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Modélisation Non-Linéaire de Certaine Série Chronologique Périodique

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By: **Walid SLIMANI**

Entitled:

Non linear Modeling of Certain Periodic Time Series

Under the direction of: Pr. Cherfaoui MOULOUD

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To my parent

Thanks

First and foremost, I would like to express my gratitude to God and blessings throughout my journey.

I would like to express my heartfelt gratitude and deep appreciation to Professor M. CHERFAOUI for his invaluable guidance.
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Abstract

This thesis focuses on the study of $GARCH$ models for financial series, particularly those with periodically time-varying coefficients. It explores various extensions of these models, including $logGARCH(1,1)$, absolute value $GARCH$, and bilinear threshold $GARCH$, with a focus on their theoretical properties, estimation methods, and empirical applications. The research provides insights into stationary solutions, estimation techniques, and practical relevance, demonstrating their effectiveness in modeling exchange rates of the Algerian Dinar against major currencies.

Keywords: Periodic $logGARCH$ model, Periodic absolute value $GARCH$ model, Periodic bilinear threshold $GARCH$ models, Strictly periodically stationary, Gaussian QML estimator, Consistency, Asymptotic Normality.

Mathematics Subject Classification: C12; C13; 62F12; 62M10.

Résumé

Cette thèse se concentre sur l'étude des modèles $GARCH$ pour les séries financières, en particulier ceux avec des coefficients variant périodiquement dans le temps. Elle explore diverses extensions de ces modèles, notamment $logGARCH(1,1)$, le modèle $GARCH$ de valeur absolue et les modèles $GARCH$ seuil bilinéaires, en mettant l'accent sur leurs propriétés théoriques, leurs méthodes d'estimation et leurs applications empiriques. La recherche offre des perspectives sur les solutions stationnaires, les techniques d'estimation et la pertinence pratique, démontrant leur efficacité dans la modélisation des taux de change du dinar algérien par rapport aux principales devises.

Mots clé : Modèle $logGARCH$ périodique, Modèle $GARCH$ de valeur absolue périodique, Modèles $GARCH$ seuil bilinéaires périodiques, Strictement périodiquement stationnaire, Estimateur QML gaussien, Cohérence, Normalité asymptotique.

Mathematics Subject Classification : C12 ; C13 ; 62F12 ; 62M10.

الملخص

هذه الرسالة تركز على دراسة نماذج GARCH للسلاسل المالية، وتحديداً تلك التي تحتوي على معاملات متغيرة بانتظام مع مرور الوقت. تستكشف الرسالة توسيعات مختلفة لهذه النماذج، بما في ذلك(logGARCH(1, 1 ونموذج GARCH لقيمة مطلقة ونماذج GARCH ذات عتبة ثنائية مع التركيز على خصائصها النظرية وأساليب التقدير والتطبيقات التجريبية. تقدم هذه البحث رؤى حول الحلول الثابتة وتقنيات التقدير والأهمية العملية، مما يظهر فعاليتها في نمذجة أسعار صرف الدينار الجزائري مقابل العملات الرئيسية.

الكلمات المفتاحية: النموذج الدوري logGARCH، النموذج الدوري لقيمة مطلقة GARCH، نماذج GARCH ذات عتبة ثنائية دورية، ثابت دوري بدقة، مقدرQML غاوسي، الثبات، الطبيعة التدريجية.

تصنيف مواضيع الرياضيات: .C12; C13; 62F12; 62M10

Doctoral formation of thesis entitled: " Non linear modeling of certain periodic time series"

Written by: **Walid SLIMANI**

Under the direction of: Prof M. CHERFAOUI and Dr I. LASCHEB

May 11, 2024

This thesis focuses on the Non-Linear Modeling of Certain Periodic Time Series. It consists of three different chapters, each chapter containing a different modeling approach for time series along with simulations. To write this thesis, I underwent a five-year doctoral training program such as

Year 01 (2019-2020):

- Advanced courses in ITC and scientific research.
- Bibliographic research.

Years 02 ans 03 (2020-2022:

- Intensive research work under the supervision of a thesis advisor.
- Data collection and analysis.
- Writing scientific publications and participation in national or international conferences.
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  1. On periodic $logGARCH(1,1)$ processes.
  2. $QMLE$ for periodic absolute value $GARCH$ models.
  3. $QMLE$ of periodic time-varying bilinear threshold GARCH models.

  Participation in national conferences
  1. On periodic exponential $GARCH$
  2. Large sample properties of Yule-Walker estimates for periodic $ARCH$ models
  3. Markov-switching threshold $ARCH$ processes : Probabilistic structure and empirical evidence.
  4. A comparative study in threshold $ARCH$ models with periodic coefficients

  Participation in international conferences
  1. On estimation in periodic threshold $ARCH(q)$ models: Gaussian $QMLE$ approach.
  2. On some probabilistic properties of GARCH(q,p) with GED innovation.
- Periodic validation of progress through research seminars and presentations.
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- Teaching at the University Mentouri as a visiting lecturer.

Year 5 (2023-2024):

- Completion of the thesis.
- Preparation for the thesis defense.
- Publication of the thesis results in the following journals: Random Operators and Stochastic Equations and Journal of Siberian Federal University Mathematics and Physics
- Teaching at the University Mentouri as a visiting lecturer.

Contents

Introduction

As the world has evolved and undergone significant changes across various sectors, banks and stock exchanges have encountered noteworthy financial fluctuations throughout the day, week, or year. These fluctuations are frequently depicted through specific mathematical models:

$$\epsilon_t = 10^r(ln(P_t) - ln(P_{t-1})), \quad t = 1, \ldots N, r > 0.$$

In this context, two primary categories of models have emerged. The first category comprises models with random coefficients, while the second category includes models with time-varying coefficients. The latter category, characterized by coefficients that change over time, has garnered greater attention from researchers and analysts. These models aim to capture the dynamic nature of financial markets and provide a more accurate representation of real-world phenomena. For the most suitable model to characterize financial series witnessed a significant breakthrough in 1982 with Engle's introduction of the $ARCH$ (Autoregressive Conditional Heteroskedasticity) model. This marked a pioneering step in financial time series analysis, enabling the modeling of changing volatility over time. Building upon Engle's work, Bollerslev [9] made substantial advancements in the $ARCH$ model four years later, refining and generalizing it to better accommodate the evolving dynamics of financial markets. These models, collectively termed $GARCH$ (Generalized Autoregressive Conditional Heteroskedasticity), have become pivotal in comprehending and forecasting financial volatility. However, as financial markets continued to exhibit extreme and unpredictable fluctuations, $GARCH$ models faced limitations. Notably, they struggled to keep pace with the rapid and volatile changes in the market. One significant limitation was the assumption of a non-negative relationship between asset values and volatility, which didn't always hold true in practical situations. This unrealistic assumption posed challenges in applying these models effectively. Recognizing the practical shortcomings, financial researchers and analysts developed more advanced iterations of the $GARCH$ model. Notable variants include $PGARCH$ (Power $GARCH$), $EGARCH$ (Exponential $GARCH$), and $logGARCH$ (Logarithmic $GARCH$). These advanced models were designed to provide a more accurate and flexible representation of financial volatility, accounting for the complex and often asymmetric nature of market fluctuations. $PGARCH$ model, which was introduced for the first time in 1993 by Ding and all [17], is widely regarded as the most efficient model and the optimal tool for modeling returns in financial assets and exchange rates. It is also instrumental in analyzing seasonal financial volatility and forecasting financial time series with proven accuracy within specific time frames. These models are non-linear and possess unique characteristics. Their role lies in replicating crucial aspects of financial time. Through empirical studies, these models have revealed their limitations in various scenarios, highlighting numerous shortcomings, including both positive and negative innovations, which have an equal impact on financial volatility. This can be at odds with the analysis of stock behavior in the stock market. Furthermore, these models struggle to effectively handle data due to their inability to adequately account for the effects of falling prices being more influential than rising prices, along with the challenges posed by financial leverage. It has been studied in numerous research studies [6], [27], [13], [20], [48], [42] and [36]. The second model is $ENGRACH$. The $EGARCH$ models, introduced by Nelson [38], are designed to ensure the non-negativity of volatility and effectively handle high values, including negative dynamics in financial data, known as "leverage effects." Nelson also provides conditions for achieving covariance stationarity in these models under specific error distribution specifications. However, these models have certain drawbacks, including computational complexity, challenges in model selection, sensitivity to assumptions, reduced interpretability of coefficients, and limited predictive power for future volatility due to the dynamic nature of financial markets. Users should be aware of these limitations and exercise caution when applying $EGARCH$ models for specific tasks. In this thesis, our primary objective is to investigate and scrutinize three distinct models. The ultimate goal of this examination is to identify the model that best reflects the various fluctuations in the market while

causing the least harm and exhibiting minimal shortcomings. The study will involve a comprehensive analysis of these models to assess their effectiveness and practicality in capturing and explaining the market's ups and downs. We seek to determine which model offers the most accurate representation of market behavior, all while minimizing any negative impacts or limitations associated with the chosen model.
By comparing and contrasting these models, we aim to contribute to a better understanding of market dynamics and ultimately provide insights that can be valuable for decision-makers and stakeholders in the financial world. This research will shed light on the strengths and weaknesses of each model, allowing us to make more informed and responsible choices when dealing with market fluctuations.
In the first chapter of the thesis, we examine the $logGARCH(1,1)$ model, which traces its roots back to the $logGARCH$model, similar to the $EGARCH$ model. It was initially discovered by Geweke [25] in 1986 and Pantula [41]. Subsequently, a more advanced version, $logGARCH(p,q)$, was developed by Francq and all [21, 22, 23, 24]. This concept has gradually gained prominence in various scientific papers, known for its ability to capture dynamic information (see Asai [4]).
Our primary focus will be on the role of this model in allowing us to handle the periodicity of information. This periodic change in information is prevalent in most economic time series. In this section, we delve into the implications and applications of this aspect of the model. The remainder of this chapter is structured as follows. We start by delve into the probabilistic properties of the periodic $logGARCH(1,1)$ model, establishing both necessary and sufficient conditions for strict periodic stationarity and periodic correlation. Furthermore, we derive a closed-form expression for the second-order moment and establish explicit conditions for the existence of higher-order moments. Then, we apply the standard quasi-maximum likelihood (QML) method to estimate the model's unknown parameters. We give numerical illustrations to complement the estimation method. Finally, we offer concluding remarks summarizing the key insights and implications drawn from our analysis.
The second chapter is dedicated to the expansion of the standard Absolute Value $GARCH$ ($AVGARCH$) model into the realm of periodically time-varying coefficients, referred to as $PAVGARCH$. In this category of models, parameters have the flexibility to transition between different regimes, allowing for the incorporation of asymmetric effects within volatility modeling. Specifically, these models describe volatility as a linear combination of the absolute value of the shock (innovation) and lagged volatility, introducing asymmetry by manipulating the absolute value of the shock.
The primary objectives of this chapter revolve around the examination of the probabilistic properties of the $PAVGARCH$ model, primarily through its vector representation. We establish necessary and sufficient conditions to ensure the model's strict periodic stationarity. Additionally, we apply the standard Quasi-Maximum Likelihood (QML) method to estimate the model's parameters, presenting conditions that guarantee the strong consistency and asymptotic normality of the QML estimator. To substantiate our theoretical findings, a series of numerical experiments is conducted to demonstrate the practical relevance of our approach. Lastly, we apply our model to analyze two foreign exchange rates, specifically the Algerian Dinar against the Euro ($Euro/Dinar$) and the American Dollar ($Dollar/Dinar$). Our empirical findings reveal that our approach not only outperforms alternative models but also fits the data effectively.
The last chapter of thesis is dedicated to enhancing the standard Bilinear Threshold $GARCH(BLTGARCH)$ model by introducing periodically time-varying coefficients, denoted as $PBLTGARCH$. These models belong to a class where parameters can transition between different regimes, allowing for the integration of asymmetric effects in modeling volatility. Asymmetric effects pertain to situations where the impact of positive and negative shocks on volatility is not symmetrical. Our first major focus is on establishing necessary and sufficient conditions for the existence of stationary solutions, but with a periodic sense in mind. This means that we're considering not just stationary solutions in the traditional sense, but those that exhibit periodic behavior over time. The second significant aspect of this paper is the development of a quasi-maximum likelihood (QML) estimation approach for the $PBLTGARCH$ model. The QML method is a statistical technique used to estimate model parameters by maximizing a likelihood function. We study the properties of the QML estimator, including its strong consistency and asymptotic normality. This means we investigate how well our estimation method works and under what conditions it provides reliable results. We specify these conditions, which include assumptions like strict stationarity and the finiteness of moments of certain orders for the error terms. To further support the practicality of our approach, we conduct a Monte Carlo study to illustrate how our estimation method performs in finite-sample scenarios, where we have limited data.
Lastly, we apply our enhanced model to analyze exchange rate data between the Algerian Dinar and the European Euro. This application demonstrates the real-world relevance and effectiveness of our model in capturing the complexities of financial data, particularly in the context of exchange rates.

Chapter 1

Cyclical $LogGARCH(1,1)$ Processes: Modeling and Analysis

Contents

This chapter was accepted in the journal Random Operators and Stochastic Equations.
This chapter focuses on exploring the probabilistic and statistical properties of a specific type of time series model ($P-logGARCH(1,1)$). The chapter begins with a brief rationale for introducing the $P-logGARCH$ model, highlighting the need for a time series model that captures periodic variations in coefficients. Subsequently, we delve into the probabilistic structure of $P-logGARCH$ models, shedding light on the mathematical framework that underpins these processes. One of the key objectives of this chapter is to establish the theoretical conditions that guarantee the strict stationarity of the $P-logGARCH(1,1)$ process in a periodic sense. This is a crucial aspect as it ensures that the model remains stable and well-behaved over time. Furthermore, we delve into the examination of higher-order moments of the solutions generated by $P-logGARCH$ models. This analysis provides insights into the distributional properties and statistical behavior of these processes, contributing to a more comprehensive understanding of their characteristics. In essence, this chapter serves as a foundational exploration of $P-logGARCH$ models, offering theoretical insights and establishing conditions that validate their use in modeling time series data with periodically time-varying coefficients.

1.1 Presentation of $P - \log GARCH$ models

Let $p, q > 0$ and let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $(\epsilon_t)_{t\in\mathbb{Z}}$ defined on $(\Omega, \mathcal{F}, P)$. If $\epsilon_t \,|\mathcal{F}_{t-1} \rightsquigarrow \mathcal{L}(0, h_t)$, and

$$\forall t \in \mathbb{Z}, \quad \epsilon_t = \eta_t h_t \text{ and } \log h_t^2 = \omega(t) + \sum_{i=1}^{q} \alpha_i(t) \log \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j(t) \log h_{t-j}^2, \quad 1 \le i \le q, \ 1 \le j \le p$$

where the functions $\omega(t)$, $\alpha_i(t)$, and $\beta_j(t)$ are periodic in t with period $s, (s \ge 1)$ (i.e $\forall t, k \in \mathbb{Z}, \quad \omega(t) = \omega(ks + t)$, $\alpha_i(t) = \alpha_i(ks + t)$ and $\beta_j(t) = \beta_j(ks + t)$. Then, $(\epsilon_t)_{t\in\mathbb{Z}}$ is called a periodic $logGARCH(p, q)$ model $(P - \log GARCH_s(p, q))$ space of orders p and q with period $s > (p \vee q)$. We proceed with the underlying assumption that the innovation sequence, denoted as $(\eta_t)_{t\in\mathbb{Z}}$, adheres to a specific set of conditions that are pivotal within the context of the analysis. These conditions establish a foundational framework for the study, ensuring that the innovations possess certain key properties.

Assumption 1.1.1. *Let $(\eta_t)_{t\in\mathbb{Z}}$ be a sequence of independent and identically distributed $(i.i.d.)$ random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with zero-mean and unit variance $(i.i.d.(0, 1))$. Let $\mathcal{F}_{t-1}$ be the σ-algebra generated by $\{\eta_{t-i}, \quad \forall i \ge 1\}$.*

To facilitate a focused and streamlined discussion, we will direct our attention to a specific model in the realm of time series analysis, namely $P - \log GARCH_s(1, 1)$. This targeted approach enables a deeper exploration of the properties and characteristics of this model, enhancing the precision of our analysis. Therefore, we have

$$\epsilon_t = \eta_t h_t \quad \text{and} \quad \log h_t^2 = \omega(t) + \alpha(t) \log \epsilon_{t-1}^2 + \beta(t) \log h_{t-1}^2, \quad t \in \mathbb{Z}, \tag{1.1}$$

This model is recognized for its simplicity, as it can be efficiently implemented with a limited number of parameters, making it accessible for practical applications. Moreover, it often exhibits a good fit to observed data. It's important to note that a high value of $\beta(t)$ indicates persistent volatility, remaining relatively stable over an extended period with gradual changes. Conversely, a high value of $\alpha(t)$ suggests that volatility is characterized by frequent and abrupt fluctuations. We transform $t \to st + \vartheta$ and we set $\epsilon_{st+\vartheta} = \epsilon_t(\vartheta)$, $h_{st+\vartheta} = h_t(\vartheta)$ and $\eta_{st+\vartheta} = \eta_t(\vartheta)$ and we denoted by $(\epsilon_t(\vartheta)) = \epsilon_t$, $h_t(\vartheta) = h_t$ and $\epsilon_t(\vartheta\eta_t(\vartheta) = \eta_t$.
Therefore, the model given by (1.1) can be expressed in an equivalent form which is given as:

$$\epsilon_t(\vartheta) = \eta_t(\vartheta) h_t(\vartheta) \text{ and } \log h_t^2(\vartheta) = \omega(\vartheta) + \alpha(\vartheta) \log \epsilon_t^2(\vartheta - 1) + \beta(\vartheta) \log h_t^2(\vartheta - 1), \quad \forall t \in \mathbb{Z}. \tag{1.2}$$

Let $1 \le \vartheta \le s$, the $\vartheta - th$ "season" for of the cycle t and $\{\omega(\vartheta), \alpha(\vartheta), \beta(\vartheta)\}$ are the model coefficients at season $\vartheta \in \{1, ..., s\}$. For convenience and if $\vartheta < 0$, we have $\epsilon_t(\vartheta) = \epsilon_{t-1}(\vartheta + s), \eta_t(\vartheta) = \eta_{t-1}(\vartheta + s)$ and $h_t(\vartheta) = h_{t-1}(\vartheta + s)$.
The non-periodic symbols (ϵ_t), (h_t) and (η_t) can be used interchangeably with their periodic counterparts $(\epsilon_t(\vartheta))$, $(h_t(\vartheta))$ and $(\eta_t(\vartheta))$. If there is no need to emphasize the seasonal aspect, the following notes on s and $(\epsilon_t)_{t\in\mathbb{Z}}$ are hereby presented.

Remark 1.1.1. *The introduction of the assumption $s > (p \vee q)$ aims to ensure that the process $(\epsilon_t)_{t\ge 1}$ in the general formulation of periodic $logGARCH(p, q)$ demonstrates dependence through consecutive observations within a given period.*

Remark 1.1.2. *In addition to the expression provided in (1.1), the process $(\epsilon_t)_{t\in\mathbb{Z}}$ can be reformulated as follows:*

$$\epsilon_t = \eta_t h_t \text{ and } \log h_t^2 = \omega'(t) + \gamma(t) \log h_t^2 + \alpha(t) z_{t-1}, \quad \forall t \in \mathbb{Z}, \tag{1.3}$$

which will be heavy used. On the other hand, if $E\left\{\left|\log \epsilon_t^2\right|\right\} < +\infty$, then the process $\left(\log \epsilon_t^2\right)_t$ admits a first-order periodic autoregressive moving average $PARMA(1, 1)$ representation.

$$\log \epsilon_t^2 = \omega''(t) + \gamma(t) \log \epsilon_{t-1}^2 - \beta(t) z_{t-1} + z_t, \forall t \in \mathbb{Z}, \tag{1.4}$$

where $z_t = \log \eta_t^2 - E\left\{\log \eta_t^2\right\}$, $\omega'(t) = \omega(t) + E\left\{\log \eta_t^2\right\}$, $\omega''(t) = \omega(t) + (1 - \beta(t)) E\left\{\log \eta_{t-1}^2\right\}$ and $\gamma(t) = \alpha(t) + \beta(t)$. In addition, $\log \eta_t^2$ can be seen as an $i.i.d.$ white noise with zero median. Therefore, if (η_t) is standard normal $E\left\{\log \eta_t^2\right\} \approx -1.27035$ and $Var\left\{\log \eta_t^2\right\} = \dfrac{\pi^2}{2}$. The models (1.1) and (1.4) have become appealing tools for investigating both symmetric volatility and distinct "seasonal" patterns in modeling financial time series that exhibit a periodic rhythm. This is characterized by the probability law governing such a series being invariant under shifts of length s.
More precisely, a process (Y_t) is said to be strictly periodically stationary with period s if, for all $h, t_1, t_2, \ldots, t_n$ in $\mathbb{Z}$ and for any collection of Borel sets $A_1, A_2, \ldots, A_n$ in $\mathbb{R}$. Therefore,

$$P\left(Y_{t_1+sh} \in A_1, Y_{t_2+sh} \in A_2, ..., Y_{t_n+sh} \in A_n\right) = P\left(Y_{t_1} \in A_1, Y_{t_2} \in A_2, ..., Y_{t_n} \in A_n\right).$$

Observe that when $s = 1$, the process is characterized as strictly stationary. In the case of a second-order process, meaning that $(Y_t) \in \mathbb{L}_2$, it is denoted as periodically correlated with a period s if both are given as $\mu(l) = E\{Y_l\} = \mu(l+s)$, and $\gamma(l,k) = E\{Y_l Y_k\} = \gamma(l+s, k+s)$. (see, for instance, Hurd and Miamee [31]).

1.2 Stationary study

The representation (1.3) or (1.4) is not stationary within "year", since the distribution depends on what "season" of the "year" it is. To tackle this issue, Gladyshev [28] demonstrated that by incorporating time-varying coefficients with periodicity, it becomes feasible to integrate seasonal patterns into a multivariate stationary process. More precisely, consider the periodic version of representation (1.3). Then,

$$\underline{\log} h_t^2 = \left(\log h_t^2(1), ..., \log h_t^2(s)\right)'. \tag{1.5}$$

Introducing the following

$$b_t(\vartheta) = \sum_{k=0}^{\vartheta-1} \left\{ \prod_{j=0}^{k-1} \gamma(\vartheta - j) \right\} W_t(\vartheta - k),$$

where $\forall \vartheta = 1, ..., s, \quad W_t(\vartheta) = \omega'(\vartheta) + \alpha(\vartheta)\ z_t(\vartheta - 1)$. Now, let define the matrix M of size $s \times s$ and the vector and the vector $\underline{B}_t$ as:

$$M = \begin{pmatrix} 0 & ... & 0 & \gamma(1) \\ 0 & ... & 0 & \gamma(1)\gamma(2) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & ... & 0 & \prod_{\vartheta=0}^{s-1} \gamma(s-\vartheta) \end{pmatrix}_{s\times s}, \quad \underline{B}_t = \begin{pmatrix} b_t(1) \\ b_t(2) \\ \vdots \\ b_t(s) \end{pmatrix}_{s\times 1}.$$

Therefore, (1.1) is equivalent to a (non unique) vector-valued AR process given by

$$\underline{\log} h_t^2 = M \underline{\log} h_{t-1}^2 + \underline{B}_t, \quad \forall t \in \mathbb{Z}. \tag{1.6}$$

Therefore, we need to establish conditions on M and $(\underline{B}t)$ such that (1.6) has a strictly stationary, causal, and ergodic solution. Consequently, $\left(\log \epsilon t^2\right) t$ will be strictly periodically stationary and periodically ergodic whenever $\left(\underline{\log} h t^2\right)_t$.
For this purpose, and since the representations (1.3) and (1.6) are valid for all integer values of t, successive substitution of such representations reveals that formal series solutions to (1.3) and to (3.10) can be given, respectively, by:

$$\log h_t^2 = \sum_{k=0}^{\infty} \left\{ \prod_{i=0}^{k-1} \gamma(t-i) \right\} W_{t-k}, \quad \text{and} \quad \underline{\log} h_t^2 = \sum_{i=0}^{\infty} M^i \underline{B}_{t-i}. \tag{1.7}$$

The challenge is to determine the conditions on $\alpha(\vartheta)$ and $\beta(\vartheta)$ for all $\vartheta = 1, ..., s$. As well as on $(\eta t)_{t\in\mathbb{Z}}$. These formal series solutions provided in (1.5) become genuine, strictly stationary solutions for both (1.3) and (1.6). In the upcoming section, we are going to explore the conditions based on (1.6) that ensure the existence of strictly periodically stationary (SPS) solutions for equations (1.5).

1.2.1 Strict periodically stationary analysis

In this specific section, we conduct a detailed analysis to identify and explore the conditions that are both necessary and sufficient to ensure the strict stationarity of the models (1.7). The solution corresponding to the equation (1.4) is termed "strictly periodic stationary (SPS)". This indicates that the solution exhibits a pattern of periodic behavior while maintaining strict stationarity.
The key tool in studying strict stationarity in $VAR(1)$ models (1.5) is the top-Lyapunov exponent (or the Cauchy root test when M is not stochastic). Indeed, $(\eta_t)_{t\in\mathbb{Z}}$ is an $i.i.d$ process with $E\left\{\log^+ ||\underline{B}_1||\right\} < \infty$, where $\forall x > 0, \quad \log^+ x = \text{Max}\{\log x, 0\}$. Now, let define $\gamma^{(s)}$ by the following way:

$$\gamma^{(s)} = \lim_{n\to\infty} \frac{1}{n} \log \|M^n\| = \log \rho(M) = \sum_{\vartheta=0}^{s-1} \log |\gamma(s-\vartheta)| \tag{1.8}$$

which is independent of the distribution of (η_t) contrary to the $P-GARCH$ models.

Now, we are in the position to state our first result. In the following proposition, we will prove that the series $\underline{\log h_t^2}$ converges absolutely almost surely and is the unique causal and ergodic solution of (1.5). It is also the unique causal, strictly periodically stationary, and periodically ergodic solution of (1.3).

Proposition 1.2.1. *Let us explore the state space representation given in* (1.5) *that is associated with* $P - LogGARCH_s(1,1)$ *models, as described in* (1.1)*, under the conditions outlined in* 1.1.1*. Assuming that* $E\left\{\log^+ ||\underline{B}_1||\right\} < \infty$*, and* $\gamma^{(s)} < 0$ *in* (3.11)*. We obtain the following results*

1. *The series* $\underline{\log h_t^2}$ *in* (1.7) *converges absolutely almost surely and is the unique causal and ergodic solution of* (1.5)*.*
2. *The series* $\log h_t^2$ *defined in* (1.7) *is the unique causal, strictly periodically stationary and periodically ergodic solution of* (1.3) *with* $\sup\limits_{1\leq\vartheta\leq s} \prod\limits_{i=0}^{k-1} |\gamma(\vartheta - i)| = O(\rho^k), \rho \in]0,1[,\ k \geq 1.$

Proof.

1. By Cauchy root test, it is not difficult to see that the series converges if and only if $\gamma^{(s)} < 0$. Moreover, simple calculation shows that $\log h_t^2$ defined in (1.7) is the unique, causal, strictly periodically stationary and periodically ergodic solution of (1.3).
2. Immediate consequence of the convergence of the series $\underline{\log h_t^2}$.

□

We introduce a second proposition, which directly follows from the first proposition.

Proposition 1.2.2. *Let* $\underline{\log h_t^2}$ *be series converges absolutely and is the the unique causal, strictly periodically stationary and periodically ergodic solution of* (3.10) *and* (1.3)*. If* $E\left\{\|\underline{B}_t\|^2\right\} < +\infty$ *and* $\sum\limits_{k=0}^{\infty} \sup\limits_{1\leq\vartheta\leq s}\left\{\prod\limits_{i=0}^{k-1} |\gamma(\vartheta - i)|^2\right\} < \infty$*. Then,* $\left(\log \epsilon_t^2\right)_t$ *is second order periodically correlated process.*

Proof. See Nelson [38]. □

We can deduce the following observation from the last proposition.

Remark 1.2.1. *If we use the representation of* (1.5) *and the fact that* $\underline{\log h_t^2}$ *is second order. Additionally, if* $\rho\left(M^{\otimes 2}\right) < 1$*, which is equivalent to* $\prod\limits_{\vartheta=0}^{s-1} \gamma^2(s-\vartheta) < 1$*. Then, the sufficient conditions in the proposition* 1.2.2 *is achieved.*

1.2.2 Example

For the $P - \log GARCH_s(1,1)$ model, after some tedious algebra we find that the necessary and sufficient condition ensuring the existence of SPS solution is that

$$\sum_{\vartheta=1}^{s} \log |\alpha(\vartheta) + \beta(\vartheta)| < 0$$

is strictly negative. Note that when $s = 1$, the strictly periodically stationary condition coincide with that of the standard $\log GARCH(1,1)$ given by Francq et al [23] (see Example 2.1 p. 36). It is worth noting that the existence of regimes which satisfy

$$\log |\alpha(\vartheta) + \beta(\vartheta)| > 0$$

does not preclude strict periodic stationarity.

It is well known that for the $P-GARCH$ type models, the strictly periodically stationary condition entails the existence of a moment of order $\tau>0$ for (ϵ_t). The following proposition shows that this is also the case for $\left|\log \epsilon_t^2\right|$ in $P-\log GARCH_s(1,1)$ model when $E\left\{\log^+\left|\log \eta_0^2\right|\ \right\}<\infty$.

Proposition 1.2.3. *Let ϵ_t be the strictly periodically stationary solution of (1.3). Assuming that $\gamma^{(s)}<0$ and $E\left\{\left|\log \eta_0^2\right|^{\tau_0}\right\}<\infty,\quad \tau_0>0$. Then, there exist $\tau>0$ such that for any $\vartheta\in\{1,...,s\}$, we have $E\left\{\left|\log h_t^2(\vartheta)\right|^{\tau}\right\}<\infty$, and $E\left\{\left|\log \epsilon_t^2(\vartheta)\right|^{\tau}\right\}<\infty$.*

Proof. From the definition of $\gamma^{(s)}$, there is a positive integer i_0 such that $\log\left\|M^{i_0}\right\|<0$. Let

$$g(t)=\left\|M^{i_0}\right\|^t.$$

Then, g is decreasing in a neighborhood of 0 because $g'(0)<0$ and since $g(0)=1$, it follows that there exist $\tau\in\,]0,1[$ such that $\left\|M^{i_0}\right\|^{\tau}<1$. Hence for any $\vartheta\in\{1,...,s\}$, we have

$$\left\|\underline{\log} h_t^2(\vartheta)\right\|\leq\sum_{i=0}^{\infty}\left\|M^i\right\|\left\|B_t\left(\vartheta-i\right)\right\|.$$

Since $\tau\in\,]0,1[$ we get

$$E\left\{\left\|\underline{\log} h_t^2(\vartheta)\right\|^{\tau}\right\}\leq\sum_{i=0}^{\infty}\left\|M^i\right\|^{\tau}E\left\{\left\|B_t\left(\vartheta-i\right)\right\|^{\tau}\right\}\leq B\left(\tau\right)\sum_{i=0}^{\infty}\left\|M^i\right\|^{\tau},$$

where

$$B\left(\tau\right)=\max_{1\leq\vartheta\leq s}E\left\{\left\|B_0\left(\vartheta-i\right)\right\|^{\tau}\right\}.$$

Now working with a multiplicative norm and using the above discussion, it follows that with

$$a=\sum_{i=0}^{i_0-1}\left\|M^i\right\|^{\tau}>0,$$

and

$$b=\left\|M^{i_0}\right\|^{\tau}\in\,]0,1[\,.$$

We have

$$\sum_{i=0}^{\infty}\left\|M^i\right\|^{\tau}\leq\sum_{i=0}^{\infty}ab^i.$$

Hence $E\left\{\left|\log h_t^2(\vartheta)\right|^{\tau}\right\}<+\infty$. Additionally, since

$$E\left\{\left|\log \epsilon_t^2(\vartheta)\right|^{\tau}\right\}<E\left\{\left|\log h_t^2(\vartheta)\right|^{\tau}\right\}+E\left\{\left|\log \eta_0^2(\vartheta)\right|^{\tau}\right\}<\infty,$$

which holds true when $\tau\leq\tau_0$. ☐

In the upcoming sections, we will delve into the moment properties of strictly periodically stationary (SPS) solutions for equation (1.3). This entails exploring and conducting a more detailed analysis of how these solutions behave in terms of statistical moments, including mean, variance, and higher-order moments, within the context of the equation (1.2) .

1.3 Moments properties of periodically correlated process

In this section, we examine the moments properties of the strictly periodically stationary solution to $P-logGARCH_s(1,1)$. For this purpose, we put

$$Z_t=\log h_t^2,\ \underline{Z}_t=\underline{\log} h_t^2,\ \gamma=\textstyle\prod_{\vartheta=1}^{s}\gamma\left(\vartheta\right),\text{ and }\gamma_{\vartheta}(r)=\prod_{i=1}^{r-1}\gamma\left(\vartheta-i+1\right).$$

We note that for any $\vartheta=1,2,...,s$

$$A_{\vartheta}=\sum_{k=0}^{\infty}\left\{\prod_{i=0}^{k-1}\gamma\left(\vartheta-i\right)\right\}\omega(\vartheta-k)=\left(1-\gamma\right)^{-1}\gamma_{\vartheta}(\vartheta)\textstyle\sum_{r=1}^{s}\gamma_s(r-\vartheta)\omega\left(\vartheta-r\right),$$

with the convention $\prod_{i=x}^{y}\gamma\left(i\right)=1$ if $x>y$. Let us introduce the following lemma which proves that the periodic version of the process $(Z_t)_t$ has a new representation.

Lemma 1.3.1. *Consider the model $P - logGARCH_s(1,1)$. Then, the periodic version of the process $(Z_t)_t$ admit the following representation:*

$$Z_t(\vartheta) = \gamma_\vartheta(\vartheta)\sum_{\tau=0}^{t-1}\gamma^\tau\sum_{r=1}^{s}\gamma_s(r-\vartheta)\left\{\alpha(\vartheta-r+1)\log\eta_{t-\tau}^2(\vartheta-r)+\omega(\vartheta-r+1)\right\}$$
$$+\gamma^t\gamma_\vartheta(\vartheta)\sum_{k=1}^{\vartheta}\gamma_s(k-\vartheta)\left\{\alpha(\vartheta-k+1)\log\eta_0^2(\vartheta-k)+\omega(\vartheta-k+1)\right\}$$
$$+\gamma^t\gamma_\vartheta(\vartheta)Z_0. \tag{1.9}$$

Proof. First, from the representation (1.3) of $\log h_t^2(\vartheta)$ it is necessary and sufficient to express in a unique way $\log h_t^2(\vartheta)$ in terms of $\log\eta_t^2(\vartheta)$and its infinite past as an infinite periodic moving average. Indeed, since

$$Z_t = \gamma(t)Z_{t-1} + W_t.$$

Then by iterating $t-$times, we get

$$Z_t = \left(\prod_{i=1}^{t}\gamma(t-i+1)\right)Z_0 + \textstyle\sum_{k=1}^{t}\left(\prod_{i=1}^{k-1}\gamma(t-i+1)\right)W_{t-k+1},$$

or in periodic version, for all $\vartheta\in\{1,...,s\}$

$$Z_t(\vartheta) = \left(\prod_{i=1}^{\vartheta+ts}\gamma(\vartheta-i+1)\right)Z_0 + \sum_{k=1}^{ts}\gamma_\vartheta(k)W_t(\vartheta-k+1) + \sum_{k=ts+1}^{\vartheta+ts}\gamma_\vartheta(k)W_t(\vartheta-k+1). \tag{1.10}$$

By using some simple calculation, we can show that

$$\left(\prod_{i=1}^{\vartheta+ts}\gamma(\vartheta-i+1)\right)Z_0 = \gamma^t\gamma_\vartheta(\vartheta)Z_0,$$

$$\sum_{k=1}^{ts}\gamma_\vartheta(k)W_t(\vartheta-k+1) = \gamma_\vartheta(\vartheta)\sum_{\tau=0}^{t-1}\gamma^\tau\sum_{r=1}^{s}\gamma_s(r-\vartheta)W_{t-\tau}(\vartheta-r+1),$$
$$\sum_{k=1+ts}^{\vartheta+ts}\gamma_\vartheta(k)W_t(\vartheta-k+1) = \gamma_\vartheta(\vartheta)\gamma^t\sum_{l=1}^{\vartheta}\gamma_s(l-\vartheta)W_0(\vartheta-l+1).$$

Now, replacing W_t by its expression and rearranging the above terms in $Z_t(\vartheta)$. We get the equation (1.9). □

In the following lemma, we study the convergence in $\mathbb{L}_2$ of the sequence $(Z_t(\vartheta))_t$ defined by (1.10) when $t\to\infty$ for each $\vartheta\in\{1,...,s\}$.

Lemma 1.3.2. *Assume that the condition $|\gamma| = \prod_{i=1}^{s}|\gamma(i)| < 1$ holds. Then the process $(Z_t(\vartheta))_t$ defined by (1.10) converges in $\mathbb{L}_2$.*

Proof. The proof follows from the propositions (1.2.2) and (1.2.1). Thus, we have

$$\sum_{k=0}^{\infty}\left\{\prod_{i=0}^{k-1}|\gamma(\vartheta-i)|^2\right\} \le \left\{\sum_{k=0}^{\infty}\left\{\prod_{i=0}^{k-1}|\gamma(\vartheta-i)|\right\}\right\}^2 \le (1-\gamma)^{-2}\left\{|\gamma_\vartheta(\vartheta)|\sum_{r=1}^{s}|\gamma_s(r-\vartheta)|\right\}^2 < +\infty.$$

This is done to arrive at the result. □

We present the following proposition which is a consequence of last lemma.

Proposition 1.3.1. *Consider the model $P - logGARCH_s(1,1)$. Then, under the conditions of propositions 1.2.2 and 1.2.1, the closed form of the second moment of $\epsilon_t(\vartheta)$, for all $\vartheta = 1,2,...,s$, is given by*

$$E\left(\epsilon_t^2(\vartheta)\right) = \exp(A_\vartheta)\prod_{k=0}^{\infty}E\left\{\exp\left(\gamma_\vartheta(\vartheta)\gamma^k\sum_{r=1}^{s}\gamma_\vartheta(r)\alpha(\vartheta-r)\log\eta_{t-k}^2(\vartheta-r-1)\right)\right\}.$$

Proof. The proof of this proposition follows essentially the same arguments as in Sadoun and Bentarzi [42]. By using the above lemmas: lemma 1.3.1 gives an expression of $(Z_t)_t$ in term of finite series of $\left(\log \eta_{t-i}^2\right)_{1\leq i\leq n}$ and $(\omega(t-i))_{1\leq i\leq n}$ and lemma 1.3.2 ensures the convergence in $\mathbb{L}_2$ of such series. Hence, in law to the following expression

$$\log h_n^2(\vartheta) = \gamma_\vartheta(\vartheta)\sum_{\tau=0}^{\infty}\gamma^\tau\sum_{r=1}^{s}\gamma_s(r-\vartheta)\alpha(\vartheta-r)\log\eta_{n-\tau}^2(\vartheta-r-1)+A_\vartheta.$$

This yields

$$h_n^2(\vartheta)=\exp(A_\vartheta)\prod_{\tau=0}^{\infty}\exp\left(\gamma_\vartheta(\vartheta)\gamma^\tau\sum_{r=1}^{s}\gamma_s(r)\alpha(\vartheta-r+1)\log\eta_{n-\tau}^2(\vartheta-r)\right).$$

Furthermore, we get

$$E\left(h_n^2(\vartheta)\right)=\exp(A_\vartheta)E\left\{\prod_{\tau=0}^{\infty}\exp\left(\gamma_\vartheta(\vartheta)\gamma^\tau\sum_{r=1}^{s}\gamma_s(r-\vartheta)\alpha(\vartheta-r)\log\eta_{n-\tau}^2(\vartheta-r-1)\right)\right\},$$

and

$$E\left(\epsilon_n^2(\vartheta)\right)=\exp(A_\vartheta)\prod_{\tau=0}^{\infty}E\left\{\exp\left(\gamma_\vartheta(\vartheta)\gamma^\tau\sum_{r=1}^{s}\gamma_s(r-\vartheta)\alpha(\vartheta-r)\log\eta_{n-\tau}^2(\vartheta-r-1)\right)\right\}.$$

Finally, the process $\log h_n(\vartheta)$ being periodically stationary of second-order. □

Corollary 1.3.1. *Let be* $\log GARCH(1,1)$ *a classical model. The process* $(\epsilon_n)_{n\in Z}$ *is a stationary in the second-order moment if and only if* $|\alpha+\beta|<1$. *Moreover the closed forms of the second unconditional moment* $E\left(\epsilon_t^2\right)$ *of such process is given by*

$$E\left(\epsilon_t^2\right)=\exp\left(\left(1-(\alpha+\beta)\right)^{-1}\omega\right)\prod_{k=1}^{\infty}E\left\{\exp\left(\alpha(\alpha+\beta)^{k-1}\log\eta_{t-k}^2\right)\right\}.$$

Proof. Straightforward and hence omitted. □

1.3.1 Conditions for the existence of higher moments

This section provides the condition ensuring the existence of the $m-$th order moments $E(\epsilon_t^m)$ and their explicit form in terms of the parameters of the model (3.8). We first establish the following basic result.

Lemma 1.3.3. *We consider the model* $P-logGARCH_s(1,1)$ *with associated state-space representation* (3.10). *Let define* $M_i^{(m)}(t)'s$ *recursively by*

$$M_0^{(1)}(t)=\underline{B}_t, M_1^{(1)}(t)=M,$$

and for $m>0$

$$M_i^{(m+1)}(t)=\underline{B}_t\otimes M_i^{(m)}(t)+M\otimes M_{i-1}^{(m)}(t),$$

with the convention

$$M_i^{(m+1)}(t)=O.$$

Then, for any positive integer m the following representation holds

$$\underline{Z}_t^{\otimes m}=\sum_{i=0}^{m}M_i^{(m)}(t)\underline{Z}_{t-1}^{\otimes i}, \tag{1.11}$$

and when $i>m$ *or* $i<0$ *we get*

$$\underline{Z}_t^{\otimes 0}=M_0^{(0)}(t)=1.$$

Proof. The formulation (1.11) is readily apparent when $m = 1$. Assuming that (1.11) holds for a certain value of $m \geq 1$, therefore, we get:

$$\begin{aligned}
\underline{Z}_t^{\otimes(m+1)} &= \sum_{i=0}^{m} \left(\underline{B}_t + M\underline{Z}_{t-1}\right) \otimes M_i^{(m)}(t)\underline{Z}_{t-1}^{\otimes i}, \\
&= \sum_{i=0}^{m} \left(\underline{B}_t \otimes M_i^{(m)}(t)\underline{Z}_{t-1}^{\otimes i} + M\underline{Z}_{t-1} \otimes M_i^{(m)}(t)\underline{Z}_{t-1}^{\otimes i}\right), \\
&= \sum_{i=0}^{m} \left(\underline{B}_t \otimes M_i^{(m)}(t)\right) \underline{Z}_{t-1}^{\otimes i} + \sum_{i=0}^{m} \left(M \otimes M_i^{(m)}(t)\right) \underline{Z}_{t-1}^{\otimes(i+1)}, \\
&= \sum_{i=0}^{m+1} \left(\underline{B}_t \otimes M_i^{(m)}(t) + M \otimes M_{i-1}^{(m)}(t)\right) \underline{Z}_{t-1}^{\otimes i}, \\
&= \sum_{i=0}^{m+1} M_i^{(m+1)}(t)\underline{Z}_{t-1}^{\otimes i}.
\end{aligned}$$

We get the result. □

The following proposition introduces the context by highlighting the importance of the $P-logGARCH_s$ model in time series analysis, specifically in the financial domain. It establishes a crucial assumption regarding the finite expected norm of $\underline{B}_t^{\otimes m}$. The next proposition's key revelation is the direct link between the moments and the parameter $\rho\left(M^{\otimes m}\right)$. It underscores that the existence of moments is contingent on $\rho\left(M^{\otimes m}\right) < 1$, and consequently, the periodic correlation structure of $\log ht^2$ within L_m is illuminated.

Proposition 1.3.2. *Consider the model $P-logGARCH_s(1,1)$ with associated state-space representation (1.5). Assuming that $E\left\{\left\|\underline{B}_t^{\otimes m}\right\|\right\} < +\infty$. Then, $\left(\underline{Z}_t\right)_t$ admits moments up to $m-$order if and only if:*

$$\rho\left(M^{\otimes m}\right) = |\gamma|^m < 1.$$

Hence $\left(\log h_t^2\right)_t$ is a periodically correlated process belonging to L_m.

Proof. Based on the expression provided in (1.11), it becomes readily apparent that

$$E\left\{\underline{Z}_t^{\otimes m}\right\} = \sum_{i=0}^{m-1} E\left\{M_i^{(m)}(t)\underline{Z}_{t-1}^{\otimes i}\right\} + M^{\otimes m} E\left\{\underline{Z}_{t-1}^{\otimes m}\right\},$$

which admits a solution whenever

$$\rho\left(M^{\otimes m}\right) < 1,$$

or equivalently

$$|\gamma|^m < 1.$$

□

The subsequent proposition serves the purpose of determining the $2m$-th moment of the process denoted as $\left(\epsilon_t\right)_{t\in\mathbb{Z}}$. This proposition plays a crucial role in quantifying the behavior and statistical properties of the process, shedding light on its moments up to the specified order of $2m$. It is an essential step in gaining a comprehensive understanding of the underlying dynamics of the process and its mathematical properties.

Proposition 1.3.3. *Consider the model $P - logGARCH_s(1,1)$. Supposing that $\mu_\eta(2m) = E\left\{\eta_t^{2m}\right\} < +\infty$, and $E\left\{\eta_t^k\right\} = 0, \quad \forall 1 \leq k < 2m$,(these conditions are obviously satisfied if η_t is a Gaussian white noise). Then, the $2m$-th moment (assuming it exists) of $\left(\epsilon_t\right)_{t\in\mathbb{Z}}$ process is given for by*

$$\begin{aligned}
E\left(\epsilon_t^{2m}(\vartheta)\right) &= \mu_\eta(2m)\exp\left(mA_\vartheta\right)\prod_{k=0}^{\infty} E\left\{\exp\left(m\gamma_\vartheta(\vartheta)\gamma^k\sum_{r=1}^{s}\gamma_s(r-\vartheta)\alpha\left(\vartheta-r+1\right)\log\eta_{t-k}^2\left(\vartheta-r\right)\right)\right\}, \\
&= \mu_\eta(2m)\exp\left(mA_\vartheta\right)\prod_{k=0}^{\infty} E\left\{\eta_0^{m\lambda_k(\vartheta)}\right\},
\end{aligned} \tag{1.12}$$

where

$$\lambda_k\left(\vartheta\right) = 2\gamma_\vartheta(\vartheta)\gamma^k\sum\nolimits_{r=1}^{s}\alpha\left(\vartheta-r+1\right)\gamma_s\left(r-\vartheta\right).$$

Proof. This is a direct consequence of proposition 1.3.2. □

Now, we present the following Remark.

Remark 1.3.1. *Proposition 1.3.3 clearly indicates that the existence of* $E\left(\epsilon_t^{2m}(\vartheta)\right)$ *depends on the distribution of* $(\eta_t)\,t$. *Hence,* $E\left(\epsilon t^{2m}(\vartheta)\right)$ *exists if and only if* $\prod_{k=0}^{\infty} E\left\{\eta_0^{m\lambda_k(\vartheta)}\right\}$ *exists and is finite.*

The following proposition provides a set of sufficient conditions (based on the distribution of (η_t)) for the existence of the unconditional moments of order $2m-th$ of (ϵ_t).

Proposition 1.3.4. *Consider the model* $P-logGARCH_s(1,1)$. *Assume that* $|\gamma|<1$, *then*

1. *If* $(\eta_t)_t$ *follows an i.i.d. generalized error distribution* (GED) *and* $\lambda_k(\vartheta)>-1$ *for* $k\in\mathbb{N}$. *Then* $E\left(\epsilon_t^{2m}(\vartheta)\right)<\infty$ *and is given by (1.12).*
2. *If* $(\eta_t)_t$ *follows an i.i.d. Student's* $t_{(v)}$*-distribution with degrees of freedom* $v>2$, $2m<v$ *and* $\lambda_k(\vartheta)>-1$ *for* $k\in\mathbb{N}$. *Then,* $E\left(\epsilon_t^{2m}(\vartheta)\right)<\infty$ *and is given by (1.12).*

Proof. Based on the equation (1.12), it is evident that in order for $E\left(\epsilon_t^{2m}(\vartheta)\right)$ to exist, it is imperative that $E\left(\eta_t^{2m}\right)$ remains finite. Hence, if $(\eta_t)_t \rightsquigarrow GED$, then

$$E\left\{|\eta_t|^{\lambda_k(\vartheta)}\right\}<+\infty, \text{ for all } \lambda_k(\vartheta)>-1 \text{ and } k\in\mathbb{N}.$$

By contrast, if

$$(\eta_t)_t \rightsquigarrow t_{(\vartheta)} \text{ with } \vartheta>2.$$

Then

$$E\left\{|\eta_t|^{\lambda_k(\vartheta)}\right\}<+\infty, \text{ for } -1<\lambda_k(\vartheta)<\vartheta \text{ and } \lambda_k(\vartheta)\neq 2.$$

So, if

$$\lambda_k(\vartheta)\in\,]-1,2[\,,$$

therefore

$$E\left\{|\eta_t|^{\lambda_k(\vartheta)}\right\}<+\infty, \quad \text{for each } k\in\mathbb{N}.$$

Additionally, since

$$\sum_{k=0}^{\infty}\left|E\left\{|\eta_0|^{\lambda_k(\vartheta)}\right\}-1\right| \text{ converges.}$$

We obtain

$$\prod_{k=0}^{\infty} E\left\{|\eta_0|^{\lambda_k(\vartheta)}\right\} \text{ converges.}$$

Hence

$$E\left(\epsilon_t^{2m}(\vartheta)\right)<\infty.$$

□

1.4 Estimation issues

In this section, we consider the Gaussian quasi-maximum likelihood estimator $(QMLE)$ to estimate the parameters in $P-logGARCHs(1.1)$ model gathered in the vector $\underline{\theta}'=(\underline{\omega}',\underline{\alpha}',\underline{\beta}')'\in\Theta\subset\mathbb{R}^{3s}$ where $\underline{\omega}=(\omega(1),\omega(2),...,\omega(s))'$, $\underline{\alpha}=(\alpha(1),\alpha(2),...,\alpha(s))'$ and $\underline{\beta}=(\beta(1),\beta(2),...,\beta(s))'$. The true parameter value denoted by $\underline{\theta}^0\in\Theta\subset\mathbb{R}^{3s}$ is unknown and therefore must be estimated. For this purpose, let $\underline{\epsilon}_t=(\epsilon_1,\epsilon_2,...,\epsilon_N)$, $N=ns$ be an observation of the causal and strictly periodically stationary solution of model (1.3) and let $h_t^2(\underline{\theta})$ be the conditional variance of ϵ_t given $\mathcal{F}_{t\text{-}1}$. The Gaussian quasi-likelihood function of $\underline{\theta}$ conditional on initial values $\epsilon_0^2,\tilde{h}_0^2(\underline{\theta})$ is given by

$$\tilde{L}_{ns}(\underline{\theta})=-(ns)^{-1}\sum_{t=1}^{n}\sum_{\vartheta=0}^{s-1}\tilde{l}_{st+\vartheta}(\underline{\theta}), \tag{1.13}$$

with

$$\tilde{l}_t(\underline{\theta})=\frac{\epsilon_t^2}{\tilde{h}_t^2(\underline{\theta})}+\log\tilde{h}_t^2(\underline{\theta}),$$

where $\log \widetilde{h}_t^2(\underline{\theta})$ is recursively defined for $t \geq 1$ by

$$\log \tilde{h}_t^2(\underline{\theta}) = \omega(t) + \alpha(t)\log \epsilon_{t-1}^2 + \beta(t)\log \tilde{h}_{t-1}^2(\underline{\theta}).$$

A Gaussian QML estimator of $\underline{\theta}$ is defined as any measurable solution $\hat{\underline{\theta}}_N$

$$\hat{\underline{\theta}}_N = \underset{\underline{\theta}\in\Theta}{\mathrm{Arg\,min}}\left(\tilde{L}_N(\underline{\theta})\right), \tag{1.14}$$

In view of the strong dependency of $\widetilde{h}_t^2(\underline{\theta})$ on initial values ϵ_0^2, let's denote $\tilde{h}_0^2(\underline{\theta})$ as $\tilde{l}_t(\underline{\theta})$ for $t \geq 1$. It is important to note that $\left(\tilde{l}_t(\underline{\theta})\right) t \geq 1$ is neither strictly periodically stationary nor periodically ergodic. Therefore, it is appropriate to replace the sequence $\left(\tilde{l}_t(\underline{\theta})\right)_{t\geq 1}$ with its strictly periodically stationary and periodically ergodic version. Hence, we elaborate on an approximate version:

$$L_{ns}(\underline{\theta}) = -\frac{1}{ns}\sum_{t=1}^{n}\sum_{\vartheta=0}^{s-1} l_{st+\vartheta}(\underline{\theta}).$$

The likelihood (1.3) is then computed with

$$l_t(\underline{\theta}) = \frac{\epsilon_t^2}{h_t^2(\underline{\theta})} + \log h_t^2(\underline{\theta}).$$

As already pointed by Francq and all [24], when some observations are equal to zero or are so close to zero, $\hat{\underline{\theta}}_N$ cannot be evaluated. Hence, a lower bound for the $|\varepsilon_t|' s$ is however imposed. 10^{-8} is often proposed as a well lower bound leaving unchanged the numerical illustration. Moreover, Sucavvat [45] have proposed four solutions for this situation among them, to treat zeros as missing values and hence an estimates issue may remedy the problem.

Note that the proofs of certain theorems are, by now, standard and follow from similar arguments used in demonstrating the strong consistency and asymptotic normality of the $P-GARCH$ models (see Aknouche and Guerbyenne [2] and/or for asymmetric $\log -GARCH$ Francq and all in [24]. The main aim here is to reveal the basic assumptions and to quantify the asymptotic distribution of our model. Hence, given several similarities between certain steps of the proof in the above references, we provide proofs only when it seems pertinent to us, and we refer to the above reference for further details.

1.4.1 Strong consistency of $QMLE$

To study the strong consistency and the asymptotic normality of QML estimator, we need to introduce some notations. Let B denote the log operator. For all $\vartheta \in \{1, ..., s\}$ we consider the polynomials

$$a_\vartheta(z) = \alpha_0(\vartheta)z^i,$$

$$b_\vartheta(z) = 1 - \beta_0(\vartheta)z^j.$$

Let $\gamma^{(s)}(\underline{\theta}_0)$ be the Lyapunov exponent defined in (1.8) evaluated at $\underline{\theta}_0$. Consider the following regularity assumptions:

A1. $\underline{\theta}_0 \in \Theta$ and Θ is compact.

A2. $\gamma^{(s)}\left(\underline{\theta}_0\right) < 0$ and $\forall\ \underline{\theta} \in \Theta$, $\prod_{\vartheta=1}^{s-1} |\beta(s-\vartheta)| < 1$.

A3. $E\left\{\left|\log \epsilon_t^2\right|\right\} < \infty$.

A4. The process $\left(\eta_t^2\right)_t$ has a nondegenerate distribution with $E\left\{\eta_0^2\right\} = 1$ and $E\left\{\left|\log \eta_0^2\right|^{\tau_0}\right\} < \infty$ for some $\tau_0 > 0$.

A5. $|\alpha_0(\vartheta)| + |\beta_0(\vartheta)| \neq 0$ for all $\vartheta \in \{1, ..., s\}$.

Several observations can be noted. Assumptions **A1**, **A2**, and **A5** closely resemble the prerequisites for ensuring the consistency of the $QMLE$ in $PGARCH$ models (as described by Aknouche and Guerbyenne [2]). Assumption **A4** is introduced to aid in identification.

We can now present our initial outcome.

Theorem 1.4.1. *Let* $\left(\hat{\underline{\theta}}_N\right)$ *be a sequence of QMLE satisfying (3.14). Then, under the assumptions* ***A1-A5****, almost surely (a.s.),* $\hat{\underline{\theta}}_N \to \underline{\theta}_0$ *as* $N \to \infty$.

Proof. We will establish the following intermediate results:

(i) $\lim_{N\to\infty}\sup_{\underline{\theta}\in\Theta}\left|\tilde{L}_N(\underline{\theta}) - L_N(\underline{\theta})\right| = 0 \ a.s.$

(ii) If $h_t^2(\underline{\theta}) = h_t^2(\underline{\theta}_0)$ $a.s.$ then $\underline{\theta} = \underline{\theta}_0$.

(iii) If $\theta \neq \underline{\theta}_0$, then $\sum_{\vartheta=1}^{s} E_{\underline{\theta}}\left\{l_{st+\vartheta}(\underline{\theta})\right\} > \sum_{\vartheta=1}^{s} E_{\underline{\theta}_0}\left\{l_{st+\vartheta}(\underline{\theta}_0)\right\}$.

(iv) For any $\underline{\theta} \neq \underline{\theta}_0$ there a neighborhood $\mathcal{V}(\underline{\theta})$ such that $\liminf_{N\to\infty}\inf_{\underline{\theta}^*\in\mathcal{V}(\underline{\theta})}\left(\tilde{L}_N(\underline{\theta}^*)\right) > \sum_{\vartheta=1}^{s} E_{\underline{\theta}_0}\left\{l_{\vartheta}(\underline{\theta}_0)\right\}$ a.s. The proof follows essentially the same arguments as Francq and Zakoïan [22] and Bibi and Aknouche [6]. (i) Iterating (1.3) that for almost all trajectories

$$\sup_{\underline{\theta}\in\Theta}\left|\log h_t^2(\underline{\theta}) - \log \tilde{h}_t^2(\underline{\theta})\right| \leq \rho^t K,$$

where $K > 0$ and $\rho \in]0,1[$. Moreover,

$$\begin{aligned}\frac{1}{t}\log\left|\frac{1}{h_t^2(\underline{\theta})} - \frac{1}{\tilde{h}_t^2(\underline{\theta})}\right| &= -\frac{1}{t}\sum_{j=0}^{t-2}\left\{\prod_{i=0}^{j-1}\gamma(t-i)\right\}W_{t-j-1} \\ &+ \frac{1}{t}\log\left|\exp\left\{-\prod_{i=0}^{t-2}\gamma(t-i)\log h_1^2(\underline{\theta})\right\} - \exp\left\{-\prod_{i=0}^{t-2}\gamma(t-i)\log \tilde{h}_1^2(\underline{\theta})\right\}\right|.\end{aligned}$$

The first component on the right-hand side of the given equation converges almost surely to zero (thanks to Cesàro's lemma). In contrast, the second component converges to $-\log\left|\prod_{\vartheta=1}^{s}\gamma(s-\vartheta)\right| < 0$ (utilizing the mean value theorem). We get

$$\frac{1}{t}\log\left|\frac{1}{h_t^2(\underline{\theta})} - \frac{1}{\tilde{h}_t^2(\underline{\theta})}\right| \leq \rho^t K.$$

Then, the first point is proved.

To prove (ii), suppose that

$$\log h_t^2(\underline{\theta}) = \log h_t^2(\underline{\theta}_0), \text{ a.s for some}\, t.$$

Then

$$\left(\frac{\alpha_{0,\vartheta}(B)}{\beta_{0,\vartheta}(B)} - \frac{\alpha_{\vartheta}(B)}{\beta_{\vartheta}(B)}\right)\log\epsilon_{st+\vartheta}^2 = \left(\frac{\omega(\vartheta)}{\beta_0(B)} - \frac{\omega_0(\vartheta)}{\beta_{0,\vartheta}(B)}\right), \text{ for all } 1 \leqslant \vartheta \leqslant s.$$

where the polynomials $\alpha_{0,\vartheta}(z) = \alpha_0(\vartheta)\,z$, $\alpha_{\vartheta}(z) = \alpha(\vartheta)\,z$,$\beta_{\vartheta}(z) = 1 - \beta(\vartheta)\,z$ and $\beta_{0,\vartheta}(z) = 1 - \beta_0(\vartheta)\,z$. So, by Assumption **A**2, the polynomial $\beta_{\vartheta}(z)$ is invertible.

Let e_t be any random variable $\sigma(\eta_{t-j}, j > 1)$ −measurable. If $\frac{\alpha_{0,\vartheta}(B)}{\beta_{0,\vartheta}(B)} \neq \frac{\alpha_{\vartheta}(B)}{\beta_{\vartheta}(B)}$, for some $\vartheta \in \{1,...,s\}$. Then, there exists some periodic coefficient $c(\vartheta)$ such that:

$$c(\vartheta)\log\epsilon_{st+\vartheta}^2 + e_{st+\vartheta-1} = 0, \ a.s,$$

which is equivalent to

$$c(\vartheta)\log\eta_{st+\vartheta}^2 + c(\vartheta)\log h_{st+\vartheta}^2 + e_{st+\vartheta-1} = 0.$$

The independence between η_t and $\left\{h_t^2, e_{t-1}\right\}$ and Assumption **A4,** implies that $c(\vartheta) = 0$ for all ϑ, which leads to a contradiction. Hence, $\alpha_0(\vartheta) = \alpha(\vartheta)$, $\beta_{0,\vartheta}(z) = \beta_{\vartheta}(z)$, $\forall |z| \leqslant 1$ and $\omega(\vartheta) = \omega_0(\vartheta)$ for all ϑ. The proofs of point (iii) as well as that of point (iv) are identical to those given by

Aknouche and all [2] for the $PGARCH$ model. We now complete the proof of Theorem 1.4.1. In view of assertions (i)-(iv), the proof of the theorem is completed by using the compactness of Θ. Indeed, for any neighborhood $\mathcal{V}(\underline{\theta}0)$ of $\underline{\theta}0$, we have:

$$\lim_{N\to\infty}\sup\inf_{\widetilde{\underline{\theta}}\in\mathcal{V}(\underline{\theta}_0)}\left\{-\frac{1}{N}\widetilde{L}_N(\widetilde{\underline{\theta}})\right\}\leq\lim_{N\to\infty}\left\{-\frac{1}{N}L_N(\underline{\theta}_0)\right\}=\lim_{N\to\infty}\left\{-\frac{1}{N}\widetilde{L}_N(\underline{\theta}_0)\right\}=\sum\nolimits_{\vartheta=1}^{s}E_{\underline{\theta}_0}\left\{l_\vartheta\left(\underline{\theta}_0\right)\right\}. \tag{1.15}$$

The compact Θ is recovered by a union of a neighborhood $\mathcal{V}(\underline{\theta}_0)$ of $\underline{\theta}_0$ and the set of neighborhoods $\mathcal{V}(\underline{\theta})$, $\underline{\theta}\in\Theta\backslash\mathcal{V}(\underline{\theta}_0)$, where $\mathcal{V}(\underline{\theta})$ fulfills the assertion (iv). Therefore, there exists a finite sub-covering of Θ by $\mathcal{V}(\underline{\theta}_0)$,$\mathcal{V}(\underline{\theta}_1)$,...,$\mathcal{V}(\underline{\theta}_k)$ such that

$$\inf_{\widetilde{\underline{\theta}}\in\mathcal{V}(\underline{\theta}_0)}\left\{-\frac{1}{N}L_N(\widetilde{\underline{\theta}})\right\}=\min_{i\in\{1,\ldots,k\}}\inf_{\widetilde{\underline{\theta}}\in\Theta\cap\mathcal{V}(\underline{\theta}_i)}\left\{-\frac{1}{N}\widetilde{L}_N(\widetilde{\underline{\theta}})\right\}.$$

From (1.15) and assertion (iv), the latter relation shows that $\widehat{\underline{\theta}}_N\in\mathcal{V}(\underline{\theta}_0)$ for N sufficiently large, which complete the proof of the theorem.

□

Now, we wish to highlight some key insights, background information, or important considerations in the following Remark.

Remark 1.4.1. *From assumptions **A1-A5**, it is clear that the above result remains true for the particular periodic logARCH case, i.e., when $\beta(\vartheta)=0$ for all ϑ.*

1.4.2 Asymptotic normality of $QMLE$

To show the asymptotic normality of $\hat{\underline{\theta}}_N$, the following additional assumptions are made.

A6. $\underline{\theta}_0\in\overset{0}{\Theta}$ where $\overset{0}{\Theta}$ denotes the interior of Θ and $k_4=E\left\{\eta_t^4\right\}<\infty$.

A7. There exists $\delta>0$ such that $E\left\{\exp\left(\delta\left|\log\eta_0^2\right|\right)\right\}<\infty$.

Condition **A6.** is a crucial assumption to obtain the asymptotic normality of $QMLE$, since it allows to validate the first-order condition on the maximizer of the log-likelihood function, while the Condition **A7.** is necessary for the existence of the limiting covariance matrix of the $QMLE$. The second main result of this section is the following.

Theorem 1.4.2. *Consider the $P-\log GARCH_s(1,1)$ model (3.6). Then, under the conditions **A1.-A7.**, as $N\to\infty$ we have $\sqrt{N}\left(\widehat{\underline{\theta}}_N-\underline{\theta}_0\right)\rightsquigarrow\mathcal{N}\left(\underline{O},\Sigma\left(\underline{\theta}_0\right)\right)$ where $\Sigma\left(\underline{\theta}_0\right)=(\kappa-1)J^{-1}$ with J is a positive definite and non-singular matrix given by*

$$J:=\sum_{\vartheta=1}^{s}E_{\underline{\theta}_0}\left\{\frac{\partial\log h_\vartheta^2\left(\underline{\theta}_0\right)}{\partial\underline{\theta}}\frac{\partial\log h_\vartheta^2\left(\underline{\theta}_0\right)}{\partial\underline{\theta}'}\right\}.$$

The proof of Theorem 1.4.2 rest classically on a Taylor series expansion of $\frac{\partial L_N}{\partial\underline{\theta}}(\underline{\theta})$ around $\underline{\theta}_0$, i.e.,

$$\underline{O}=(N)^{-\frac{1}{2}}\sum_{t=1}^{N}\frac{\partial l_t\left(\widehat{\underline{\theta}}_N\right)}{\partial\underline{\theta}}=(N)^{-\frac{1}{2}}\sum_{t=1}^{N}\frac{\partial l_t\left(\underline{\theta}_0\right)}{\partial\underline{\theta}}+\left((N)^{-1}\sum_{t=1}^{N}\frac{\partial^2 l_t\left(\underline{\theta}^*\right)}{\partial\underline{\theta}\partial\underline{\theta}'}\right)(N)^{\frac{1}{2}}\left(\widehat{\underline{\theta}}_N-\underline{\theta}_0\right)$$

where $\underline{\theta}^*$ is such that $\|\underline{\theta}^*-\underline{\theta}\|\leq\left\|\widehat{\underline{\theta}}_N-\underline{\theta}\right\|$. The gradient vector $\frac{\partial\log h_{st+\vartheta}^2\left(\underline{\theta}\right)}{\partial\underline{\theta}}$ and Hessian matrix $\frac{\partial^2\log h_{st+\vartheta}^2\left(\underline{\theta}\right)}{\partial\underline{\theta}\partial\underline{\theta}'}$ are given by

$$\left.\begin{aligned}\frac{\partial\log h_{st+\vartheta}^2\left(\underline{\theta}\right)}{\partial\underline{\theta}}&=\frac{\partial\omega\left(\vartheta\right)}{\partial\underline{\theta}}+\frac{\partial\alpha\left(\vartheta\right)}{\partial\underline{\theta}}\log\epsilon_{st+\vartheta-1}^2+\frac{\partial\beta\left(\vartheta\right)}{\partial\underline{\theta}}\log h_{st+\vartheta-1}^2\left(\underline{\theta}\right)+\beta(\vartheta)\frac{\partial\log h_{st+\vartheta-1}^2\left(\underline{\theta}\right)}{\partial\underline{\theta}}\\&=\underline{e}\left(\vartheta\right)+\underline{e}\left(\vartheta+s\right)\log\epsilon_{st+\vartheta-1}^2+\underline{e}\left(\vartheta+2s\right)\log h_{st+\vartheta-1}^2\left(\underline{\theta}\right)+\beta(\vartheta)\frac{\partial\log h_{st+\vartheta-1}^2\left(\underline{\theta}\right)}{\partial\underline{\theta}}\\\frac{\partial^2\log h_{st+\vartheta}^2\left(\underline{\theta}\right)}{\partial\underline{\theta}\partial\underline{\theta}'}&=\left(\underline{e}\left(\vartheta+2s\right)\frac{\partial\log h_{st+\vartheta-1}^2\left(\underline{\theta}\right)}{\partial\underline{\theta}'}\right)'+e\left(\vartheta+2s\right)\frac{\partial\log h_{st+\vartheta-1}^2\left(\underline{\theta}\right)}{\partial\underline{\theta}'}+\beta(\vartheta)\frac{\partial^2\log h_{st+\vartheta-1}^2\left(\underline{\theta}\right)}{\partial\underline{\theta}\partial\underline{\theta}'}\end{aligned}\right\}\tag{3.4}$$

where $\underline{e}\left(j\right)$ denotes a $3s\times1$ unit vector whose entries are all zeros except the one in the j row. Again, we will split the proof of Theorem 1.4.2 into several intermediate results gathered in following lemma.

Lemma 1.4.1. *Under the conditions* ***A1.–A7.****, we have*

1. $(N)^{-\frac{1}{2}}\sum_{t=1}^{N}\frac{\partial l_t(\underline{\theta}_0)}{\partial\underline{\theta}}\rightsquigarrow\mathcal{N}(\underline{O},(\kappa-1)J)$ as $N\to\infty$,

2. $\left\|\sum_{t=1}^{N}\frac{\partial^2 l_t\left(\widetilde{\underline{\theta}}_N\right)}{\partial\underline{\theta}\partial\underline{\theta}'}-J\right\|$ converges $a.s.$ to 0 for any consistency estimate $\tilde{\underline{\theta}}_N$ of $\underline{\theta}_0$ and J is invertible,

3. $(N)^{-\frac{1}{2}}\left\|\sum_{t=1}^{N}\frac{\partial l_t(\underline{\theta}_0)}{\partial\underline{\theta}}-\sum_{t=1}^{N}\frac{\partial\widetilde{l}_t(\underline{\theta}_0)}{\partial\underline{\theta}}\right\|$ converges $a.s.$ to 0 as $N\to\infty$.

Proof. Let us prove the point 1. Indeed, we have

$$\frac{\partial L_{ns}(\underline{\theta})}{\partial\underline{\theta}}=-(ns)^{-1}\sum_{k=1}^{n}\left\{\sum_{\vartheta=1}^{s}\left(1-\frac{\epsilon^2_{st+\vartheta}}{h^2_{st+\vartheta}(\underline{\theta})}\right)\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}\right\}=(ns)^{-1}\sum_{k=1}^{n}\left\{\sum_{\vartheta=1}^{s}\left(1-\eta^2_{st+\vartheta}\right)\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}\right\}.$$

Since $\log h_t^2$ is independent of η_t^2 with $E\left\{\eta_t^2\right\}=1$, the central limit theorem for the martingale difference $\sum_{\vartheta=1}^{s}\left(1-\eta^2_{st+\vartheta}\right)\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}$ applies whenever $(1-k_4)\sum_{\vartheta=1}^{s}E\left\{\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}'}\right\}$ exists. For any $\underline{\theta}\in\overset{0}{\Theta}$, the random vector $\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}$ is strictly periodically stationary solution of the first equation in (3.4), the Assumption **A2**. entails that $\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}$ is a linear combination of $\log\epsilon^2_{st+\vartheta-1}$ and $\log h^2_{st+\vartheta-1}(\underline{\theta})$. Proposition 1.3.3 and 1.3.4 ensures that, for any $m>0$ there exists a neighborhood $\mathcal{V}$ of $\underline{\theta}_0$ such that $E\left\{\sup_{\vartheta}\left|\log h^2_{st+\vartheta}(\underline{\theta})\right|^m\right\}<+\infty$ for all $\vartheta\in\{1,...,s\}$ and $\log\varepsilon^2_{st+\vartheta}$ admit moment of any order. Thus, for any $m>0$ there exists $\mathcal{V}$ such that $E\left\{\sup_{\mathcal{V}}\left\|\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}\right\|^m\right\}<+\infty$.

We turn to prove the point 2. Then, we have

$$\frac{\partial^2 l_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}\partial\underline{\theta}'}=\left(1-\frac{\eta^2_{st+\vartheta}h^2_{st+\vartheta}(\underline{\theta}_0)}{h^2_{st+\vartheta}(\underline{\theta})}\right)\frac{\partial^2\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}\partial\underline{\theta}'}+\frac{\eta^2_{st+\vartheta}h^2_{st+\vartheta}(\underline{\theta}_0)}{h^2_{st+\vartheta}(\underline{\theta})}\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}'}.\quad(3.5)$$

From (3.4) and by Assumption **A2.**, we get

$$\frac{\partial^2\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}\partial\underline{\theta}'}$$
$$=\sum_{k=0}^{\infty}\left\{\prod_{i=0}^{k-1}\beta(\vartheta-i)\right\}\left(\left(\underline{e}(\vartheta-k+2s)\frac{\partial\log h^2_{st+\vartheta-k-1}(\underline{\theta})}{\partial\underline{\theta}'}\right)'+\underline{e}(\vartheta-k+2s)\frac{\partial\log h^2_{st+\vartheta-k-1}(\underline{\theta})}{\partial\underline{\theta}'}\right).$$

Since, we can always choose a neighborhood $\mathcal{V}$ such that $\sup_{\vartheta}\left\|\frac{\partial\log h^2_{st+\vartheta-k-1}(\underline{\theta})}{\partial\underline{\theta}'}\right\|\in\mathbb{L}_m$. This entails that $\frac{\partial^2\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}\partial\underline{\theta}'}$ is integrable. Moreover, Cauchy-Schwarz inequality applied on the right-hand side of (3.5) yield the integrability of $\sup_{\vartheta}\frac{\partial^2 l_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}\partial\underline{\theta}'}$ Hence, $\sup_{\vartheta}\left\|\frac{\partial^2 L_t\left(\widehat{\underline{\theta}}_N\right)}{\partial\underline{\theta}\partial\underline{\theta}'}-E\left\{\frac{\partial^2 l_t(\underline{\theta})}{\partial\underline{\theta}\partial\underline{\theta}'}\right\}\right\|\to 0$ a.s. The invertibility of J follows by the same arguments as in Francq and Zakoian [22]. Finally, we prove point 3 from (3.4). We have

$$\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}-\frac{\partial\log\widetilde{h}^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}=\underline{e}(\vartheta+2s)\left(\log h^2_{st+\vartheta}(\underline{\theta})-\log\widetilde{h}^2_{st+\vartheta}(\underline{\theta})\right)$$
$$+\beta(\vartheta)\left(\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}-\frac{\partial\log\widetilde{h}^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}\right).$$

So, it can be shown that under **A2.,** there exists $\rho\in\,]0,1[$ and $K>0$ such that

$$\left\|\frac{\partial\log h^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}-\frac{\partial\log\widetilde{h}^2_{st+\vartheta}(\underline{\theta})}{\partial\underline{\theta}}\right\|\leq K\rho^t,\quad\text{for all }\vartheta\in\{1,...,s\}.$$

Point 3 easily follows. ☐

1.5 Monte Carlo experiment

In this section, we describe the performance of the finite sample properties of the $QMLE$ of the unknown vector $\underline{\theta}$ of parameters involved in $P-logGARCH_s(1,1)$ model based on Monte Carlo experiments. To this end, we simulate $T = 500$ replications for different moderate sample sizes $n \in \{100, 200, 400\}$ via some innovations processes. For instance, standardized innovations $\mathcal{N}(0,1)$, Student $t_{(5)}$ and GED with shape parameter $\tau = 1$ (Laplace distribution). These innovations are nondegenerated and satisfies of course the assumption **A**4.

Remark 1.5.1. *Let introduce the probability distribution function*

$$F(x) = dF_{\mu,\sigma,\tau}(x) = \frac{\exp\left\{-\frac{1}{2}\left|\frac{x-\mu}{\sigma}\right|^{1/\tau}\right\}}{2^{\tau+1}\sigma\Gamma(\tau+1)}dx,$$

where μ, σ and τ are respectively the location, scale and shape parameters belonging to $\mathbb{R}$, $]0,\infty[$ and in $]0,\infty[$.

The GED is a symmetrical unimodal member of the exponential family with the probability distribution function $F(x)$. It incorporates a variety of distributions, for instance, Normal Distribution for $\tau = 0.5$, double exponential or Laplace distribution for $\tau = 1$. The kurtosis of such distribution is

$$K = \frac{\Gamma(\tau)\Gamma(5\tau)}{\{\Gamma(3\tau)\}^2}.$$

Therefore, as τ increased the density gets flatter, while when $\tau \to 0$ the distribution inclines towards the uniform. The standardized version is obtained by setting

$$\mu = 0 \text{ and } \sigma = \sqrt{2^{-2\tau}\Gamma(\tau)/\Gamma(3\tau)}.$$

(See Marin [37] and Nelson [38]).

Additionally, we deduce the following Remark.

Remark 1.5.2. *It is worth noting that all odd moments of GED clearly vanish by symmetry. Meanwhile, the even moments $\mu_k(\tau)$ are given by*

$$\mu_k(\tau) = 2^{k\tau}\sigma^k\frac{\Gamma(\tau(k+1))}{\Gamma(\tau)}.$$

Furthermore, since $|\Gamma(x)|$ is bounded when $x \in [1,2]$, then $\mu_k(\tau)$ is bounded if the shape parameter $\tau \in \left[\frac{1}{k+1}, \frac{2}{k+1}\right]$.

We conclude this section with the following.

Remark 1.5.3. *The $t_{(v)}$ is a unimodal and symmetric distribution with density*

$$f_{\mu,\beta,v}(x) = \frac{\Gamma(\frac{v+1}{2})}{\Gamma\left(\frac{v}{2}\right)\sqrt{v\beta\pi}}(1+\frac{(x-\alpha)^2}{\beta v})^{-\left(\frac{v+1}{2}\right)},$$

where α, β and v are the location, scale and shape parameters. Hence for standardization it is required that mean$(t_{(v)}) = \alpha = 0$, Var$\left(t_{(v)}\right) = \dfrac{\beta v}{v-2} = 1$ thus $\beta = \dfrac{v-2}{v}$, $v > 2$.

The parameter vector $\underline{\theta}$ adheres to strict periodic stationarity, detailed at the end of each table. Empirical results, produced using MATLAB scripts, provide insights into model performance. The tables present the average parameter estimates across N simulations, with columns displaying the results. To evaluate the Quasi Maximum Likelihood Estimator ($QMLE$) performance, we furnish the root mean square error ($RMSE$) for each $\hat{\theta}_n(i)$, where $i = 1, ..., s$ (enclosed in brackets). Moreover, we incorporate the asymptotic distributions of $\hat{\theta}_n(\vartheta)$, where $\vartheta = 1, ..., s$, across the N simulations, accompanied by boxplot summaries, aligned with the corresponding table

1.5.1 Standard $\log GARCH(1,1)$ model

Our initial practical illustration, aiming to elucidate our theoretical analysis, involves the standard $\log GARCH(1,1)$ model. The parameter vector for this model is denoted as $\underline{\theta} = (\omega, \alpha, \beta)'$, chosen to subject the condition $\gamma = \log|\alpha + \beta| < 0$. The outcomes of our simulations, conducted under various innovations in accordance with two distinct Models, are detailed in Table 1.1.

n	$\mathcal{N}(0,1)$ $\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$t_{(5)}$ $\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	GED $\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$
100	1.0437	0.5103	0.3220	1.0206	0.5006	0.3373	0.9869	0.4910	0.3426
	(0.0220)	(0.0023)	(0.0051)	(0.0216)	(0.0023)	(0.0050)	(0.0225)	(0.0032)	(0.0063)
200	1.0281	0.5066	0.3325	1.0137	0.4994	0.3429	0.9921	0.4939	0.3458
	(0.0126)	(0.0014)	(0.0029)	(0.0130)	(0.0015)	(0.0030)	(0.0138)	(0.0019)	(0.0037)
300	1.0211	0.5050	0.3370	1.0090	0.5007	0.3441	0.9926	0.4955	0.3487
	(0.0092)	(0.0010)	(0.0023)	(0.0092)	(0.0012)	(0.0024)	(0.0101)	(0.0014)	(0.0027)
Model(1) : $\underline{\theta} = (1.00, 0.50, 0.35)'$									
100	1.0011	−0.7423	0.1396	0.9914	−0.7467	0.1513	0.9712	−0.7550	0.1639
	(0.0149)	(0.0023)	(0.0051)	(0.0180)	(0.0024)	(0.0056)	(0.0343)	(0.0035)	(0.0077)
200	1.0028	−0.7447	0.1434	0.9942	−0.7487	0.1528	0.9828	−0.7536	0.1594
	(0.0098)	(0.0014)	(0.0031)	(0.0117)	(0.0015)	(0.0037)	(0.0219)	(0.0022)	(0.0044)
300	1.0014	−0.7456	0.1459	0.9944	−0.7476	0.1519	0.9862	−0.7525	0.1573
	(0.0070)	(0.0011)	(0.0024)	(0.0082)	(0.0013)	(0.0029)	(0.0167)	(0.0016)	(0.0032)
Model(2): $\underline{\theta} = (1.00, -0.75, 0.15)'$									

Table 1.1: Results of estimating the $\log GARCH(1,1)$ according to different innovations

The asymptotic distribution of the sequences $(\sqrt{n}(\hat{\underline{\theta}}_n(i) - \underline{\theta}(i)))_{n\geq 1})$, $i = 1, ..., 3$ followed by their boxplot summary associated to different innovations of Model(1) of Table 1.1 are shown in Figure 1.1.

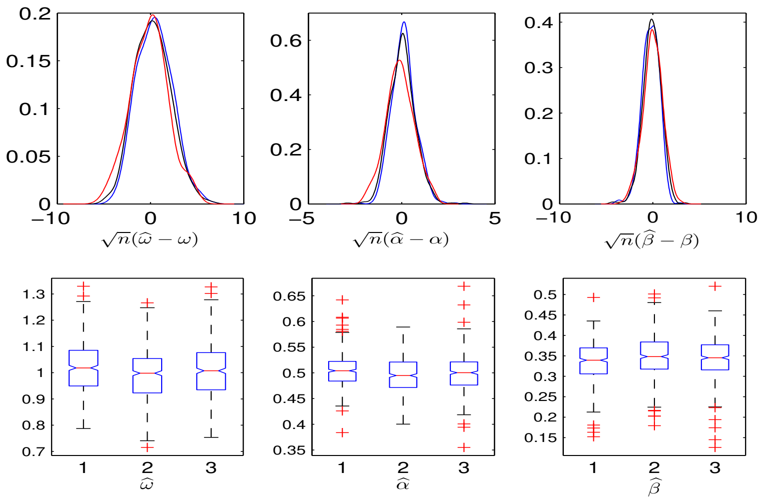

Figure 1.1: Top panels: the asymptotic distribution of $\sqrt{n}(\underline{\hat{\theta}}_n(i) - \underline{\theta}(i))$ associated to $Normal$ (red curve), GED (bleu curve) and $Student$ (blackline curve). Bottom panels: Boxplot summary of $\underline{\hat{\theta}}_n(i)$, $i = 1, ..., 3$ (1 for $Normal$, 2 for GED and 3 for $Student$) according to Model(1) of Table 1.1.

Now, a few comments can be made: By inspecting the results presented in table 1.1 for model(1) and model(2), it becomes apparent that the performance of the Quasi Maximum Likelihood Estimator (QML) is significantly less robust under $t_{(5).}$ and GED innovations compared to $\mathcal{N}(0,1)$. Moreover, a general trend emerges where the root mean square error ($RMSE$) associated with diverse innovations tends to diminish with increasing sample sizes. This trend is visually reinforced by the asymptotic distribution plots in figure 1.1, revealing fatter tails (positive kurtosis or leptokurtic) for these distributions. Additionally, figure 1.1, depicting boxplots of the $QMLE$ outcomes under various innovations, underscores noteworthy dissimilarities in elementary statistics, with $\mathcal{N}(0,1)$ exhibiting fewer outliers compared to other innovations.

1.5.2 Periodic $\log GARCH_s(1,1)$ model

The second example of our Monte Carlo experiment is devoted to estimate the periodic $P{-}\log GARCH_s(1,1)$ model with $s=2$ .i.e., $\epsilon_t = \eta_t h_t$ and

$$\log h_{2t+\vartheta}^2(\vartheta) = \omega(\vartheta) + \alpha(\vartheta)\log \epsilon_{2t+\vartheta}^2(\vartheta-1) + \beta(\vartheta)\log h_{2t+\vartheta}^2(\vartheta-1).$$

This situation is raised in modelling some daily returns when we suspect the so-called "Monday effect" (opening price) of day-of-the week seasonality (see for instance Franses and Raap (2000)). The vector of parameters to be estimated is thus $\underline{\theta} = (\underline{\omega}', \underline{\alpha}', \underline{\beta}')'$ where $\underline{\omega}' = (\omega(1), \omega(2))$, $\underline{\alpha}' = (\alpha(1), \alpha(2))$ and $\underline{\beta}' = (\beta(1), \beta(2))$, are chosen to ensure the the strict periodic stationarity condition of our model. To this end, we suggest that $\log|\alpha(1)+\beta(1)| + \log|\alpha(2)+\beta(2)| < 0$. So, the results of simulation according to two models are given in Table 1.2 below.

n	ϑ	$\mathcal{N}(0,1)$ $\underline{\hat{\omega}}$	$\mathcal{N}(0,1)$ $\underline{\hat{\alpha}}$	$\mathcal{N}(0,1)$ $\underline{\hat{\beta}}$	$t_{(5)}$ $\underline{\hat{\omega}}$	$t_{(5)}$ $\underline{\hat{\alpha}}$	$t_{(5)}$ $\underline{\hat{\beta}}$	GED $\underline{\hat{\omega}}$	GED $\underline{\hat{\alpha}}$	GED $\underline{\hat{\beta}}$
200	1	.9374(.1326)	.7617(.0177)	− .1765(.0491)	.9803(.1928)	.7623(.0215)	− .1723(.0624)	.9667(.2466)	.7556(.0209)	− .1754(.0590)
	2	.9449(.1094)	− .7392(.0189)	.1448(.0375)	.9843(.1656)	− .7398(.0202)	.1491(.0513)	.9419(.1814)	− .7580(.0194)	.1595(.0485)
300	1	.9545(.0819)	.7548(.0100)	− .1577(.0253)	.9903(.1260)	.7590(.0122)	− .1624(.0329)	.9863(.1670)	.7526(.0113)	− .1614(.0353)
	2	.9736(.0683)	− .7406(.0111)	.1431(.0237)	.9946(.1048)	− .7428(.0145)	.1481(.0301)	.9557(.1107)	− .7586(.0118)	.1602(.0271)
400	1	.9893(.0537)	.7605(.0064)	− .1685(.0177)	.9919(.0715)	.7558(.0092)	− .1604(.0238)	.9855(.7542)	.7542(.0085)	− .1611(.0229)
	2	.9845(.0450)	− .7441(.0067)	.1434(.0154)	.9941(.0774)	− .7417(.0100)	.1437(.0227)	.9865(.7514)	− .7514(.0087)	.1540(.0191)
Model (1): $\underline{\omega} = (1.00, 1.00)$, $\underline{\alpha} = (0.75, -0.75)$, $\underline{\beta} = (-0.15, 0.15)$										
200	1	.9408(.1356)	.7566(.0158)	.5046(.0423)	.9389(.2068)	.7504(.0186)	.5247(.0566)	.9765(.2272)	.7434(.0192)	.5076(.0462)
	2	.9658(.1177)	.7627(.0183)	.3402(.0282)	.9992(.1856)	.7589(.0209)	.3428(.0347)	.9177(.2142)	.7305(.0205)	.3725(.0326)
300	1	.9625(.0806)	.7546(.0084)	.5041(.0221)	.9707(.1317)	.7522(.0106)	.5124(.0289)	.9777(.1605)	.7463(.0105)	.5085(.0283)
	2	.9848(.0747)	.7611(.0108)	.3410(.0174)	.9973(.1202)	.7560(.0143)	.3466(.0218)	.9549(.1274)	.7381(.0123)	.3616(.0189)
400	1	.9836(.0559)	.7564(.0056)	.4987(.0162)	.9736(.0983)	.7519(.0082)	.5051(.0250)	.9828(.1119)	.7503(.0078)	.5008(.0197)
	2	.9914(.0497)	.7585(.0066)	.3422(.0112)	.9991(.0862)	.7582(.0096)	.3426(.0159)	.9890(.0859)	.7462(.0087)	.3541(.0137)

Model (1): $\underline{\omega} = (1.00, 1.00)$, $\underline{\alpha} = (0.75, -0.75)$, $\underline{\beta} = (-0.15, 0.15)$

Table 1.2: Results of estimating the $P - \log GARCH_2(1,1)$ according to different innovations

The asymptotic distribution of the sequences $(\sqrt{n}(\hat{\underline{\theta}}_n(i) - \underline{\theta}(i)))_{n\geq 1})$, $i = 1, ..., 6$ followed by their boxplot summary associated to different innovations of Model(1) of Table 1.2 are shown in Figure 1.2.

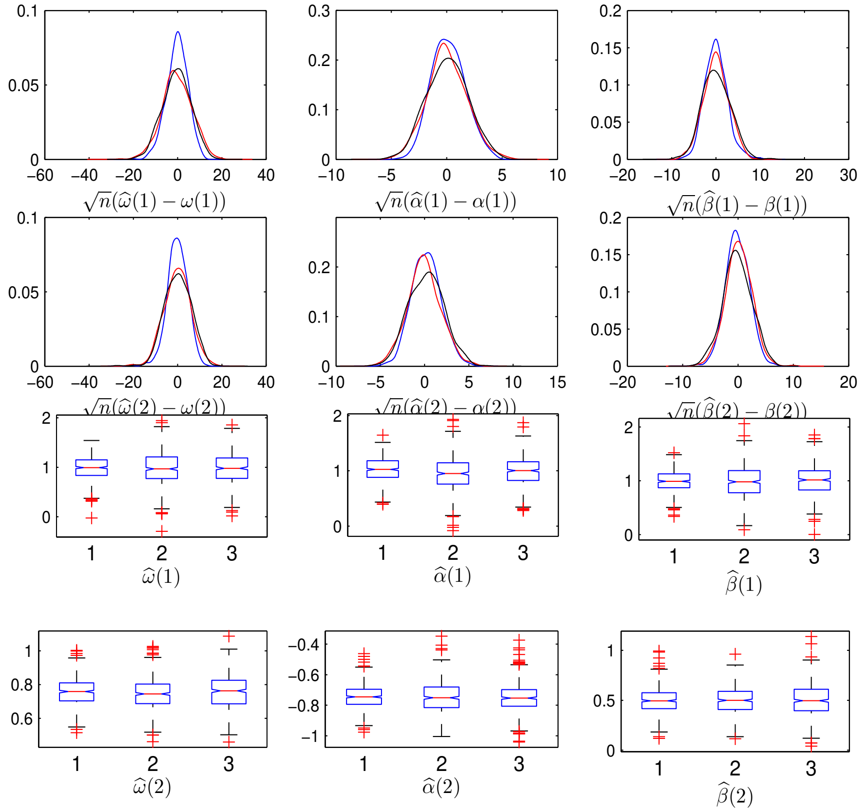

Figure 1.2: Top panels: the asymptotic distribution of $\sqrt{n}(\hat{\underline{\theta}}_n(i) - \underline{\theta}(i))$ associated to *Normal* (red curve), *GED* (bleu curve) and *Student* (blackline curve). Bottom panels: Boxplot summary of $\hat{\underline{\theta}}_n(i)$, $i = 1, ..., 6$ (1 for *Normal*, 2 for *GED* and 3 for *Student*) according to Model(1) of Table 1.2.

1.5.3 Comments

A concise analysis of the Monte Carlo experiment results reveals the following:

1. Table 1.2 showcase parameter estimates for the $P - \log GARCH_s(1,1)$ model with $s = 2$, fitted using Model 1 and Model 2. The estimations, based on 500 independent simulations with various innovations, indicate that the performance of the Quasi Maximum Likelihood Estimator (QML) associated with $\mathcal{N}(0,1)$ and GED innovations is relatively subpar compared to $t_{(5)}$ innovations. This trend persists even when varying the degree of freedom v for $t_{(v)}$ (results not presented here). In general, it is evident that the parameters for these models are well-estimated, showing no significant deviations across three types of error innovations: $\mathcal{N}(0,1)$, $t_{(5)}$, and GED. However, some estimates have moderately high standard deviations, attributed to the relatively small sample size n.

2. Regarding the asymptotic kernel distribution of $\sqrt{n}(\widehat{\theta}_n(i) - \theta(i))$ for $i = 1, \ldots, 6$, as illustrated in figure 1.2, it is apparent that the $P - \log GARCH_2(1,1)$ model generates kernels with significantly flatter

3. Notably, assumptions **A1** to **A5** crucial for ensuring consistency are clearly satisfied. Additionally, the conditions **A6** and **A7**, necessary for establishing asymptotic normality, are also met. Moreover, upon examining the boxplots presented in figure 1.2, it becomes evident that the elementary statistics of the Quasi Maximum Likelihood Estimator ($QMLE$) exhibit significant differences under various innovations. Specifically, the $t_{(5)}$ distribution displays fewer outliers compared to the others, as expected due to the fatter tails of the $t_{(5)}$ distribution.

1.6 Case study

In recent decades, many researchers addressed the question of day-of-the-week seasonality in returns and volatility. In particular, it was observed that in many stock markets, the Monday returns are often lower than those of other days, this is referred to as the Monday effect (see for instance Franses and Paap [20]. In this section, we consider the series of the daily exchange rates of the Algerian Dinar with respect to European currency ($Euro$) denoted by $y_t^{(e)}$ and the American Dollar denoted by $y_t^{(d)}$.

1.6.1 Data description

We consider returns series $r_t^{(e)} = 100\times(\log\left(y_t^{(e)}\right) - \log\left(y_{t-1}^{(e)}\right))$ and $r_t^{(d)} = 100\times(\log\left(y_t^{(d)}\right) - \log\left(y_{t-1}^{(d)}\right))$ of daily exchange rates of Algerian dinar with respect to the Euro and the Dollar. The observation cover the period from January $3, 2000$ to September $29, 2011$. Since there are some weeks comprise less than five observations (due to legal holidays), we remove the entire weeks with less than five data available rather than estimating the "pseudo-missing" observations by an ad-hoc method. Thus, the final length of transformed data is 3055 observations uniformly distributed on 611 weeks. The elementary statistics in data are summarized in Table 1.3 below

Series	mean	median	mode	skewness	kurtosis	JBtes	LB(Q(12),Q(24))
$y_t^{(e)}$	88.61181	91.09945	69.7347	-0.518144	2.132958	232.4666	$10^4 \times (3.6065, 7.0724)$
$y_t^{(d)}$	73.45113	73.12610	79.9396	-0.600469	3.764200	258.0098	$10^4 \times (3.5888, 7.0054)$

Table 1.3: Summary Statistics for daily spot prices $y_t^{(e)}$ and $y_t^{(d)}$.

In Table 1.3 the difference between means, medians and modes implies that the series are not symmetric. The high kurtosis computed in these series, being leptokurtic, implies that the distribution of the series have fatter tails, and a more sensitive peak around the mean, when compared to the normal distribution. JBtes (Jarque-Bera test) and LB(Q(12), Q(24)) for normality and autocorrelation tests show that both returns are neither normally distributed nor serially correlated for the instance 12 and 24 lags. Moreover, the results shown in Table 1.4 examine the effect of heteroscedasticity in the series $\left(r_t^{(e)}\right)$ and $\left(r_t^{(d)}\right)$.

	$r_t^{(e)}$				$r_t^{(d)}$			
lags	10	15	20	25	10	15	20	25
$ARCH$ (text) statistics	152.3993	200.3745	244.6458	266.6962	245.6729	249.3297	344.0355	346.1818
Critical value	18.3070	24.9958	31.4104	37.6525	18.3070	24.9958	31.4104	37.6525
$P-value$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 1.4: $ARCH$ effect test of daily returns $\left(r_t^{(e)}\right)$ and $\left(r_t^{(d)}\right)$.

The results of Table 1.4 can be summarized as: since the p-value is less than 0.05, the $ARCH$ statistics is greater than the critical value at 95% confidence level. These imply that there is a strong evidence for rejecting the null hypothesis of no $ARCH$ effect. The rejection indicates the existence of $ARCH$ effects in the returns series and therefore the variance of such a returns is not constant. The test was implemented in $MATLAB$ with "*archtest*" function for the returns. Figure 1.3 displays the plots of the series (y_t) and its returns (r_t) corresponding to foreign exchange of EUR/DZD (series superscripted by (e)) and those corresponding to USD/DZD (series superscripted by (d)).

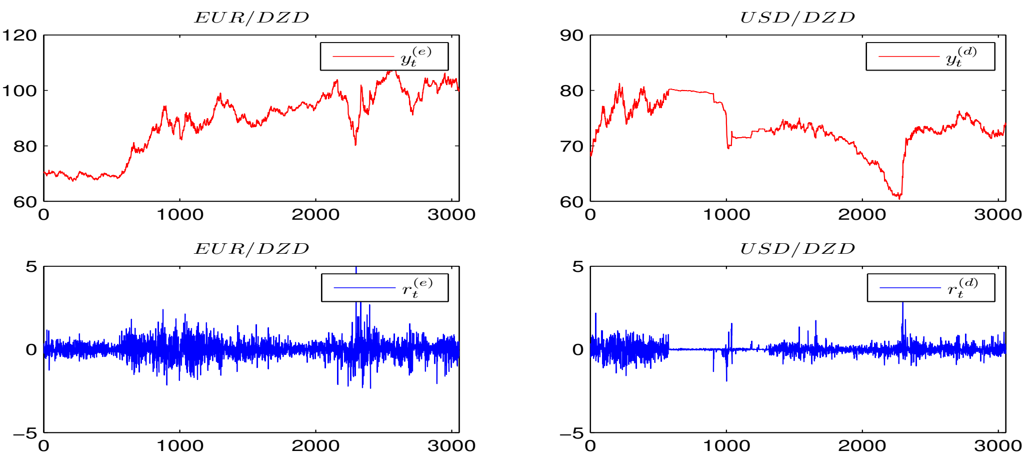

Figure 1.3: Left panel displays the series y_t and r_t corresponding to EUR/DZD. Right panel display similar series correspont to USD/DZD.

Figure 1.4 displays the sample autocorrelations functions (ACF) of the series $(r_t)_{t\geq 1}$, $\left(r_t^2\right)_{t\geq 1}$ and $(|r_t|)_{t\geq 1}$ computed at 30 lags.

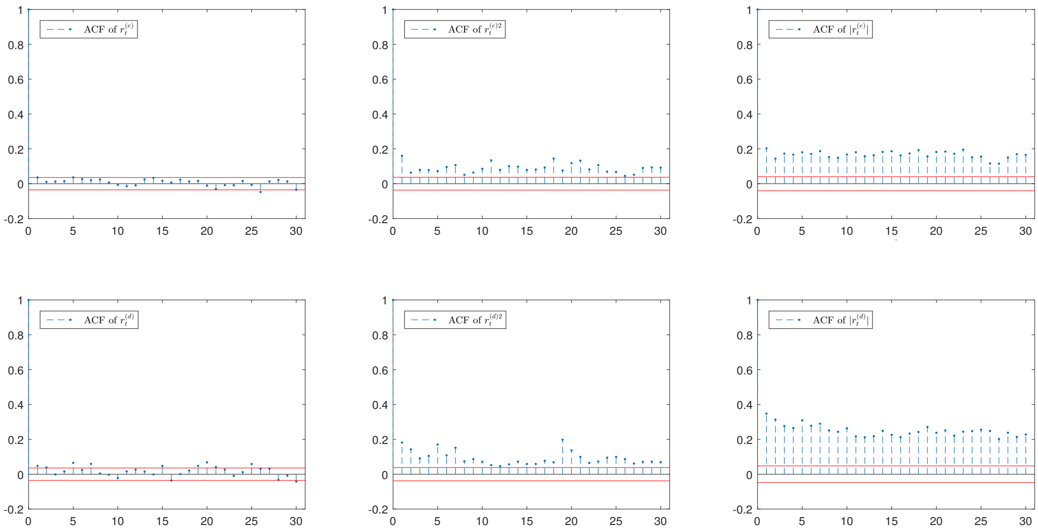

Figure 1.4: Top panel: Sample autocorrelations of returns associated to Euro superscripted by (e). Bottom panel: Sample autocorrelations of returns associated to dollar superscripted by (d).

From Figure 1.4, we can see that the log returns (r_t) show no evidence of serial correlation, but the squared and absolute returns are positively autocorrelated. Also, the decay rates of the sample autocorrelations of $\left(r_t^2\right)$ and $(|r_t|)$ appear to be violated compared with the correlation associated to an $ARMA$ process suggesting possibly a non linear behavior for modelling purpose.

1.6.2 Modeling

The $P-\log GARCH_s\,(1,1)$ and its competitor $P-EGARCH_s\,(1,1)$ model are used to depict more stylized facts for some financial series. These models are particular models from: $\epsilon_t = \eta_t h_t$ and $\log h_t^2 = \omega(t) + \mu\,(t)\, g_t\,(\eta_{t-1}) + \beta(t) \log h_{t-1}^2,\ t \in \mathbb{Z}$, where $g_t\,(\eta_t)$ is well defined function of η_t and some periodic parameters $\omega(t), \mu(t), \alpha(t), \gamma(t)$ and $\beta(t)$. For example setting $g_t\,(\eta_{t-1}) = \alpha(t)\eta_{t-1} + \gamma(t)\,(|\eta_{t-1}| - E\,\{|\eta_{t-1}|\})$ yield an extension of $EGARCH\,(1,1)$ model proposed by Nelson [38] to the periodic one. On the other hand, setting $g_t\,(\eta_{t-1}) = \alpha(t) \log \eta_{t-1}^2 = \alpha(t)\left(\log \epsilon_{t-1}^2 - \log h_{t-1}^2\right)$ we obtain the periodic $\log GARCH_s\,(1,1)$. For ease of exposition, we suppose that $\mu\,(t) = 1$ for all $t \in \mathbb{Z}$, i.e.,

$$\epsilon_t = \eta_t h_t \text{ and } \log h_t^2 = \omega(t) + \alpha(t)\eta_{t-1} + \gamma(t)\,(|\eta_{t-1}| - E\,\{|\eta_{t-1}|\}) + \beta(t) \log h_{t-1}^2,\ t \in \mathbb{Z}$$

where $\omega(t), \alpha(t), \gamma(t)$ and $\beta(t)$ are defined in a similar way as in periodic $\log GARCH_s\,(1,1)$. Additionally, for any $\vartheta = 1, ..., s$, $\alpha(\vartheta)$ is the leverage effect parameter whenever it is negative and $\prod_{\vartheta=1}^{s} |\beta\,(\vartheta)| < 1$ to ensure the strict periodic stationarity condition. We propose a Sunday and Monday effect on price of the Dinar against the Euro and Dollar, i.e. when the parameters $\omega(t), \alpha(t), \gamma(t)$ and $\beta(t)$ are defined as in Model(2). The tables below establish the estimation of $P-\log GARCH_2\,(1,1)$ and $P-EGARCH_2\,(1,1)$ models fitted on $\left(r_t^{(e)}\right)$ and $\left(r_t^{(d)}\right)$. The column $P-value$ gives the $p-values$ of the Wald test for the nullity of the components of the vector $\underline{\theta}$.

The first results of parameters estimation and the estimated standard deviation (results displayed into brackets) when the innovation is $\mathcal{N}(0,1)$ are presented in Table 1.5

	$P-\log GARCH_2(1,1)$				$P-EGARCH_2(1,1)$				
$\mathcal{N}(0,1)$	$\underline{\omega}$	$\underline{\alpha}$	$\underline{\beta}$	$P-value$	$\underline{\omega}$	$\underline{\alpha}$	$\underline{\gamma}$	$\underline{\beta}$	$P-value$
$r_t^{(e)}$	0.0942	0.0152	1.0377	0.0000	0.0928	-0.0263	0.1399	1.0702	0.0000
	(0.0234)	(0.0036)	(0.0828)		(0.0911)	(0.0141)	(0.0218)	(0.0498)	
	0.0174	0.0176	0.9088		-0.2664	0.0191	0.1563	0.9407	
	(0.0236)	(0.0052)	(0.0371)		(0.0495)	(0.0095)	(0.0210)	(0.0290)	
$r_t^{(d)}$	0.0729	0.0390	0.7832	0.0000	-0.0660	0.0061	0.2177	0.7509	0.0000
	(0.0317)	(0.0034)	(0.0250)		(0.0698)	(0.0154)	(0.0233)	(0.0171)	
	0.0371	0.0048	1.0566		0.0649	0.0307	0.2962	1.1500	
	(0.0251)	(0.0053)	(0.0262)		(0.0638)	(0.0142)	(0.0183)	(0.0174)	

Table 1.5: Parameters estimation of $P-\log GARCH_2\,(1,1)$ and $P-EGARCH_2\,(1,1)$ models fit to $\left(r_t^{(e)}\right)$ and $\left(r_t^{(d)}\right)$ according to $\mathcal{N}(0,1)$ innovation.

The plots the squared returns and the estimated volatilities associated to $\mathcal{N}(0,1)$ innovation are showed in Figure 1.5 below.

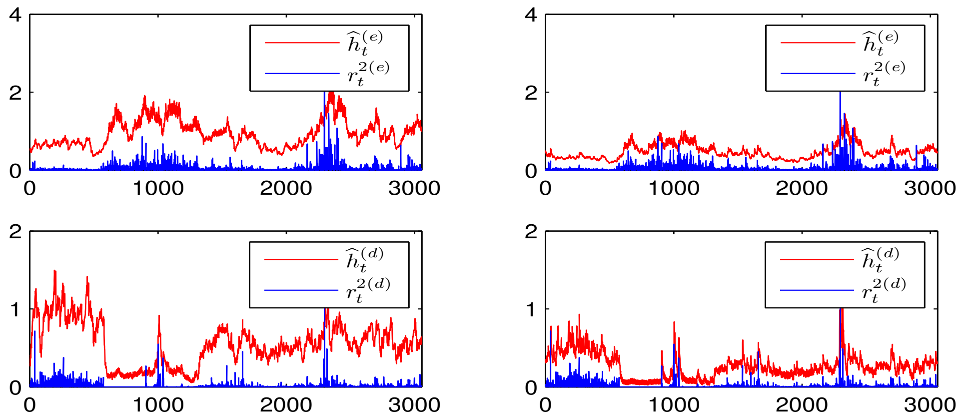

Figure 1.5: Dark blue: squared returns, light red: volatilities estimates according to $P-\log GARCH_2\,(1,1)$ (left) and to $P-EGARCH_2\,(1,1)$ (right) models with $\mathcal{N}(0,1)$ innovation.

The second results of parameters estimation of the series $\left(r_t^{(e)}\right)$ and $\left(r_t^{(d)}\right)$ according to $P-\log GARCH_2\left(1,1\right)$ and $P-EGARCH_2\left(1,1\right)$ when the innovation is $t_{(5)}$ are presented in 1.6.

$t_{(5)}$	$P-\log GARCH_2(1,1)$ ω	α	β	$P-value$	$P-EGARCH_2(1,1)$ ω	α	γ	β	$P-value$
$r_t^{(e)}$	0.0920	0.0309	1.0107	0.0000	0.0146	−0.0044	0.1960	1.0126	0.0000
	(0.0881)	(0.0086)	(0.0959		(0.0793)	(0.0335)	(0.0543)	(0.0806	
	0.0065	0.0342	0.9265		−0.0631	0.0267	0.2034	0.9826	
	(0.0590)	(0.0081)	(0.0553)		(0.0538)	(0.0235)	(0.0469)	(0.0522)	
$r_t^{(d)}$	0.0772	0.1247	0.8637	0.0000	−0.0849	0.0231	0.6098	0.9109	0.0000
	(0.1118)	(0.0147)	(0.0393)		(0.0907)	(0.0438)	(0.0712)	(0.0311)	
	0.1087	0.0691	0.9126		−0.0225	0.0831	0.3734	1.0462	
	(0.0786)	(0.0146)	(0.0287)		(0.0705)	(0.0315)	(0.0495)	(0.0240)	

Table 1.6: Parameters estimation of $P-\log GARCH_2\left(1,1\right)$ and $P-EGARCH_2\left(1,1\right)$ models fitted on $\left(r_t^{(e)}\right)$ and $\left(r_t^{(d)}\right)$ according to $t_{(5)}$ innovation.

The plot the squared returns and the estimated conditional variance (regarded as an estimate of $\left(\left\{r_t^{(*)}\right\}^2\right)$) associated to $t_{(5)}$ innovation are shown in 1.6 below.

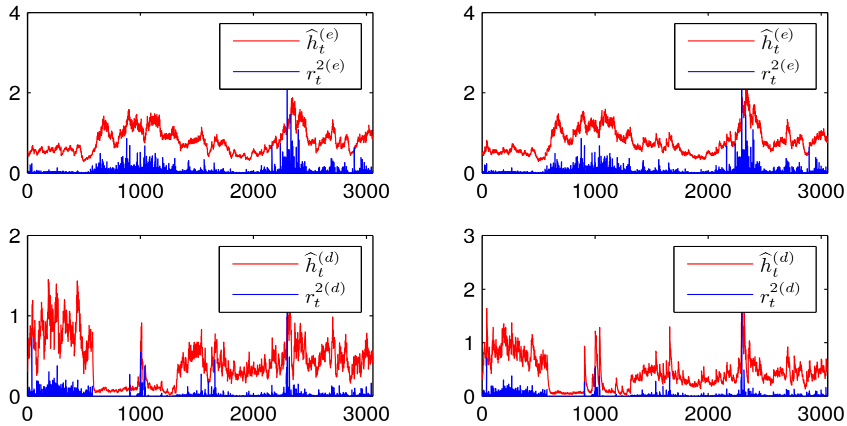

Figure 1.6: Dark blue: squared returns, light red: volatilities estimates according to $P-\log GARCH_2\left(1,1\right)$ (left) and to $P-EGARCH_2\left(1,1\right)$ (right) models with *student* innovation.

	$P-\log GARCH_2(1,1)$				$P-EGARCH_2(1,1)$				
GED	ω	α	β	$P-value$	ω	α	γ	β	$P-value$
$r_t^{(e)}$	0.0877	0.0298	0.9957	0.0000	0.0068	−0.0056	0.2523	1.0119	0.0000
	(0.1407)	(0.0078)	(0.0830)		(0.0815)	(0.0537)	(0.0872)	(0.0988)	
	−0.0514	0.0363	0.9341		−0.1574	0.0329	0.2655	0.9821	
	(0.0846)	(0.0061)	(0.0533)		(0.0599)	(0.0378)	(0.0767)	(0.0636)	
$r_t^{(d)}$	0.0617	0.1105	0.8479	0.0000	−0.0620	0.0428	0.7254	0.8741	0.0000
	(0.1126)	(0.0119)	(0.0314)		(0.0989)	(0.0660)	(0.1056)	(0.0350)	
	0.0418	0.0533	0.9470		−0.0269	0.0992	0.4716	1.0696	
	(0.0780)	(0.0117)	(0.0243)		(0.0799)	(0.0502)	(0.0742)	(0.0290)	

Table 1.7: Parameters estimation of $P-\log GARCH_2(1,1)$ and $P-EGARCH_2(1,1)$ models fit to $\left(r_t^{(e)}\right)$ and $\left(r_t^{(d)}\right)$ according to GED innovation.

The third results of parameters estimation of the series $\left(r_t^{(e)}\right)$ and $\left(r_t^{(d)}\right)$ according to $P-\log GARCH_2(1,1)$ and $P-EGARCH_2(1,1)$ models when the innovation is GED are presented in the 1.7.
The plots the squared returns and the estimated volatilities (regarded as an estimate of $\left(\left\{r_t^{(*)}\right\}^2\right)$) associated to GED innovation are showed in 1.7 below.

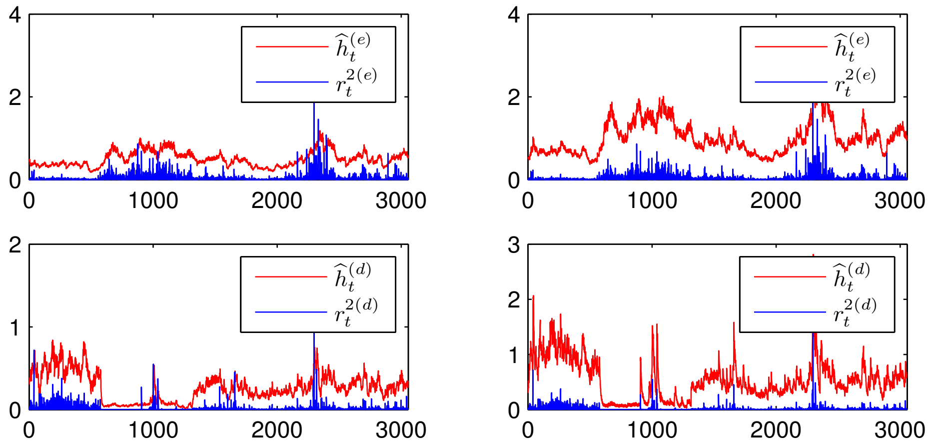

Figure 1.7: Dark blue: squared returns, light red: volatilities estimates according to $P-\log GARCH_2(1,1)$ (left) and to $P-EGARCH_2(1,1)$ (right) models with GED innovation.

The top-Lyapunov exponent associated to estimated volatilities $\widehat{r}_t^{(e)}$ and $\widehat{r}_t^{(d)}$ noted $\gamma_e^{(5)}$ and $\gamma_d^{(5)}$ respectively according to $P-\log GARCH_2(1,1)$ and $P-EGARCH_2(1,1)$ models with different innovations are summarized in Table 1.8 below.

	$P-\log GARCH_2(1,1)$			$P-EGARCH_2(1,1)$		
Series	$\mathcal{N}(0,1)$	$t_{(5)}$	GED	$\mathcal{N}(0,1)$	$t_{(5)}$	GED
$-\gamma_e^{(5)}$	0.1266	0.1808	0.3241	0.0478	0.0275	0.0306
$-\gamma_d^{(5)}$	0.1226	0.2011	0.1188	0.1536	0.0511	0.0673

Table 1.8: The top-Lyapunov exponent associated to $\widehat{r}_t^{(e)}$ and $\widehat{r}_t^{(d)}$ according to

The above table shows clearly that the estimated volatilities $\widehat{r}_t^{(e)}$ and $\widehat{r}_t^{(d)}$ are strictly periodically stationary. However, the zone of strictly periodically stationary for $P-\log GARCH_2(1,1)$ model fitted to $\left(\widehat{r}_t^{(e)}\right)$ is less restrictive when the innovation is GED, contrary to that fitted on $\left(\widehat{r}_t^{(d)}\right)$ when the zone becomes more interesting. On the other hand, the zone of strictly periodically stationary for $P-EGARCH_2(1,1)$ model fitted to $\widehat{r}_t^{(e)}$ and/or $\widehat{r}_t^{(d)}$ is more large when the innovation is $t_{(5)}$.

1.6.3 Comments

Tables 1.5, 1.6 and 1.7 display the estimation of $P-\log GARCH_2(1,1)$ and $P-EGARCH_2(1,1)$ models fitted to daily returns $\left(r_t^{(e)}\right)$ and $\left(r_t^{(d)}\right)$ according to $\mathcal{N}(0,1)$, $t_{(5)}$ and GED innovations. For all models, the persistence parameters $\beta(\vartheta)$, $\vartheta=1,2$ are very high in contrary to the intercept parameters $\omega(\vartheta)$, $\vartheta=1,2$ which are closed to zeros. Moreover, the null hypothesis $H0 : \underline{\theta} = \underline{O}$ is statistically significant rejected at the 99% confidence level. This significance confirms that there is evidence of the existence of an effect $ARCH$. Additionally, from Table 1.8, the estimated models satisfy also the assumptions **A1.−A7.** used to show, the consistency and asymptotic normality. Furthermore, the negativity of parameters $\alpha(\vartheta), \vartheta=1,2$ in $P-EGARCH_2(1,1)$ implies the presence of leverage effect. So, according to this criterion the $P-EGARCH_2(1,1)$ is preferred for the $\left(y_t^{(e)}\right)_t$ series. Figure 1.5, 1.6 and 1.7 represent the plots of the volatilities (plots in red) estimates of the series of returns $\left(r_t^{(e)}\right)$ and $\left(r_t^{(d)}\right)$ according to $P-\log GARCH_2(1,1)$ (left panels) and $P-EGARCH_2(1,1)$ (right panels) models with different innovations and compared with the appropriate squared returns (plots in blue). We can see from these plots that the estimated volatilities according to different innovations move together. It also demonstrates that a large piece of returns (positive or negative) leads to a high volatility and a small piece of returns leads to a low volatility, indicating volatility clustering. In particular, the period between 2000 and 2002 is characterized by low volatility levels compared to the period between 2009 and 2010 for both series. In addition, a high volatility cluster beginning in 2005 is observed and is mainly due to the global financial crisis. After this period of uncertainty, a cluster of low volatility is observed during 3 years. Other high volatility cluster is detected and could be related to the devaluation of the Dinar. Finally, the conditional volatility seems to be more stable after 2010. Our empirical results demonstrate that $P-\log GARCH_2(1,1)$ is able to capture very large volatilities of the series $\left(r_t^{(e)}\right)$ observed for instance from 2009 to 2010 for all innovations unless maybe for the GED innovation. On the other hand, the $P-EGARCH_2(1,1)$ model fitted to $\left(r_t^{(d)}\right)$ and/or $\left(r_t^{(e)}\right)$ do not capture exactly the same empirical properties.

Chapter 2

Quasi Maximum Likelihood Estimation for Periodic Absolute Value GARCH Models

Contents

This chapter was published by Slimani and all [43] in the journal Random Operators and Stochastic Equations.
The primary objectives of this chapter encompass the examination of the probabilistic characteristics of the $PAVGARCH$ model through its vector representation. We delve into the exploration of the necessary and sufficient conditions that ensure strict stationarity in a periodic sense for the $PAVGARCH$ model. The second aim of this article is to apply the standard quasi-maximum likelihood (QML) method for parameter estimation within the model. Consequently, we establish conditions that guarantee both strong consistency and asymptotic normality for the QML estimator of the model's parameters. Subsequently, we provide a series of numerical experiments to illustrate the practical relevance of our theoretical findings. Finally, we apply our model to the analysis of two foreign exchange rates: the Algerian Dinar against the European currency Euro ($Euro/Dinar$) and the American currency Dollar ($Dollar/Dinar$). Our empirical work demonstrates that our approach not only outperforms but also fits the data effectively. Before we proceed, we will introduce some notations and definitions.

2.1 Periodic $AVGARCH$ model and state-space representation

Definition 2.1.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. The process $(\varepsilon_n)_{n\in\mathbb{Z}}$ is called a periodic $AVGARCH_s(p,q)$ process with period $s > 0$ abbreviated by $(PAVGARCH_s(p,q))$ if*

$$\varepsilon_n = h_n e_n \text{ and } h_n = \alpha_0(s_n) + \sum_{i=1}^{q}\alpha_i(s_n)|\varepsilon_{n-i}| + \sum_{j=1}^{p}\beta_j(s_n)h_{n-j}, \tag{2.1}$$

where $|\varepsilon_n| = \varepsilon_n^+ + \varepsilon_n^-$ with $\varepsilon_n^+ = \max(\varepsilon_n, 0)$, $\varepsilon_n^- = \min(\varepsilon_n, 0)$.

Let introduce $\Delta(k)$ as:

$$\Delta(k) : = \{sn + k, \quad n \in \mathbb{Z}, \quad s > 0\}, \quad k \in \mathbb{Z},$$

with that refers to the state or “season” of the periodic cycle at time n. We put the set $\mathbb{S} = \{1, ..., s\}$ and we define s_n as:

$$s_n : = \sum_{k=1}^{s} k\mathbb{I}_{\Delta(k)}(n).$$

In equation (2.1), $(s_n)_n$ represents a periodic sequence consisting of positive integers. This sequence operates within a finite state space. The innovation sequence $(e_n)_{n\in\mathbb{Z}}$ is assumed to adhere to the following condition.

Assumption 2.1.1. *Let $(e_n)_{n\in\mathbb{Z}}$ be a sequence of independent and identically distributed $(i.i.d.)$ random variables defined in probability space $(\Omega, \mathcal{F}, P)$. These random variables possess zero mean and unit variance. Additionally, it's essential to note that e_k is independent of (ε_n) for all instances where k is greater than n.*

Now, by transforming $n = st + \upsilon$ and by setting $\varepsilon_{st+\upsilon} = \varepsilon_t(\upsilon)$, $h_{st+\upsilon} = h_t(\upsilon)$ and $e_{st+\upsilon} = e_{st+\upsilon}(\upsilon)$, the model (2.1) is rewritten in the following periodic version:

$$\varepsilon_t(\upsilon) = h_t(\upsilon)e_t(\upsilon) \text{ and } h_t(\upsilon) = \alpha_0(\upsilon) + \sum_{i=1}^{q}\alpha_i(\upsilon)|\varepsilon_t(\upsilon - i)| + \sum_{j=1}^{p}\beta_j(\upsilon)h_t(\upsilon - j), \tag{2.2}$$

where the coefficients $\alpha_0(\upsilon)$, $\alpha_i(\upsilon)$ and $\beta_j(\upsilon)$ with $i \in \{1, ..., q\}$ and $j \in \{1, ..., p\}$ are positive with $\alpha_0(\upsilon) > 0$ for any $\upsilon \in \mathbb{S}$.

In the equation (2.2), the notation $\varepsilon_t(\upsilon)$ (and similarly $e_t(\upsilon)$ and $h_t(\upsilon)$) refers to ε_t (or e_t and h_t during the υ-th "season" within the cycle t). To simplify, $\varepsilon_t(\upsilon) = \varepsilon_{t-1}(\upsilon + s)$, $h_t(\upsilon) = h_{t-1}(\upsilon + s)$ and $e_t(\upsilon) = e_{t-1}(\upsilon + s)$, if $\upsilon < 0$.

The non-periodic notations υ_t, e_t and h_t can be used interchangeably with their periodic counterparts $\varepsilon_t(\upsilon)$, $e_t(\upsilon)$ and $h_t(\upsilon)$ whenever there's no specific emphasis on seasonality required.

Remark 2.1.1. *Beside the representation (2.2), the model $PAVGARCH_s(1.1)$ can be rewritten as*

$$\varepsilon_t(\upsilon) = h_t(\upsilon)e_t(\upsilon) \text{ and } h_t(\upsilon) = \alpha_0(\upsilon) + g_{(\upsilon)}(e_t(\upsilon - 1))h_t(\upsilon - 1), \tag{2.3}$$

where $g_{(\upsilon)}(e_t(\upsilon - 1)) = \alpha_1(\upsilon)|e_t(\upsilon - 1)| + \beta_1(\upsilon)$ and the coefficients $\alpha_0(\upsilon)$, $\alpha_1(\upsilon)$ and $\beta_1(\upsilon)$ are positive with $\alpha_0(\upsilon) > 0$ for any $\upsilon \in \mathbb{S}$.

2.1.1 State-space representation

In this section, we delve into a pivotal aspect of this chapter. Our primary objective here is to explore and determine a causal solution to address the problem we have been examining. In essence, we are investigating a resolution that not only provides answers but also pinpoints the causes and factors contributing to our problem. This process involves a comprehensive analysis to uncover the underlying causes and their interrelationships, enabling us to identify effective solutions.

We start by defining for each $\upsilon \in S$ the $r = (q+p)-$random vectors as

$$\underline{e}_t(\upsilon) := \left(\alpha_0(\upsilon)|e_t(\upsilon)|, \underline{O}'_{(q-1)}, \alpha_0(\upsilon), \underline{O}'_{(p-1)}\right)', \quad \underline{\varepsilon}_t(\upsilon) := \left(|\varepsilon_t(\upsilon)|, ..., |\varepsilon_t(\upsilon-q+1)|, h_t(\upsilon), ..., h_t(\upsilon-p+1)\right)',$$

$q-$vector as:

$$\underline{\alpha}_{1:q}(\upsilon) := (\alpha_1(\upsilon), ..., \alpha_q(\upsilon))',$$

and $p-$vector by:

$$\underline{\beta}_{1:p}(\upsilon) := (\beta_1(\upsilon), ..., \beta_p(\upsilon))'$$

and $r \times r-$random matrix in the following way

$$M_{\upsilon(e_t(\upsilon))} = \begin{pmatrix} \underline{\alpha}_{1:q-1}(\upsilon)|e_t(\upsilon)| & \alpha_q(\upsilon)|e_t(\upsilon)| & \underline{\beta}_{1:p-1}(\upsilon)|e_t(\upsilon)| & \beta_p(\upsilon)|e_t(\upsilon)| \\ I_{((q-1)\times(q-1))} & \underline{O}_{((q-1)\times 1)} & \underline{O}_{(q-1)\times p)} & \\ \underline{\alpha}_{1:q-1}(\upsilon) & \alpha_q(\upsilon) & \underline{\beta}_{1:p-1}(\upsilon) & \beta_p(\upsilon) \\ \underline{O}_{(p-1)\times q} & & I_{((p-1)\times(p-1))} & \underline{O}_{(p-1)\times 1} \end{pmatrix}_{r\times r}. \tag{2.4}$$

The equation (2.2) can be reformulated and expressed in a state-space form. This state-space representation is given as:

$$\underline{\varepsilon}_t(\upsilon) = M_\upsilon(e_t(\upsilon))\underline{\varepsilon}_t(\upsilon-1) + \underline{e}_t(\upsilon). \tag{2.5}$$

The equation (2.5) shares the same defining equation as independent periodic distribution ($i.p.d$) random coefficient autoregressive models recently introduced by Aknouche and Guerbyenne [2]. In this section, our main focus lies in seeking a causal solution for the equation (2.5). In other words, we aim to find the solution $(\varepsilon_t)_t$ for model (2.1), where εt is a measurable function of e_{t-i} for all $i \geq 0$. To achieve this, we employ an iterative approach by applying equation (2.5) a total of s times, resulting in the following transformation:

$$\underline{\varepsilon}_t(s) = \left\{\prod_{\upsilon=0}^{s-1} M_{s-\upsilon}(e_t(s-\upsilon))\right\}\underline{\varepsilon}_{t-1}(s) + \sum_{k=1}^{s}\left\{\prod_{\upsilon=0}^{s-k-1} M_{s-\upsilon}(e_t(s-\upsilon))\right\}\underline{e}_t(k).$$

We put $\underline{\varepsilon}(t) = \underline{\varepsilon}_t(s)$, therefore the last equation can be rewritten as

$$\underline{\varepsilon}(t) = H(\underline{e}_t)\underline{\varepsilon}(t-1) + \underline{\eta}(\underline{e}_t), \tag{2.6}$$

where

$$\underline{e}_t = (e_t(s), e_t(s-1), ..., e_t(1))',$$

and

$$H(\underline{e}_t) = \left\{\prod_{\upsilon=0}^{s-1} M_{s-\upsilon}(e_t(s-\upsilon))\right\}, \ \underline{\eta}(\underline{e}_t) = \sum_{k=1}^{s}\left\{\prod_{\upsilon=0}^{s-k-1} M_{s-\upsilon}(e_t(s-\upsilon))\right\}\underline{e}_t(k).$$

However, equations akin to (2.6) have been thoroughly examined in the existing literature, as exemplified by Bougerol and Picard [11, 12], along with the extensive references cited therein.

2.1.2 Strict periodic stationarity

In this section, we establish the necessary conditions to demonstrate the strict stationarity of the model presented by the equation (2.6). The existence of a causal solution for the equation (2.1) is essentially equivalent to the existence of a causal solution for (2.6). This equivalence is quite apparent: any causal solution for the (2.1) leads. Through the transformation provided by (2.5) to a causal solution for equation (2.6) and vice versa. In other words, each component of a stationary solution of the dual process $((\ \underline{\varepsilon}'_t(1), \ldots, \underline{\varepsilon}'_t(s))_{t\in\mathbb{Z}}$ (as detailed by Gladyshev [28]) is one of the solutions to the equation (2.6)). In the following section, we explore the necessary and sufficient conditions required to ensure the strict stationarity of the models presented in the equation (2.6). The corresponding solution of the equation (2.5), known as strictly periodic stationary (SPS), is the primary focus of our investigation. Our key tool in studying the strict stationarity of the equation (2.6) lies in the top Lyapunov exponent, denoted as $\gamma_L^{(s)}(H)$, associated with the sequence of random matrices $(H_t)_t$ defined within the equation (2.6) and expressed as:

$$\gamma_L^{(s)}(H) := \inf_{t>0}\left\{\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{t-1} H(\underline{e}_{t-j})\right\|\right\}\right\} \overset{a.s.}{=} \lim_{t\to\infty}\left\{\frac{1}{t}\log\left\|\prod_{j=0}^{t-1} H(\underline{e}_{t-j})\right\|\right\}. \tag{2.7}$$

The final portion of the above equation can be readily demonstrated through the application of Kingman's subadditive ergodic theorem (see Kingnaan [34]). Additionally, the existence of $\gamma_L^{(s)}(H)$ is ensured by the fact that

$$E\log^+\|H(\underline{e}_t)\| \leq E\|H(\underline{e}_t)\| < +\infty,$$

where $\log^+(x) = \max(\log x, 0)$, for any $x > 0$.

2.1.3 Example

For the $PAVGARCH_s(1,1)$ model, after some tedious algebra we find that the necessary and sufficient condition ensuring the existence of SPS solution is that

$$\sum_{\vartheta=1}^{s} E\left\{\log\left\{a_1(\upsilon)|e_0| + \beta_1(\upsilon)\right\}\right\} < 0$$

is strictly negative. It is worth noting that the existence of regimes which satisfy

$$E\left\{\log\left\{a_1(\upsilon)|e_0| + \beta_1(\upsilon)\right\}\right\} > 0$$

does not preclude strict periodic stationarity.

Theorem 2.1.1. *The solution of* (2.6) *is represented by the series:*

$$\underline{\varepsilon}(t) = \sum_{k\geq 1}\left\{\prod_{j=0}^{k-1} H\left(\underline{e}_{t-j-1}\right)\right\}\underline{\eta}\left(\underline{e}_{t-k-1}\right) + \underline{\eta}\left(\underline{e}_{t-1}\right). \tag{2.8}$$

This solution exists if and only if the top Lyapunov exponent $\gamma_L^{(s)}(H)$ is strictly negative, the series (2.8) converges absolutely almost surely, and it constitutes the unique ergodic solution process for (2.6). Consequently, the equation (2.5) becomes a strictly periodic stationary (SPS) process and possesses a causal solution in the form of the series:

$$\underline{\varepsilon}_t(\upsilon) = \sum_{k\geq 0}\left\{\prod_{j=0}^{k-1} M_{\upsilon-i}\left(e_t(\upsilon-i)\right)\right\}\underline{e}_t(\upsilon-k). \tag{2.9}$$

This series also converges absolutely almost surely, and the process $(\varepsilon t(\upsilon))t \in \mathbb{Z}$ is established as the unique, causal, SPS, and periodically ergodic solution to Equation (2.1).

In the following theorem we built a condition on $\gamma_L^{(s)}(H)$ in order to find a positive δ.

Theorem 2.1.2. *Assuming that $\gamma_L^{(s)}(H) < 0$. Then, there is $\delta > 0$ such that $E(h_t^\delta) < \infty$, and $E(|\varepsilon_t|^\delta) < \infty$, $\forall t$.*

Proof. see Slimani and all [43]. □

In the context of the $PAVGARCH_s(1,1)$ model, two critical observations shed light on the behavior of this stochastic process.

Proposition 2.1.1. *For $PAVGARCH_s(1.1)$, the following assertions hold.*

1. *If $\gamma_L^{(s)}(H) > 0$, almost surely $h_t \to +\infty$ at an exponential rate, i.e., $\rho^t h_t \to +\infty$, and $\rho^t\varepsilon_t^2 \to +\infty$, as $t \to +\infty$, for any $\rho > e^{-\gamma_L^{(s)}(\Gamma)}$.*
2. *If $\gamma_L^{(s)}(H) = 0$, in distribution $h_t \to +\infty$, and $\varepsilon_t^2 \to +\infty$ as $t \to +\infty$.*

Proof. The proof some results by Slimani and all [43]. First, perform s iterations of the equation (2.3) to obtain the following equality:

$$h_t(\upsilon) = \alpha_0(s-k)\left\{\sum_{i=0}^{s-1}\left\{\prod_{i=0}^{k-1} g_{(s-i)}\left(e_t(s-i-1)\right)\right\}\right\} + \left\{\prod_{i=0}^{s-1} g_{(s-i)}\left(e_t(s-i-1)\right)\right\} h_t(0). \tag{2.10}$$

Now, we put

$$\omega\left(\underline{e}_t\left(1\right)\right)=\alpha_0\left(s-k\right)\left\{\sum_{i=0}^{s-1}\left\{\prod_{i=0}^{k-1}g_{(s-i)}\left(e_t\left(s-i-1\right)\right)\right\}\right\},$$

$$c\left(\underline{e}_t\left(0\right)\right)=\left\{\prod_{i=0}^{s-1}g_{(s-i)}\left(e_t\left(s-i-1\right)\right)\right\},$$

$$h\left(t+1\right)=h_t\left(s\right)$$

and rewriting (2.10) as

$$h\left(t+1\right)=c\left(\underline{e}_t\left(0\right)\right)h\left(t\right)+\omega\left(\underline{e}_t\left(1\right)\right),$$

with $\underline{e}_t\left(l\right)=\left(e_{st+l},...,e_{st+s-1}\right)$. Note that $c(\underline{e}_t(0))$ represents a sequence of independent and identically distributed (i.i.d.) non-negative random variables, and they are also independent of $h(k)$ for any $k<t$. Using this notation, the proof essentially follows the same rationale as outlined Francq and Zakoïan [22]. □

2.2 QML estimation

In this section, we lay out a set of essential hypotheses. These hypotheses serve as the foundation upon which we establish the conditions necessary to rigorously demonstrate the convergence of our estimations.

Let define the following vectors:

$$\underline{\alpha}':=\left(\underline{\alpha}_0',\underline{\alpha}_1',...,\underline{\alpha}_q'\right),\quad \beta':=\left(\underline{\beta}_1',...,\underline{\beta}_q'\right),\ \underline{\theta}'\left(v\right):=\left(\alpha_0\left(v\right),\alpha_1\left(v\right),...,\alpha_q\left(v\right),\beta_1\left(v\right),...,\beta_p\left(v\right)\right),v\in\mathbb{S},$$

with

$$\underline{\alpha}_i':=\left(\alpha_i\left(1\right),...,\alpha_i\left(s\right)\right),\quad \underline{\beta}_k':=\left(\beta_k\left(1\right),...,\beta_k\left(s\right)\right),\text{ for all }0\leq i\leq q,1\leq k\leq p.$$

We present the following quasi-maximum likelihood estimator ($QMLE$) for the $PAVGARCH_s$ parameter gathered in vector

$$\underline{\theta}':=\left(\underline{\alpha}',\underline{\beta}'\right):=\left(\underline{\theta}'\left(1\right),...,\underline{\theta}'\left(s\right)\right)\in\Theta\subset]0,+\infty[^s\times[0,+\infty[^{s(q+p)}.$$

The true parameter value denoted by $\underline{\theta}_0':=\left(\underline{\alpha}_0',\underline{\beta}_0'\right)\in\Theta\subset]0,+\infty[^s\times[0,+\infty[^{s(q+p)}$. Since the value of $underline\theta_0'$ is unknown, it becomes imperative to estimate it. To achieve this, we examine a specific realization: $\{\varepsilon_1,...,\varepsilon_n;\quad n=sN\}$, from the unique causal SPS and PE solution of (2.2). Let $h_t^2\left(\underline{\theta}\right)$ be the conditional variance of ε_t given $\mathcal{F}_{t-1}$. The Gaussian likelihood function of $\underline{\theta}\in\Theta$ conditional on initial values $\varepsilon_0,...,\varepsilon_{1-q},h_0,...,h_{1-p}$ is given by

$$\tilde{L}_n\left(\underline{\theta}\right)=\left\{\prod_{t=1}^{n}\frac{1}{\left(2\pi\tilde{h}_t^2\left(\underline{\theta}\right)\right)^{\frac{1}{2}}}\right\}\exp\left\{-\sum_{t=1}^{n}\frac{\varepsilon_t^2}{2\tilde{h}_t^2\left(\underline{\theta}\right)}\right\},\tag{2.11}$$

where $\widetilde{h}_t^2\left(\underline{\theta}\right)$ is recursively defined as

$$t\geq1,\quad \tilde{h}_t\left(\underline{\theta}\right)=\alpha_0\left(t\right)+\sum_{i=1}^{q}\alpha_i\left(t\right)\left|\varepsilon_{t-i}\right|+\sum_{j=1}^{p}\gamma_j\left(t\right)\tilde{h}_{t-j}\left(\underline{\theta}\right).$$

A $QMLE$ of $\underline{\theta}$ is defined by; For any measurable solution $\hat{\underline{\theta}}$

$$\hat{\underline{\theta}}_n=Arg\max_{\underline{\theta}\in\Theta}\tilde{L}_n\left(\underline{\theta}\right)=Arg\min_{\underline{\theta}\in\Theta}\left(\tilde{I}_n\left(\underline{\theta}\right)\right),\tag{2.12}$$

where (ignoring the constants)

$$\tilde{I}_n\left(\underline{\theta}\right)=\left(sN\right)^{-1}\sum_{t=1}^{N}\sum_{v=0}^{s-1}\tilde{l}_{st+v}\left(\underline{\theta}\right),\quad\text{with }\tilde{l}_t\left(\underline{\theta}\right)=\frac{\varepsilon_t^2}{\tilde{h}_t^2\left(\underline{\theta}\right)}+\log\tilde{h}_t^2\left(\underline{\theta}\right).$$

In view of the strong dependency of $\widetilde{h}_t(\underline{\theta})$ on initial values $\varepsilon_0, ..., \varepsilon_{1-q}, h_0, ..., h_{1-p}$. $\left(\tilde{l}_t(\underline{\theta})\right)_{t\geq 1}$ is not a SPS nor a periodically ergodic (PE) process. Therefore, it will be more convenient to work with an unobserved SPS and PE version $I_n(\underline{\theta})$ of the likelihood (2.11) i.e,

$$I_n(\underline{\theta}) = (sN)^{-1}\sum_{t=1}^{N}\sum_{\upsilon=0}^{s-1} l_{st+\upsilon}(\underline{\theta}), \text{ with } l_t(\underline{\theta}) = \frac{\varepsilon_t^2}{h_t^2(\underline{\theta})} + \log h_t^2(\underline{\theta}).$$

In the upcoming sections, we will provide the conditions that guarantee the strong consistency and asymptotic normality of $\hat{\underline{\theta}}$. Our approach draws substantial inspiration from Aknouche and Bibi [1].

2.2.1 Strong consistency of $QMLE$

This section plays a crucial role in our chapter as it is dedicated to the construction of necessary hypotheses. These hypotheses are instrumental in demonstrating the convergence of our estimators, a fundamental aspect of our research. Let's take into account the following assumptions regarding regularities.

B0. $\underline{\theta}_0 \in \Theta$ and Θ is a compact subset of $\mathbb{R}^{s(1+q+p)}$.

B1. Let L denote the lag operator and consider the polynomials

$$\mathcal{A}_{0,\upsilon}(\boldsymbol{z}) = \sum_{i=1}^{q}\alpha_{0,i}(\upsilon)\boldsymbol{z}^i,$$

$$\mathcal{B}_{0,\upsilon}(\boldsymbol{z}) = 1 - \sum_{i=1}^{p}\beta_{0,i}(\upsilon)\boldsymbol{z}^i,$$

with the convention $\mathcal{A}_{0,\upsilon}(\boldsymbol{z}) = 0$ if $q = 0$ and $\mathcal{B}_{0,\upsilon}(\boldsymbol{z}) = 1$ if $p = 0$, for all $\upsilon \in \{1, ..., s\}$.

B2. If $p > 0$, $\mathcal{A}_{0,\upsilon}(\boldsymbol{z})$ have no common roots with $\mathcal{B}_{0,\upsilon}(\boldsymbol{z})$ for all υ. Moreover, $\mathcal{A}_{0,\upsilon}(\mathbf{1}) \neq 0$ and $\alpha_{0,q}(\upsilon) + \beta_{0,p}(\upsilon) \neq 0$ for all $\upsilon \in \mathbb{S}$.

B3. $\gamma_L^{(s)}(H_0) < 0$ where $\gamma_L^{(s)}(H_0)$ is the top-Lyapunov exponent associated with the random matrix $H(\underline{e}_t)$ evaluate under the true value $\underline{\theta}_0$ and

$$\rho\left(\prod_{\upsilon=1}^{s}A_\upsilon\right) < 1, \text{ with } A_\upsilon = \begin{pmatrix} \underline{\beta}_{1:p-1}(\upsilon) & \beta_p(\upsilon) \\ I_{(p-1)} & \underline{O}_{(p-1)} \end{pmatrix}.$$

B4. $(e_t)_{t\in\mathbb{Z}}$ is non-degenerate and $P(e_t > 0) \in (0, 1)$.

We assume the compactness of Θ to leverage various results from real analysis. Assumption **B2** is crucial for ensuring identifiability, while assumption **B3** not only guarantees the existence of a finite moment for the true value $\underline{\theta}_0$ but also establishes the presence of a strong, positive solution (SPS) and a partial equilibrium (PE) solution for the model in the equation (2.2), as well as ensuring causality in the solution of $h_t(\underline{\theta})$. Additionally, assumption **B4** is introduced to facilitate identifiability and also to ensure that the process (ε_t) has positive and negative values with positive probabilities. These assumptions in place, we are now prepared to present our first significant result.

Theorem 2.2.1. *Let $\left(\hat{\underline{\theta}}_N\right)$ be a sequence of QMLE satisfying (2.12). Then, under 2.1.1 and the assumptions **B0-B4**, $\hat{\underline{\theta}}_N \to \underline{\theta}_0$ almost surely (a.s.) when $N \to \infty$.*

To prove Theorem 2.2.1, we formulate the following technical assertions gathered in the next lemma.

Lemma 2.2.1. *Under Assumptions **B0-B4**, we obtain*

1. $\limsup\limits_{N\to\infty}\sup\limits_{\underline{\theta}\in\Theta}\left|\tilde{L}_{sN}(\underline{\theta}) - L_{sN}(\underline{\theta})\right| = 0$ *a.s.*

2. *There exists $t \in Z$ such that $h_t(\underline{\theta}) = h_t(\underline{\theta}_0)$ a.s. $\Rightarrow \underline{\theta} = \underline{\theta}_0$.*

3. $\sum_{\vartheta=1}^{s} E_{\underline{\theta}_0}\{l_{st+\vartheta}(\underline{\theta}_0)\} < \infty$ *and if $\theta \neq \underline{\theta}_0$. Then*

$$\sum_{\vartheta=1}^{s} E_{\underline{\theta}}\{l_{st+\vartheta}(\underline{\theta})\} > \sum_{\vartheta=1}^{s} E_{\underline{\theta}_0}\{l_{st+\vartheta}(\underline{\theta}_0)\}.$$

For any $\underline{\theta} \neq \underline{\theta}_0$ there a neighborhood $\mathcal{V}(\underline{\theta})$ such that

$$a.s.\lim_{N\to\infty} \inf_{\underline{\theta}^*\in\Theta} \left(\tilde{L}_{sN}(\underline{\theta}^*)\right) > \sum_{\vartheta=1}^{s} E_{\underline{\theta}_0} \left\{ l_\vartheta(\underline{\theta}_0) \right\}.$$

Proof. The proof follows essentially the same arguments as in Aknouche and Bibi [1]. □

In the upcoming section, we provide the asymptotic normality of $\hat{\underline{\theta}}_n$.

2.2.2 Asymptotic normality of $QMLE$

In order to establish the asymptotic normality of $\hat{\underline{\theta}}_n$, we introduce the following additional assumptions:

B5. $\underline{\theta}_0 \in \mathring{\Theta}$, with $\mathring{\Theta}$ denotes the interior of Θ.

B6. $1 < \kappa = E\left\{e_t^4\right\} < \infty$.

Assumption **B5** is indispensable for the Quasi Maximum Likelihood Estimator (QMLE) and supports the validation of the first-order condition for maximizing the log-likelihood. On the other hand, Assumption **B6** is essential for the existence of the asymptotic covariance matrix for the QMLE. The second key result in this section is as follows:

Theorem 2.2.2. *Suppose that $(\varepsilon_t, t \in Z)$ is generated by the equations (2.2). Then, under the assumptions **B0-B6** we obtain*

$$\sqrt{sN}\left(\hat{\underline{\theta}}_{sN} - \underline{\theta}_0\right) \rightsquigarrow \mathcal{N}\left(\underline{O}, (\kappa-1)\, J^{-1}\right) \text{ as } N \to \infty,$$

where the matrix J given by

$$J := \sum_{\upsilon=1}^{s} E_{\underline{\theta}_0}\left\{\frac{\partial^2 l_{st+\vartheta}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)\right\} = \sum_{\upsilon=1}^{s} E_{\underline{\theta}_0}\left\{\frac{1}{h_{st+\upsilon}^2(\underline{\theta}_0)} \frac{\partial h_{st+\upsilon}}{\partial\underline{\theta}}(\underline{\theta}_0) \frac{\partial h_{st+\upsilon}}{\partial\underline{\theta}'}(\underline{\theta}_0)\right\}.$$

The proof of Theorem 2.2.2 rests classically on a Taylor series expansion of $\frac{\partial L_{sN}}{\partial\underline{\theta}}(\underline{\theta})$, around $\underline{\theta}_0$ which is given by:

$$\underline{O} = (sN)^{-\frac{1}{2}} \sum_{t=1}^{sN} \frac{\partial l_t}{\partial\underline{\theta}}(\hat{\underline{\theta}}_{sN}) = (sN)^{-\frac{1}{2}} \sum_{t=1}^{sN} \frac{\partial l_t}{\partial\underline{\theta}}(\underline{\theta}_0) + \left((sN)^{-1} \sum_{t=1}^{sN} \frac{\partial^2 l_t}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}^*)\right)(sN)^{\frac{1}{2}}\left(\hat{\underline{\theta}}_{sN} - \underline{\theta}_0\right),$$

where the coordinates of $\underline{\theta}^*$ are between the corresponding entries of $\hat{\underline{\theta}}_{sN}$ and those of $\underline{\theta}_0$. The theorem will follow straightforwardly. To achieve this, we establish the following intermediate results, which are presented in the following lemma.

Lemma 2.2.2. *Under assumptions **B0-B6**, we obtain*

1. $\sum_{\upsilon=1}^{s} E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\Theta}\left\|\frac{\partial l_{st+\vartheta}}{\partial\underline{\theta}}(\underline{\theta}_0)\frac{\partial l_{st+\vartheta}}{\partial\underline{\theta}'}(\underline{\theta}_0)\right\|\right\} < \infty$ *and* $\sum_{\upsilon=1}^{s} E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\Theta}\left\|\frac{\partial^2 l_{st+\vartheta}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)\right\|\right\} < \infty$.

2. J *is invertible and* $\sum_{\upsilon=1}^{s} Var_{\underline{\theta}_0}\left\{\frac{\partial l_{st+\vartheta}}{\partial\underline{\theta}}(\underline{\theta}_0)\right\} = (\kappa-1)J$.

3. *There is a neighborhood $\mathcal{V}(\underline{\theta}_0)$ of $\underline{\theta}_0$ such that*

$$\sum_{\upsilon=1}^{s} E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)}\left\|\frac{\partial^3 l_{st+\vartheta}}{\partial\theta_i\partial\theta_j\partial\theta_k}(\underline{\theta}_0)\right\|\right\} < \infty, \quad \forall\, i,j,k \in \{1,...,s(1+q+p)\}.$$

4. $p\lim\left\|(sN)^{-\frac{1}{2}}\sum_{t=1}^{N}\sum_{\upsilon=1}^{s}\left(\frac{\partial\tilde{l}_{st+\vartheta}}{\partial\underline{\theta}}(\underline{\theta}_0) - \frac{\partial l_{st+\vartheta}}{\partial\underline{\theta}}(\underline{\theta}_0)\right)\right\| = 0.$

5. $p\lim \sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)} \left\| (sN)^{-1}\sum_{t=1}^{N}\sum_{\upsilon=1}^{s}\left(\frac{\partial^2\tilde{l}_{st+\vartheta}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)-\frac{\partial^2 l_{st+\vartheta}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)\right)\right\| = 0$

6. $(sN)^{-\frac{1}{2}}\sum_{t=1}^{sN}\frac{\partial l_t}{\partial\underline{\theta}}(\underline{\theta}_0) \rightsquigarrow \mathcal{N}\left(\underline{O},(\kappa-1)\,J\right)$ *as* $N\to\infty$ *and almost surely*

$$\lim_{n\to\infty}\left((sN)^{-1}\sum_{t=1}^{sN}\frac{\partial^2 l_t}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\tilde{\theta}})\right) = J.$$

Proof. see Francq and Zakoïan [21] and Bibi [5] . □

Remark 2.2.1. *The asymptotic properties of Quasi Maximum Likelihood Estimator (QMLE) are also valid for the specific periodic integrated AVGARCH model derived from the* $PAVGARCH_s$ *model when the parameters are constrained to be on the boundary of the second-order periodic stationarity domain. This is attributed to the strict inclusion of the latter domain within the strict stationarity one.*

2.3 Monte Carlo experiment

This section evaluates the finite sample properties of the Quasi Maximum Likelihood Estimator (QMLE) for unknown parameters in the $PAVGARCH_s(1,1)$ model through Monte Carlo experiments. With 500 replications across varying sample sizes $n\in\{1000,2000,3000\}$, we consider innovations like standardized normal, Student $t_{(5)}$, and Generalized Exponential Distribution (GED) (see Remarks 1.5.3,1.5.2 and 1.5.1). The parameter vector $\underline{\theta}$ adheres to strict periodic stationarity, detailed at the end of each table. Empirical results, produced using MATLAB scripts, provide insights into model performance. The tables present the average parameter estimates across N simulations, with columns displaying the results. To evaluate the Quasi Maximum Likelihood Estimator ($QMLE$) performance, we furnish the root mean square error ($RMSE$) for each $\hat{\theta}_n(i)$, where $i=1,...,s$ (enclosed in brackets). Moreover, we incorporate the asymptotic distributions of $\hat{\theta}_n(\vartheta)$, where $\vartheta=1,...,s$, across the N simulations, accompanied by boxplot summaries, aligned with the corresponding table.

2.3.1 Standard $AVGARCH$ model

Our initial practical illustration, aiming to elucidate our theoretical analysis, involves the standard $AVGARCH(1,1)$ model. The parameter vector for this model is denoted as $\underline{\theta}=(a_0,\alpha_1,\beta_1)'$, chosen to subject the condition

$$\gamma_L = E\left\{\log\left\{a_1\left|e_0\right|+\beta_1\right\}\right\}<0.$$

The outcomes of our simulations, conducted under various innovations in accordance with two distinct Models, are detailed in Table 2.1

	$\mathcal{N}(0,1)$			$t_{(5)}$			GED		
n	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$
1000	1.0162	0.4996	0.2445	1.0232	0.4997	0.2423	1.0119	0.4962	0.2474
	(0.0206)	(0.0018)	(0.0040)	(0.0267)	(0.0034)	(0.0061)	(0.0263)	(0.0044)	(0.0064)
2000	1.0024	0.4980	0.2508	1.0083	0.5002	0.2476	1.0090	0.4963	0.2476
	(0.0098)	(0.0009)	(0.0020)	(0.0146)	(0.0016)	(0.0032)	(0.0134)	(0.0021)	(0.0031)
3000	1.0012	0.5000	0.2502	1.0056	0.5022	0.2472	1.0064	0.4971	0.2488
	(0.0062)	(0.0006)	(0.0013)	(0.0091)	(0.0012)	(0.0020)	(0.0091)	(0.0013)	(0.0022)
Model(1) : $\underline{\theta}=(1.00,0.50,0.25)'$									
1000	1.0145	0.4485	0.1443	1.0217	0.4483	0.1413	1.0074	0.4451	0.1492
	(0.0200)	(0.0018)	(0.0059)	(0.0247)	(0.0033)	(0.0079)	(0.0233)	(0.0043)	(0.0079)
2000	1.0020	0.4475	0.1511	1.0047	0.4495	0.1491	1.0070	0.4461	0.1476
	(0.0094)	(0.0009)	(0.0028)	(0.0140)	(0.0016)	(0.0045)	(0.0122)	(0.0021)	(0.0042)
3000	1.0006	0.4496	0.1507	1.0030	0.4517	0.1483	1.0045	0.4468	0.1492
	(0.0061)	(0.0006)	(0.0019)	(0.0086)	(0.0011)	(0.0028)	(0.0085)	(0.0013)	(0.0030)

Model(2): $\underline{\theta}=(1.00,0.45,0.15)'$

Table 2.1: Results of estimating the $AVGARCH(1,1)$ according to different innovations

The asymptotic distribution of the sequences $(\sqrt{n}(\underline{\hat{\theta}}_n(i) - \underline{\theta}(i)))_{n\geq 1})$, $i = 1, ..., 3$ followed by their boxplot summary associated to different innovations of Model(1) of Table 2.1 are shown in Figure 2.1.

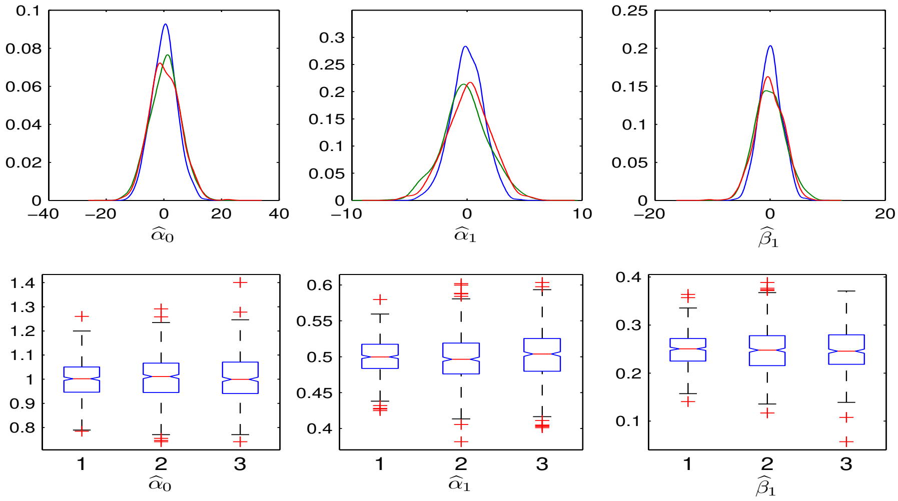

Figure 2.1: Top panels: Asymptotic kernels distribution of $\sqrt{n}(\underline{\hat{\theta}}_n(i) - \underline{\theta}(i))$. Bottom panels: Box plot summary of $\underline{\hat{\theta}}_n(i)$, $i = 1, ..., 3$ associated to *Normal* (blue curve), *GED* (green curve) and *Student* (red curve) innovations according to Model (1) of Table 2.1.

Through a comprehensive analysis of the outcomes presented in table 2.1 for model (1) and model (2), it becomes apparent that the performance of the Quasi Maximum Likelihood Estimator (QML) is significantly less robust under $t_{(5)}$. and GED innovations compared to $\mathcal{N}(0, 1)$. Moreover, a general trend emerges where the root mean square error ($RMSE$) associated with diverse innovations tends to diminish with increasing sample sizes. This trend is visually reinforced by the asymptotic distribution plots in figure(1) (Top panels), revealing fatter tails (positive kurtosis or leptokurtic) for these distributions. Additionally, figure(1) (Bottom panels) , depicting boxplots of the $QMLE$ outcomes under various innovations, underscores noteworthy dissimilarities in elementary statistics, with $\mathcal{N}(0, 1)$ exhibiting fewer outliers compared to other innovations.

2.3.2 Periodic $AVGARCH$ model

The second example of our Monte Carlo experiment is devoted to estimate the periodic $AVGARCH_s(1, 1)$ model with $s = 2$ .i.e., $\epsilon_t = \eta_t h_t$ and

$$h_{2t+v}(\vartheta) = \alpha_0(v) + \alpha_1(v)\epsilon_{2t+v}(v-1) + \beta_1(v)h_{2t+v}(v-1).$$

This situation is raised in modelling some daily returns when we suspect the so-called "Monday effect" (opening price) of day-of-the week seasonality (see for instance Franses and Paap [20]. The vector of parameters to be estimated is thus $\underline{\theta} = (\underline{\alpha}'_0, \underline{\alpha}'_1, \underline{\beta}'_1)'$ where $\underline{\alpha}'_0 = (\alpha_0(1), \alpha_0(2))$, $\underline{\alpha}'_1 = (\alpha_1(1), \alpha_1(2))$, $\underline{\beta}'_1 = (\beta_1(1), \beta_1(2))$, are chosen to ensure the SPS condition of our model. To this end, we suggest that

$$\sum_{\vartheta=1}^{2} E\{\log(a_1(\vartheta)|e_0| + \beta_1(\vartheta))\} < 0.$$

So, the results of simulation according to two models are given in Table 2.2 below.

		$\mathcal{N}(0,1)$			$t_{(5)}$			GED		
n	ϑ	$\underline{\hat{\alpha}}_0$	$\underline{\hat{\alpha}}_1$	$\underline{\hat{\beta}}_1$	$\underline{\hat{\alpha}}_0$	$\underline{\hat{\alpha}}_1$	$\underline{\hat{\beta}}_1$	$\underline{\hat{\alpha}}_0$	$\underline{\hat{\alpha}}_1$	$\underline{\hat{\beta}}_1$
1000	1	0.9487(0.1547)	0.5009(0.0109)	0.2697(0.0319)	0.9457(0.2056)	0.4940(0.0182)	0.2803(0.0483)	0.9313(0.2194)	0.4905(0.0217)	0.2920(0.0533)
	2	0.9826(0.1013)	0.5535(0.0091)	0.1551(0.0181)	0.9472(0.1378)	0.5573(0.0154)	0.1725(0.0283)	0.9252(0.1668)	0.5482(0.0198)	.01889(0.0353)
2000	1	0.9813(0.0780)	0.4995(0.0052)	0.2577(0.0170)	0.9818(0.1158)	0.5030(0.0089)	0.2596(0.0256)	0.9597(0.1293)	0.4907(0.0111)	0.2734(0.0309)
	2	0.9852(0.0533)	0.5515(0.0043)	0.1541(0.0092)	0.9836(0.0780)	0.5579(0.0085)	0.1551(0.0151)	0.9732(0.0847)	0.5501(0.0098)	0.1623(0.0182)
3000	1	0.9799(0.0560)	0.5015(0.0034)	0.2586(0.0117)	0.9794(0.0754)	0.5043(0.0061)	0.2604(0.0173)	0.9777(0.0775)	0.4919(0.0070)	0.2633(0.0188)
	2	0.9931(0.0393)	0.5512(0.0029)	0.1509(0.0069)	0.9901(0.0532)	0.5537(0.0052)	0.1520(0.0105)	0.9830(0.0605)	0.5530(0.0062)	0.1566(0.0128)
Model(1): $\underline{\alpha}_0 = (1.00, 1.00)$, $\underline{\alpha}_1 = (0.50, 0.50)$, $\underline{\beta}_1 = (0.25, 0.15)$										
1000	1	1.0137(0.1119)	0.9967(0.0211)	0.4968(0.0171)	0.9683(0.1676)	0.9943(0.0362)	0.5172(0.0320)	1.0472(0.1895)	0.9978(0.0423)	0.4871(0.0333)
	2	0.9590(0.0976)	1.1670(0.0241)	0.5017(0.0124)	0.8919(0.1456)	1.1972(0.0478)	0.5140(0.0207)	0.9476(0.1812)	1.1767(0.0529)	0.5102(0.0230)
2000	1	1.0036(0.0923)	1.0059(0.0110)	0.4945(0.0128)	0.9247(0.0966)	1.0042(0.0188)	0.5302(0.0180)	1.0542(0.1044)	0.9917(0.0217)	0.4814(0.0191)
	2	0.9884(0.0930)	1.1624(0.0114)	0.4959(0.0072)	0.9032(0.0914)	1.1743(0.0226)	0.5129(0.0108)	0.9711(0.1074)	1.1642(0.0249)	0.5034(0.0125)
3000	1	1.0003(0.0614)	1.0098(0.0069)	0.4955(0.0084)	0.9501(0.0623)	1.0068(0.0125)	0.5208(0.0125)	1.0638(0.0704)	0.9883(0.0137)	0.4786(0.0123)
	2	1.0026(0.0616)	1.1593(0.0074)	0.4932(0.0052)	0.9368(0.0634)	1.1659(0.0141)	0.5051(0.0077)	0.9620(0.0705)	1.1608(0.0156)	0.5068(0.0079)

Model(2): $\underline{\alpha}_0 = (0.50, 0.50)$, $\underline{\alpha}_1 = (1.00, 1.15)$, $\underline{\beta}_1 = (0.50, 0.50)$

Table 2.2: Results of estimating the $PAVGARCH_2(1,1)$ according to different innovations

The asymptotic distribution of the sequences $(\sqrt{n}(\hat{\underline{\theta}}_n(i) - \underline{\theta}(i)))_{n\geq 1})$, $i = 1, ..., 6$ associated to different innovations of Model (1) of Table 2.2 are shown in Figure 2.2.

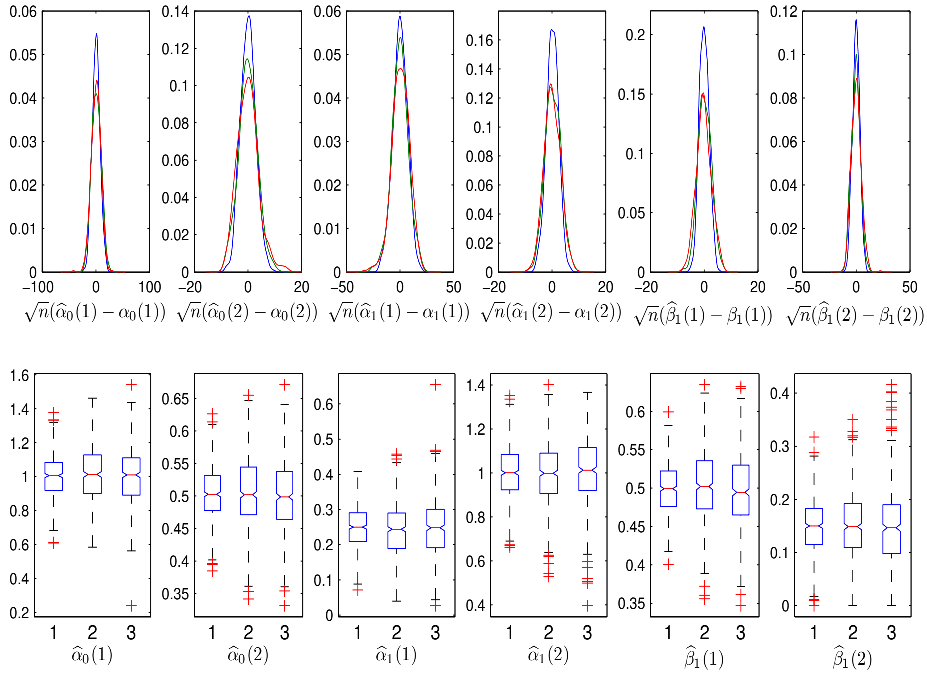

Figure 2.2: Top panels: Asymptotic kernels distribution of $\sqrt{n}(\hat{\underline{\theta}}_n(i) - \underline{\theta}(i))$. Bottom panels: Boxplot summary of $\hat{\underline{\theta}}_n(i)$, $i = 1, ..., 6$ associated to *Normal* (blue curve), *GED* (green curve) and *Student* (red curve) innovations according to Model (1) of Table 2.2.

2.3.3 Comments

A brief examination of the results from our Monte Carlo experiments reveals the following: Tables 2.1 and 2.2 provide parameter estimates for the $PAVGARCH_s(1,1)$ model with $s = 2$, fitted to Model(1) and Model(2), generated through 500 independent simulations with different innovations. First and foremost, it is evident that the performance of the Quasi Maximum Likelihood Estimator (QML) is notably weaker when applied to $t_{(5)}$ and GED innovations compared to $\mathcal{N}(0,1)$. In general, it is clear that the parameters associated with these models are well-estimated, with no significant deviations in estimate values observed for the three different innovation errors: $\mathcal{N}(0,1)$, $t_{(5)}$, and GED. Notably, some estimates have moderate standard deviations. In table 2.2, the model was simulated following a $PAVGARCH_2(1,1)$ model, where the parameters of the five regimes in model(1) satisfy the condition

$$E\left\{\log\left(a_1(\vartheta)\left|e_0\right| + \beta_1(\vartheta)\right)\right\} < 0, \quad \text{for } \vartheta = 1, ..., 2.$$

In Table 2.2, the second regimes in Model(2) are explosive in the sense that

$$E\left\{\log\left(a_1(\vartheta)\left|e_0\right| + \beta_1(\vartheta)\right)\right\} > 0,$$

but the Strong Periodic Stationary (SPS) property of the model is ensured. The results, in general, align satisfactorily with the asymptotic theory. In Figure 2.2, regarding the asymptotic kernels distribution of $\sqrt{n}(\widehat{\theta}n(i) - \theta(i))$ for $i = 1, ..., 6$, it is apparent that the $PAVGARCH_2(1,1)$ model produces flatter ("platykurtic") kernels. It's noteworthy that the assumptions **A1 - A5** required for consistency are clearly satisfied. Furthermore, the assumptions **A6** and **A7** needed for asymptotic normality are also met. Lastly, the boxplots displayed in Figure 2.2 reveal substantial dissimilarities in the elementary statistics of $QMLE$ under different innovations, with $\mathcal{N}(0,1)$ showing fewer outliers compared to the others.

2.4 Application

In this section, we apply our model for modelling the foreign exchange rates of Algerian Dinar with respect to European currency (EUR/DZD) denoted by $y_t^{(e)}$ and the American Dollar (USD/DZD) denoted by $y_t^{(d)}$ already analyzed by Hamdi and Souam [29] via a mixture periodic $GARCH$ models. We consider returns series

$$r_t^{(e)} = 100 \times (\log\left(y_t^{(e)}\right) - \log\left(y_{t-1}^{(e)}\right)) \text{ and } r_t^{(d)} = 100 \times (\log\left(y_t^{(d)}\right) - \log\left(y_{t-1}^{(d)}\right)),$$

of daily exchange rates of Algerian dinar against the Euro. The observation cover the period from January $3, 2000$ to September $29, 2011$. Since there are some weeks comprise less than five observations (due to legal holidays), we remove the entire weeks with less than five data available rather than estimating the "pseudo-missing" observations by an ad-hoc method. Thus, the final length of transformed data is 3055 observations uniformly distributed on 611 weeks. Figure 2.3 displays the plots of the series (y_t) and its returns (r_t) corresponding to foreign exchange of EUR/DZD (series superscripted by (e)) and those corresponding to USD/DZD (series superscripted by (d)).

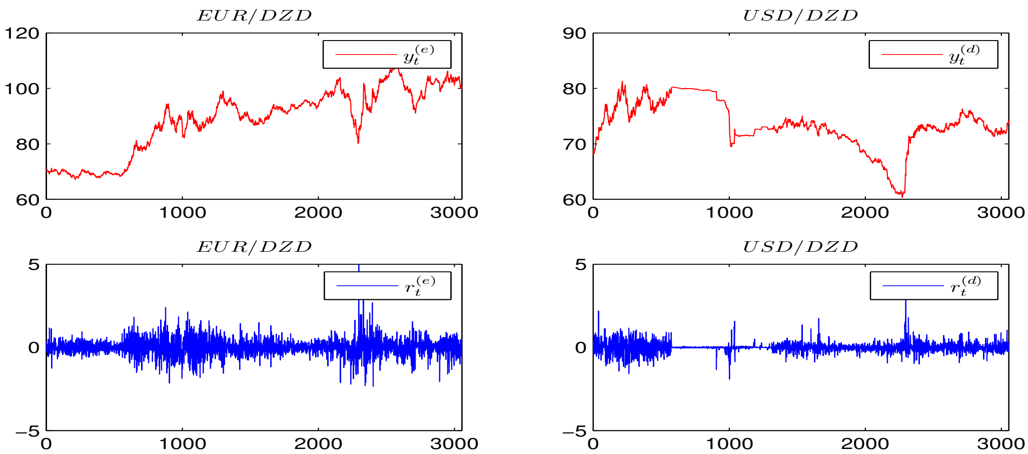

Figure 2.3: Left panel displays the series y_t and r_t corresponding to EUR/DZD. Right panel display similar series correspont to USD/DZD.

The elementary statistics in data are summarized in Table 2.3 below

Series	mean	median	mode	skewness	kurtosis	JBtes	LB(Q(12),Q(24))
$y_t^{(e)}$	88.61181	91.09945	69.7347	−0.518144	2.132958	232.4666	$10^4 \times (3.6065, 7.0724)$
$y_t^{(d)}$	73.45113	73.12610	79.9396	−0.600469	3.764200	258.0098	$10^4 \times (3.5888, 7.0054)$

Table 2.3: Summary Statistics for daily spot prices $y_t^{(e)}$ and $y_t^{(d)}$.

In Table 2.3 the difference between means, medians and modes implies that the series are not symmetric. The high kurtosis computed in these series, being leptokurtic, implies that the distribution of the series have fatter tails, and a more sensitive peak around the mean, when compared to the normal distribution. JBtes (Jarque-Bera test) and LB(Q(12), Q(24)) for normality and autocorrelation tests show that both returns are neither normally distributed nor serially correlated for the instance 10 and 25 lags. Moreover, the results shown in Table 2.4 examine the effect of heteroscedasticity in the series $\left(r_t^{(e)}\right)$ and $\left(r_t^{(d)}\right)$.

	$r_t^{(e)}$				$r_t^{(d)}$			
lags	10	15	20	25	10	15	20	25
$ARCH$ statistics	152.3993	200.3745	244.6458	266.6962	245.6729	249.3297	344.0355	346.1818
Critical value	18.3070	24.9958	31.4104	37.6525	18.3070	24.9958	31.4104	37.6525
$P-value$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 2.4: $ARCH$ effect test of daily returns $\left(r_t^{(e)}\right)$ and $\left(r_t^{(d)}\right)$.

The results of Table 2.4 can be summarized as: since the $p-$value is less than 0.05, the $ARCH$ statistics is greater than the critical value at 95% confidence level. These imply that there is a strong evidence for rejecting the null hypothesis of no $ARCH$ effect. The rejection indicates the existence of $ARCH$ effects in the returns series and therefore the variance of such a returns is not constant. The test was implemented in $MATLAB$ with "*archtest*" function for the returns. Figure 2.4 displays the sample autocorrelations functions (ACF) of the series $(r_t)_{t\geq 1}$, $\left(r_t^2\right)_{t\geq 1}$ and $(|r_t|)_{t\geq 1}$ computed at 30 lags.

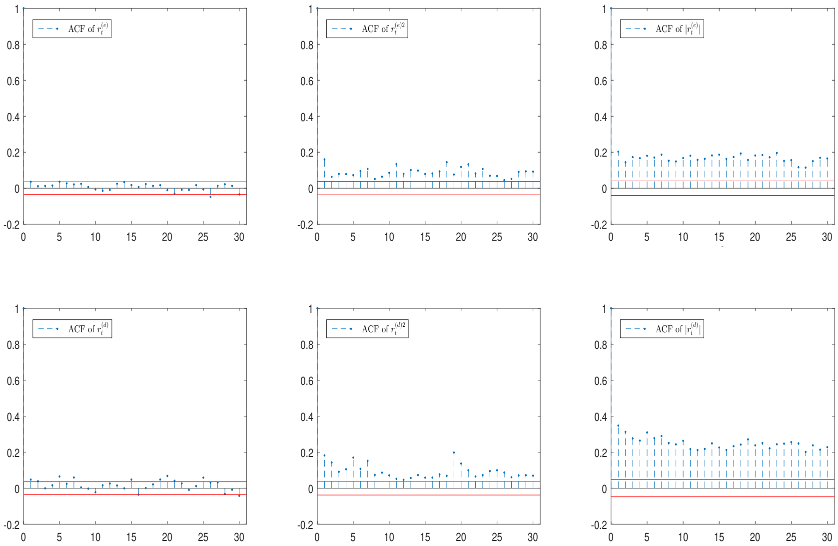

Figure 2.4: Top panel: Sample autocorrelations of returns associated to Euro superscripted by (e). Bottom panel: Sample autocorrelations of returns associated to dollar superscripted by (d).

From Figure 2.4, we can see that the log returns (r_t) show no evidence of serial correlation, but the squared and absolute returns are positively autocorrelated. Also, the decay rates of the sample autocorrelations of $\left(r_t^2\right)$ and $(|r_t|)$ appear to be violated compared with the correlation associated to an $ARMA$ process suggesting possibly a non linear behavior for modelling purpose.

2.4.1 Modeling

The first attempt will be modeling the series $(r_t)_{t\geq 1}$ by a standard $AVGARCH(1,1)$ model and the second attempt is to look for a model able to cover the day-of -week seasonality in return (r_t) (see for instance Franses and Paap [20]). So, in order to analyze the seasonality, we fitted the following simple $PAVGARCH_5(1,1)$ model for each series or equivalently. Hence, we estimate its volatility process $(h_t)_{t\geq 1}$ through five periodic effects,

$$r_t = h_t e_t \text{ and } h_t = \alpha_0\left(t\right) + \alpha_1\left(t\right)\left|r_{t-1}\right| + \beta_1\left(t\right)h_{t-1} \tag{14}$$

The parameters estimates of volatility $AVGARCH\,(1,1)$ $\left((\hat{h}_t^{(s)})_{t\geq 1}\right)$ and the parameters estimates of five-regimes (intra-day) of $(\hat{h}_t^{(p)})_{t\geq 1}$ to model (14) according to EUR/DZD and USD/DZD when the innovation is $\mathcal{N}(0,1)$ are presented in Table 2.5

$\mathcal{N}(0,1)$		EUR/DZD $\hat{\alpha}_0$	EUR/DZD $\hat{\alpha}_1$	EUR/DZD $\hat{\beta}_1$	USD/DZD $\hat{\alpha}_0$	USD/DZD $\hat{\alpha}_1$	USD/DZD $\hat{\beta}_1$
$(\hat{h}_t^{(s)})_{t\geq 1}$	-	0.0024 (0.0014)	0.0607 (0.0096)	0.9491 (0.0085)	0.0050 (0.0047)	0.1213 (0.0359)	0.8959 (0.0390)
	Days	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$
$(\hat{h}_t^{(p)})_{t\geq 1}$	Sunday	0.0001 (0.0292)	0.0444 (0.0215)	1.0657 (0.0941)	0.0233 (0.0252)	0.0250 (0.1252)	0.9006 (0.0691)
	Monday	0.0002 (0.0452)	0.0895 (0.0297)	0.9827 (0.1102)	0.0036 (0.0405)	0.1144 (0.0868)	0.8965 (0.3345)
	Tuesday	0.0098 (0.0317)	0.0783 (0.0034)	0.8701 (0.1082)	0.0001 (0.0110)	0.1159 (0.0539)	0.8957 (0.0789)
	Wednesday	0.0027 (0.0356)	0.0413 (0.0160)	0.8757 (0.0947)	0.0070 (0.0192)	0.1127 (0.0884)	0.8972 (0.0396)
	Thursday	0.0029 (0.0264)	0.0677 (0.0179)	0.940 (0.0826)	0.0138 (0.0417)	0.1148 (0.0716)	0.8984 (0.0443)

Table 2.5: Parameters estimation of $AVGARCH\,(1,1)$ and $PAVGARCH\,(1,1)_5$ models fit to EUR/DZD and USD/DZD according to $\mathcal{N}(0,1)$ innovation.

The plots of the squared returns and the estimated volatilities according to $AVGARCH\,(1,1)$ and $PAVGARCH\,(1,1)_5$ associated to $\mathcal{N}(0,1)$ innovation are showed in Figure 2.5 below.

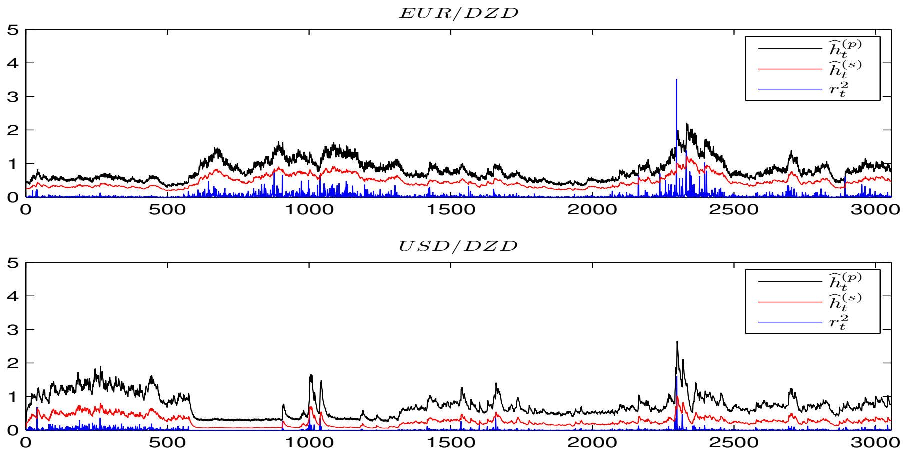

Figure 2.5: Blue: squared returns, Red: volatilities estimates according to $AVGARCH(1,1)$ and Black : volatilities estimates $PAVGARCH_5(1,1)$ according to EUR/DZD(top) and USD/DZD(bottom) with innovation $\mathcal{N}(0,1)$.

The second results of parameters estimates of volatility $AVGARCH\,(1,1)$ $\left((\hat{h}_t^{(s)})_{t\geq1}\right)$ and $PAVGARCH\,(1,1)_5$ $\left((\hat{h}_t^{(p)})_{t\geq1}\right)$ according to EUR/DZD and USD/DZD when the innovation is $t_{(5)}$ are presented in Table 2.6

$t_{(5)}$		EUR/DZD $\hat{\alpha}_0$	EUR/DZD $\hat{\alpha}_1$	EUR/DZD $\hat{\beta}_1$	USD/DZD $\hat{\alpha}_0$	USD/DZD $\hat{\alpha}_1$	USD/DZD $\hat{\beta}_1$
$(\hat{h}_t^{(s)})_{t\geq1}$	-	0.0024 (0.0012)	0.0642 (0.0086)	0.9485 (0.0076)	0.0007 (0.0003)	0.2161 (0.0287)	0.8438 (0.0203)
	Days	$\underline{\hat{\alpha}}_0$	$\underline{\hat{\alpha}}_1$	$\underline{\hat{\beta}}_1$	$\underline{\hat{\alpha}}_0$	$\underline{\hat{\alpha}}_1$	$\underline{\hat{\beta}}_1$
$(\hat{h}_t^{(p)})_{t\geq1}$	Sunday	0.0030 (0.0432)	0.2329 (0.0549)	0.7132 (0.0969)	0.0001 (0.0295)	0.2882 (0.1484)	0.9194 (0.2909)
	Monday	0.1037 (0.0534)	0.2179 (0.0526)	0.7024 (0.1310)	0.0001 (0.0499)	0.2104 (0.0892)	0.8351 (0.2877)
	Tuesday	0.0165 (0.1246)	0.1226 (0.0420)	0.8459 (0.0690)	0.0014 (0.0446)	0.2224 (0.0531)	0.7563 (0.3071)
	Wednesday	0.0012 (0.0527)	0.1177 (0.0468)	0.9914 (0.3911)	0.0011 (0.0421)	0.1807 (0.0816)	0.8222 (0.3003)
	Thursday	0.1115 (0.0403)	0.2914 (0.0618)	0.6970 (0.0744)	0.0009 (0.0228)	0.2450 (0.0534)	0.8529 (0.2232)

Table 2.6: Parameters estimation of $AVGARCH\,(1,1)$ and $PAVGARCH\,(1,1)_5$ models fit to EUR/DZD and USD/DZD according to $t_{(5)}$ innovation.

The plots of the squared returns and the estimated volatilities according to $AVGARCH\,(1,1)$ and $PAVGARCH\,(1,1)_5$ associated to $t_{(5)}$ innovation are showed in Figure 2.6 below.

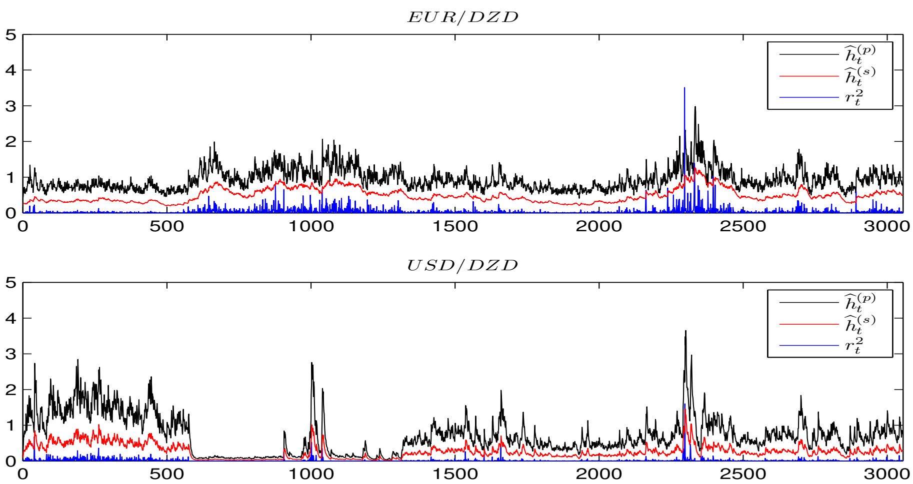

Figure 2.6: Blue: squared returns, Red: volatilities estimates according to $AVGARCH(1,1)$ and Black : volatilities estimates $PAVGARCH_5(1,1)$ according to EUR/DZD(top) and USD/DZD(bottom) with innovation

The third results of parameters estimates of volatility $AVGARCH\,(1,1)$ $\left((\hat{h}_t^{(s)})_{t\geq 1}\right)$ and $PAVGARCH\,(1,1)_5$ $\left((\hat{h}_t^{(p)})_{t\geq 1}\right)$ according to EUR/DZD and USD/DZD when the innovation is GED are presented in Table 2.7

GED		EUR/DZD $\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	USD/DZD $\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$
$(\hat{h}_t^{(s)})_{t\geq 1}$	-	0.0025 (0.0013)	0.0668 (0.0091)	0.9480 (0.0077)	0.0013 (0.0009)	0.1905 (0.0365)	0.8645 (0.0261)
	Days	$\underline{\hat{\alpha}}_0$	$\underline{\hat{\alpha}}_1$	$\underline{\hat{\beta}}_1$	$\underline{\hat{\alpha}}_0$	$\underline{\hat{\alpha}}_1$	$\underline{\hat{\beta}}_1$
$(\hat{h}_t^{(p)})_{t\geq 1}$	Sunday	0.0778 (0.0443)	0.1723 (0.0754)	0.8355 (0.1196)	0.0015 (0.0450)	0.3099 (0.868)	0.8759 (0.2382)
	Monday	0.0001 (0.0339)	0.2463 (0.0982)	0.8165 (0.1130)	0.0037 (0.0483)	0.2165 (0.0609)	0.7748 (0.2581)
	Tuesday	0.0285 (0.0464)	0.2209 (0.0834)	0.7945 (0.1182)	0.0003 (0.0741)	0.2995 (0.1215)	0.7240 (0.3979)
	Wednesday	0.0063 (0.0278)	0.1807 (0.0794)	0.7761 (0.0784)	0.0003 (0.0665)	0.2169 (0.1051)	0.8402 (0.3763)
	Thursday	0.0276 (0.0362)	0.2151 (0.0933)	0.7853 (0.1171)	0.0065 (0.0080)	0.2998 (0.0859)	0.8473 (0.1265)

Table 2.7: Parameters estimation of $AVGARCH\,(1,1)$ and $PAVGARCH\,(1,1)_5$ modelsfit to EUR/DZD and USD/DZD according to GED innovation.

The plots of the squared returns and the estimated volatilities according to $AVGARCH\,(1,1)$ and $PAVGARCH\,(1,1)_5$ associated to GED innovation are showed in Figure 2.7 below.

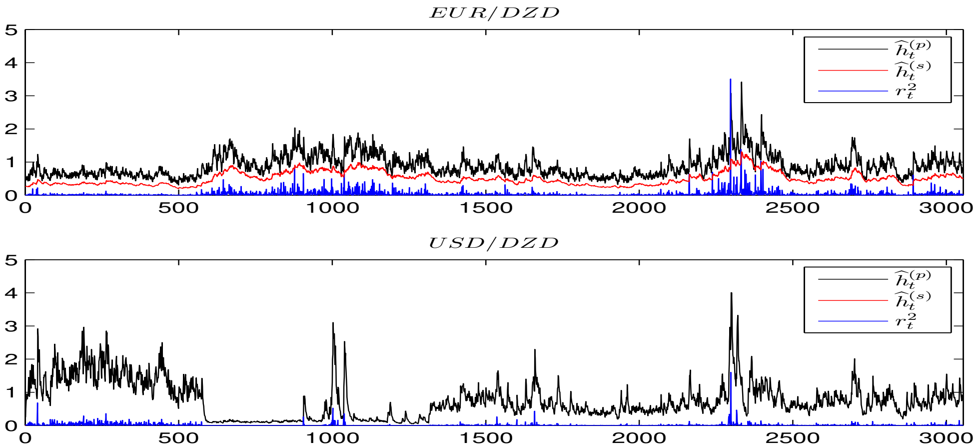

Figure 2.7: Blue: squared returns, Red: volatilities estimates according to $AVGARCH(1,1)$ and Black : volatilities estimates $PAVGARCH_5(1,1)$ according to EUR/DZD(top) and USD/DZD(bottom) with innovation GED.

2.4.2 Comments

In Tables 2.5, 2.6 and 2.7, we present the estimation results for $AVGARCH\,(1,1)$ and $PAVGARCH_5\,(1,1)$ models according to $\mathcal{N}(0,1)$, $t_{(5)}$ and GED innovations, reflect some characteristics of "spurious" $GARCH$ effects. In particular, the components of $\underline{\hat{\alpha}}_0$ are close to zeros will that the components of $\underline{\hat{\beta}}_1$ are close to ones with moderate $RMSE$. Figures 2.5, 2.6 and 2.7 represents the plots of the volatilities estimates according to $AVGARCH\,(1,1)$ model (plots in red) and $PAVGARCH_5\,(1,1)$ model (plots in black) with different innovations associated to Euro and dollar and compared with the appropriate squared returns (plots in blue). We can see from these plots reveal synchronized movements of estimated volatilities for different innovations and highlight volatility clustering, where large returns lead to high volatility and vice versa. Notably, periods of low volatility are observed between 2000 and 2002 and after 2010, while high volatility clusters coincide with the global financial crisis and other economic events.

Chapter 3

Estimation of Periodic Time-Varying Bilinear Threshold GARCH Models Using QML

Contents

This chapter was published by Slimani and all [44] in the journal Journal of Siberian Federal University. Mathematics and Physics.

this chapter has made significant contributions to the field of time series analysis and financial modeling. The establishment of necessary and sufficient conditions for the existence of stationary solutions, coupled with the introduction of a quasi-maximum likelihood (QML) estimation approach for the PTBLTGARCH model, enhances our understanding of the model's behavior. The Markovian representation of the PTBLGARCHs model, along with the delineation of conditions for a Strict Periodic Stationary (SPS) solution, constitutes a notable advancement. Section 3 provides a comprehensive examination of the strong consistency and asymptotic normality of the Quasi-Maximum Likelihood Estimator (QMLE), underscoring the robustness of the proposed methodology. Numerical demonstrations in Section 4, including a Monte Carlo study, showcase the finite-sample properties of the QMLE. Moreover, the practical application of the model to analyze exchange rates between the Algerian Dinar and the European Euro in Section 5 adds real-world relevance and insights. The chapter concludes in Section 6, summarizing key findings and contributions, thereby consolidating its significance in the realm of financial econometrics.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let the innovation process $(e_t, t \in \mathcal{Z})$ be a sequence independent and identically distributed with zero mean and unit variance $(i.i.d\,(0,1))$ defined on the same probability space. We define the time-varying coefficients "volatility"process $(h_t, t \in \mathcal{Z})$ satisfy the recursion

$$h_t^2 = \alpha_0(t) + \sum_{i=1}^{q} \alpha_i(t)\,\varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j(t)\,h_{t-j}^2, \tag{3.1}$$

where $(\alpha_i(t), 0 \le i \le q)$ and $(\beta_j(t), 0 \le j \le p)$ are non negative periodic functions with period s with $\alpha_0(t) > 0$.

Generally, by $PGARCH_s$ process, we mean a discrete-time strictly stationary process $(\varepsilon_t, t \in \mathcal{Z})$,$\mathcal{Z} = \{0, \pm 1, \pm 2, ...\}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ and satisfying the factorization

$$\varepsilon_t = h_t e_t. \tag{3.2}$$

The study, conducted by Rodriguez and Ruiz [46], investigated five prominent models utilized for modeling time-invariant asymmetric volatility processes with a leverage effect. These models encompass the Generalized Quadratic ARCH (GQARCH), Threshold GARCH (TGARCH), GJR-GARCH (GJR), Exponential GARCH (EGARCH), and Asymmetric Power GARCH (APGARCH) models. Given their significance in modeling, forecasting, and capturing volatility asymmetry, these models prove suitable for addressing leverage effects. Additionally, a noteworthy inclusion in this exploration is the Bilinear Threshold GARCH (BLTGARCH) model, recently introduced by Choi and all [15], featuring time-invariant coefficient volatility processes.

$$h_t^2 = \alpha_0 + \sum_{i=1}^{q} \left(\alpha_i \varepsilon_{t-i}^{+2} + \beta_i \varepsilon_{t-i}^{-2}\right) + \sum_{k=1}^{d} \left(b_k \varepsilon_{t-k}^{+} + \omega_k \varepsilon_{t-k}^{-}\right) h_{t-k} + \sum_{j=1}^{p} \gamma_j h_{t-j}^2, \tag{3.3}$$

where $\varepsilon_n^+ = \max(\varepsilon_n, 0)$, $\varepsilon_n^- = \min(\varepsilon_n, 0)$, $\varepsilon_n^{+2} = (\varepsilon_n^+)^2$, $\varepsilon_n^{-2} = (\varepsilon_n^-)^2$, and $d = p \wedge q$.

This chapter addresses the fundamental issue of non-stationarity in BLTGARCH models. In these models, the parameters exhibit periodicity with a period denoted by "s". Consequently, we introduce a periodic BLTGARCH(q,d,p) model, called $\text{PBLTGARCH}_\text{s}$, defined by equation (3.2).

$$h_t^2 = \alpha_0(t) + \sum_{i=1}^{q} \left(\alpha_i(t)\,\varepsilon_{t-i}^{+2} + \beta_i(t)\,\varepsilon_{t-i}^{-2}\right) + \sum_{k=1}^{d} \left(b_k(t)\,\varepsilon_{t-k}^{+} + \omega_k(t)\,\varepsilon_{t-k}^{-}\right) h_{t-k} + \sum_{j=1}^{p} \gamma_j(t)\,h_{t-j}^2. \tag{3.4}$$

In (3.4), the functions $(\alpha_i(t), 0 \le i \le q)$, $(\beta_i(t), 1 \le i \le q)$, $(b_k(t), 1 \le k \le d)$, $(\omega_k(t), 1 \le k \le d)$ and $(\gamma_j(t), 1 \le j \le p)$ are periodic with period $s \ge 1$. Moreover, $(\alpha_i(t), 0 \le i \le q)$, $(\beta_i(t), 1 \le i \le q)$, $(\gamma_j(t), 1 \le j \le p)$ are non negative sequences with $\alpha_0(.) > 0$, except the functions $(b_k(t), 1 \le k \le d)$, $(\omega_k(t), 1 \le k \le d)$ which have values in $(-\infty, +\infty)$.

So, by transforming t into $t \to st + v$ and setting $\varepsilon_t(v) = \varepsilon_{st+v}$,$h_t(v) = h_{st+v}$, and $e_t(v) = e_{st+v}$.

The equation (3.4) has an equivalent periodic form, given as:

$$\begin{aligned} h_t^2(v) = {} & \alpha_0(v) + \sum_{i=1}^{q} \left(\alpha_i(v)\,\varepsilon_t^{+2}(v-i) + \beta_i(v)\,\varepsilon_t^{-2}(v-i)\right) \\ & + \sum_{k=1}^{d} \left(b_k(v)\,\varepsilon_t^{+}(v-k) + \omega_k(v)\,\varepsilon_t^{-}(v-k)\right) h_t(v-k) + \sum_{j=1}^{p} \gamma_j(v)\,h_t^2(v-j). \end{aligned} \tag{3.5}$$

In (3.5), the notation $\varepsilon_t(v)$ refers to ε_t during the $v-th$ "season" $v \in \mathbb{S} = \{1, ..., s\}$ of cycle t, and for the convenient we set $\varepsilon_t(v) = \varepsilon_{t-1}(v+s)$, $h_t(v) = h_{t-1}(v+s)$ and $e_t(v) = e_{t-1}(v+s)$, if $v < 0$. The non-periodic notations $(\varepsilon_t), (e_t)$ and (h_t) will be used interchangeably with their periodic counterparts $(\varepsilon_t(v))$, $(e_t(v))$ and $(h_t(v))$ whenever emphasizing seasonality is not required. It is important to note that since h_t^2 represents the conditional variance of ε_t given information up to time $t-1$, the positivity of the functions $(\alpha_i(t), 0 \le i \le q)$, $(\beta_i(t), 1 \le i \le q)$ and $(\gamma_j(t), 1 \le j \le p)$ ensures the positivity of h_t^2 in the $PTGARCH_s$ model. However, this does not hold true in $PBLTGARCH_s$, even when $b_k(\cdot) \ge 0$ and $\omega_k(\cdot) \ge 0$, due to the penultimate term in the quation (3.5). Therefore, the positivity of h_t^2 needs to be studied on a case-by-case basis (see Nelson [39]). For the sake of simplicity, we assume throughout this chapter that

Assumption 3.0.1. *Almost surely* $(a.s)$, $h_t^2 > 0$.

Remark 3.0.1. *The $PBLTGARCHs(q,d,p)$ model encompasses various other models, including:*

1. *Standard $BLTGARCH(q,d,p)$: This model is obtained by assuming $s=1$ (see Choi and all [15]).*
2. *Periodic $TGARCHs(p,q)$: This model is obtained by setting $b_k(\upsilon)=\omega_k(\upsilon)=0$ for $\upsilon\in\mathbb{S}$ (see Bibi [5]).*
3. *Periodic $BLGARCHs(q,d,p)$: This model is obtained by setting $\alpha_i(\upsilon)=\beta_i(\upsilon)$ and $b_k(\upsilon)=-\omega_k(\upsilon)$ for $\upsilon\in\mathbb{S}$ (see Bibi and Ghezal [7]).*

3.1 Probabilistic properties of $PBLTGARCHs(p,q,d)$

Similar to numerous time series models, the derivation of an equivalent Markovian representation for equations (3.2) to (14) proves advantageous, streamlining the analytical process. To achieve this, we introduce a vector denoted by r, with a dimensionality of $r=(p+2q+2d)$.

$$\underline{\varepsilon}_t' := \left(h_t^2,...,h_{t-p+1}^2,\varepsilon_t^{+2},\varepsilon_t^{-2},...,\varepsilon_{t-q+1}^{+2},\varepsilon_{t-q+1}^{-2},h_te_t^+,h_te_t^-,...,h_{t-d+1}e_{t-d+1}^+,h_{t-d+1}e_{t-d+1}^-\right),$$

$$\underline{H}_0' := \left(1,\underline{O}_{(r-1)}'\right),\quad \underline{H}_1' := \left(\underline{O}_{(p)}',1,-1,\underline{O}_{(r-p-2)}'\right),$$

and

$$\underline{\eta}_t(e_t) := \underline{\alpha}_{0,p+1}(t)\,e_t^{+2}+\underline{\alpha}_{0,p+2}(t)\,e_t^{-2}+\underline{\alpha}_{0,r-2d+1}(t)\,e_t^{+}+\underline{\alpha}_{0,r-2d+2}(t)\,e_t^{-}+\underline{\alpha}_{0,1}(t),$$

in which the $j-th$ entry of $\underline{\alpha}_{0,j}(t)$ is $\alpha_0(t)$ and all other elements are 0.

We obtain the following state-space representation $\varepsilon_t^2=\underline{H}_1'\underline{\varepsilon}_t$ and $h_t^2=\underline{H}_0'\underline{\varepsilon}_t$. We obtain

$$\underline{\varepsilon}_t = A_t(e_t)\,\underline{\varepsilon}_{t-1}+\underline{\eta}_t(e_t),\ t\in\mathbb{Z}. \tag{3.6}$$

Such that

$$A_t(e_t) := A_1(t)\,e_t^{+2}+A_2(t)\,e_t^{-2}+A_3(t)\,e_t^{+}+A_4(t)\,e_t^{-}+A_5(t),$$

where $(A_j(t),1\le j\le 5)$ are appropriate $r\times r-$periodic matrices easily obtained and uniquely determined by

$$\{\alpha_i(t),\ \beta_i(t),\ b_k(t),\ \omega_k(t),\ \gamma_j(t),1\le i,k,j\le q\vee p\}.$$

Now, by iteratively applying equation (3.6) a total of s times, we obtain the following equation:

$$\underline{\varepsilon}_{(t+1)s} = H(\underline{e}_t)\,\underline{\varepsilon}_{ts}+\underline{\eta}(\underline{e}_t),\ t\in\mathbb{Z}. \tag{3.7}$$

Since, we have $\underline{e}_{t+1}=\left(e_{(t+1)s},...,e_{st+1}\right)'$,

$$H(\underline{e}_t)=\left\{\prod_{j=0}^{s-1}A_{(t+1)s-j}\left(e_{(t+1)s-j}\right)\right\},\underline{\eta}(\underline{e}_t)=\sum_{k=0}^{s-1}\left\{\prod_{j=0}^{s-1}A_{(t+1)s-j}\left(e_{(t+1)s-j}\right)\right\}\underline{\eta}_{(t+1)s-k}\left(e_{(t+1)s-k}\right).$$

Put $\underline{\varepsilon}_{ts}=\underline{\varepsilon}(t)$ (if there is no confusion). Then, the equation (3.7) has a new form

$$\underline{\varepsilon}(t) = H\left(\underline{e}_{t-1}\right)\underline{\varepsilon}(t-1)+\underline{\eta}\left(\underline{e}_{t-1}\right),\quad t\in\mathbb{Z}. \tag{3.8}$$

It's important to note that $H(\underline{e}_t)$ represents a sequence of independent and identically distributed (i.i.d.) random matrices, which are independent of $\underline{\varepsilon}(k)$ for $k\le t$, while $\underline{\eta}(\underline{e}_t)$ is a sequence of i.i.d. vectors. Consequently, the existence of what is referred to as strictly periodically stationary (SPS) and periodic ergodic (PE) solutions in equations (3.2) to (3.5) is equivalent to the existence of strictly stationary and ergodic solutions in equation (3.8). This connection is given by Bougerol and Picard [12], and also examined by Horst [30].

$$\underline{\varepsilon}(t) = \sum_{k\ge 1}\left\{\prod_{i=0}^{k-1}H\left(\underline{e}_{t-i-1}\right)\right\}\underline{\eta}\left(\underline{e}_{t-k-1}\right)+\underline{\eta}\left(\underline{e}_{t-1}\right). \tag{3.9}$$

The formation of a distinctive, strictly stationary, and ergodic solution of the equation (3.8) hinges on the requirement that the top Lyapunov exponent $\gamma(H)$ aligns with the strictly stationary and ergodic sequence of random matrices.

$$H = \left(H\left(\underline{e}_t\right), t \in \mathbb{Z}\right).$$

This condition holds if and only if

$$\gamma\left(H\right) := \inf_{t>0}\left\{\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{t-1}H\left(\underline{e}_{t-j-1}\right)\right\|\right\}\right\} \overset{a.s.}{=} \lim_{t\to\infty}\left\{\frac{1}{t}\log\left\|\prod_{j=0}^{t-1}H\left(\underline{e}_{t-j-1}\right)\right\|\right\}. \tag{3.10}$$

The condition that $\gamma(H) < 0$ is crucial to ensure the existence of a strictly stationary and ergodic solution. Fortunately, the assurance of $\gamma(H)$ can be established based on the following rationale: The expectation of the positive logarithm of the norm of $H(\underline{e}_t)$, denoted as $\log^+\|H(\underline{e}_t)\|$, is bounded, such that $E\left[\log^+\|H(\underline{e}_t)\|\right] \leq E\left[\|H(\underline{e}_t)\|\right] < \infty$. Here, $\log^+(x)$ represents the function that takes the maximum of $\log(x)$ and 0. Moreover, the right-hand side of the equation (3.10) can be rigorously justified by employing Kingman's subadditive ergodic theorem [34].

In the following theorem we are going to ensure that the equation. (3.8) admits a unique, strictly stationary, causal and ergodic solution given by the series (3.9) and the equations (3.5) (3.2) have unique SPE and PE. unique, SPS

Theorem 3.1.1. *Assuming $\gamma(H)$ corresponding to $PBLTGARCHs(q,d,p)$ models is strictly negative. Then, the equation. (3.8) admits a unique, strictly stationary, causal and ergodic solution given by the series (3.9). Therefore, the equation. (3.5) and hence (3.2) admits a unique, SPS causal and PE solution given by*

$$h_t^2 = \underline{H}_0'\underline{\varepsilon}_t,$$

or

$$\varepsilon_t = e_t\left\{\underline{H}_1'\underline{\varepsilon}_t\right\}^{\frac{1}{2}},$$

where $\underline{\varepsilon}_t$ is given by the series equation (3.9).

We deduce the following result.

Corollary 3.1.1. *If $\gamma(H) < 0$ and $E\{|e_0|^{2\delta}\} < \infty$ for some $\delta > 0$. Then, there is $\delta^* \in]0,1]$ such that $E(h_t^{\delta^*}) < \infty$ and $E({\varepsilon_t}^{\delta^*}) < \infty$.*

Proof. The proof follows essentially the same arguments as in Bibi and Ghezal [8]. □

Remark 3.1.1. *Aknouche and Guerbyenne [2], the others have studied the conditions ensuring the existence and the uniqueness of SPS and PE solution of equations (3.2) and (3.5) using directly the equation (3.6) by showing that*

$$\inf_{t>0}\left\{\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{ts-1}A_{ts-j}\left(e_{ts-j}\right)\right\|\right\}\right\}. \tag{3.11}$$

This is a sufficient condition for that the equation (3.6) has a unique, causal, SPS and PE solution given by

$$\underline{\varepsilon}_t = \sum_{k\geq 1}\left\{\prod_{i=0}^{k-1}A_{t-i}\left(e_{t-i}\right)\right\}\underline{\eta}_{t-k}\left(e_{t-k}\right) + \underline{\eta}_t\left(e_t\right). \tag{3.12}$$

Bibi and Ghezal [8] have shown that the series equations (3.9) and (3.12) coincide a.s. whenever the condition equation (3.11) holds true.

Remark 3.1.2. *It's essential to emphasize that the condition $\gamma_L^{(s)}(H) < 0$ ensures a form of global stability for the model (3.6). However, when $\gamma_L^{(s)}(H) \geq 0$, the model (3.6) is regarded as unstable, and consequently, it lacks a strictly periodically stationary (SPS) solution. To illustrate this, let's consider the $PBLAARCH_s(1.1)$ model, which is defined by*

$$\varepsilon_t(v) = h_t(v)\,e_t(v),$$

and

$$h_t^2(\upsilon) = \alpha_0(\upsilon) + \alpha_1(\upsilon)\left|e_t^2(\upsilon-1)\right| h_t^2(\upsilon-1) + b_1(\upsilon)\left|e_t(\upsilon-1)\right| h_t(\upsilon-1).$$

It is straightforward to demonstrate that

$$\gamma_L^{(s)}(H) = E\left(\log\left(\prod_{\upsilon=0}^{s-1}\left(\left|\alpha_1(\upsilon)\left|e_0^2\right| + b_1(\upsilon)\left|e_0\right|\right|\right)\right)\right) \geq 0.$$

Therefore, the presence of certain, albeit not all, "stable regimes" is established. (i.e., $E\left\{\log\left(\left|\alpha_1(\upsilon)\left|e_0^2\right| + b_1(\upsilon)\left|e_0\right|\right|\right)\right\} < 0$) does not guarantees the existence of SPS solution. More generally we have the following convergence of the volatility to infinity for $PBLAARCH_s(1,1)$ process encompassing (3.7).

3.1.1 Example

For the $PBLTGARCH_s(1,1,1)$ model, after some tedious algebra we find that the necessary and sufficient condition ensuring the existence of SPS solution is that

$$\sum_{\upsilon=1}^{s} E\left\{\log\left\{\left|\alpha_1(\upsilon)e_0^{+2} + \beta_1(\upsilon)e_0^{-2} + b_1(\upsilon)e_0^{+} + \omega_1(\upsilon)e_0^{-} + \gamma_1(\upsilon)\right|\right\}\right\}$$

is strictly negative. It is worth noting that the existence of regimes which satisfy

$$E\left\{\log\left\{\left|\alpha_1(\upsilon)e_0^{+2} + \beta_1(\upsilon)e_0^{-2} + b_1(\upsilon)e_0^{+} + \omega_1(\upsilon)e_0^{-} + \gamma_1(\upsilon)\right|\right\}\right\} > 0,$$

does not preclude strict periodic stationarity.

3.2 QML estimation

In this section , we consider the quasi-maximum likelihood estimator ($QMLE$) for estimating the parameters of $PBLTGARCH_s$ model gathered in vector

$$\underline{\theta}' = \left(\underline{\theta}_1, ..., \underline{\theta}_{s(1+2q+2d+p)}\right) := \left(\underline{\alpha}', \underline{\beta}', \underline{b}', \underline{\omega}', \underline{\gamma}'\right) \in \Theta \subset \mathbb{R}^{s(1+2q+2d+p)},$$

where $\underline{\alpha}' := \left(\underline{\alpha}_0', \underline{\alpha}_1', ..., \underline{\alpha}_q'\right)$, $\underline{\beta}' := \left(\underline{\beta}_1', ..., \underline{\beta}_q'\right)$, $\underline{b}' := \left(\underline{b}_1', ..., \underline{b}_d'\right)$, $\underline{\omega}' := \left(\underline{\omega}_1', ..., \underline{\omega}_d'\right)$, $\underline{\gamma}' := \left(\underline{\gamma}_1', ..., \underline{\gamma}_p'\right)$ with $\underline{\alpha}_i' := (\alpha_i(1), ..., \alpha_i(s))$, $\underline{\beta}_i' := (\beta_i(1), ..., \beta_i(s))$, $\underline{b}_k' := (b_k(1), ..., b_k(s))$ $\underline{\omega}_k' := (\omega_k(1), ..., \omega_k(s))$, $\underline{\gamma}_j' := (\gamma_j(1), ..., \gamma_j(s)), \forall 0 \leq i \leq q,\ 1 \leq k \leq d,\ 1 \leq j \leq p$. The true parameter value denoted by $\underline{\theta}_0 \in \Theta \subset \mathbb{R}^{s(1+2q+2d+p)}$ is unknown and therefore it must be estimated. For this purpose, consider a realization $\{\varepsilon_1, ..., \varepsilon_n; n = sN\}$ from the unique, causal, SPS and PE solution of (3.2) and (3.5) and let $h_t^2(\underline{\theta})$ be the conditional variance of ε_t given $\mathcal{F}_{t-1}$where

$$\mathcal{F}_t := \sigma(\varepsilon_\tau; \tau \leq t).$$

The Gaussian $log-$likelihood function of $\underline{\theta} \in \Theta$ conditional on some initial values $\varepsilon_0, ..., \varepsilon_{1-q}, h_0, ..., h_{1-p}$ which are generated by equations (3.2)-(3.5) is given up to an additive constant by

$$\tilde{L}_{Ns}(\underline{\theta}) = -(Ns)^{-1}\sum_{t=1}^{N}\sum_{\upsilon=0}^{s-1}\tilde{l}_{st+\upsilon}(\underline{\theta}), \tag{3.13}$$

with

$$\tilde{l}_t(\underline{\theta}) = \frac{\varepsilon_t^2}{\tilde{h}_t^2(\underline{\theta})} + \log\tilde{h}_t^2(\underline{\theta}),$$

where $\tilde{h}_t^2(\underline{\theta})$ is recursively defined, for $t \geq 1$ by

$$\tilde{h}_t^2(\underline{\theta}) = \alpha_0(t) + \sum_{i=1}^{q}\left(\alpha_i(t)\varepsilon_{t-i}^{+2} + \beta_i(t)\varepsilon_{t-i}^{-2}\right) + \sum_{k=1}^{d}\left(b_k(t)\varepsilon_{t-k}^{+} + \omega_k(t)\varepsilon_{t-k}^{-}\right)\tilde{h}_{t-k}(\underline{\theta}) + \sum_{j=1}^{p}\gamma_j(t)\tilde{h}_{t-j}^2(\underline{\theta}).$$

A $QMLE$ of $\underline{\theta}$ is defined as any measurable solution $\hat{\underline{\theta}}_{Ns}$ of

$$\hat{\underline{\theta}}_{Ns} = Arg\max_{\underline{\theta}\in\Theta}\tilde{L}_{Ns}(\underline{\theta}) = Arg\min_{\underline{\theta}\in\Theta}\left(-\tilde{L}_{Ns}(\underline{\theta})\right). \tag{3.14}$$

Given the pronounced dependency of $\tilde{h}_t^2(\underline{\theta})$ on initial values $\varepsilon_0, ..., \varepsilon_{1-q}, h_0, ..., h_{1-p}$, the sequence $\left(\tilde{l}_t(\underline{\theta})\right) t \geq 1$ does not exhibit strictly periodically stationary (SPS) or periodically ergodic (PE) properties. Therefore, it is more convenient to operate with an unobserved SPS and PE version. Consequently, we use an approximate version

$$\tilde{L}Ns = -(NS)^{-1}\sum_{v=0}^{s-1} l_{st+v}(\underline{\theta}),$$

of the likelihood (3.13), while

$$l_t(\underline{\theta}) = \frac{\varepsilon_t^2}{h_t^2(\underline{\theta})} + \log h_t^2(\underline{\theta}).$$

Before exploring the remarks, it's essential to note Bibi and Ghezal's [7] and Bibi's [5] research on Quasi-Maximum Likelihood Estimation $(QMLE)$ for periodic time-varying bilinear $BLGARCH$ models. Their work forms the basis for our analysis of the asymptotic properties of $QMLE$ in the $PBLTGARCH_s(p, q, d)$ model, shedding light on our model's distinct features and contributions in the realm of time series modeling.

Remark 3.2.1. *Bibi and Ghezal [7] have established the Quasi-Maximum Likelihood Estimator ($QMLE$) for periodic time-varying bilinear $BLGARCH$ models, specifically when $\alpha_i(t) = \beta_i(t)$ and $b_k(t) = -\omega_k(t)$. It is therefore worthwhile to extend the investigation of the asymptotic properties of the $QMLE$ to the $PBLTGARCH_s(p, q, d)$ model.*

Remark 3.2.2. *Bibi [5] explored the asymptotic properties of the $QMLE$ for a broad category of periodic $PTGARCH$ models ($b_k = \omega_k = 0, 1 \leq k \leq d$). However, it's important to note that their class of models does not encompass the one we have proposed.*

In the next section, we provide conditions that guarantee both the strong consistency and asymptotic normality of $\hat{\underline{\theta}}$. Our methodology draws significant inspiration from Aknouche and Bibi [1] and Francq and Zakoïan [22].

3.2.1 Strong consistency of $QMLE$

Consider the following regularities assumptions:

C0. $\underline{\theta}_0 \in \Theta$ and Θ is a compact subset of $\mathbb{R}^{s(1+2q+2d+p)}$.
C1. Let L denote the lag operator and consider the polynomials

$$\mathcal{A}_{0,v}(\boldsymbol{z}) = \sum_{i=1}^{q}\alpha_{0,i}(v)\boldsymbol{z}^i,\ \mathcal{B}_{0,v}(\boldsymbol{z}) = \sum_{i=1}^{q}\beta_{0,i}(v)\boldsymbol{z}^i,$$

$$\mathcal{H}_{0,v}(\boldsymbol{z}) = \sum_{k=1}^{d} b_{0,k}(v)\left[\boldsymbol{z}^k, \boldsymbol{z}^k\right],\ \mathcal{D}_{0,v}(\boldsymbol{z}) = \sum_{k=1}^{d}\omega_{0,k}(v)\left[\boldsymbol{z}^k, \boldsymbol{z}^k\right],$$

$$\mathcal{C}_{0,v}(\boldsymbol{z}) = 1 - \sum_{i=1}^{p}\gamma_{0,i}(v)\boldsymbol{z}^i,$$

where $\left[L^k, L^k\right]\left[h_t, \varepsilon_t^+\right] = h_{t-k}\varepsilon_{t-k}^+$, and $\left[L^k, L^k\right]\left[h_t, \varepsilon_t^-\right] = h_{t-k}\varepsilon_{t-k}^-$ with the convention $\mathcal{A}_{0,v}(\boldsymbol{z}) = 0$, $\mathcal{B}_{0,v}(\boldsymbol{z}) = 0$, if$q = 0$, $\mathcal{H}_{0,v}(\boldsymbol{z}) = 0$, $\mathcal{D}_{0,v}(\boldsymbol{z}) = 0$, if $d = 0$, and $\mathcal{C}_{0,v}(\boldsymbol{z}) = 1$, if $p = 0$, $\forall v \in \{1, ..., s\}$.
C2. If $p > 0$, $\mathcal{A}_{0,v}(\boldsymbol{z})$, $\mathcal{B}_{0,v}(\boldsymbol{z})$, $\mathcal{H}_{0,v}(\boldsymbol{z})$, and $\mathcal{D}_{0,v}(\boldsymbol{z})$, have no common roots with $\mathcal{C}_{0,v}(\boldsymbol{z})$. Moreover,

$$\mathcal{A}_{0,v}(1) + \mathcal{B}_{0,v}(1) + \mathcal{H}_{0,v}(1) + \mathcal{D}_{0,v}(1) \neq 0$$

and

$$\alpha_{0,q}(v) + \beta_{0,q}(v) + |b_{0,d}(v)| + |\omega_{0,d}(v)| + \gamma_{0,p}(v) \neq 0, \quad \forall\, v \in \mathbb{S}.$$

C3. $\gamma(H(\underline{\theta}_0)) < 0$ with $H(\underline{\theta}_0)$ instead of H to emphasize that the unknown parameter is $\underline{\theta}_0$ and $\forall\underline{\theta} \in \Theta$,

$$\sup_{\underline{\theta}\in\Theta}\gamma(|\Omega|) < 0,$$

where $\gamma(|\Omega|)$ is the Lyapunov exponent associated with the random matrices $(|\Omega|, t \in \mathbb{Z})$

$$|\Omega| = \begin{pmatrix} \zeta_{1:p-1}(e_t) & \gamma_p(t) + \left|e^+_{t-p} b_p(t)\right| + \left|e^+_{t-p}\omega_p(t)\right| \\ I_{(p-1)} & \underline{O}_{(p-1)} \end{pmatrix}$$

with

$$\zeta_{1:p}(e_t) = \left(\gamma_1(t) + \left|e^+_{t-1}b_1(t)\right| + \left|e^-_{t-1}\omega_1(t)\right|, ..., \gamma_p(t) + \left|e^+_{t-p}b_p(t)\right| + \left|e^+_{t-p}\omega_p(t)\right|\right)'.$$

C4. $(e_t)_{t\in\mathbb{Z}}$ is non degenerate and $P(e_t \neq 0) > 0$.

The compactness of Θ is a crucial assumption, facilitating the use of real analysis results. Assumption **C2** is vital for model identifiability, while **C3** ensures the existence of strictly periodically stationary (SPS) and periodically ergodic (PE) solutions in Equation (3.7), along with finite moments. Additionally, **C3** guarantees a causal solution for $h_t(\underline{\theta})$.
Assumption **C4** is introduced primarily for identifiability purposes and it further ensures that the process (ε_t) takes both positive and negative values with a positive probability. These foundational assumptions in place, we can now proceed to state our initial result. This result forms the basis for further exploration and analysis.

Theorem 3.2.1. *Consider a sequence $(\hat{\underline{\theta}}_{Ns})$, where each element is a $(QMLE)$. We assume that the assumption **C0-C4** are are verified. Then, $\hat{\underline{\theta}}_{Ns}$ exhibits strong consistency, meaning that as N tends to infinity, $\hat{\underline{\theta}}_{Ns}$ converges to $\hat{\underline{\theta}}_0$. This convergence forms a fundamental property underpinning the robustness of our estimators.*

To validate the assertions outlined in theorem 3.2.1, we undertake a systematic process, progressively establishing specific technical assertions. These crucial assertions are encapsulated within the subsequent lemma, serving as the foundational building blocks upon which the theorem relies. Through this meticulous approach, we clarify the underpinnings of our theorem, thereby ensuring its credibility and rigor.

Lemma 3.2.1. *Under the assumptions **C0-C4**, we can establish the following properties:*

1. $\lim_{N\to\infty} \sup_{\underline{\theta}\in\Theta} \left|\tilde{L}_{Ns}(\underline{\theta}) - L_{Ns}(\underline{\theta})\right| = 0$ *a.s.*
2. *There is* $t \in \mathbb{Z}$ *such that* $h^2_t(\underline{\theta}) = h^2_t(\underline{\theta}_0)$ *a.s.* $\Rightarrow \underline{\theta} = \underline{\theta}_0$.
3. $\sum_{v=1}^{s} E_{\underline{\theta}_0}\{l_{st+v}(\underline{\theta}_0)\} < \infty$. *If* $\theta \neq \underline{\theta}_0$. *Then,* $\sum_{v=1}^{s} E_{\underline{\theta}}\{l_{st+v}(\underline{\theta})\} > \sum_{\vartheta=1}^{s} E_{\underline{\theta}_0}\{l_{st+v}(\underline{\theta}_0)\}$.
4. *For any* $\underline{\theta} \neq \underline{\theta}_0$ *there a neighborhood* $\mathcal{V}(\underline{\theta})$ *such that a.s.*

$$\liminf_{N\to\infty} \inf_{\tilde{\underline{\theta}}\in\Theta} \left(\tilde{L}_{sN}(\tilde{\underline{\theta}})\right) > \sum_{\vartheta=1}^{s} E_{\underline{\theta}_0}\{l_v(\underline{\theta}_0)\}.$$

Proof. The proof follows essentially the same arguments by Aknouche and Bibi [1]. □

3.2.2 Asymptotic normality of $QMLE$

To establish the asymptotic normality of $\hat{\underline{\theta}}_{Ns}$, we introduce the following supplementary assumptions:

C5. $\underline{\theta}_0 \in \mathring{\Theta}$, where $\mathring{\Theta}$ denotes the interior of Θ.

C6. $\kappa_4 = E\left\{e^4_t\right\} < \infty$.

C7. For all $v \in \mathbb{S}$, $b_i(v)\, e^+_t(v) + \omega_i(v)\, e^-_t(v) + \gamma_i(v) \geq 0$, almost surely, for $i = 1, ..., p$.

Assumption **C5** plays a crucial role in the Quasi-Maximum Likelihood Estimator $(QMLE)$ by facilitating the validation of the first-order condition for maximizing the log-likelihood. Meanwhile, Assumption **C6** is vital for ensuring the existence of the asymptotic covariance matrix of the $QMLE$. Additionally, **C7** guarantees the positivity of h^2_t and provides the basis for bounding various derivatives of h^2_t.

Our second primary result in this section is as follows:

Theorem 3.2.2. *Assuming that* $(\varepsilon_t, t \in Z)$ *is generated by the equations 3.7, under the fulfillment of Assumptions* ***C0.-C7.****, we obtain the following:*

$$\sqrt{Ns}\left(\underline{\hat{\theta}}_{Ns} - \underline{\theta}_0\right) \rightsquigarrow \mathcal{N}\left(\underline{O}, (\kappa_4 - 1)\, J^{-1}\right) \text{ as } N \to \infty,$$

where the matrix J given by

$$J := \sum_{\upsilon=1}^{s} E_{\underline{\theta}_0}\left\{\frac{\partial^2 l_{st+\vartheta}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)\right\} = \sum_{\upsilon=1}^{s} E_{\underline{\theta}_0}\left\{\frac{1}{h^4_{st+\upsilon}(\underline{\theta}_0)}\frac{\partial h^2_{st+\upsilon}}{\partial\underline{\theta}}(\underline{\theta}_0)\frac{\partial h^2_{st+\upsilon}}{\partial\underline{\theta}'}(\underline{\theta}_0)\right\}.$$

Proof. The proof of the theorem 3.2.2 rests classically on a Taylor series expansion of $\frac{\partial L_{sN}}{\partial\underline{\theta}}(\underline{\theta})$ around $\underline{\theta}_0$ which is given by

$$\underline{O} = (Ns)^{-\frac{1}{2}}\sum_{t=1}^{Ns}\frac{\partial l_t}{\partial\underline{\theta}}(\underline{\hat{\theta}}_{Ns}) = (Ns)^{-\frac{1}{2}}\sum_{t=1}^{Ns}\frac{\partial l_t}{\partial\underline{\theta}}(\underline{\theta}_0) + \left((Ns)^{-1}\sum_{t=1}^{Ns}\frac{\partial^2 l_t}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\tilde{\theta}})\right)(Ns)^{\frac{1}{2}}\left(\underline{\hat{\theta}}_{Ns} - \underline{\theta}_0\right).$$

where the coordinates of $\underline{\tilde{\theta}}$ are between the corresponding entries of $\underline{\hat{\theta}}_{Ns}$ and those of $\underline{\theta}_0$. □

Now, we prove the intermediate results gathered in the next lemma.

Lemma 3.2.2. *Under assumptions* ***A0-A6****, we have:*

(a) $\sum_{\upsilon=1}^{s} E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\Theta}\left\|\frac{\partial l_{st+\upsilon}}{\partial\underline{\theta}}(\underline{\theta}_0)\frac{\partial l_{st+\upsilon}}{\partial\underline{\theta}'}(\underline{\theta}_0)\right\|\right\} < \infty$ *and* $\sum_{\upsilon=1}^{s} E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\Theta}\left\|\frac{\partial^2 l_{st+\upsilon}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)\right\|^2\right\} < \infty.$

(b) J *is invertible and* $\sum_{\upsilon=1}^{s} Var_{\underline{\theta}_0}\left\{\frac{\partial l_{st+\upsilon}}{\partial\underline{\theta}}(\underline{\theta}_0)\right\} = (\kappa_4 - 1)J.$

(c) There is a neighborhood $\mathcal{V}(\underline{\theta}_0)$ *of* $\underline{\theta}_0$ *such that* $\sum_{\upsilon=1}^{s} E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)}\left\|\frac{\partial^3 l_{st+\vartheta}}{\partial\theta_i\partial\theta_j\partial\theta_k}(\underline{\theta}_0)\right\|\right\} < \infty$*, for all* $i, j, k \in \{1, ..., s(1 + 2q + 2d + p)\}$.

(d) $p\lim\left\|(Ns)^{-\frac{1}{2}}\sum_{t=1}^{N}\sum_{\upsilon=1}^{s}\left(\frac{\partial\tilde{l}_{st+\vartheta}}{\partial\underline{\theta}}(\underline{\theta}_0) - \frac{\partial l_{st+\vartheta}}{\partial\underline{\theta}}(\underline{\theta}_0)\right)\right\| = 0.$

(e) $p\lim\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)}\left\|(Ns)^{-1}\sum_{t=1}^{N}\sum_{\upsilon=1}^{s}\left(\frac{\partial^2\tilde{l}_{st+\vartheta}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0) - \frac{\partial^2 l_{st+\vartheta}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)\right)\right\| = 0.$

(f) $(Ns)^{-\frac{1}{2}}\sum_{t=1}^{sN}\frac{\partial l_t}{\partial\underline{\theta}}(\underline{\theta}_0) \rightsquigarrow \mathcal{N}\left(\underline{O}, (\kappa_4 - 1)\, J\right)$ *as* $N \to \infty$ *and almost surely* $\lim_{N\to\infty}\left((Ns)^{-1}\sum_{t=1}^{Ns}\frac{\partial^2 l_t}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\tilde{\theta}})\right) \overset{a.s}{=} J.$

Proof. The proof follows essentially the same arguments by Francq and Zakoïan [22] and Aknouche and Bibi [1]. □

3.2.3 Example

Let us apply the foregoing results to the $PBLTARCH_s(1,1)$ generated by (3.2) with

$$h^2_{st+\upsilon}(\underline{\theta}) = \alpha_0(\upsilon) + \sum_{i=1}^{q}\left(\alpha_i(\upsilon)\varepsilon^{+2}_{st+\upsilon-1} + \beta_i(\upsilon)\varepsilon^{-2}_{st+\upsilon-1}\right) + \sum_{k=1}^{d}\left(b_k(\upsilon)\varepsilon^{+}_{st+\upsilon-1} + \omega_k(\upsilon)\varepsilon^{-}_{st+\upsilon-1}\right)h_{st+\upsilon-1}(\underline{\theta}).$$

It is not difficult to see that a sufficient condition for the SPS solution for $PBLTARCH_s(1,1)$ is

$$\prod_{\upsilon=0}^{s-1}\left(\left|\alpha_1(\upsilon)e_0^{+2} + \beta_1(\upsilon)e_0^{-2} + b_1(\upsilon)e_0^{+} + \omega_1(\upsilon)e_0^{-}\right|\right) < 1.$$

We suppose that

$$\underline{\theta}_0 = \left(\underline{\alpha}'_{(0)}, \underline{\beta}'_{(0)}, \underline{b}'_{(0)}, \underline{\omega}'_{(0)}\right)'.$$

Belong to some compact Θ, then $\hat{\underline{\theta}}_{Ns}$ is however strongly consistent. Moreover, if $\underline{\theta} \in \mathring{\Theta}$, then from Theorem 3.2.2, $\hat{\underline{\theta}}_{Ns}$ is asymptotically Gaussian with mean $\underline{O}$ and asymptotic variance–covariance matrix given by

$$J = diag\{J_{\upsilon}, 1 \leq \upsilon \leq s\} \ \text{with} J_{\upsilon} = E_{\underline{\theta}_0}\left\{\frac{1}{h^4_{st+\upsilon}}\underline{\Sigma}_{\upsilon}, \underline{\Sigma}'_{\upsilon}\right\},$$

where

$$\underline{\Sigma}_{\upsilon} = (\underline{\Sigma}_{1\upsilon}, \underline{\Sigma}_{2\upsilon}, \underline{\Sigma}_{3\upsilon}, \underline{\Sigma}_{4\upsilon}, \underline{\Sigma}_{5\upsilon}),$$

and

$$\underline{\Sigma}_{1\upsilon} = \frac{\partial h^2_{st+\upsilon}}{\partial \alpha_0(\upsilon)}(\underline{\theta}_0), \underline{\Sigma}_{2\upsilon} = \frac{\partial h^2_{st+\upsilon}}{\alpha_1(\upsilon)}(\underline{\theta}_0),$$

$$\underline{\Sigma}_{3\upsilon} = \frac{\partial h^2_{st+\upsilon}}{\beta_1(\upsilon)}(\underline{\theta}_0), \underline{\Sigma}_{4\upsilon} = \frac{\partial h^2_{st+\upsilon}}{b_1(\upsilon)}(\underline{\theta}_0), \underline{\Sigma}_{5\upsilon} = \frac{\partial h^2_{st+\upsilon}}{\omega_1(\upsilon)}(\underline{\theta}_0), \text{ for each } \upsilon \in \mathbb{S}.$$

In the works of Chan [14] and Jensen and Rahbak [32], the groundwork has been established for the consistency and asymptotic normality of the Quasi-Maximum Likelihood Estimator ($QMLE$) in non-stationary time-invariant $GARCH$ models. Their findings are particularly applicable when $\gamma(H(\underline{\theta}_0) \geq 0$. Consequently, there is significant potential to extend the asymptotic properties of $QMLE$ to encompass situations where all regimes demonstrate explosive behavior in the context of $PBLTGARCHs$.

3.3 Monte Carlo experiment

In this section, we evaluate the finite sample properties of the Quasi-Maximum Likelihood Estimator ($QMLE$) for unknown parameters in the $BLTGARCH_s(1,1,1)$ model using Monte Carlo experiments. To achieve this, we conducted 500 replications with varying sample sizes, specifically $n \in \{1000, 3000\}$, and employed two different distributions for innovations: the standard normal distribution $\mathcal{N}(0,1)$ and the Student's t distribution with 5 and 15 degrees of freedom. The parameter vector $\underline{\theta}$ was carefully chosen to satisfy the strict periodically stationary condition. All empirical results were obtained through the implementation of custom scripts in the MATLAB computing language.
In the tables below, each column represents the average of parameter estimates across the N simulations. To assess the performance of the $QMLE$, we report the Root Mean Square Error ($RMSE$) for each $\hat{\theta}_n(i)$, with the results presented in brackets. Additionally, the asymptotic distributions of $\hat{\theta}_n(\vartheta)$, where ϑ ranges from 1 to s, are displayed for each N simulation, accompanied by a summary in the form of boxplots following the respective table.

3.3.1 Standard $BLTGARCH$ model

The first example illustrating our theoretical analysis is the standard $BLTGARCH(1,1,1)$ model, its vector of parameters $\underline{\theta} = (\alpha_0, \alpha_1, \beta_1, b_1, \omega_1, \gamma_1)'$ is chosen to subject the condition

$$\gamma_L = E\left\{\log\left|\alpha_1 e_0^{+2} + \beta_1 e_0^{-2} + b_1 e_0^{+} + \omega_1 e_0^{-} + \gamma_1\right|\right\} < 0.$$

The results of simulation according to two models for $\underline{\theta}$ are given in table 3.1

	$\mathcal{N}(0,1)$		$t_{(5)}$		$t_{(15)}$	
Parameters	1000	3000	1000	3000	1000	3000
$\hat{\alpha}_0$	1.0061 (0.0061)	0.9958 (0.0042)	1.0052 (0.0052)	0.9946 (0.0054)	1.0084 (0.0084)	0.9979 (0.0021)
$\hat{\alpha}_1$	0.4915 (0.0085)	0.4986 (0.0014)	0.5064 (0.0064)	0.5011 (0.0011)	0.4970 (0.0030)	0.5000 (0.0009)
$\hat{\beta}_1$	0.3382 (0.0118)	0.3498 (0.0002)	0.3366 (0.0134)	0.3574 (0.0074)	0.3334 (0.0166)	0.3485 (0.0015)
$\hat{b}_1$	0.2560 (0.0060)	0.2513 (0.0013)	0.2454 (0.0046)	0.2501 (0.0001)	0.2456 (0.0044)	0.2516 (0.0016)
$\hat{\omega}_1$	0.3570 (0.0070)	0.3482 (0.0018)	0.3620 (0.0120)	0.3419 (0.0081)	0.3616 (0.0116)	0.3509 (0.0009)
$\hat{\gamma}_1$	0.1493 (0.0007)	0.1513 (0.0013)	0.1489 (0.0011)	0.1526 (0.0026)	0.1497 (0.0003)	0.1504 (0.0004)
Model(1) : $\underline{\theta} = (1.00, 0.50, 0.35, 0.25, 0.35, 0.15)'$						
$\hat{\alpha}_0$	0.9978 (0.0103)	0.9976 (0.0051)	0.9976 (0.0243)	1.0019 (0.0137)	0.4927 (0.0283)	0.4927 (0.0283)
$\hat{\alpha}_1$	0.4946 (0.0225)	0.4944 (0.0119)	0.5125 (0.0557)	0.4901 (0.0310)	0.4927 (0.0283)	0.4927 (0.0283)
$\hat{\beta}_1$	0.3427 (0.0151)	0.3473 (0.0077)	0.3516 (0.0401)	0.3509 (0.0238)	0.4927 (0.0283)	0.4927 (0.0283)
$\hat{b}_1$	0.0022 (0.0434)	0.0066 (0.0220)	−0.0156 (0.0910)	0.0110 (0.0611)	0.4927 (0.0283)	0.4927 (0.0283)
$\hat{\omega}_1$	0.0045 (0.0341)	0.0017 (0.0180)	0.0019 (0.0807)	−0.0024 (0.0501)	0.4927 (0.0283)	0.4927 (0.0283)
$\hat{\gamma}_1$	0.1531 (0.0037)	0.1504 (0.0021)	0.1504 (0.0081)	0.1503 (0.0054)	0.4927 (0.0283)	0.4927 (0.0283)
Model(2) : $\underline{\theta} = (1.00, 0.50, 0.35, 0.0, 0.0, 0.15)'$						

Table 3.1: Average and $RMSE$ of 500 simulations of $QMLE$ for standard $BLTGARCH(1,1,1)$.

The overlaying of asymptotic distribution of the kernels associated to the sequences $\left(\sqrt{n}\left(\underline{\hat{\theta}}_n(i) - \underline{\theta}(i)\right)\right)_{n\geq 1}$, $i = 1, ..., 6$ for different innovations according to model(1) of Table 3.1 are shown in Figure 3.1

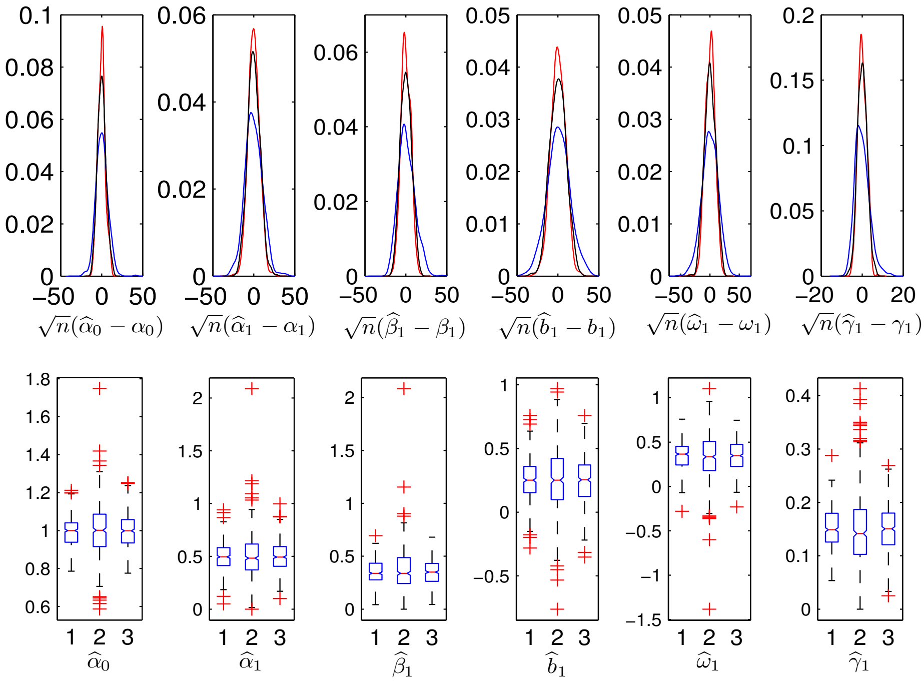

Figure 3.1: Top panels: the asymptotic distribution of $\sqrt{n}(\underline{\hat{\theta}}_n(i) - \underline{\theta}(i))$ associated to $Normal$ (red curve), $t_{(5)}$ (bleu curve) and $t_{(15)}$ (blackline curve). Bottom panels: Boxplot summary of $\underline{\hat{\theta}}_n(i)$, $i = 1, ..., 6$ (1 for $Normal$, 2 for $t_{(5)}$ and (3) for $t_{(15)}$) according to Model(1) of Table 3.1.

Several observations can be drawn from the results presented in table 3.1 and the accompanying figures in figure 3.1. Firstly, it is evident that the performance of the Quasi Maximum Likelihood Estimation (QMLE) associated with innovations following a $t_{(5)}$ distribution is notably poorer compared to those following a $\mathcal{N}(0, 1)$ distribution. This highlights the sensitivity of the estimation method to the choice of the underlying distribution. Secondly, there is a consistent trend of decreasing Root Mean Square Error (RMSE) as the sample size increases. This implies that larger sample sizes tend to yield more accurate parameter estimates. Additionally, the top panel of Figure 3.1 indicates that the asymptotic variance associated with $t_{(5)}$ innovations is slightly higher than that of $\mathcal{N}(0, 1)$ innovations. Furthermore, the bottom panels of figure 3.1, represented by boxplots, reveal substantial differences in the statistical characteristics of the QMLE results, particularly that the $\mathcal{N}(0, 1)$ distribution exhibits fewer outliers compared to $t_{(5)}$. These insights emphasize the importance of both the choice of distribution for the innovations and the sample size in the context of QMLE performance.

3.3.2 Periodic $BLTGARCH$ model

The second example of our Monte Carlo experiment here is devoted to estimate the periodic $BLTGARCH_s(1, 1, 1)$ model with $s = 2$ according to standard $\mathcal{N}(0, 1)$ and *student* $t_{(5)}$ as innovations distributions. The vector of parameters to be estimated is thus $\underline{\theta} = (\underline{\alpha}'_0, \underline{\alpha}'_1, \underline{\beta}'_1, \underline{b}'_1, \underline{\omega}'_1, \underline{\gamma}'_1)'$ where $\underline{\alpha}'_0 = (\alpha_0(1), \alpha_0(2))$, $\underline{\alpha}'_1 = (\alpha_1(1), \alpha_1(2))'$, etc... are subjected to two models Model(1) and Model(2). The results of simulation according to two models(1) and (2) are given in Table 3.2 below.

n	ϑ	$\mathcal{N}(0,1)$ 1000	$\mathcal{N}(0,1)$ 3000	$t_{(5)}$ 1000	$t_{(5)}$ 3000	$t_{(15)}$ 1000	$t_{(15)}$ 3000
$\underline{\hat{\alpha}}_0$	1	0.9888 (0.0264)	0.9953 (0.0133)	0.9561 (0.0739)	0.9921 (0.0280)	0.9626 (0.0734)	0.9912 (0.0208)
	2	0.9944 (0.0286)	0.9928 (0.0134)	0.9515 (0.0728)	0.9721 (0.0343)	0.9870 (0.0861)	0.9884 (0.0200)
$\underline{\hat{\alpha}}_1$	1	0.4999 (0.0335)	0.4947 (0.0162)	0.5048 (0.0918)	0.4971 (0.0427)	0.5053 (0.0796)	0.4968 (0.0268)
	2	0.5008 (0.0414)	0.4973 (0.0203)	0.4905 (0.0944)	0.5038 (0.0598)	0.5289 (0.0995)	0.5068 (0.0338)
$\underline{\hat{\beta}}_1$	1	0.3631 (0.0468)	0.3562 (0.0242)	0.3661 (0.1020)	0.3596 (0.0608)	0.3963 (0.1184)	0.3648 (0.0421)
	2	0.3359 (0.0324)	0.3411 (0.0154)	0.3464 (0.0725)	0.3468 (0.0352)	0.3639 (0.0746)	0.3436 (0.0228)
$\underline{\hat{b}}_1$	1	−0.2607 (0.0688)	−0.2473 (0.0333)	−0.2760 (0.1567)	−0.2482 (0.0940)	−0.2793 (0.1546)	−0.2479 (0.0568)
	2	−0.0027 (0.0805)	0.0058 (0.0392)	0.0084 (0.1868)	−0.0054 (0.1054)	−0.0623 (0.1706)	−0.0039 (0.0626)
$\underline{\hat{\omega}}_1$	1	0.3240 (0.0977)	0.3412 (0.0500)	0.3198 (0.2002)	0.3459 (0.1247)	0.2759 (0.2127)	0.3323 (0.0851)
	2	0.0126 (0.0720)	0.0087 (0.0348)	−0.0030 (0.1636)	−0.0093 (0.0814)	−0.0457 (0.1681)	0.0021 (0.0497)
$\underline{\hat{\gamma}}_1$	1	0.1598 (0.0093)	0.1527 (0.0043)	0.1793 (0.0280)	0.1549 (0.0110)	0.1773 (0.0273)	0.1559 (0.0074)
	2	0.1578 (0.0090)	0.1530 (0.0044)	0.1807 (0.0284)	0.1680 (0.0124)	0.1734 (0.0231)	0.1582 (0.0072)
Model(1) : $\underline{\theta} = (1.00, 1.00, 0.50, 0.50, 0.35, 0.35, -0.25, 0.00, 0.35, 0.00, 0.15, 0.15)'$							
$\underline{\hat{\alpha}}_0$	1	0.9844 (0.0554)	0.9935 (0.0242)	0.9821 (0.1133)	0.9933 (0.0586)	0.9643 (0.1261)	0.9899 (0.0382)
	2	1.0391 (0.1631)	1.0152 (0.0764)	0.9648 (0.3207)	0.9962 (0.1817)	1.0457 (0.3405)	1.0161 (0.1196)
$\underline{\hat{\alpha}}_1$	1	0.5023 (0.0231)	0.4959 (0.0119)	0.5112 (0.0662)	0.5023 (0.0350)	0.5078 (0.0590)	0.5078 (0.0191)
	2	0.4819 (0.0640)	0.4870 (0.0327)	0.5140 (0.1545)	0.5203 (0.0872)	0.5101 (0.1497)	0.4889 (0.0517)
$\underline{\hat{\beta}}_1$	1	0.2571 (0.0108)	0.2541 (0.0055)	0.2757 (0.0321)	0.2512 (0.0148)	0.2527 (0.0268)	0.2568 (0.0092)
	2	0.4179 (0.0600)	0.4324 (0.0305)	0.4290 (0.1037)	0.4434 (0.0883)	0.4489 (0.1363)	0.4375 (0.0449)
$\underline{\hat{b}}_1$	1	0.2459 (0.0324)	0.2550 (0.0166)	0.2316 (0.0893)	0.2544 (0.0496)	0.2414 (0.0795)	0.2441 (0.0259)
	2	0.1794 (0.1533)	0.1748 (0.0765)	0.1381 (0.3424)	0.1219 (0.1929)	0.1336 (0.3187)	0.1610 (0.1151)
$\underline{\hat{\omega}}_1$	1	0.1404 (0.0204)	0.1470 (0.0103)	0.1175 (0.0569)	0.1495 (0.0261)	0.1425 (0.0520)	0.1411 (0.0174)
	2	0.1905 (0.1395)	0.1739 (0.0759)	0.1477 (0.2891)	0.1490 (0.1913)	0.1335 (0.3097)	0.1695 (0.1089)
$\underline{\hat{\gamma}}_1$	1	0.1532 (0.0018)	0.1502 (0.0008)	0.1587 (0.0061)	0.1518 (0.0032)	0.1570 (0.0050)	0.1521 (0.0015)
	2	0.7415 (0.0185)	0.7419 (0.0098)	0.7734 (0.0532)	0.7584 (0.0315)	0.7504 (0.0488)	0.7443 (0.0158)
Model(2) : $\underline{\theta} = (1.00, 1.00, 0.50, 0.50, 0.25, 0.45, 0.25, 0.15, 0.15, 0.15, 0.15, 0.75)'$							

Table 3.2: Average and $RMSE$ of 500 simulations of $QMLE$ for $PBLTGARCH_2(1,1,1)$.

The overlaying of asymptotic distribution of the kernels associated to the sequences $\left(\sqrt{n}\left(\underline{\hat{\theta}}_n(i) - \underline{\theta}(i)\right)\right)_{n\geq 1}$, $i = 1, ..., 12$ according to model(1) of Table 3.2 are shown in Figure3.2

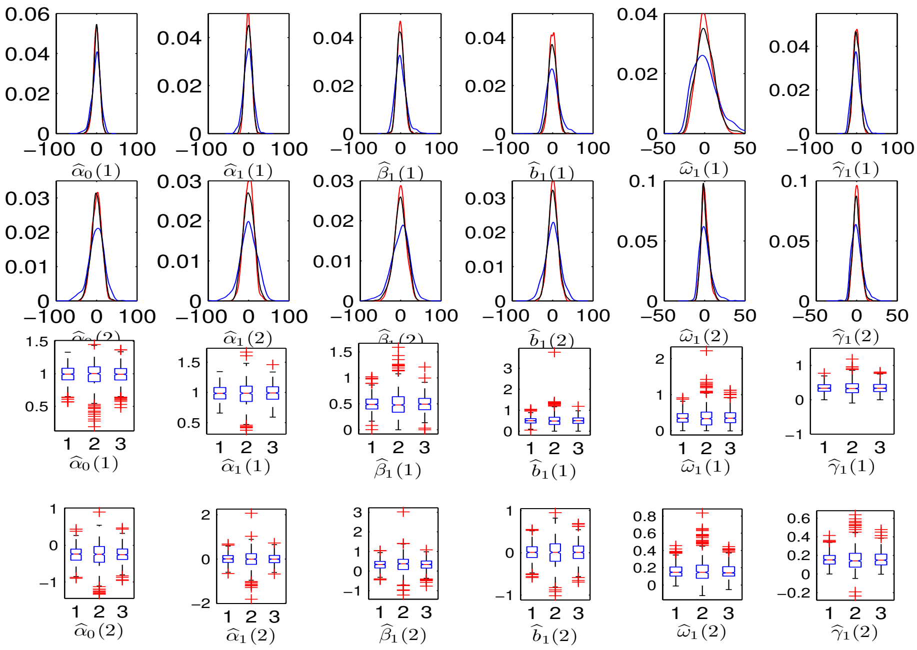

Figure 3.2: Top panels: the asymptotic distribution of $\sqrt{n}(\hat{\underline{\theta}}_n(i) - \underline{\theta}(i))$ associated to $Normal$ (red curve), $t_{(5)}$ (bleu curve) and $t_{(15)}$ (blackline curve). Bottom panels: Boxplot summary of $\hat{\underline{\theta}}_n(i)$, $i = 1, ..., 12$ (1 for $Normal$, 2 for $t_{(5)}$ and (3) for $t_{(15)}$) according to Model(1) of table 3.2.

3.3.3 Comments

Upon analyzing the outcomes of the Monte Carlo experiment outlined in Table 3.2, several observations can be made: The table furnishes parameter estimates for the $PBLTGARCH_s(1,1,1)$ model with $s = 2$, fitted to two distinct scenarios: Model(1) and Model(2), utilizing innovations generated from the standard $\mathcal{N}(0,1)$ distribution and the Student's t distribution with 5 and 15 degrees of freedom respectively. These estimates are derived from 500 independent simulations. Primarily, it is evident that the performance of the Quasi-Maximum Likelihood Estimates ($QMLE$) associated with the $t_{(5)}$ innovations and $t_{(15)}$ innovations is notably inferior when compared to those obtained from the standard $\mathcal{N}(0,1)$ distribution. Overall, the model parameters are accurately estimated, showing minimal deviations in estimated values when using either the $\mathcal{N}(0,1)$ or $t_{(5)}$ and $t_{(15)}$ innovations. Additionally, many of the estimated values exhibit moderate standard deviations. The results presented in Table 3.2 pertain to a $PBLTGARCH_s(1,1,1)$ model with parameters specifically chosen for Model(1) and Model(2). In Model(1), the parameters for the first regime are such that

$$E\left\{\log\left|\alpha_1(\vartheta)\, e_0^{+2} + \beta_1(\vartheta)\, e_0^{-2} + b_1(\vartheta)\, e_0^{+} + \omega_1(\vartheta)\, e_0^{-} + \gamma_1(\vartheta)\right|\right\} < 0, \text{ for } \vartheta = 1, ..., 2.$$

In Model (2), the parameters for the second regime are chosen such that

$$E\left\{\log\left|\alpha_1(2)\, e_0^{+2} + \beta_1(2)\, e_0^{-2} + b_1(2)\, e_0^{+} + \omega_1(2)\, e_0^{-} + \gamma_1(2)\right|\right\} > 0.$$

However, it is worth noting that the SPS property of the model is maintained.

3.4 Empirical Application

To assess the efficacy of our proposed model on real financial time series, we employ it to characterize the foreign exchange rates of the Algerian Dinar against the Euro, denoted as y_t. This financial dataset, previously investigated by Hamdi and Souam [29] using a mixture of periodic $GARCH$ models, We consider returns series

$$\left(r_t = 100 \times (\log(y_t) - \log(y_{t-1}))\right)_{t\geq 1}$$

of daily exchange rates of Algerian dinar against the Euro. The observation cover the period from January 3, 2000 to September 29, 2011. Since there are some weeks comprise less than five observations (due to legal holidays), we remove the entire weeks with less than five data available rather than estimating the "pseudo-missing" observations by an ad-hoc method. Thus, the final length of transformed data is 3055 observations uniformly distributed on 611 weeks. Figure 3.3 displays the plots of the series (y_t) and its returns (r_t).

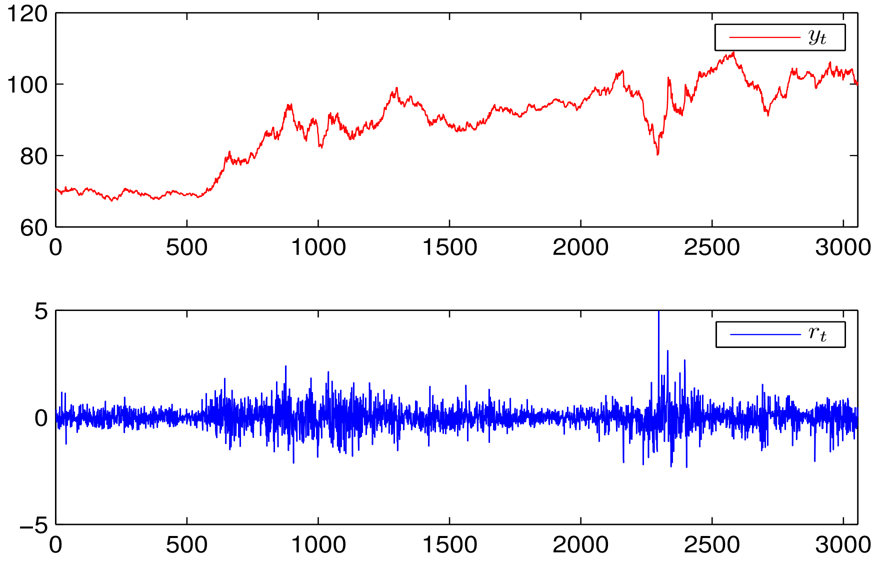

Figure 3.3: The plots of the series(y_t) , squaared r_t and absolute (r_t).

Some elementary statistics of the series $(y_t)_{t\geq 1}$ and its returns $(r_t)_{t\geq 1}$, squared return $\left(r_t^2\right)_{t\geq 1}$ absolute return $(|r_t|)_{t\geq 1}$.

Series	Means	Std.Dev	Median	Skewness	Kurtosis
y_t	88.6118	11.5755	91.0995	−0.5181	2.1330
r_t	0.0118	0.5043	0.0123	0.3536	8.9678
r_t^2	0.2543	0.7193	0.0652	16.1027	464.3694
$\|r_t\|$	0.3575	0.3557	0.2554	2.6956	18.4307

Table 3.3: Elementary statistics of the series $(y_t)_{t\geq 1}$, $(r_t)_{t\geq 1}$,$\left(r_t^2\right)_{t\geq 1}$ and $(|r_t|)_{t\geq 1}$.

The Table 3.3 presents statistical summary of the series $(y_t)_{t\geq 1}$, $(r_t)_{t\geq 1}$, $\left(r_t^2\right)_{t\geq 1}$ and $(|r_t|)_{t\geq 1}$ with summary measures of normality test results. The return $(r_t)_{t\geq 1}$ exhibits non-zero skewness and leptokurtic while $\left(r_t^2\right)_{t\geq 1}$ and $(|r_t|)_{t\geq 1}$ exhibits significant skewness and kurtosis, indicating that their distribution is more peaked with a thicker tails than normal distribution. Moreover, the results shown in Table 3.3 examine the effect of heteroscedasticity in the series $(r_t)_{t\geq 1}$

lags	10	15	20	25
$ARCH$ statistics	152.3993	200.3745	244.6458	266.6962
Critical value	18.3070	24.9958	31.4104	37.6525
$P-value$	0.0000	0.0000	0.0000	0.0000

Table 3.4: $ARCH$ effect test of daily returns (r_t).

The results of Table 3.4 can be summarized as: since the $p-$value is less than 0.05, the $ARCH$ statistics is greater than the critical value at 95% confidence level. These imply that there is a strong evidence for rejecting the null hypothesis of no $ARCH$ effect. The rejection indicates the existence of $ARCH$ effects in the returns series and therefore the variance of such a returns is not constant. The test was implemented in $MATLAB$ with "*archtest*" function for the returns. Figure(6) displays the sample autocorrelations functions (ACF) of the series $(r_t)_{t\geq 1}$, $\left(r_t^2\right)_{t\geq 1}$ and $(|r_t|)_{t\geq 1}$ computed at 40 lags.

Figure 3.4, depicts the sample autocorrelations functions of the series $(r_t)_{t\geq 1}$, $\left(r_t^2\right)_{t\geq 1}$ and $(|r_t|)_{t\geq 1}$, one can observe that $(r_t)_{t\geq 1}$ show no evidence of serial correlation, but the $\left(r_t^2\right)_{t\geq 1}$ and $(|r_t|)_{t\geq 1}$ are positively autocorrelated.

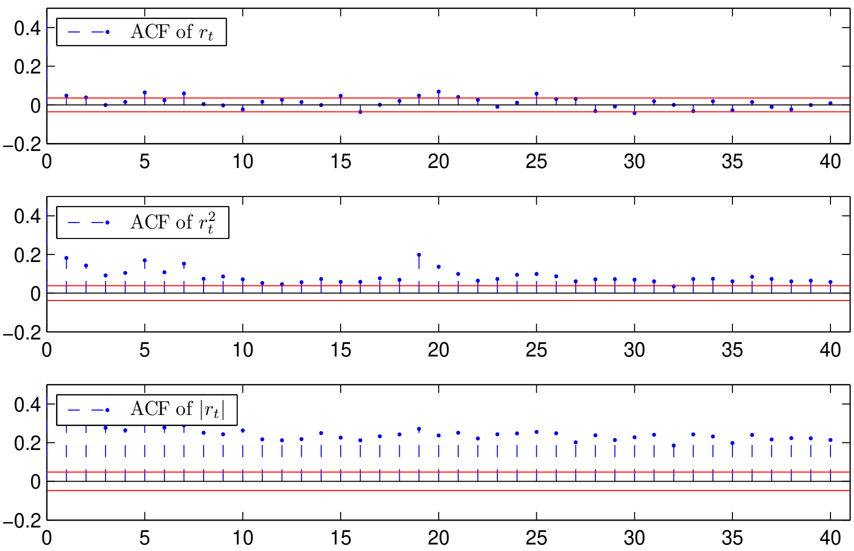

Figure 3.4: The ACF of the returns and of their squared and absolute series

3.4.1 Modeling with standard $BLTGARCH$ model

The first attempt will be modeling the series $(r_t)_{t\geq1}$ by a standard $BLTGARCH(1,1,1)$ model. The parameters estimates of volatility $(\hat{h}_t^{(s)})_{t\geq1}$ to $BLTGARCH(1,1,1)$ with their $RMSE$ are given in Table 3.5 below.

Parameters	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{b}_1$	$\hat{\omega}_1$	$\hat{\gamma}_1$
$\left(\hat{h}_t^{(s)}\right)_{t\geq1}$	0.0007	0.0304	0.0591	0.0276	0.0283	0.9540
	(0.0005)	(0.0176)	(0.0224)	(0.0439)	(0.0430)	(0.0175)

Table 3.5: Parameters estimates and their $RMSE$ of the volatilities $(\hat{h}_t^{(s)})_{t\geq1}$.

The plot of the estimated volatility $(\hat{h}_t^{(s)})_{t\geq1}$ and the squared return are showed in Figure 3.5 below.

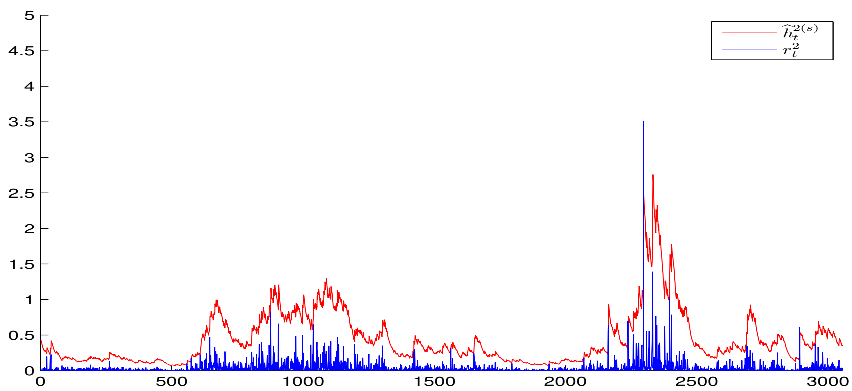

Figure 3.5: Dark blue: squared returns, light red: volatilities estimates according to Standard $BLTGARCH(1,1,1)$ (left) and to Periodic $BLTGARCH_5(1,1,1)$ (right)

3.4.2 Modeling with $PBLTGARCH$ model

The second attempt is to look for a model able to cover the day-of -week seasonality in return (r_t) (see for instance Franses and Paap [20]). So, in order to analyze the seasonality, we fitted the following simple $PBLTGARCH_5(1,1,1)$ model for each series or equivalently. Hence, we estimate its volatility process $(h_t^2)_{t\geq 1}$ through five periodic effects, $r_t = h_t e_t$ and

$$h_t^2 = \alpha_0\,(t) + \left(\alpha_1\,(t)\, r_{t-1}^{+2} + \beta_1\,(t)\, r_{t-1}^{-2}\right) + \left(b_1\,(t)\, r_{t-1}^{+} + \omega_1\,(t)\, r_{t-1}^{-}\right) h_{t-k} + \gamma_1\,(t)\, h_{t-1}^2 \tag{14}$$

The parameters estimates of five-regimes (intra-day) of $(\hat{h}_t^{(p)})_{t\geq 1}$ and their $RMSE$ according to model (14) are given in Table 3.6 .

days	$\underline{\hat{\alpha}}_0$	$\underline{\hat{\alpha}}_1$	$\underline{\hat{\beta}}_1$	$\underline{\hat{b}}_1$	$\underline{\hat{\omega}}_1$	$\underline{\hat{\gamma}}_1$
Sunday	0.0001	0.0145	0.0032	0.0165	0.0520	1.1826
	(0.0320)	(0.0329)	(0.0812)	(0.0926)	(0.1234)	(0.1894)
Monday	0.0010	0.0082	0.0419	0.0685	0.0831	1.0009
	(0.0296)	(0.0563)	(0.0588)	(0.2913)	(0.1429)	(0.1326)
Tuesday	0.0001	0.0015	0.0376	0.1162	0.0318	0.8504
	(0.0289)	(0.0651)	(0.0171)	(0.0611)	(0.0662)	(0.1156)
Wednesday	0.0025	0.0869	0.0648	0.0659	0.1768	0.7941
	(0.0142)	(0.0322)	(0.0345)	(0.1136)	(0.0951)	(0.0955)
Thursday	0.0002	0.0082	0.0645	0.0909	0.0229	0.9803
	(0.0160)	(0.0799)	(0.1260)	(0.2751)	(0.3544)	(0.2810)

Table 3.6: Parameters estimates and their $RMSE$ of the volatilities $(\hat{h}_t^{(p)})$.

The plots of estimated volatilitie $(\hat{h}_t^{(p)})_{t\geq 1}$ and the squared return are showed in Figure 3.6 below.

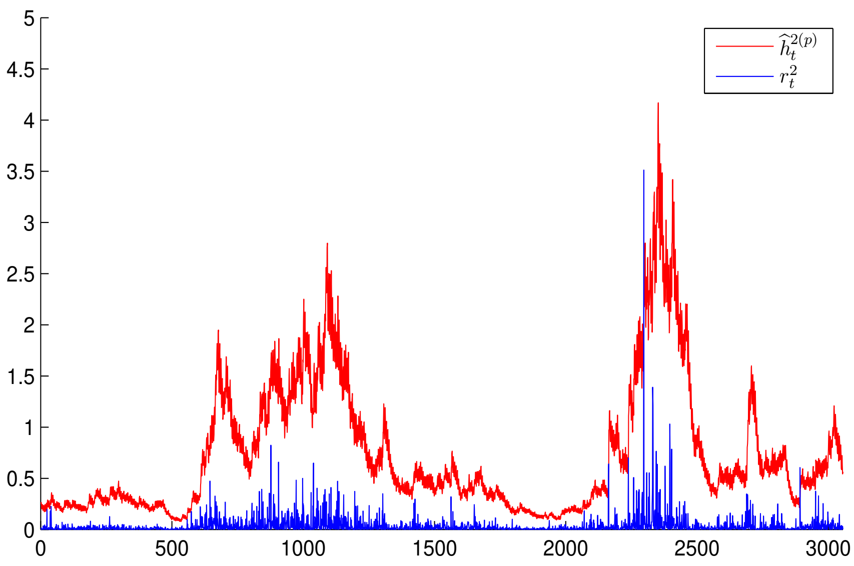

Figure 3.6: Dark blue: squared returns, light red: volatilities estimates according to Standard $BLTGARCH(1,1,1)$ (left) and to Periodic $BLTGARCH_5(1,1,1)$ (right)

comments

Now, a few comments can be made: In Table 3.5 and Table 3.6 , it is clearly seen that the components of $\underline{\hat{\alpha}}_0$ are close to 0 will that the components of $\underline{\hat{\gamma}}_1$ are close to 1 with moderate $RMSE$. Additionally, Figure 3.5 and Figure 3.6 represent the plots of the volatilities estimates (plots in red) according to $(\hat{h}_t^{(s)})_{t\geq 1}$ and $(\hat{h}_t^{(p)})_{t\geq 1}$ and compared with the appropriate $\left(r_t^2\right)_{t\geq 1}$ (plots in blue). Moreover, it seems that it is very difficult to distinguish between the volatilities $(\hat{h}_t^{(s)})_{t\geq 1}$ and $(\hat{h}_t^{(p)})_{t\geq 1}$ except perhaps that the volatilities $(\hat{h}_t^{(p)})_{t\geq 1}$ is more fluctuated than $(\hat{h}_t^{(s)})_{t\geq 1}$.

3.5 Conclusion

In this thesis we were interested in Nonlinear Modeling of Certain Periodic Time Series:

- **The first chapter**: Beside the probabilistic structure and the conditions ensuring the existence of higher-order moments, this paper studies also the asymptotic properties of the quasi-maximum likelihood estimators of $P-logGARCH_s(1,1)$ model. So, in the first part we have given the necessary and sufficient conditions for the existence of strictly periodically stationary solution followed by its moment properties of such a model. This chapter presents for the second part, the strong consistency and the asymptotic normality of the QML estimator under mild conditions. The theoretical results are illustrated in third part by a Monte Carlo experiment through some usual innovations.

- **The second chapter**: In this chapter we established some probabilistic and statistical properties of the $PAVGARCH(p,q)$ model. So, we have given the necessary and sufficient conditions for the existence of a strictly periodically stationary solution based on the negativity of the top-Lyapunov exponent. Moreover, this chapter presents the strong consistency and the asymptotic normality of the QME under mild conditions. Finally, the theoretical results are illustrated by a Monte Carlo experiment through some usual innovations and an application to the exchange rate of Algerian Dinar against the European currency (Euro) and the the American currency (Dollar) showing its performance and its efficiency.

- **The third chapter**: In this chapterr, we focus on the theoretical and asymptotic properties of the $PBLTGARCH(q,d,p)$ model. Indeed, for the first part, we have given the necessary and sufficient conditions for the existence of a strictly periodically stationary solution based on the negativity of the top-Lyapunov exponent. This chapter presents for the second part, the strong consistency and the asymptotic normality of the QML estimator under mild conditions. The theoretical results are illustrated in third part by a Monte Carlo experiment through some usual innovations and an application to the exchange rate of Algerian Dinar against the European currency (Euro) showing its performance and its efficiency.

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