2}\right)\mathbb{E}\left(\exp|W_1|\exp\left(W_1\right)\right)\right)$, *then,* $Y_0 = 1.5424$

The following figure shows the Y and Z process trajectories for Example 2.

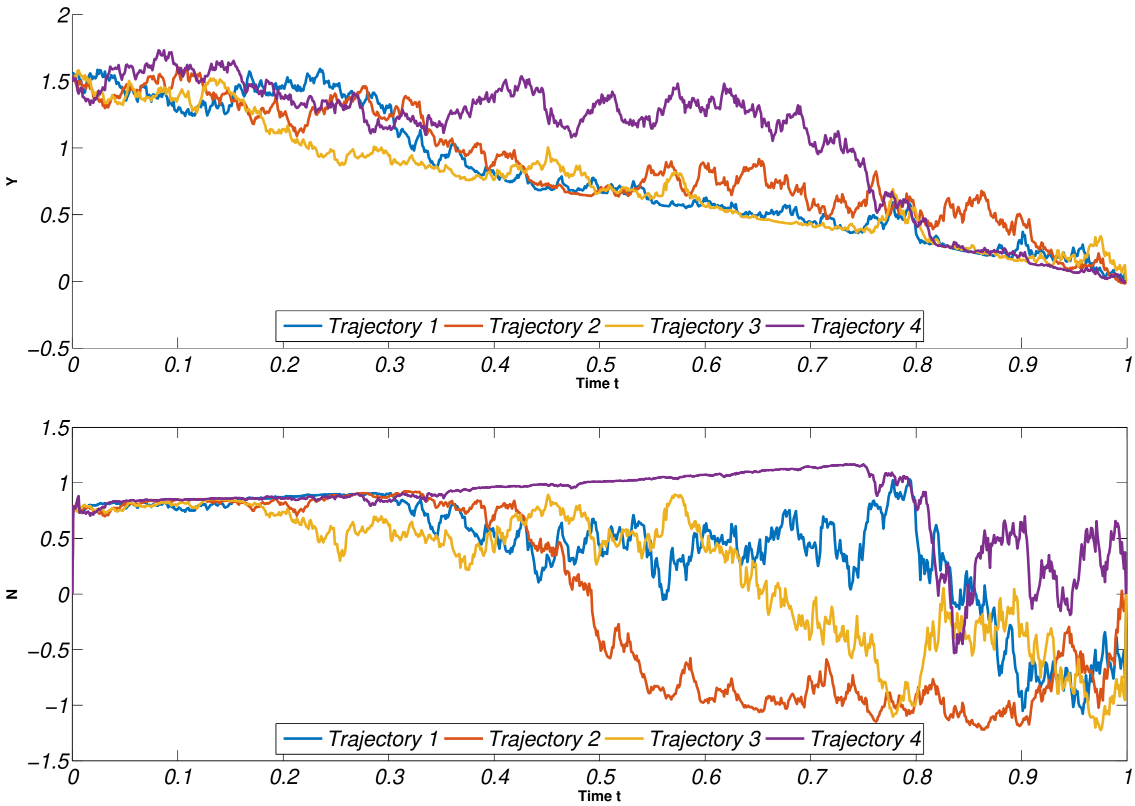

Figure 2.2: The trajectories of the solution of Q-BSDE

Example 2.3

We consider the Q-BSDE as follows

$$Y_t = \exp(W_1) + \int_t^1 \left(Z_r + \mathcal{X}_{[0,2]}\left|Z_r\right|^2\right) \mathrm{d}r - \int_t^1 Z_r \mathrm{d}W_r, \quad 0 \leq t \leq 1, \tag{2.34}$$

In this example, we use Zvenkin's transformation, which can be computed as follows:

$$\int_0^y f(t)\mathrm{d}t = \begin{cases} 0 & \textit{if} \;\; y \leq 0 \\ \int_0^y \mathcal{X}_{[0,2]}(t)\mathrm{d}t = \int_0^y \mathrm{d}t = y & \textit{if} \;\; 0 < y \leq 2 \\ \int_0^2 \mathrm{d}t = 2 & \textit{if} \;\; 2 < y \end{cases}$$

and

$$g(y) = \exp(\int_0^y 2f(t)\mathrm{d}t) = \begin{cases} 1 & \textit{if} \;\; y \leq 0 \\ e^{2y} & \textit{if} \;\; 0 < y \leq 2 \\ e^4 & \textit{if} \;\; 2 < y \end{cases}$$

and finally

$$F(x) = \int_0^x g(y)\mathrm{d}y = \begin{cases} x & \textit{if} \;\; x \leq 0 \\ \dfrac{e^{2x}-1}{2} & \textit{if} \;\; 0 < x \leq 2 \\ F(2) + \int_2^x g(y)\mathrm{d}y = \dfrac{e^4-1}{2} + e^4(x-2) & \textit{if} \;\; 2 < x \end{cases}$$

and the inverse is given by

$$F^{-1}(z) = \begin{cases} z & \textit{if} \;\; z \leq 0 \\ \dfrac{\ln(2z+1)}{2} & \textit{if} \;\; 0 < z \leq \dfrac{e^4-1}{2} \\ e^{-4}\left(z + \frac{3}{2}e^4 + \frac{1}{2}\right) & \textit{if} \;\; z > \dfrac{e^4-1}{2}. \end{cases}$$

After transformation $\bar{Y}_t = F(Y_t)$ we find the following linear BSDE

$$\bar{Y}_t = F(\xi) + \int_t^1 \bar{Z}_s \mathrm{d}s - \int_t^1 \bar{Z}_s \mathrm{d}W_s, \tag{2.35}$$

with $\bar{Z}_t = g(Y_t)Z_t$.

We apply the Monte Carlo method, with 10000000 *samples, to calculate the exact solution of Q-BSDE (2.34)* $Y_0 = F^{-1}\left(\bar{Y}_0\right) = F^{-1}\left(\exp\left(-\frac{1}{2}\right)\mathbb{E}\left(F\left(\xi\right)\exp\left(W_1\right)\right)\right)$*, then,* $Y_0 = 4.4804$*. The numerical results obtained for Q-BSDE (2.34) are*

n	200	400	600	800	1000
Y^n	*4.4876*	*4.5087*	*4.4857*	*4.4780*	*4.5149*

The following figure shows the Y and Z process trajectories for Example 3.

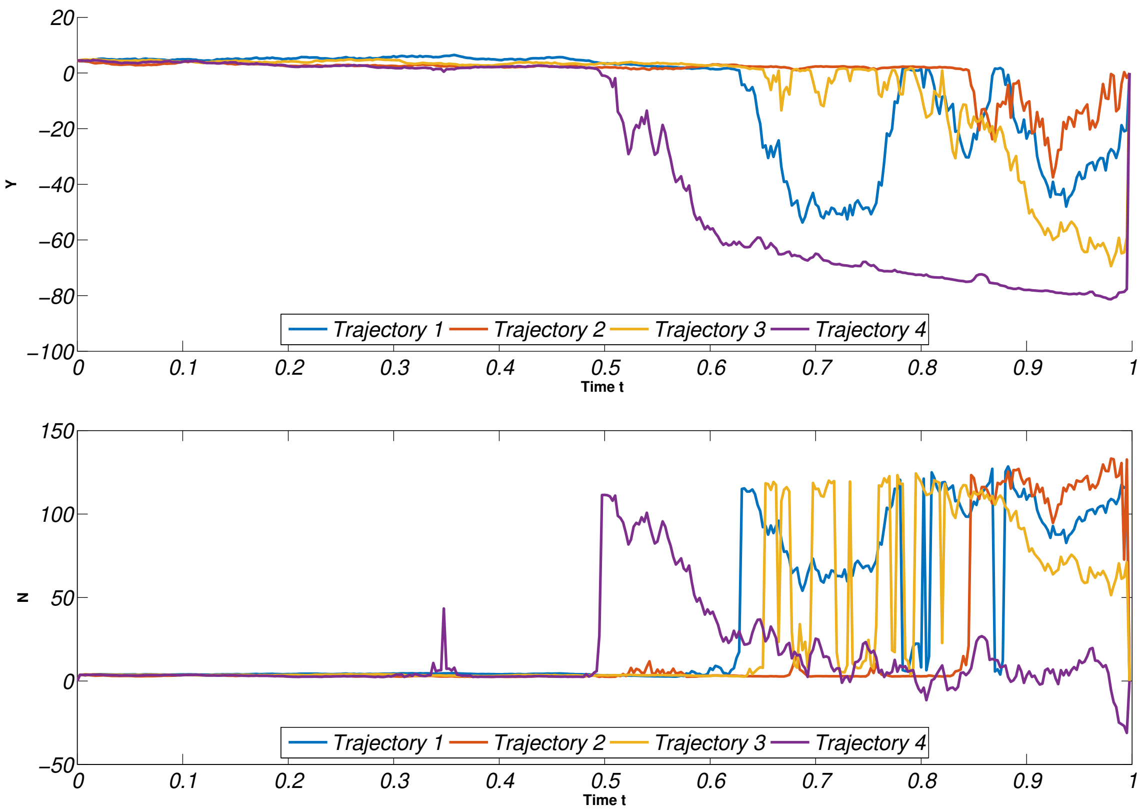

Figure 2.3: The trajectories of the solution of Q-BSDE

Chapter 3

The Maximum Principle for Optimal Control of Diffusion with Non-Smooth Coefficients via Malliavin Calculus

3.1 Introduction

This chapter presents the main results of the second topic of this thesis; we focus on investigating both the necessary and sufficient conditions of optimality for a class of controlled stochastic differential equations taking the following form:

$$\begin{cases} \mathrm{d}X_t &= b(t,X_t,u_t)\mathrm{d}t + \sigma\left(t,X_t\right)\mathrm{d}W_t, \\ X_0 &= x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $b : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, are given deterministic functions, $(W_t)_{t\in[0,T]}$ is a d-dimensional Brownian motion, x is the initial state at time 0 and u stands for the control process. We aim to extend the result established by Mezerdi and Yekhlef to the global Lipschitz setting. Our analytical approach relies on the Rademacher Theorem, which states that nearly all points within the domain of a Lipschitz function exhibit differentiability. Moreover, the Theorem specifies that these points of differentiability possess Borel-measurable and uniformly bounded derivatives. The proof strategy involves the following steps:

1) We approximate the original non-regular control problem through a sequence of perturbed regular problems.

2) We establish the necessary conditions for near-optimality by utilizing the Malliavin calculus.

3) We achieve the desired outcome by taking the limits.

3.2 Problem Formulation and Auxiliary Lemmas

3.2.1 Problem Statement

We aim to investigate necessary and sufficient optimality conditions verified by $\hat{u}$ associated with the **problem A** as already described in the introduction. To achieve this, we need to consider the following assumptions.

Assumptions 1

(A1) ℓ, b are continuously differentiable with respect to u and their first derivatives are bounded.

(A2) b and σ are bounded and $b, \sigma, \partial_u \ell, \partial_u b$ are Lipschitz functions in x

(A3) ℓ and g are twice continuously differentiable in x, with bounded first and second derivatives.

(A4) There exists $\lambda > 0$ such that for $\forall\ \xi \in \mathbb{R}^d$, $\xi^* \sigma \sigma^* \xi \geq \lambda |\xi|^2$,

(A5) $\mathbb{E}|\partial_u b(t, \hat{X}_t, \hat{u}_t)|^2$ is finite.

With these assumptions in mind, it is evident that equation (3.1) fulfills the standard Lipschitz and linear growth conditions. Therefore, it possesses one and only one solution such that for any $p \geq 1$, $\mathbb{E}[\sup_{t \leq T} |X_t|^p]$ is finite.

Given that the functions b and σ exhibit Lipschitz continuity solely in x, Rademacher's Theorem verifies the existence of derivatives almost everywhere (as per the Lebesgue measure). Let $\partial_x b(t, x, a)$ and $\partial_x \sigma(t, x)$ denote any Borel measurable functions satisfying for each (t, a):

$$\begin{aligned} \frac{\partial b}{\partial x}(t, x, a) &= \partial_x b\,(t, x, a) \quad \mathrm{d}x \text{ a.e.} \\ \frac{\partial \sigma}{\partial x}(t, x) &= \partial_x \sigma(t, x) \quad \mathrm{d}x \quad \text{a.e.} \end{aligned}$$

It is evident that the derivatives of Lipschitz functions are bounded by the Lipschitz constant M almost everywhere. Suppose that the mapping:

$$a \mapsto \partial_x b(t, x, a) \text{ is continuous uniformly in } (t, x). \tag{3.2}$$

Let us recall Krylov's inequality and Ekeland's variational principle, which will be used in the sequel.

Lemma 3.1 (*Krylov's inequality*)

*Let $(W_t, t \geq 0)$ be a d-dimensional Brownian motion defined on a filtered and completed probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$. Given a stochastic process X that can be written in integral form as $X_t = x + \int_0^t b(s)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}W_s$ where $b : \Omega \times \mathbb{R}^+ \to \mathbb{R}^d$ and $\sigma : \Omega \times \mathbb{R}^+ \to \mathbb{R}^d \otimes \mathbb{R}^d$ are $\mathcal{F}_t$-adapted and bounded processes such that σ satisfies the assumption **(A4)** Then, for any Borel function*

$$\varphi : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R} \text{ with support in } [0,\ T] \times B(0, R)$$

the following inequality holds:

$$\int_0^T \mathbb{E}|\varphi(t, X_t)|\mathrm{d}t \quad \leq \quad C\left(\int_0^T \int_{B(0,R)} |\varphi(t,x)|^{d+1}\mathrm{d}t\mathrm{d}x\right)^{\frac{1}{d+1}},$$

Here C is a constant and $B(0, R)$ is the ball, of $\mathbb{R}^d$, centered at 0 with radius R.

The following Lemma is often referred to as Ekeland's variational principle. It provides a method for establishing near-optimal necessary conditions.

Lemma 3.2 (*Ekeland's variational principle*)

On a complete metric space (U, d) we consider a lower semi-continuous function $J : U \to \mathbb{R} \cup \{+\infty\}$ assumed to be bounded from below such that for each $\varepsilon > 0$ there exists $u \in U : J(u) \leq \inf \{J(v); v \in U\} + \varepsilon$, then there exists u_ε such that:

(i) $J(u_\varepsilon) \leq J(u)$,

(ii) $d(u, u_\varepsilon) \leq \sqrt{\varepsilon}$,

(iii) $J(v) + \sqrt{\varepsilon} d(v, u_\varepsilon) \leq J(u_\varepsilon), \ \forall \ v \in U$.

3.2.2 Some Auxiliary Findings

In this subsection, we will outline some fundamental results concerning the SMP theory. To begin with, we endowed the set $\mathcal{U}^{\mathcal{G}}$ by the following metric d. For $u,\ v \in \mathcal{U}^{\mathcal{G}}$, $d(u, v) = (\int_0^T \mathbb{E}|u_t - v_t|^p \mathrm{d}t)^{\frac{1}{p}}$, for some $p \geq 2$.

Lemma 3.3

(i) $(\mathcal{U}^{\mathcal{G}}, d)$ *is a complete metric space,*

(ii) Let u, $v \in \mathcal{U}^{\mathcal{G}}$, X^u *and* X^v *the solution of* (3.1) *corresponding to* u *and* v*, then* :

$$\mathbb{E}\sup_{t\leq T}|X_t^u - X_t^v|^p \leq K d^p(u,v),$$

(iii) The cost functional $\mathcal{J} : (\mathcal{U}^{\mathcal{G}}, d) \longrightarrow \mathbb{R}$ *is continuous.*

Proof: i) See [26] and [21].

ii) The convexity property of the mapping $x \mapsto |x|^p$ for $p > 1$ together with Hölder's inequality, lead to

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}|X_t^u - X_t^v|^p &\leq C\int_0^T \mathbb{E}|b(t,X_t^u,u_t) - b(t,X_t^v,v_t)|^p \mathrm{d}t \\ &\quad + C\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t (\sigma(s,X_s^u) - \sigma(s,X_s^v)\,\mathrm{d}W_s\right|^p\end{aligned}$$

Exploiting the Lipschitz continuity of the coefficients b,σ, and using Hölder's and BDG inequalities, one can derive:

$$\mathbb{E}\sup_{t\leq T}|X_t^u - X_t^v|^p \leq C\left[\int_0^T \mathbb{E}|X_t^u - X_t^v|^p \mathrm{d}t\right] + C\int_0^T \mathbb{E}|u_t - v_t|^p \mathrm{d}t.$$

Gronwall's inequality implies the existence of a positive constant C (which may vary from one line to another) such that:

$$\mathbb{E}\sup_{t\leq T}|X_t^u - X_t^v|^p \leq C d^p(u,v).$$

iii) Similar arguments demonstrate the continuity of $\mathcal{J}$. ■

We must outline the following processes to establish the Hamiltonian function:

$$\begin{aligned}K(t) :&= \partial_x g(X_T) + \int_t^T \partial_x \ell(s,X_s,u_s)\mathrm{d}s, \\ D_t K(t) :&= D_t\partial_x g(X_T) + \int_t^T D_t\partial_x\ell(s,X_s,u_s)\mathrm{d}s, \\ \mathcal{H}_0(s,x,u) :&= K(s)b(s,x,u) + D_s K(s)\sigma(s,x), \\ G(t,s) :&= \exp\left(\int_t^s \{\partial_x b(r,X_r,u_r) - \frac{1}{2}(\partial_x\sigma)(\partial_x\sigma)^*(r,X_r)\}\mathrm{d}r + \int_t^s \partial_x\sigma(r,X_r)\mathrm{d}W_r\right),\end{aligned}$$

and

$$Y(t) := K(t) + \int_t^T \partial_x\mathcal{H}_0(s,X_s,u_s)G(t,s)\mathrm{d}s, \tag{3.3}$$

where D stands for the Malliavin derivative.

Since $\partial_x b$ and $\partial_x \sigma$ are bounded, it clear that for fixed t, $G(t,\cdot)$ uniquely satisfy is the following linear SDE

$$\begin{cases} \mathrm{d}G(t,s) &= \partial_x b(s,X_s,u_s)G(t,s)\mathrm{d}s + \partial_x\sigma(s,X_s)G(t,s)\mathrm{d}W_s, \\ G(t,t) &= 1. \end{cases} \tag{3.4}$$

which satisfies, for any $p>0$

$$\mathbb{E}\sup_{0\le t\le s\le T}|G(t,s)|^p < \infty. \tag{3.5}$$

Let $\partial_x b_i$ and $\partial_x\sigma_i$ be versions of $\partial_x b$ and $\partial_x\sigma$ for $i=1,\ 2$. We denote by G_1 and G_2 respectively Y_1 and Y_2 solutions of (3.4) respectively (3.3) corresponding to $\partial_x b_i$ and $\partial_x\sigma_i$ for $i=1,\ 2$.

The purpose of the next Lemma is to show that the two versions G_1 and G_2 are indistinguishable, as well as Y_1 and Y_2.

Lemma 3.4

For any $p\ge 2$ the following proprieties are satisfied

(i) $\mathbb{E}\sup_{0\le s\le t\le T}|G_1(t,s)-G_2(t,s)|^p = 0,$

(ii) $\mathbb{E}\sup_{0\le t\le T}|Y_1(t)-Y_2(t)|^p = 0.$

Proof (i) : Let $\partial_x b_1$, $\partial_x b_2$ and $\partial_x\sigma_1$, $\partial_x\sigma_2$ be respectively two Borel versions of the generalized derivatives of b and σ with respect to x, that is for each $t\in[0,\ T]$, $v\in\mathbb{R}$,

$$\partial_x b_1(t,x,v) \;=\; \partial_x b_2(t,x,v) \text{ and } \partial_x\sigma_1(t,x,v) \;=\; \partial_x\sigma_2(t,x,v) \quad \mathrm{d}x \text{ a.e.}$$

From the inequality $|e^x-e^y|\le(e^x+e^y)|x-y|$ we deduce

$$\begin{aligned} |G_1(t,s)-G_2(t,s)| \;\le\; (G_1(t,s)+G_2(t,s))\Big[\int_t^s |\partial_x b_1-\partial_x b_2|(r,X_r,u_r)\mathrm{d}r \\ +\frac{1}{2}\int_t^s \left|(\partial_x\sigma_1)^2-(\partial_x\sigma_2)^2\right|(r,X_r)\mathrm{d}r \\ +\Big|\int_t^s (\partial_x\sigma_1-\partial_x\sigma_2)(r,X_r)\mathrm{d}W_r\Big|\Big]. \end{aligned} \tag{3.6}$$

since $\partial_x\sigma_1$, and $\partial_x\sigma_2$ are bounded, we get

$$\mathbb{E}\sup_{0\le t\le s\le T}|G_i(t,s)|^p < \infty \quad \text{for} \quad i=1,\ 2.$$

We can confidently assert, while preserving generality, that the support of the functions $\partial_x b_i$ and $\partial_x \sigma_i$ are included in $[0,\ T] \times B(0, R)$ for $i = 1, 2$, applying Krylov's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le s\le T}|G_1(t,s)-G_2(t,s)|^p\right] \qquad (3.7)\\
\le\ & C\left\{\left(\int_0^T\int_{B(0,R)}\sup_{a\in U}|\partial_x b_1(r,x,a)-\partial_x b_2(r,x,a)|^{2p(d+1)}\mathrm{d}x\mathrm{d}r\right)^{\frac{1}{d+1}}\right.\\
&+\left(\int_0^T\int_{B(0,R)}|\partial_x\sigma_1(r,x)-\partial_x\sigma_2(r,x)|^{2p(d+1)}\mathrm{d}x\mathrm{d}r\right)^{\frac{1}{d+1}}\\
&\left.+\left(\int_0^T\int_{B(0,R)}|\partial_x\sigma_1(r,x)-\partial_x\sigma_2(r,x)|^{2(d+1)}\mathrm{d}x\mathrm{d}r\right)^{\frac{p}{d+1}}\right\}^{\frac{1}{2}}.
\end{aligned}$$

Since $\partial_x b_1(t,x,v) = \partial_x b_2(t,x,v)$ and $\partial_x\sigma_1(t,x) = \partial_x\sigma_2(t,x)$ $\mathrm{d}x$ a.e., we get the desired result. ■

Proof (ii) : By Hölder's inequality, we have:

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}|Y_1(t)-Y_2(t)|^p\\
\le\ & \mathbb{E}\int_0^T\sup_{0\le t\le s\le T}|\partial_x\mathcal{H}_0^1(s,X_s,u_s)G_1(t,s)-\partial_x\mathcal{H}_0^2(s,X_s,u_s)G_2(t,s)|^p\mathrm{d}s\\
\le\ & C\mathbb{E}\int_0^T|\partial_x b_1(s,X_s,u_s)-\partial_x b_2(s,X_s,u_s)|^p|K_1(s)|^p\sup_{0\le t\le s\le T}|G_1(t,s)|^p\mathrm{d}s\\
&+C\mathbb{E}\int_0^T|K(s)\partial_x b_2(s,X_s,u_s)|^p\sup_{0\le t\le s\le T}|G_1(t,s)-G_2(t,s)|^p\mathrm{d}s\\
&+C\mathbb{E}\int_0^T|\partial_x\sigma_1(s,X_s)-\partial_x\sigma_2(s,X_s)|^p|D_sK(s)|^p\sup_{0\le t\le s\le T}|G_1(t,s)|^p\mathrm{d}s\\
&+C\mathbb{E}\int_0^T|\partial_x\sigma_2(s,X_s)|^p|D_sK(s)|^p\sup_{0\le t\le s\le T}|G_1(t,s)-G_2(t,s)|^p\mathrm{d}s.
\end{aligned}$$

Then, by using Cauchy-Schwartz inequality, K, $\partial_x\sigma_2$, $\partial_x b_2$ are bounded, Krylov's inequality, the previous Assertion (i) and (3.5), one can obtain

$$\mathbb{E}\sup_{0\le t\le T}|Y_1(t)-Y_2(t)|^p = 0.$$

■

3.2.3 Near-Optimality Conditions for a Sequence of Perturbed Control Problems

Using convolution, we will smooth out the coefficients b and σ. Assume that φ is an $\mathbb{R}^d$-valued non-negative function that has one continuous derivative and a compact support, such that:

$$\int_{\mathbb{R}^d} \varphi(z)\mathrm{d}z = 1.$$

Then the functions b_n and σ_n defined by

$$b_n(t,x,u) = n^d b(t,\cdot,u) * \varphi(n\cdot)(x) \text{ and } \sigma_n(t,x) = n^d \sigma(t,\cdot) * \varphi(n\cdot)(x)$$

are smooth functions.

Lemma 3.5

a) *The Borel measurable functions b_n and σ_n are bounded and Lipschitz in x.*

b) $\forall\ (t,x,u) \in [0,\ T] \times \mathbb{R}^d \times U$,

(i)

$$|(b_n - b)(t,x,u)| + |(\sigma_n - \sigma)(t,x)| \leq \frac{C}{n} = \varepsilon_n,$$

(ii)

$$|(\partial_u b_n - \partial_u b)(t,x,u)| \leq \frac{C}{n} = \varepsilon_n,$$

where the constant C is positive.

c) *σ_n and b_n are continuously differentiable functions in both x and u. Moreover, $\forall\ t \in [0,\ T]$, such that $\rho = x,\ u$, we have*

$$\lim_{n\to+\infty} \partial_\rho \sigma_n(t,x) = \partial_\rho \sigma(t,x) \quad \mathrm{d}x \text{ a.e.},$$

and

$$\lim_{n\to+\infty} \partial_\rho b_n(t,x,a) = \partial_\rho b(t,x,a) \quad \mathrm{d}x \text{ a.e.},$$

d) $\int\int_{[0,\ T]\times B(0,R)} \sup_{u\in U} |(\partial_x b_n - \partial_x b)(t,x,u)|\mathrm{d}t\mathrm{d}x \to_{n\to+\infty} 0.$

Proof: The proofs of the assertions a), b-(i), c) with $\rho = x$ and d) are carried out in [7].

To prove b-(ii) and c) with $\rho = u$, it suffices to remark that

$$\partial_u b_n(t,x,a) = n^d \int \partial_u b(t, x-y, a)\varphi(ny)\mathrm{d}y.$$

■

Now we define a family of modified control problems, where for each integer n, the original non-smooth coefficients b and σ are replaced by their respective smooth counterparts b_n and σ_n. We denote by X_t^n the solution of the following SDE:

$$\begin{cases} \mathrm{d}X_t^n &= b_n\left(t, X_t^n, u_t\right)\mathrm{d}t + \sigma_n\left(t, X_t^n\right)\mathrm{d}W_t, \\ X_0^n &= x, \end{cases} \tag{3.8}$$

The associated cost is defined as follows:

$$\mathcal{J}_n(u) = \mathbb{E}\left[\int_0^T \ell\left(t, X_t^n, u_t\right)\mathrm{d}t + g(X_t^n)\right]. \tag{3.9}$$

The subsequent outcome provides estimations that establish the relationship between the original control problem and its perturbed versions.

Lemma 3.6

Let $u \in \mathcal{U}^{\mathcal{G}}$ and X, X^n represent respectively the solution of (3.1) and (3.8) corresponding to u. Then, for any $p \geq 2$, there exist a positive constant C_p, such that the upcoming estimates hold:

a) $\mathbb{E}\sup_{0\leq t\leq T}|X_t - X_t^n|^p \leq C_p(\varepsilon_n)^p$, $\varepsilon_n = \frac{c}{n}$.

b) $|\mathcal{J}_n(u) - \mathcal{J}(u)| \leq C_p\varepsilon_n$.

Proof: The proof of the previous Lemma lies on the condition **(A2)** in Assumptions 1, and the use of Lemma 3.5, Hölder's and Burkholder-Davis-Gundy inequalities. It is rather standard, so we dropped it here. ■

Assume that the control $\hat{u}$ is optimal for the original problem (3.1) and (0.10), that is:

$$\mathcal{J}\left(\hat{u}\right) = \min_{u\in\mathcal{U}^{\mathcal{G}}}\left\{\mathcal{J}(u)\right\}.$$

Here we point out that $\hat{u}$ is not optimal for the new regularized control problem described by (3.8) and (3.9). Indeed, according to Lemma 3.6, there exists a sequence δ_n of positive real numbers converging to 0 such that:

$$\mathcal{J}_n(\hat{u}) = \min_{u\in\mathcal{U}^{\mathcal{G}}} \mathcal{J}_n(u) + \delta_n$$

. Now, we can express the perturbed near-optimal control problem in the following manner.

Problem $\mathbf{A}^{\delta_n}$: For each integer n, we want to find $u \in \mathcal{U}^{\mathcal{G}}$ such that $\hat{u}$ nearly minimizing the cost functional (3.9) subject to (3.8).

Remark 3.7

*As δ_n tends to 0, **Problem $\mathbf{A}^{\delta_n}$** transforms back into **Problem A**.*

Since $\hat{u}$ is near-optimal for (3.8) and (3.9), and the Assumptions in [44] are satisfied according to the Assumptions 1, this ensures the smoothness of the functions; therefore, we can return to the result previously established in [44], using Malliavin calculus. Specifically, the SMP for near optimality is given by the following Proposition.

In what follows we will denote by the processes $\hat{K}$, $\hat{X}$, $\hat{G}$, and $\hat{Y}$ by K, X, G, and Y when they are associated with the optimal control $\hat{u}$.

Proposition 3.8 (*Necessary Condition for near Optimality*)

Assume that $\hat{u}^n \in \mathcal{U}^{\mathcal{G}}$ is near optimal control for the cost $\mathcal{J}_n$. Then, for all bounded $\mathcal{G}_t$-measurable random variable α, we can write:

$$\mathbb{E}\Big[\partial_u \mathcal{H}_n(t, \hat{X}_t^n, \hat{u}_t^n, \hat{Y}^n(t))\alpha\Big] = O(\delta_n), \tag{3.10}$$

alternatively, we can express this as:

$$\mathbb{E}\Big[\partial_u \mathcal{H}_n(t, \hat{X}_t^n, \hat{u}_t^n, \hat{Y}^n(t))\Big|\mathcal{G}_t\Big] = O(\delta_n), \text{ for a.e. } (t,\ \omega), \tag{3.11}$$

where the family of Hamiltonian functions $\mathcal{H}_n : [0,T]\times\mathbb{R}\times U\times\mathbb{R} \to \mathbb{R}$, is defined by:

$$\mathcal{H}_n(t,x,u,Y) = \ell(t,x,u) + Y(t)\ b_n(t,x,u), \tag{3.12}$$

for each integer n.

Proof: Since that $\hat{u}$ is δ_n-optimal for the **Problem $\mathbf{A}^{\delta_n}$**, so by applying Lemma 3.2 to the continuous functional $\mathcal{J}_n$, there exists an admissible control $\hat{u}^n \in \mathcal{U}^{\mathcal{G}}$ such that

i) $\mathcal{J}_n(\hat{u}^n) \le \mathcal{J}_n(u)$, for any $u \in \mathcal{U}^{\mathcal{G}}$,

ii) $d(\hat{u}^n, \hat{u}) \leq (\delta_n)^{\frac{1}{2}}$,

iii) $\hat{u}^n$ minimizes the new cost functional $\mathcal{J}_{n,\delta}(u) := \mathcal{J}_n(u) + (\delta_n)^{\frac{1}{2}} d(u, \hat{u}^n)$

Now, let us define the perturbed controls as follows: for $v^n \in \mathcal{U}^{\mathcal{G}}$ such that v^n is a bounded $\mathcal{G}_t$-adapted process, there exists $\delta > 0$ such that

$$u^{n,\theta} = \hat{u}^n + \theta v^n \in \mathcal{U}^{\mathcal{G}} \text{ for all } \theta \in [-\delta, \delta]. \tag{3.13}$$

The fact that

$$\lim_{\theta \to 0} \frac{1}{\theta} \left(\mathcal{J}_{n,\delta}\left(u^{n,\theta}\right) - \mathcal{J}_{n,\delta}\left(\hat{u}^n\right) \right) = 0,$$

and a simple computation leads to

$$\lim_{\theta \to 0} \frac{1}{\theta} \left(\mathcal{J}_n(u^{n,\theta}) - \mathcal{J}_n\left(\hat{u}^n\right) \right) = O\left(\delta_n\right).$$

We use the same arguments as in [44], for the expression on the left-hand side of the above equation, one can show that, for any $\alpha = \alpha(\omega)$ bounded and $\mathcal{G}_t$-measurable, we have

$$\mathbb{E}[\partial_u \mathcal{H}_n(t, \hat{X}_t^n, \hat{u}_t^n, \hat{Y}^n(t))\alpha] = O\left(\delta_n\right).$$

■

Before proceeding to the subsequent regular sufficient condition for optimality, a pivotal tool in establishing our second main result, it is imperative to introduce the following process

$$\hat{Z}_t^n = \lim_{\theta \to 0} \frac{1}{\theta} \left(X_t^{n,\theta} - \hat{X}_t^n \right)$$

such that $X^{n,\theta}$ the solution associated to the perturbed control defined as follow

$$u^{n,\theta} = \hat{u}^n + \theta v^n$$

where v^n is some bounded $\mathcal{G}_t$-adapted process and $\theta \in [-\delta, \delta]$. Obviously, the process $\hat{Z}^n$ satisfies the following equation

$$\begin{cases} d\hat{Z}_t^n &= \Big[\partial_x b_n(t, \hat{X}_t^n, \hat{u}_t^n)\hat{Z}_t^n + \partial_u b_n(t, \hat{X}_t^n, \hat{u}_t^n)v_t^n \Big] dt \\ & \quad + \partial_x \sigma_n(t, \hat{X}_t^n)\hat{Z}_t^n dW_t, \\ \hat{Z}_0^n &= 0, \end{cases}$$

We also need to define the subsequent quantities. For any t, $h \in (0, T)$ such that $t+h \leq T$, we set

$$\begin{aligned} I_1 &= \mathbb{E}\left[\int_t^T \left\{\hat{K}^n(s)\partial_x b_n(s, \hat{X}_s^n, \hat{u}_s^n)\hat{Z}_s^n + D_s\hat{K}^n(s)\partial_x\sigma_n(s, \hat{X}_s^n)\hat{Z}_s^n\right\} \mathrm{d}s\right], \\ I_2 &= \mathbb{E}\Big[\int_t^{t+h} \left\{\hat{K}^n(s)\partial_u b_n(s, \hat{X}_s^n, \hat{u}_s^n) + D_s\hat{K}^n(s)\partial_u\sigma_n(s, \hat{X}_s^n) + \partial_u\ell(s, \hat{X}_s^n, \hat{u}_s^n)\right\} \alpha \mathrm{d}s\Big]. \end{aligned}$$

Proposition 3.9 (*Sufficient Condition for near Optimality*)

Suppose there exists $\hat{u}^n \in \mathcal{U}^{\mathcal{G}}$ such that the condition (3.11) holds true. Then, $\hat{u}^n$ is near optimal for ***Problem*** $\mathbf{A}^{\delta_n}$.

Proof: Suppose $\hat{u}^n \in \mathcal{U}^{\mathcal{G}}$ satisfies (3.11). Then, reversing the proof's steps of the SMP outlined in Mezerdi *et al.* [44], we can establish the validity of the following inequality:

$$I_1 + I_2 \leq O(\delta_n), \tag{3.14}$$

Then, for all $v_\alpha(s) \in \mathcal{U}^{\mathcal{G}}$ of the form

$$v_\alpha(s, \omega) = \alpha(\omega)\mathcal{X}_{(t,t+h]}(s)$$

For any t, h within the interval $[0,\ T]$ where $t + h \leq T$, and for some bounded $\mathcal{G}_t$-measurable α, we have the following inequality

$$\begin{aligned} 0 \geq \mathbb{E}\Big[\int_0^T &\{\hat{K}^n(t)\{\partial_x b_n(t, \hat{X}_t^n, \hat{u}_t^n)\hat{Z}_t^n + \partial_u b_n(t, \hat{X}_t^n, \hat{u}_t^n)v_t\} \\ &+ D_t\hat{K}^n(t)\partial_x\sigma_n(t, \hat{X}_t^n)\hat{Z}_t^n + \partial_u\ell(t, \hat{X}_t^n, \hat{u}_t^n)v_t\}\mathrm{d}t\Big], \end{aligned} \tag{3.15}$$

is satisfied for all linear combinations of such v_α. Considering that all bounded $v \in \mathcal{U}^{\mathcal{G}}$ can be pointwise approximated by linear combinations bounded in (t, ω), it consequently implies that the inequality (3.15) remains valid for all bounded $v \in \mathcal{U}^{\mathcal{G}}$. Hence, by reversing the steps in the proof of Lemma 4.3 in [44], we deduce that

$$\lim_{\theta \to 0} \frac{1}{\theta}(\mathcal{J}_n(u^{n,\theta}) - \mathcal{J}_n(\hat{u}^n)) \leq O(\delta_n).$$

■

3.3 Stochastic Maximum Principle

This section aims to derive the NCO and the SCO for Problem A mentioned earlier, where the coefficients b and σ are Lipschitz continuous but not necessarily differentiable in x. In the next subsection, we will summarize and prove some approximations that will aid in establishing these conditions.

3.3.1 Some Convergence Results

Lemma 3.10

*For any $p \geq 1$, under **(A1),(A2)** and **(A5)** we have*

(i) $\lim_{n\to+\infty} \mathbb{E}(\int_0^T |\partial_x b_n(s, \hat{X}_s^n, \hat{u}_s^n) - \partial_x b(s, \hat{X}_s, \hat{u}_s)|^p \mathrm{d}s) = 0$, *and* $\lim_{n\to+\infty} \mathbb{E} \int_0^T |\partial_x \sigma_{j,n}(s, \hat{X}_s^n) - \partial_x \sigma_j(s, \hat{X}_s)|^p \mathrm{d}s = 0$

(ii) $\lim_{n\to+\infty} \mathbb{E}|\partial_u b_n(s, \hat{X}_s^n, \hat{u}_s^n) - \partial_u b(s, \hat{X}_s, \hat{u}_s)|^p = 0.$

Proof: Firstly, we prove the assertion **(i),** we give the proof for b, and the same method is applied for σ. It should be noted that in (i), it is possible to substitute $\hat{u}_s^n$ with $\hat{u}_s$, due to the fact that $d(\hat{u}_s^n, \hat{u}_s)$ tends to 0 as n approaches to $+\infty$. Let $n_0 \geq 1$, then:

$$\begin{aligned}
&\lim_{n\to+\infty} \mathbb{E}\left(\int_0^T |\partial_x b_n(s, \hat{X}_s^n, \hat{u}_s^n) - \partial_x b(s, \hat{X}_s, \hat{u}_s)|^p \mathrm{d}s\right) \\
\leq\ & \overline{\lim}_{n\to+\infty} C_p \left\{ \mathbb{E}\left(\int_0^T |\partial_x b_n(s, \hat{X}_s^n, \hat{u}_s) - \partial_x b_{n_0}(s, \hat{X}_s^n, \hat{u}_s)|^p \mathrm{d}s\right)\right. \\
&+ \mathbb{E}\left(\int_0^T |\partial_x b_{n_0}(s, \hat{X}_s^n, \hat{u}_s) - \partial_x b_{n_0}(s, \hat{X}_s, \hat{u}_s)|^p \mathrm{d}s\right) \\
&\left. + \mathbb{E}\left(\int_0^T |\partial_x b_{n_0}(s, \hat{X}_s^n, \hat{u}_s) - \partial_x b(s, \hat{X}_s, \hat{u}_s)|^p \mathrm{d}s\right)\right\} \\
=\ & C_p \left(J_1^n + J_2^n + J_3^n\right).
\end{aligned}$$

Consider a continuous function $w(t,x)$, which satisfies that $w(t,x) = 0$ for $t^2 + |x|^2 \geq 1$, $w(0,0) = 1$. Then, for $N > 0$

$$\begin{aligned}
\overline{\lim}_{n\to+\infty} J_1^n \ \leq\ & N C_p \left\{ \mathbb{E}\left(\int_0^T \left(1 - w\left(\frac{t}{R}, \frac{\hat{X}_t}{R}\right)\right) \mathrm{d}t\right)\right. \\
&\left. + \overline{\lim}_{n\to+\infty} \mathbb{E}\left(\int_0^T w\left(\frac{t}{R}, \frac{\hat{X}_t^n}{R}\right) \sup_{a\in U} |\partial_x b_n(t, \hat{X}_t^n, a) - \partial_x b_{n_0}(t, \hat{X}_t^n, a)|^p \mathrm{d}t\right)\right\}.
\end{aligned}$$

Then, according to Krylov's inequality, we have:

$$\begin{aligned}
\overline{\lim}_{n\to+\infty} J_1^n \ \leq\ & N C_p \left\{ \mathbb{E}\left(\int_0^T \left(1 - w\left(\frac{t}{R}, \frac{\hat{X}_t}{R}\right)\right) \mathrm{d}t\right)\right. \\
&\left. + N \overline{\lim}_{n\to+\infty} \left| \sup_{a\in U} |\partial_x b_n(t,x,a) - \partial_x b_{n_0}(t,x,a)|^p \right|_{d+1,R} \right\}.
\end{aligned}$$

where $|\cdot|_{d+1,R}$ denotes the norm in $L^{d+1}([0,\ T]\times B(0,R))$. Based on Lemma 3.5, the final expression on the right-hand side approaches 0 as both n_0 and n tend to infinity. Additionally, as R tends to infinity, we find that $\overline{\lim}_{n\to+\infty}J_1^n = 0$. Similarly, by estimating J_3^n, we also have $\overline{\lim}_{n\to+\infty}J_3^n = 0$.

Finally, $\lim_{n\to+\infty} J_2^n = 0$, due to the continuity of $\partial_x b_{n_0}(s,x,a)$ in (x,a) and the uniform convergence in probability of $\hat{X}_t^n$ to $\hat{X}_t$. Hence

$$\lim_{n\to+\infty}\mathbb{E}\left(\int_0^T |\partial_x b_n(s,\hat{X}_s^n,\hat{u}_s^n)-\partial_x b(s,\hat{X}_s,\hat{u}_s)|^p \mathrm{d}s\right)=0.$$

Secondly, we return to **(ii)**, we have

$$\begin{aligned}\mathbb{E}|\partial_u b_n(t,\hat{X}_t^n,\hat{u}_t^n)-\partial_u b(t,\hat{X}_t,\hat{u}_t)|^p \ \leq\ & C\mathbb{E}|\partial_u b_n(t,\hat{X}_t^n,\hat{u}_t^n)-\partial_u b(t,\hat{X}_t^n,\hat{u}_t^n)|^p \\ & +C\mathbb{E}|\partial_u b(t,\hat{X}_t^n,\hat{u}_t^n)-\partial_u b(t,\hat{X}_t,\hat{u}_t^n)|^p \\ & +C\mathbb{E}|\partial_u b(t,\hat{X}_t,\hat{u}_t^n)-\partial_u b(t,\hat{X}_t,\hat{u}_t)|^p,\end{aligned} \tag{3.16}$$

According to Lemma 3.5, the first term converges to 0. Utilizing the fact that $\partial_u b$ is Lipschitz in x and u, $d(\hat{u}^n,\hat{u})$ converges to 0 and Lemma 3.3, we show that the second and third terms in the right-hand side of (3.16) converge to 0. ■

Lemma 3.11

*If **(A2)** is satisfied then, for any $p\geq 2$, we have*

(i)

$$\mathbb{E}\sup_{0\leq t\leq s\leq T}\left|\hat{G}_n(t,s)-\hat{G}(t,s)\right|^p \underset{n\to+\infty}{\to} 0$$

(ii)

$$\mathbb{E}\sup_{0\leq r\leq t\leq T}|D_r\hat{X}_t^n - D_r\hat{X}_t|^p \underset{n\to+\infty}{\to} 0.$$

Proof (i) : : Since $|e^x-e^y|\leq(e^x+e^y)|x-y|$, by Cauchy-Schwartz, Burkholder-Davis-Gundy inequalities, $\mathbb{E}\sup_{0\leq s\leq t\leq T}|\hat{G}(t,s)|^p$ is finite, the fact that $(a^2-b^2)=(a-b)(a+b)$, $\partial_x\sigma_n$

and $\partial_x\sigma$ are bounded functions. Thanks to Lemma 3.10, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le s\le T}\left|\hat{G}_n(t,s)-\hat{G}(t,s)\right|^p\\
\le\ & C_p\mathbb{E}\int_0^T|\partial_x b_n(r,\hat{X}_r^n,\hat{u}_r^n)-\partial_x b(r,\hat{X}_r,\hat{u}_r)|^{2p}\mathrm{d}r\\
&+C_p\mathbb{E}\int_0^T\left(|\partial_x\sigma_n(r,\hat{X}_r^n)|^{2p}+|\partial_x\sigma(r,\hat{X}_r)|^{2p}\right)|(\partial_x\sigma_n(r,\hat{X}_r^n)-\partial_x\sigma(r,\hat{X}_r))|^{2p}\mathrm{d}r\\
&+C_p\mathbb{E}\Big(\int_0^T|\partial_x\sigma_n(r,\hat{X}_r^n)-\partial_x\sigma(r,\hat{X}_r)|^2\mathrm{d}r\Big)^{\frac{p}{2}}\\
&\underset{n\to+\infty}{\to}0.
\end{aligned}$$

■

Proof (ii) : : We know that, the Malliavin derivative $\{(D_r\hat{X}_t)\}_{0\le r\le t\le T}$ obeys the subsequent linear stochastic differential equation

$$D_r\hat{X}_t=\hat{\sigma}(r,\hat{X}_r)+\int_r^t\partial_x\hat{\sigma}(s,\hat{X}_s)D_r\hat{X}_s\mathrm{d}W_s+\int_r^t\partial_x\hat{b}(s,\hat{X}_s,\hat{u}_s)D_r\hat{X}_s\mathrm{d}s.$$

Therefore, according to Itô's formula,

$$D_r\hat{X}_t=\sigma(r,\hat{X}_r)\exp\left\{\int_r^t\partial_x\sigma(s,\hat{X}_s)\mathrm{d}W_s+\int_r^t\partial_x b(s,\hat{X}_s,\hat{u}_s)-\frac{1}{2}\partial_x\sigma(s,\hat{X}_s)\mathrm{d}s\right\},$$

By the same steps as the previous assertion (i), we obtain

$$|D_r\hat{X}_t^n-D_r\hat{X}_t|\ \le\ |\sigma_n(r,\hat{X}_r^n)||\hat{G}_n(r,t)-\hat{G}(r,t)|+|\hat{G}(r,t)||\sigma_n(r,\hat{X}_r^n)-\sigma(r,\hat{X}_r)|.$$

Thanks to assertion (i), $\mathbb{E}\sup_{0\le r\le t\le T}|\hat{G}(r,t)|^p<\infty$ and σ_n bounded function and Lemma 3.6, we successfully attain the desired result. ■

Lemma 3.12

*Under **(A2)** and **(A3)**, for any $p\ge 2$, we have*

(i)

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\hat{K}^n(t)-\hat{K}(t)|^p\Big]\to 0,\ \ n\to+\infty.$$

(ii)

$$\mathbb{E}\Big[\sup_{0\le s\le t\le T}|D_s\hat{K}^n(t)-D_s\hat{K}(t)|^p\Big]\to 0, n\to+\infty.$$

(iii)

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\hat{Y}^n(t)-\hat{Y}(t)|^p\Big]\underset{n\to+\infty}{\to}0, n\to+\infty.$$

Proof (i) : Given that $\partial_x g$ is Lipschitz continuous at x, and applying Hölder's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|\hat{K}^n(t)-\hat{K}(t)|^p\right]\\
\le\ & C_p\left(\mathbb{E}|\partial_x g(\hat{X}_T^n)-\partial_x g(\hat{X}_T)|^p+\int_0^T\mathbb{E}|\partial_x\ell(s,\hat{X}_s^n,\hat{u}_s^n)-\partial_x\ell(s,\hat{X}_s,\hat{u}_s)|^p\mathrm{d}s\right)\\
\le\ & C_p\left(\mathbb{E}\sup_{0\le t\le T}|\hat{X}_t^n-\hat{X}_t|^p+\int_0^T\mathbb{E}|\partial_x\ell(s,\hat{X}_s^n,\hat{u}_s^n)-\partial_x\ell(s,\hat{X}_s,\hat{u}_s)|^p\mathrm{d}s\right).
\end{aligned}$$

Thanks to the continuity of $\partial_x\ell$, Lemma 3.6 as well as the convergence of $d(\hat{u}^n,\hat{u})$ to 0 as n goes to $+\infty$, we deduce

$$\lim_{n\to+\infty}\mathbb{E}\sup_{0\le t\le T}|\hat{K}^n(t)-\hat{K}(t)|^p=0,$$

■

Proof (ii) : We have

$$D_t\hat{K}(t)=\partial_{xx}g(\hat{X}_T)D_t\hat{X}_T+\int_t^T\partial_{xx}\ell(s,\hat{X}_s,\hat{u}_s)D_t\hat{X}_s\mathrm{d}s,$$

and for each integer n,

$$D_t\hat{K}^n(t)=\partial_{xx}g(\hat{X}_T^n)D_t\hat{X}_T^n+\int_t^T\partial_{xx}\ell(s,\hat{X}_s^n,\hat{u}_s^n)D_t\hat{X}_s^n\mathrm{d}s,$$

then,

$$\begin{aligned}
|D_t\hat{K}^n(t)-D_t\hat{K}(t)| \le\ & |\partial_{xx}g(\hat{X}_T^n)||D_t\hat{X}_T-D_t\hat{X}_T|\\
&+|\partial_{xx}g(\hat{X}_T^n)-\partial_{xx}g(\hat{X}_T)||D_t\hat{X}_T|\\
&+\int_t^T|\partial_{xx}\ell(s,\hat{X}_s^n,\hat{u}^n)||D_t\hat{X}_s^n-D_t\hat{X}_s|\mathrm{d}s\\
&+\int_t^T|\partial_{xx}\ell(s,\hat{X}_s^n,\hat{u}^n)-\partial_{xx}\ell(s,\hat{X}_s,\hat{u})||D_t\hat{X}_s|\mathrm{d}s.
\end{aligned}$$

Employing the convexity property of the mapping $a\mapsto|x|^p$, for $p>1$ and Cauchy-Schwartz inequality, $\mathbb{E}\sup_{0\le t\le s\le T}|D_t\hat{X}_s|^p$ is finite, Lemma 3.11 and Lemma 3.6, we get the desired result. ■

Proof (iii) : By employing classical estimates once more, along with Hölder's inequality, we deduce

$$\begin{aligned}
\mathbb{E}\sup_{t\le T}|\hat{Y}^n(t)-\hat{Y}(t)|^p \le\ & C\Big(\mathbb{E}\sup_{t\le T}|\hat{K}^n(t)-\hat{K}(t)|^p\\
&+\mathbb{E}\sup_{t\le T}\int_t^T|\partial_x\mathcal{H}_0^n(r,\hat{X}_r^n,\hat{u}_r^n)\hat{G}_n(t,r)-\partial_x\mathcal{H}_0(r,\hat{X}_r,\hat{u}_r)\hat{G}(t,r)|^p\mathrm{d}r\Big).
\end{aligned} \tag{3.17}$$

We start by estimating the second term in (3.17), indeed

$$\begin{aligned}
&\int_t^T |\partial_x \mathcal{H}_0^n(r,\hat{X}_r^n,\hat{u}_r^n)\hat{G}_n(t,r)-\partial_x\mathcal{H}_0(r,\hat{X}_r,\hat{u}_r)\hat{G}(t,r)|^p\mathrm{d}r \qquad (3.18)\\
\leq\ & \int_t^T |\partial_x b_n(r,\hat{X}_r^n,\hat{u}_r^n)-\partial_x b(r,\hat{X}_r,\hat{u}_r)|^p|\hat{K}^n(r)\hat{G}_n(t,r)|^p\mathrm{d}r\\
&+\int_t^T |\partial_x\sigma_n(r,\hat{X}_r^n)-\partial_x\sigma(r,\hat{X}_r)|^p|D_r\hat{K}^n(r)\hat{G}_n(t,r)|^p\mathrm{d}r\\
&+\int_t^T |\partial_x b(r,\hat{X}_r,\hat{u}_r)|^p|\hat{G}_n(t,r)|^p|\hat{K}^n(r)-\hat{K}(r)|^p\mathrm{d}r\\
&+\int_t^T |\partial_x\sigma(r,\hat{X}_r^n)|^p|\hat{G}_n(t,r)|^p|D_r\hat{K}^n(r)-D_r\hat{K}(r)|^p\mathrm{d}r\\
&+\int_t^T |\hat{K}(r)\partial_x b(r,\hat{X}_r,\hat{u}_r)|^p|\hat{G}_n(t,r)-\hat{G}(t,r)|^p\mathrm{d}r\\
&+\int_t^T |\partial_x\sigma(r,\hat{X}_r^n)|^p|D_r\hat{K}(r)|^p|\hat{G}_n(t,r)-\hat{G}(t,r)|^p\mathrm{d}r.
\end{aligned}$$

By Cauchy-Schwartz inequality and the fact that $\hat{K}^n$ is bounded,

$$\mathbb{E}\sup_{0\leq t\leq r\leq T}|\hat{G}(t,r)|^p<\infty \quad\text{and}\quad \mathbb{E}\int_0^T|D_r\hat{K}^n(r)|^p\mathrm{d}r<\infty$$

and thanks to Lemma 3.10, the first and the second terms of the right-hand-side of (3.18) converge to 0, when n goes to $+\infty$. For the third and the fourth terms, we use the boundedness of $\partial_x\sigma$ and $\partial_x b$, the fact that $\mathbb{E}\sup_{0\leq t\leq r\leq T}|\hat{G}(t,r)|^p$ if finite and Lemma 3.12. Finally the two last terms tends to 0, according to Lemma 3.11, keeping in mind that $\hat{K}$, $\partial_x\sigma$ and $\partial_x b$ are bounded. ■

Lemma 3.13

*Under **(A2)** in Assumptions 1, we have: $\mathbb{E}|\hat{Y}_t|^p$ is finite.*

Proof: Given the boundedness of $\hat{K}(t)$, the finiteness of $\mathbb{E}\sup_{0\leq t\leq r\leq T}|\hat{G}(t,r)|^{2p}$ and Hölder's inequality implies

$$\begin{aligned}
\mathbb{E}|\hat{Y}_t|^p &\leq \mathbb{E}|\hat{K}_t|^p+\mathbb{E}\Big|\int_t^T\partial_x\mathcal{H}_0(r,\hat{X}_r,\hat{u}_r)\hat{G}(t,r)\mathrm{d}r\Big|^p \qquad (3.19)\\
&\leq C_p\Big(1+\Big(\mathbb{E}\int_t^T|\partial_x\mathcal{H}_0(r,\hat{X}_r,\hat{u}_r)|^{2p}\mathrm{d}r\Big)^{\frac{1}{2}}\Big)<\infty.
\end{aligned}$$

It is noteworthy that $\mathbb{E}\int_t^T|\partial_x\mathcal{H}_0(r,\hat{X}_r,\hat{u}_r)|^{2p}\mathrm{d}r$ is finite due to the boundedness of $\hat{K}(t)$, b_x and σ_x and along with $\mathbb{E}\int_t^T|D_r\hat{K}(r)|^{2p}\mathrm{d}r$ is finite. ■

3.3.2 Optimal Variational Principle

This subsection explores the fundamental results regarding the necessary and sufficient optimality conditions for controlled SDE with irregular coefficients and a non-degenerate diffusion matrix.

Necessary Condition for Optimality

Theorem 3.14 (***The SMP***)

Suppose $\hat{u}$ represents an admissible control that minimizes the cost functional $\mathcal{J}$ over $\mathcal{U}^{\mathcal{G}}$. Then, we obtain:

$$\mathbb{E}\Big[\partial_u \mathcal{H}(t, \hat{X}_t, \hat{u}_t, \hat{Y}(t))\Big|\mathcal{G}_t\Big] = 0, \text{ for a.e. } (t,\ \omega), \tag{3.20}$$

where

$$\mathcal{H}(t, \hat{X}_t, \hat{u}_t, \hat{Y}(t)) = \ell(t, \hat{X}_t, \hat{u}_t) + \hat{Y}(t)\ b(t, \hat{X}_t, \hat{u}_t),$$

is the usual Hamiltonian with $\hat{Y}$ defined by (3.3).

Proof: Since $\hat{u}^n$ is δ_n-optimal for **Problem** $\mathbf{A}^{\delta_n}$, then by invoking Proposition 3.8, we get

$$\mathbb{E}[\partial_u \mathcal{H}_n(t, \hat{X}_t^n, \hat{u}_t^n, \hat{Y}^n(t))\alpha] = O(\delta_n).$$

The desired result will be obtained by passing to the limits when n goes to ∞. Indeed, by Cauchy-Schwartz inequality and keeping in mind that α is bounded, we obtain

$$\begin{aligned} & \mathbb{E}\Big[\Big|\Big(\partial_u \mathcal{H}_n(t, \hat{X}_t^n, \hat{u}_t^n, \hat{Y}^n(t)) - \partial_u \mathcal{H}(t, \hat{X}_t, \hat{u}_t, \hat{Y}(t))\Big)\alpha\Big|\Big] \\ \leq\ & C\Big(\mathbb{E}\Big[|\partial_u \ell(t, \hat{X}_t^n, \hat{u}_t^n) - \partial_u \ell(t, \hat{X}_t, \hat{u}_t)|\Big] \\ & + \Big(\mathbb{E}|\hat{Y}^n(t) - \hat{Y}(t)|^2\Big)^{\frac{1}{2}} \Big(\mathbb{E}|\partial_u b_n(t, \hat{X}_t^n, \hat{u}_t^n)|^2\Big)^{\frac{1}{2}} \\ & + \Big(\mathbb{E}|\hat{Y}(t)|^2\Big)^{\frac{1}{2}} \Big(\mathbb{E}|\partial_u b_n(t, \hat{X}_t^n, \hat{u}_t^n) - \partial_u b(t, \hat{X}_t, \hat{u}_t)|^2\Big)^{\frac{1}{2}}\Big). \end{aligned} \tag{3.21}$$

Since $\partial_u \ell(t,\cdot,\cdot)$ is continuous, $\hat{X}_t^n$ uniformly converges in probability to $\hat{X}_t$ and since $d(\hat{u}^n, \hat{u})$ converges to 0 as n approaches infinity it follows that the first term in the right-hand side of (3.21) converge to 0. By using Lemma 3.12 and the fact that $\partial_u b_n$ are square-integrable, the second term tends toward zero. Ultimately, using Lemmas 3.10 and 3.13 makes it evident that the third term approaches zero. Then, by (3.21) we get

$$\mathbb{E}[\partial_u \mathcal{H}(t, \hat{X}_t, \hat{u}_t, \hat{Y}(t))\alpha] = 0. \tag{3.22}$$

Since this is valid for any bounded $\mathcal{G}_t$-measurable random variable α, it follows that

$$\mathbb{E}[\partial_u \mathcal{H}(t, \hat{X}_t, \hat{u}_t, \hat{Y}(t)) \mid \mathcal{G}_t] = 0, \text{ for a.e. } (t, \ \omega).$$

This ends the proof. ■

Sufficient Optimality Condition

The following Lemmas are required to establish sufficient optimality conditions.

Lemma 3.15

*Under **(H1)** in Assumptions 1, we have:*

i) $\mathbb{E}|b_n(t, \hat{X}_t^n, u_t) - b(t, \hat{X}_t, u_t)|^2 \leq O(\delta_n)$,

ii) $\mathbb{E}|\ell \ (t, \hat{X}_t^n, u_t) - \ell(t, \hat{X}_t, u_t)|^2 \leq O(\delta_n)$.

Proof: Firstly, we prove **(i)**, we have

$$\begin{aligned} \mathbb{E}\left|b_n(t, \hat{X}_t^n, u_t) - b(t, \hat{X}_t, u_t)\right|^2 &\leq C\mathbb{E}\left|b_n(t, \hat{X}_t^n, u_t) - b(t, \hat{X}_t^n, u_t)\right|^2 \\ &\quad +C\mathbb{E}\left|b(t, \hat{X}_t^n, u_t) - b(t, \hat{X}_t, u_t)\right|^2 \\ &\leq O\left(\delta_n\right). \end{aligned} \tag{3.23}$$

Thanks to Lemma 3.5, we have $|(b_n - b)(t, \hat{X}_t^n, u_t)| \leq \frac{C}{n} = \varepsilon_n$, the Lipschitz continuity of b in x and Lemma 3.6, the second term on the right-hand-side of (3.23) are bounded by $O(\delta_n)$.

The proof of the assertion **(ii)** relies on Assumption **(A2)** and Lemma 3.6. ■

Lemma 3.16

For each integer n, we have

i) $\lim_{n\to+\infty} \mathcal{J}_n(\hat{u}^n) = \mathcal{J}(\hat{u})$

ii) $\lim_{n\to+\infty} \mathcal{J}_n(u) = \mathcal{J}(u)$

Proof: For the first assertion i). Notice that g and ℓ are Lipschitz functions in x and u. Then, by invoking Lemma 3.6, the definition of the distance d with $p = 2$, and the assertion (ii) in the proof of Proposition 3.8, one can easily check that

$$\begin{aligned}
&\lim_{n\to+\infty} \mathbb{E}|\mathcal{J}_n(\hat{u}_t^n) - \mathcal{J}(\hat{u}_t)|^2 \\
\leq\ & \lim_{n\to+\infty} \left(\mathbb{E}\int_0^T |\ell(t,\hat{X}_t^n,\hat{u}_t^n) - \ell(t,\hat{X}_t,\hat{u}_t)|^2 \mathrm{d}t + \mathbb{E}\left|g(\hat{X}_T^n) - g(\hat{X}_T)\right|^2\right) \\
\leq\ & \lim_{n\to+\infty} C\left(d^2(\hat{u}_t^n,\hat{u}_t) + \mathbb{E}\sup_{0\leq t\leq T}|\hat{X}_t^n - \hat{X}_t|^2\right) \ = \ 0.
\end{aligned}$$

For the second assertion ii), for any $u(\cdot)$ an admissible control, by the Lemma 3.6, we have

$$|\mathcal{J}_n(u) - \mathcal{J}(u)| \leq C_p\, \varepsilon_n, \quad \varepsilon_n = \frac{C}{n}$$

where M and C are positive constants, then

$$\lim_{n\to+\infty} \mathbb{E}|\mathcal{J}_n(u) - \mathcal{J}(u)|^2 = 0.$$

■

Theorem 3.17 (*Sufficient Optimality Condition*)

*Assume that $\mathcal{H}$ is convex function in u, if there exists $\hat{u} \in \mathcal{U}^{\mathcal{G}}$ is a critical point such that the necessary condition of optimality (3.20) holds. Then, the control $\hat{u}$ is optimal for the **Problem A**.*

Proof: Assume that $\hat{u}$ minimize the Hamiltonian function, that is ,

$$\mathcal{H}\left(t,\, \hat{X}_t,\, \hat{u},\, \hat{Y}(t)\right) = \min_{u\in\mathcal{U}^{\mathcal{G}}} \mathcal{H}\left(t,\, \hat{X}_t,\, u,\, \hat{Y}(t)\right),$$

Obviously, $\hat{u}$ satisfies the necessary condition for optimality (3.20).

Note that $\hat{u}$ does not necessarily satisfy the necessary condition for optimality for the perturbed control problem (3.8) and (3.9). For each $A \in \mathcal{G}_t$, $n \in \mathbb{N}$, set $\mathcal{I}_n(\hat{u}) = \mathbb{E}[\mathcal{H}_n(t,\hat{X}_t^n,\hat{u}_t,\hat{Y}_t^n)\mathcal{X}_A]$, then a simple computation, taking account of Lemma 3.12 and 3.15, leads to

$$\mathcal{I}_n(\hat{u}) = \min_{u\in\mathcal{U}^{\mathcal{G}}} \mathcal{I}_n(u) + O\left(\delta_n\right).$$

where $O\left(\delta_n\right)$ is a sequence of positive real numbers that converges to 0. Now, applying Ekeland's variational principle Lemma 3.2 to $\mathcal{I}_n$ there exists an admissible control $\hat{u}^n$ such that

i) $\mathcal{I}_{n,\delta}(\hat{u}_t^n) \leq \mathcal{I}_{n,\delta}(u)$, for any $u \in \mathcal{U}^{\mathcal{G}}$,

ii) $d(\hat{u}^n,\hat{u}) \leq \sqrt{\delta_n}$,

iii) $\hat{u}^n$ minimizes the function $\mathcal{I}_{n,\delta}(u) := \mathcal{I}_n(u) + \sqrt{\delta_n} d(u, \hat{u}^n)$.

Since $\mathcal{I}_{n,\delta}$ is a convex function in u, we conclude that

$$\partial_u \mathcal{I}_{n,\delta}(\hat{u}_t^n) = 0,$$

The use of the perturbation (3.13) and the definition of the derivative leads to,

$$\mathbb{E}\left[\partial_u \mathcal{H}_n(t, \hat{X}_t^n, \hat{u}_t^n, \hat{Y}^n(t))\mathcal{X}_A\right] = O\left(\sqrt{\delta_n}\right)$$

Since the last equality holds true for each $A \in \mathcal{G}_t$, we get

$$\mathbb{E}[\partial_u \mathcal{H}_n(t, \hat{X}_t^n, \hat{u}_t^n, \hat{Y}^n(t))|\mathcal{G}_t] = O\left(\sqrt{\delta_n}\right), \text{ for a.e. } (t,\ \omega), \tag{3.24}$$

Then, by using Proposition 3.9, we get

$$\mathcal{J}_n(\hat{u}^n) \leq \mathcal{J}_n(u) + O(\delta_n)$$

Now, passing to the limit and according to Lemma 3.16, we get

$$\mathcal{J}(\hat{u}) = \min_{u \in \mathcal{U}^{\mathcal{G}}} \mathcal{J}(u),$$

this means that $\hat{u}$ is an optimal control for the cost function $\mathcal{J}$. ■

3.4 Application to Quadratic SDE

In this section, we shall study a control problem associated with a stochastic differential equation exhibiting an irregular drift term and may have a quadratic term. Clearly, this corresponds to a specific form of an irregular stochastic control problem. More precisely, we consider the SDE

$$\begin{cases} \mathrm{d}X_t &= b(t, X_t, u_t)\mathrm{d}t + f(X_t)\sigma^2(X_t)\,\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \\ X_0 &= x \in \mathbb{R}. \end{cases} \tag{3.25}$$

Here u stands for an $\mathbb{F}$-adapted control process enjoying suitable properties and the bounded function f is supposed to be an integrable function over the whole real line.

The corresponding cost functional to (3.25) given by:

$$\mathcal{J}(u) = \mathbb{E}\left[g(X_T)\right]. \tag{3.26}$$

Note that the coefficients b, σ, and g are given maps defined as in Section 2. Furthermore, we assume that they satisfy Assumption 1. Hence, according to Theorem 4.5 in [24], the equation (3.25) enjoys the uniqueness and existence property.

We aim to derive the NCO and SCO for the irregular control problem (3.25). The main tool is to find a space transformation that eliminates the singular and quadratic part $f(X_t)\sigma^2(X_t)$. It turns out the function Ψ defined, for every $x \in \mathbb{R}$, by

$$\Psi(x) = \int_0^x \exp\left(-2\int_0^y f(t)\mathrm{d}t\right)\mathrm{d}y. \tag{3.27}$$

plays this role. In fact the function Ψ is the solution of the second order differential equation $\Psi''(x) + 2f(x)\Psi'(x) = 0$, for a.e. x vanishing at 0. Moreover, it satisfies the following properties

(i) Ψ is a one-to-one function. Both Ψ and its inverse Ψ^{-1} belong to $\mathcal{W}_1^2(\mathbb{R})$ where $\mathcal{W}_1^2(\mathbb{R})$ the space of continuous function g defined on $\mathbb{R}$ such that both g and its generalized derivatives g_l' and g_l'' are locally integrable on $\mathbb{R}$.

(ii) There exists a constant c_f depending only on f such that for any x, $y \in \mathbb{R}$

$$e^{-2c_f}|x-y| \leq |\Psi(x) - \Psi(y)| \leq e^{2c_f}|x-y|,$$

$$e^{-2c_f}|x-y| \leq |\Psi^{-1}(x) - \Psi^{-1}(y)| \leq e^{2c_f}|x-y|.$$

We introduce the following process

$$\begin{aligned}
K_t :&= \partial_x g(X_T),\\
D_t K_t :&= D_t \partial_x g(X_T),\\
\mathcal{H}_0(s, X_s, u_s) :&= K(s)\Psi'(X_s)\, b(s, X_s, u_s) + D_s K(s)\Psi'(X_s)\,\sigma(X_s),\\
G(t,s) :&= \exp\Big(\int_t^s \Big\{\partial_x(\Psi'(X_r)b(r, X_r, u_r)) - \frac{1}{2}\Big(\partial_x(\Psi'(X_r)\sigma(X_r))\Big)^2\Big\}\mathrm{d}r\\
&\qquad + \int_t^s \partial_x(\Psi'(X_r)\sigma(X_r))\mathrm{d}W_r\Big),
\end{aligned}$$

and

$$Y_t := K_t + \int_t^T \partial_x \mathcal{H}_0(s, X_s, u_s)G(t,s)\mathrm{d}s. \tag{3.28}$$

3.4.1 Necessary and Sufficient Conditions for Optimality

From the necessary and sufficient condition of optimality for SDE (3.31), we deduce the optimality necessary and sufficient condition for the Q-SDE (3.25).

Theorem 3.18

Assume that Assumption 1 is in force.

(i) If $\hat{u}$ is a minimizer of the cost functional $\mathcal{J}(u)$, then, we have:

$$\mathbb{E}\Big[\partial_u \mathcal{H}(t, \hat{X}_t, \hat{u}_t, \hat{Y}(t))\Big|\mathcal{G}_t\Big] \;=\; 0, \text{ for a.e. } (t,\ \omega), \tag{3.29}$$

where

$$\mathcal{H}(t, \hat{X}_t, \hat{u}_t, \hat{Y}(t)) \;= \hat{Y}(t)\ \Psi'(\hat{X}_t) b(t, \hat{X}_t, \hat{u}_t),$$

such that $\hat{Y}$ is the optimal solution of equation (3.28).

(ii) Conversely, if $\hat{u}$ satisfies (3.29) and $\mathcal{H}$ is convex function in u, then $\hat{u}$ is optimal for the cost (3.26).

Proof: First, we prove (i). Let X be a solution to (3.25), applying Itô-Krylov's formula to $\Psi(X_t)$ leads to

$$\Psi(X_t) = \Psi(x) + \int_0^t \Psi'(X_s) b(s, X_s, u_s) \mathrm{d}s + \int_0^t \Psi'(X_s)\sigma(X_s)\mathrm{d}W_s, \tag{3.30}$$

For any $0 \le s \le T$, we set $x_s = \Psi(X_s)$ and define the new coefficients b_1 and σ_1 by:

$$b_1(s, x, u) = \Psi'(\Psi^{-1}(x)) b(s, \Psi^{-1}(x), u) \;\text{ and }\; \sigma_1(x) = \Psi'(\Psi^{-1}(x))\sigma(\Psi^{-1}(x)).$$

By these notations the SDE (3.30), becomes

$$x_t = x + \int_0^t b_1(s, x_s, u_s)\mathrm{d}s + \int_0^t \sigma_1(x_s)\mathrm{d}W_s. \tag{3.31}$$

The cost corresponding to (3.31) is given by:

$$\mathcal{J}_1(u) = \mathbb{E}\left[g_1(X_T)\right] = \mathbb{E}\left[g\left(\Psi^{-1}(x_T)\right)\right].$$

Obviously, if $\hat{u} \in \mathcal{U}^{\mathcal{G}}$ minimizes the cost $\mathcal{J}$ over $\mathcal{U}^{\mathcal{G}}$, it minimizes the cost $\mathcal{J}_1$. Note that if the coefficients b_1, σ_1 and g_1 satisfy all the conditions of Assumption 1, then, by invoking Theorem 3.14, there exist the following processes,

$$\begin{aligned}
K_1(t):&=\partial_x g_1(x_t) = \partial_x g\left(\Psi^{-1}(x_t)\right)\\
D_tK_1(t):&=D_t\partial_x g_1(x_t)\\
\mathcal{H}_0^1(s,x,u):&=K_1(s)b_1(s,x,u)+D_sK_1(s)\sigma_1(x),\\
\partial_x\mathcal{H}_0^1(s,x,u):&=K_1(s)\partial_x b_1(s,x,u)+D_sK_1(s)\partial_x\sigma_1(x),\\
G_1(t,s):&=\exp\left(\int_t^s\{\partial_x b_1(r,x_r,u_r)-\frac{1}{2}\left(\partial_x\sigma_1\right)^2(x_r)\}\mathrm{d}r+\int_t^s\partial_x(\sigma_1(x_r))\mathrm{d}W_r\right),
\end{aligned}$$

and

$$Y_1(t):=K_1(t)+\int_t^T\partial_x\mathcal{H}_0^1(s,x_s,u_s)G_1(t,s)\mathrm{d}s.$$

such that, for a. e (t,ω) , we have:

$$\mathbb{E}[\partial_u\mathcal{H}^1\left(t,\hat{x}_t,\hat{u}_t,\hat{Y}_1(t)\right)|\mathcal{G}_t]=0, \tag{3.32}$$

where

$$\begin{aligned}
\mathcal{H}^1(t,\hat{x}_t,\hat{u}_t,\hat{Y}_1(t)) &= \hat{Y}_1(t)b_1(t,\hat{x}_t,\hat{u}_t)\\
&= \hat{Y}(t)\Psi'(\hat{X}_t)b(t,\hat{X}_t,\hat{u}_t)\\
&= \mathcal{H}(t,\hat{X}_t,\hat{u}_t,\hat{Y}(t)).
\end{aligned}$$

This implies the NCO (3.29) holds.

Now, we proceed to prove (ii). It is quite clear that if $\hat{u}\in\mathcal{U}^{\mathcal{G}}$ verifies the NCO (3.29), then (3.32) also is satisfied. Hence, Theorem 3.17 shows that $\hat{u}$ is optimal for $\mathcal{J}_1$. Consequently, it is optimal for $\mathcal{J}$. ■

Conclusion and Further Questions

In the first part of this thesis, we have provided new results for the Malliavin smoothness of the solution of Quadratic BSDE and the analysis of their numerical schemes, deepening and extending existing literature.

In Chapter 1, we have extended the results in Hu *et al.* [34] to Q-BSDE, whose generator has the following form

$$h(r, y) + h_1(r)z + f(y)z^2,$$

where h is a given bounded and global Lipschitz function in y, h_1 is bounded deterministic function. We have provided an $\mathbb{L}^q$ ($q \geq 2$)-existence and uniqueness theorem along with the $\mathbb{L}^p$-Hölder continuity for the solutions of such Q-BSDEs systems. in Chapter 2, we constructed explicit, implicit, and fully discrete schemes for numerically solving Q-BSDE.

In the second part, we have crucially improved and generalized the results presented in Meyer-Brandis *et al.* [43] and Mezerdi *et al.* [44]. Our findings investigate Pontrayagin's variational inequalities under partial information for controlled diffusion processes with Lipschitz and non-smooth coefficients. Those results allow for more general choices of the coefficients for the controlled process models in the real world of applications, for example, but not limited to, portfolio optimization, option pricing, optimal consumption and saving, optimal investment, motion planning, and sensor fusion. In particular, we have extended and expanded our main results to break into the non-Lipschitz framework and include some controlled SDEs with quadratic drift.

Further Questions:

Noting that the findings we have presented covered only some particular cases. The results of Chapters 1 and 2 demand that the non-quadratic part of the generator be linear with respect to the Brownian component z, while the results of Chapter 3 require that the diffusion matrix σ be uniformly elliptic. Therefore, there are still open questions that we will address here.

Concerning Chapters 1 and 2, there are some pending questions. We have treated some specific cases, but the more general case when the generator has the following form

$$h(r, y, z) + f(y)z^2.$$

still an open problem. Notice that fully explicit or implicit schemes for the case when the generator has the form

$$\alpha(s) + \beta(s)z + f(y)|z|^2,$$

or satisfies the Lipschitz property in both y and z plus the quadratic term $f(y)|z|^2$ are not solved yet. We will focus on analyzing numerical schemes for these last cases in future research projects and hope we can handle all the problems that are still open.

Concerning Chapter 3, their findings may be extended, based on the Bouleau-Hirsch flow propriety, to the case when the diffusion matrix σ is degenerate. For further details on this subject, we refer to [3] and [19]. See Figure 2.1.

Bibliography

[1] Hanine Azizi and Nabil Khelfallah. The maximum principle for optimal control of BSDEs with locally Lipschitz coefficients. *Journal of Dynamical and Control Systems*, 28(3):565–584, 2022.

[2] Fouzia Baghery, Nabil Khelfallah, Brahim Mezerdi, and Isabelle Turpin. On optimal control of forward–backward stochastic differential equations. *Afrika Matematika*, 28(7):1075–1092, 2017.

[3] Khaled Bahlali, Boualem Djehiche, and Brahim Mezerdi. On the stochastic maximum principle in optimal control of degenerate diffusions with Lipschitz coefficients. *Applied Mathematics and Optimization*, 56:364–378, 2007.

[4] Khaled Bahlali, M'hamed Eddahbi, and Youssef Ouknine. Quadratic BSDE with $\mathbb{L}^2$-terminal data: Krylov's estimate, Itô–Krylov's formula and existence results. *Annals of Probability: An Official journal of the Institute of Mathematical Statistics*, 45(4):2377–2397, 2017.

[5] Khaled Bahlali, Omar Kebiri, Nabil Khelfallah, and Hadjer Moussaoui. One dimensional BSDEs with logarithmic growth application to PDEs. *Stochastics*, 89(6-7):1061–1081, 2017.

[6] Khaled Bahlali, Nabil Khelfallah, and Brahim Mezerdi. Necessary and sufficient conditions for near-optimality in stochastic control of FBSDEs. *Systems & Control Letters*, 58(12):857–864, 2009.

[7] Khaled Bahlali, Brahim Mezerdi, and Youssef Ouknine. The maximum principle for optimal control of diffusions with non-smooth coefficients. *Stochastics: An International Journal of Probability and Stochastic Processes*, 57(3-4):303–316, 1996.

[8] V Bally. An approximation scheme for BSDEs and applications to control and nonlinear PDEs. *preprint*, 1995.

[9] Alain Bensoussan. Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions. *Stochastics: An International Journal of Probability and Stochastic Processes*, 9(3):169–222, 1983.

[10] Jean-Michel Bismut. Conjugate convex functions in optimal stochastic control. *Journal of Mathematical Analysis and Applications*, 44(2):384–404, 1973.

[11] Jean-Michel Bismut. An introductory approach to duality in optimal stochastic control. *SIAM Review*, 20(1):62–78, 1978.

[12] Saliha Bougherara and Nabil Khelfallah. The maximum principle for partially observed optimal control of FBSDE driven by Teugels Martingales and independent Brownian motion. *Journal of Dynamical and Control Systems*, 24:201–222, 2018.

[13] El Mountasar Billah Bouhadjar, Nabil Khelfallah, Mhamed Eddahbi, et al. One-dimensional bsdes with jumps and logarithmic growth. *Axioms*, 13(6):354, 2024.

[14] Nicolas Bouleau and Francis Hirsch. On the derivability, with respect to initial data, of the solution of a stochastic differential equation with Lipschitz coeffcients. In *Séminaire de Théorie du Potentiel n° 9*, pages 39–57. Springer, 1988.

[15] Mezerdi Brahim. Necessary conditions for optimality for a diffusion with a non-smooth drift. *Stochastics: An International Journal of Probability and Stochastic Processes*, 24(4):305–326, 1988.

[16] Philippe Briand, Bernard Delyon, and Jean Mémin. Donsker-type theorem for BSDEs. *Electronic Communications in Probability [Electronic only]*, 6:1–14, 2001.

[17] Jean-Francois Chassagneux and Adrien Richou. Numerical simulation of quadratic BSDEs. *Annals of Applied Probability: An Official Journal of the Institute of Mathematical Statistics*, 26(1):262–304, 2016.

[18] Jared Chessari, Reiichiro Kawai, Yuji Shinozaki, and Toshihiro Yamada. Numerical methods for backward stochastic differential equations: A survey. *Probability Surveys*, 20:486–567, 2023.

[19] Farid Chighoub, Boualem Djehiche, Brahim Mezerdi, and VL Girko. The stochastic maximum principle in optimal control of degenerate diffusions with non-smooth coefficients. *Random Operators and Stochastic Equations*, 17(1), 2009.

[20] Frank H Clarke. *Optimization and non-smooth analysis.* Society for Industrial and Applied Mathematics, 1990.

[21] Boualem Djehiche, Brahim Mezerdi, and Farid Chighoub. The stochastic maximum principle in optimal control of degenerate diffusions with non-smooth coefficients. *Random Operators and Stochastic Equations*, 17:35–53, 2009.

[22] Salima Doubbakh, Nabil Khelfallah, Mhamed Eddahbi, and Anwar Almualim. Malliavin regularity of non-Markovian quadratic BSDEs and their numerical schemes. *Axioms*, 12(4):366, 2023.

[23] Jim Douglas Jr, Jin Ma, and Philip Protter. Numerical methods for forward-backward stochastic differential equations. *The Annals of Applied Probability*, 6(3):940–968, 1996.

[24] Mhamed Eddahbi, Lassaad Mchiri, and Mohamed Rhaima. Numerical solution of quadratic SDE with measurable drift. *Filomat*, 36(15):5263–5278, 2022.

[25] Ivar Ekeland. On the variational principle. *Journal of Mathematical Analysis and Applications*, 47(2):324–353, 1974.

[26] Ivar Ekeland. Nonconvex minimization problems. *Bulletin of the American Mathematical Society*, 1(3):443–474, 1979.

[27] Nicole El Karoui, Shige Peng, and Marie Claire Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997.

[28] Halina Frankowska. The first order necessary conditions for nonsmooth variational and control problems. *SIAM Journal on Control and Optimization*, 22(1):1–12, 1984.

[29] Emmanuel Gobet, Jean-Philippe Lemor, and Xavier Warin. A regression-based monte carlo method to solve backward stochastic differential equations1. *The Annals of Applied Probability*, 15(3):2172–2202, 2005.

[30] Emmanuel Gobet and Plamen Turkejiev. Approximation of backward stochastic differential equations using malliavin weights and least-squares regression. *Bernoulli*, pages 530–562, 2016.

[31] Dalila Guerdouh, Nabil Khelfallah, and Josep Vives. Optimal control strategies for the premium policy of an insurance firm with jump diffusion assets and stochastic interest rate. *Journal of Risk and Financial Management*, 15(3):143, 2022.

[32] Said Hamadène. Equations différentielles stochastiques rétrogrades: le cas localement Lipschitzien. *Annales de l'IHP Probabilités et Statistiques*, 32(5):645–659, 1996.

[33] Ulrich G Haussmann. General necessary conditions for optimal control of stochastic systems. *Stochastic Systems: Modeling, Identification and Optimization, II*, pages 30–48, 1976.

[34] Yaozhong Hu, David Nualart, and Xiaoming Song. Malliavin calculus for backward stochastic differential equations and application to numerical solutions. *The Annals of Applied Probability*, 21(6), 2011.

[35] Peter Imkeller and Gonçalo Dos Reis. Path regularity and explicit convergence rate for BSDE with truncated quadratic growth. *Stochastic Processes and their Applications*, 120(3):348–379, 2010.

[36] Omar Kebiri, Lara Neureither, and Carsten Hartmann. Adaptive importance sampling with forward-backward stochastic differential equations. In *Stochastic Dynamics Out of Equilibrium: Institut Henri Poincaré, Paris, France, 2017*, pages 265–281. Springer, 2019.

[37] Magdalena Kobylanski. Backward stochastic differential equations and partial differential equations with quadratic growth. *The Annals of Probability*, 28(2):558–602, 2000.

[38] Harold Kushner. Necessary conditions for continuous parameter stochastic optimization problems. *SIAM Journal on Control*, 10(3):550–565, 1972.

[39] Jin Ma, Philip Protter, and Jiongmin Yong. Solving forward-backward stochastic differential equations explicitly—a four step scheme. *Probability Theory and Related Fields*, 98(3):339–359, 1994.

[40] Jin Ma and JM Yong. Forward-backward stochastic differential equations and their applications-introduction. pages 1–24. Springer, 1999.

[41] Imène Madoui, Mhamed Eddahbi, and Nabil Khelfallah. Quadratic BSDEs with jumps and related PIDEs. *Stochastics*, 94(3):386–414, 2022.

[42] Paul Malliavin. Stochastic calculus of variation and hypoelliptic operators. In *Proc. Intern. Symp. SDE Kyoto 1976*, pages 195–263. Kinokuniya, 1978.

[43] Thilo Meyer-Brandis, Xunyu Zhou, and Bernt Øksendal. A stochastic maximum principle via Malliavin calculus. *arXiv preprint arXiv:0911.3720*, 2009.

[44] Brahim Mezerdi and Samia Yakhlef. A stochastic maximum principle for mixed regular-singular control problems via Malliavin calculus. *Afrika Matematika*, 27(3-4):409–426, 2016.

[45] GN Milstein and Michael V Tretyakov. Numerical algorithms for forward-backward stochastic differential equations. *SIAM Journal on Scientific Computing*, 28(2):561–582, 2006.

[46] David Nualart. *The Malliavin calculus and related topics*, volume 1995. Springer, 2006.

[47] Carlo Orrieri. A stochastic maximum principle with dissipativity conditions. *Discrete and Continuous Dynamical Systems*, 35(11):5499–5519, 2015.

[48] Etienne Pardoux and Shige Peng. Adapted solution of a backward stochastic differential equation. *Systems & Control Letters*, 14(1):55–61, 1990.

[49] Shige Peng. A general stochastic maximum principle for optimal control problems. *SIAM Journal on Control and Optimization*, 28(4):966–979, 1990.

[50] Adrien Richou. Numerical simulation of bsdes with drivers of quadratic growth. *Annals of Applied Probability: An Official Journal of the Institute of Mathematical Statistics*, 21(5):1933–1964, 2011.

[51] Jiongmin Yong and Xun Yu Zhou. *Stochastic controls: Hamiltonian systems and HJB equations*, volume 43. Springer Science & Business Media, 1999.

[52] Jianfeng Zhang. A numerical scheme for BSDEs. *The Annals of Applied Probability*, 14(1):459–488, 2004.