Ministère de l'Enseignement Supérieur et de la Recherche Scientifique
Université Mohamed Khider - Biskra
Faculté des Sciences et de la Technologie
Département de Génie Électrique

# *Thèse de Doctorat*

*En vue de l'obtention du diplôme de docteur LMD en génie électrique*

# *Contribution à la commande des systèmes non linéaires décrits par les multi-modèles flous*

Présentée et soutenue publiquement par: **Rabiaa HOUILI**

Soutenue publiquement le:24 /11/2024

**Devant le jury composé de:**

| | | | |
|---|---|---|---|
| Ramzi SAADI | professeur | Université de Biskra | Président |
| Abdenacer TITAOUINE | professeur | Université de Biskra | Rapporteur |
| Mohamed Yacine HAMMOUDI | professeur | Université de Biskra | Co-Rapporteur |
| Raihane MECHGOUG | professeur | Université de Biskra | Examinateur |
| Djilani BENATTOUS | professeur | Université d'El-Oued | Examinateur |
| Amel HADRI-HAMIDA | professeur | Université de Biskra | Invitée |

*Année Universitaire 2024/2025*

Ministry of Higher Education and Scientific Research

Mohamed Khider University - Biskra

Faculty of Sciences and Technology

Department of Electrical Engineering

# *Doctoral Thesis*

*In view of obtaining the LMD doctoral degree in electrical engineering*

# *Contribution to the control of nonlinear systems described by fuzzy multi-model*

Presented and publicly defended by: **Rabiaa HOUILI**

Publicly defended on: 24/11/2024

**Before a jury composed of:**

| | | | |
|---|---|---|---|
| Ramzi SAADI | Professor | University of Biskra | President |
| Abdenacer TITAOUINE | Professor | University of Biskra | Thesis Supervisor |
| Mohamed Yacine HAMMOUDI | Professor | University of Biskra | Co-Supervisor |
| Raihane MECHGOUG | Professor | University of Biskra | Examiner |
| Djilani BENATTOUS | Professor | University of El-Oued | Examiner |
| Amel HADRI-HAMIDA | Professor | University of Biskra | Invited |

*Academic Year 2024/2025*

# Contribution à la commande des systèmes non linéaires décrits par les multi-modèles flous

Rabiaa HOUILI

December 6, 2024

# Dedication

*I dedicate this work to my beloved parents, who have made the greatest sacrifices for our education. May Allah protect them,*

*To my cherished grandparents,*

*To my husband, Choaibe, and to my twin girls, Rokaya and Khadija,*

*To my brothers, my sisters, and their children,*

*And to all my friends, both in my social and academic life.*

# Acknowledgments

Foremost, my heartfelt gratitude is extended to Allah, the Most Gracious and Merciful, for endowing me with the knowledge, strength, courage, patience, and serenity that have been my pillars throughout my academic journey. I deeply appreciate His infinite mercy and grace, which have been instrumental in the completion of this work. His benevolence has been a constant guiding light, for which my thankfulness knows no bounds.

I owe profound thanks to my thesis director, Professor Abdennacer Titaouine, and my Co-director, Mohamed Yacine Hammoudi of the LMSE Laboratory at the University of Biskra. Their unwavering support, availability, and patience have been crucial to the success of my thesis.

My sincere appreciation goes to Mr. Boumehraz Mohamed, head of the Laboratory of Energy Systems Modeling (LMSE), and Mrs. Betka, the laboratory engineer, for fostering an ideal research and experimental work environment. This work wouldn't have reached fruition without their assistance and the collective support of the LMSE members.

I extend my gratitude to the esteemed members of my jury: Professor Ramzi Saadi from the University of Biskra, who honored me by chairing my thesis; Professor Riahane Mechgoug, also from the University of Biskra; and Professor Djilani Benattous, from the University of El Oued. Their willingness to participate in my jury is deeply appreciated. Additionally, Professor Amel Hadri-Hamida from the University of Biskra has been invited

I cannot conclude without expressing my heartfelt thanks to my family my parents, husband, and brothers and my sisters and friends, whose unwavering love and support have lightened the burdens of this journey.

Finally, to everyone who has offered help, encouragement, or information, and to all who have believed in me, giving me the energy to persevere and succeed, I offer my sincerest thanks

# *List of Publications*

## Journal Publications

- **Houili, R.**, Hammoudi, M. Y., Benbouzid, M, & Titaouine, A. (2023). Observer-Based Controller Using Line Integral Lyapunov Fuzzy Function for TS Fuzzy Systems: Application to Induction Motors,11(3), 374

- Hamdi, W., Saadi, R., Hammoudi, M. Y., **Houili ,R.**, Benbouzid, M., Himeur, Y., ... & Mansoor, W. (2024). Enhanced State and Unknown Input Estimation for Synchronous Reluctance Motor Using Takagi-Sugeno Fuzzy Proportional Multi-Integral Observer. IEEE Access, 12, 42908-42920.

## International Conferences

- **Houili, R.**, Hammoudi, M.Y., &Titaouine, A. Harmonic elimination and reduction in pwm inverter: with three-phase induction motors using iterative numerical method. The 2nd World Energy Storage Conference, December 18-21, 2022 Istanbul, Turkiye (WESC-2022)

- **Houili, R.**, Hammoudi, M.Y.,Betka,A., &Titaouine, A. Stochastic optimization algorithms for parameter identification of three phase induction motors with experimental verification. International Conference on Advances in Electronics, Control and Communication Systems, Blida, Algeria, March 6-7, 2023 (ICAECCS2023.),IEEE

- **Houili, R.**, Hammoudi, M.Y., &Titaouine, A. Control Design of Takagi-Sugeno fuzzy systems via Line Integral Lyapunov Function. The 1st International Conference on Renewable Solutions for Ecosystems: Towards a Sustainable Energy Transition ,May 06-07, 2023, Djelfa, Algeria (ICRSEtoSET 2023),IEEE

- **Houili, R.**, Hammoudi, M.Y., &Titaouine, A. Line integral fuzzy lyapunov function based control design of Takagi-Sugeno fuzzy system using mean value theorem. the First International Conference on Advances in Electrical and Computer Engineering, on May 15-16, 2023, University of Echahid Cheikh Larbi Tebessi, Tebessa, Algeria, (ICAECE'2023),IEEE
- **Houili, R.**, Hammoudi, M.Y.,Betka,A.,Bacha,M., &Titaouine, A. Hybrid Particle Swarm Optimization and Manta Ray Foraging Optimization for Parameter Estimation of Induction Motors. The 3rd International Electronic Conference on Processes-Green and Sustainable Process Engineering and Process Systems Engineering, on May 29-31, 2024,Proceedings 2024,mdpi 105(1), 86;
- Bacha, M., Terki, A., **Houili ,R.** An Intelligent Diagnosis and Fault Detection Model Based on Fuzzy Logic for Photovoltaic Panels. The 3rd International Electronic Conference on Processes-Green and Sustainable Process Engineering and Process Systems Engineering, on May 29-31,2024 - sciforum.net;

# National Conferences

- **Houili, R.**, Hammoudi, M.Y.,Betka,A., &Titaouine, A. Stochastic optimization estimation and experimental verifcation for parameter identication applied to induction motor drives. The first National Applied Mathematics Seminar, Biskra, May 14-15, 2023( NAMS'23)
- **Houili, R.**, Hammoudi, M.Y.,Bacha,M., &Titaouine, A. New state feedback controller of ts fuzzy systems with unmeasured premise variables: Application to single-link flexible joint robot.The 1ST National Conference on Mechanics and Materials ,06 - 07 December 2023, Boumerdes University- Algeria(NCMM'2023)
- **Houili, R.**, Hammoudi, M.Y.,Bacha,M., &Titaouine, A. Enhanced Identification and Simulation of Photovoltaic Systems.The 1ST National Conference on Matter Science ,20 December 2023, Djelfa University- Algeria(NCSM2023)
- Bacha, M., Terki, A., **Houili ,R.**, Bacha, M.,& Youcef, I. Analyzing the influence of partial shading on photovoltaic system Performance. The 1ST National Confer-

ence on Mechanics and Materials ,06 - 07 December 2023, Boumerdes University-Algeria(NCMM'2023)

- Bacha, M., Terki, A., **Houili ,R.**,Zabia Djallel Eddine, , Youcef, I,& Bacha, M. Study of the Effect of Partial shading on photovoltaic panels. The 1ST National Conference on Emergent Technologies in Electrical Engineering ,16 - 17 December 2023, Ferhat Abbas University Setif 1- Algeria(NCETEE'2023)

- Bacha, M., Terki, A., **Houili ,R.**, Bacha, M.,& Youcef, I. Analysis of the Impact of Bypass Diode Failure onSolar PV Module Performance. The 1ST National Conference on Matter Science ,20 December 2023, Djelfa University- Algeria(NCSM2023)

# Abstract

Several challenges in controlling nonlinear systems, represented by fuzzy Takagi-Sugeno models, arise due to the complexity of modeling and synthesizing controllers. To address these challenges, utilizing a multi-model approach proves to be an effective strategy.This thesis explores the control of nonlinear systems represented by interconnected Takagi-Sugeno fuzzy multi-models and The thorough investigation of control strategies for nonlinear systems. We achieve this representation through nonlinear sector decomposition, enabling us to reformulate the system as polytopes without any loss of information.The investigation contrasts the conventional quadratic method with an advanced nonquadratic technique. The latter utilizes a line-integral Lyapunov function for stability analysis and state feedback control. Moreover, the mean value theorem is used to express the error dynamics in a way that reduces the conservatism of the bounded terms assumptions. The stability conditions are formulated as Bilinear Matrix Inequalities. To address these, we propose an iterative algorithm based on linear matrix inequalities, which transforms the Bilinear form into a sets of linear matrix inequalities. The thesis further delves into controlling nonlinear systems with TS fuzzy systems, particularly when premise variables are not measurable. To demonstrate the practicality and effectiveness of the proposed methods, It includes both numerical and practical examples to clarify the achieved results.

**Key words:** Non-linear system, Takagi-Sugeno multi model, controller, Linear Matrix Inequality, Line integral,lyapunov function, Quadratic lyapunov function..

# Résumé

Plusieurs défis se posent dans le contrôle des systèmes non linéaires, représentés par des modèles flous Takagi-Sugeno, en raison de la complexité de la modélisation et de la synthèse des contrôleurs. Pour relever ces défis, l'utilisation d'une approche multi-modèle s'avère être une stratégie efficace. Cette thèse explore le contrôle des systèmes non linéaires représentés par des multi-modèles flous Takagi-Sugeno interconnectés, ainsi qu'une investigation approfondie des stratégies de contrôle pour les systèmes non linéaires. Nous obtenons cette représentation par décomposition sectorielle non linéaire, ce qui nous permet de reformuler le système en polytopes sans perte d'information. L'étude compare la méthode quadratique conventionnelle à une technique avancée non quadratique. Cette dernière utilise une fonction de Lyapunov à intégrale linéaire pour l'analyse de stabilité et le contrôle par retour d'état. De plus, le théorème de la valeur moyenne est utilisé pour exprimer la dynamique des erreurs de manière à réduire le conservatisme des hypothèses relatives aux termes bornés. Les conditions de stabilité sont formulées sous forme d'inégalités matricielles bilinéaires. Pour y faire face, nous proposons un algorithme itératif basé sur des inégalités matricielles linéaires, qui transforme la forme bilinéaire en un ensemble d'inégalités matricielles linéaires.La thèse explore également le contrôle des systèmes non linéaires avec des systèmes flous Takagi-Sugeno, en particulier lorsque les variables de prémisse ne sont pas mesurables. Pour démontrer la praticité et l'efficacité des méthodes proposées, elle inclut à la fois des exemples numériques et pratiques afin d'éclaircir les résultats obtenus.

**Mots-Clés:** Système non linéaire, multi-modèle de Takagi-Sugeno,contrôleur,inégalités matricielles linéaire , fonction de Lyapunov intégrale de ligne, fonction de Lyapunov quadratique.

# Table of Contents

# List of Figures

# Nomenclature

| Abbreviation | |
|---|---|
| BMI | Bilinear Matrix Inequality |
| LMI | Linear Matrix Inequality |
| LPV | Linear Parameter Varying |
| LILF | Line Integral Lyapunov Function |
| MVT | Mean Value Theorem |
| MM | Multi-modèles |
| NMPV | Non-measurable premise variables |
| NQLF | Non-Quadratic Lyapunov Function |
| QLF | Quadratic Lyapunov Function |
| TS | Takagi-Sugeno |

# General Introduction

## General presentation

In the realm of control theory, modeling a process is both an essential and foundational task. A physical system can be depicted through two primary types of representations: implicit and explicit, with the latter being more commonly utilized. These models aim to closely mirror reality, connecting output and input variables via a state vector that outlines the system's evolution.

Real-world processes frequently exhibit nonlinearity, a characteristic stemming either from the complexity of the phenomena being described or from the nature of the feedback loops employed. This complexity necessitates the development of representation methods tailored for linear models. Among these methods, the Takagi-Sugeno (TS) fuzzy multi-model representation stands out. It involves deriving a set of polytopes linked through nonlinear functions, all adhering to the convex sum property. Each sub-model within this framework captures the behavior of the nonlinear system within a specific operational zone. In the field of TS fuzzy models, two primary types are extensively discussed in literature. The first type is known as coupled TS models, which can be derived through four distinct methods. The identification approach is the first method, often applied when the nonlinear system is challenging to describe analytically[BMR99], [Gas00]. The second method involves linearizing the system around various operating points[MSH98], offering a different perspective on the system's dynamics. The third method, known as the convex polytopic transformation or the sectors nonlinearities transformation[KTIT92],[TW04], [Bez13], converts the system's nonlinear terms without losing information, providing a more accurate representation compared to the first two methods. This approach, notable for its efficiency in preserving the integrity of the original model, will be the focus in this thesis. Lastly, the neural approach offers an alternative by leveraging neural networks for model approximation [EDBB10] ,[CB12].

The second category of TS fuzzy models is referred to as heterogeneous TS fuzzy multi-models [Fil91]. This framework is particularly suited for complex systems that undergo structural changes based on their operating conditions. In this setup, each sub-model operates within its unique state space, evolving independently from the others. This modeling approach introduces a level of flexibility to the identification process, accommodating the dynamic nature of complex systems. The activation functions, which are nonlinear, rely on variables termed as decision variables or premise variables. These variables may be either measurable (VDM) or non-measurable (VDNM), adding a layer of adaptability to the model based on the availability and observability of system parameters.

However, each solution comes with its unique challenges. Within the TS-LMI framework, the LMI conditions derived are clear but can lead to conservativeness. This implies that failing to meet certain conditions doesn't definitively mean a solution doesn't exist. Consequently, there is a significant push towards either ensuring the necessary conditions are met or, at a minimum, diminishing the level of conservativeness.

Following the modeling phase, estimating the state variables of the system is a crucial, if not indispensable, step for synthesizing control laws or for diagnosing industrial processes. This estimation is carried out through a dynamic system, often referred to as a state estimator or observer [BOU23].

The importance of fuzzy multi-models in controller design cannot be overstated. They facilitate a more nuanced understanding of nonlinear systems, enabling the development of controllers that can dynamically adjust to varying system states and external disturbances. This adaptability is crucial for maintaining optimal performance and stability in a wide range of applications, from robotics and automotive systems to energy management and process control in industrial settings. Furthermore, the use of fuzzy multi-models supports the implementation of Parallel Distributed Compensation (PDC) strategies [Ham15], which offers an alternative for systems represented by Takagi-Sugeno multi-models. It relies on linear controllers designed for each linear subsystem, ensuring the closed-loop stability of the nonlinear system through a common Lyapunov function for all subsystems. These Lyapunov functions can be either quadratic or non-quadratic in nature, leading to a set of Linear Matrix Inequalities (LMIs) that can be solved easily with optimization tools.

The integration of fuzzy multi-models into $H^\infty$ control strategies underscores a vital advancement in the pursuit of robustness in nonlinear system control. These models facilitate the precise delineation of a system's behavior across various operational regimes, thereby enabling the $H^\infty$ controllers to tailor their response to any disturbances or uncertainties with heightened accuracy. This tailored approach ensures that the system's performance criteria are met, even under adverse conditions, by minimizing the worst-case scenario's impact on system stability and performance.

The continuous advancement of these models, and the challenges they present, highlight the ongoing effort to refine methodologies. The primary aim is to enhance existing capabilities and overcome the inherent conservatism found in earlier approaches. To mitigate this conservatism, various strategies have been devised. This thesis centers on a particularly innovative strategy known as the Line Integral Lyapunov Function, this approach offers more flexible conditions for controller design.

# Contributions

The main objective of this thesis is to improve the control process of nonlinear systems modeled by a Takagi-Sugeno (T-S) multi-model, focusing on reducing the conservatism of existing methods. The major contributions of this work are detailed below:

- The main contribution of this study lies in the design of a new controller for Takagi-Sugeno fuzzy systems. Based on the differential mean value theorem and a line-integral Lyapunov function, we propose an innovative approach that significantly reduces the conservatism inherent in traditional methods, particularly those relying on disagreement terms and quadratic Lyapunov functions. The effectiveness of this method is illustrated through a series of numerical examples and a practical application to a single-link flexible manipulator robot.
- The second contribution lies in the formulation of an iterative algorithm based on Linear Matrix Inequalities (LMI) to solve the bilinear constraints that arise from the combination of the mean value theorem and the line-integral Lyapunov function. This algorithm solves these constraints using linear optimization solvers, providing improved results compared to existing Bilinear Matrix Inequality (BMI) solvers.

# Organization of the thesis

This thesis is structured into four main chapters, with each one addressing a distinct facet of the research topic

**Chapter 1:**The thesis begins with an introduction, followed by a detailed review of the current state of the art in nonlinear system control. It explores various control strategies, with a particular emphasis on Input-Output Feedback Linearization, Backstepping Control Design, and Nonlinear Adaptive Control which stands out for its robustness in handling parameter uncertainties.

**Chapter 2:**This chapter thoroughly examines the multi-model methodology, shedding light on its crucial role in capturing the complex dynamics of nonlinear systems. It explores a variety of multi-model frameworks and their derivation methods, showcasing their utility in modeling nonlinear systems. Central to the discussion is the emphasis on stability analysis, with a particular focus on Lyapunov's method for assessing stability in both quadratic and non-quadratic contexts. Notably, the chapter delves into the stability analysis of Takagi-Sugeno fuzzy models, laying the groundwork for securing the system's stability and reliability.

**Chapter 3:** The second chapter focuses on employing quadratic Lyapunov theory, leading to the establishment of conditions ensuring the state error's convergence to zero, expressed through Linear Matrix Inequalities (LMI). This facilitates the determination of controller gains. In the chapter's latter half, a state feedback control law, crafted through Parallel Distributed Compensation (PDC), is introduced to maintain the system-controller's stability in a closed-loop setting amidst disturbances. Additionally, the chapter delves into state reconstruction involving the Differential Mean Value Theorem and outlines the architecture of the Proportional Integral (PI) controller.

**Chapter 4:** This chapter introduces sophisticated techniques for designing controllers for continuous-time nonlinear systems. It features a standout section on developing controllers within the Takagi-Sugeno framework, tailored to scenarios involving unmeasurable variables. By utilizing a line integral Lyapunov function and leveraging the differential mean value theorem, the chapter effectively addresses state feedback controller issues. In conclusion, the chapter demonstrates significant improvements over previous research, emphasizing the enhanced stability offered by these controllers, which is validated through dynamic applications .

Chaptre 1

# Control of Non-linear Systems

## 1.1 Introduction

This chapter provides a comprehensive exploration of control strategies for nonlinear systems, which are essential in fields such as robotics, automotive systems, and electrical machines. It examines the strengths and limitations of various control techniques, including input-output feedback linearization and its limitations in addressing all nonlinearities. The chapter emphasizes the transformation of systems into linear equivalents without approximation. Additionally, it discusses backstepping control design, illustrating its systematic approach to stabilizing nonlinear systems.

Obtaining an exact model for a nonlinear system is often complex. Model inaccuracies can stem from uncertainties related to the process itself, such as poorly known or difficult-to-identify parameters, the omission of certain system dynamics, or overly simplified modeling choices. These inaccuracies are classified into two categories: parametric uncertainties and neglected dynamics. The first type directly affects the model, while the second concerns the estimation of the system's orders.

Nonlinear adaptive control techniques are designed to address these issues. The design of the control law takes into account a nominal model of the process as well as the associated parametric uncertainties. The structure of the controller, whether adaptive in nonlinear control or not, includes a nominal component (e.g., state feedback) as well as additional elements to optimally compensate for model uncertainties.

## 1.2 Control of Non-linear Systems: An Overview

In the field of process control, methods can be categorized based on the nature of the systems being controlled. The first category deals with linear stationary systems and can be broken down into three main phases:

### 1.2.1 Classical Frequency Methods (1930-60)

Classical Frequency Methods, spanning roughly from the 1930s to the 1960s, represent one of the earliest systematic approaches to control system analysis and design. These methods focus on understanding system behavior through the frequency domain, leveraging mathematical tools and graphical representations to assess stability, robustness, and performance. Key aspects of these methods include

**1. Nyquist Criterion**: A graphical method that uses the Nyquist plot, which maps the frequency response of an open-loop system, to assess the stability of the corresponding closed-loop system. It helps identify potential stability issues and determine gain and phase margins.

**2. Bode Plots**: Named after Hendrik Bode, these plots graphically represent a system's frequency response in terms of magnitude and phase across a range of frequencies. They offer insights into the system's gain and phase characteristics, guiding design modifications.

**3. Root Locus Analysis**: Developed by Walter Evans, this technique helps visualize the path of the system's poles in the complex plane as a system parameter (typically gain) varies. It aids in understanding how changes in gain affect stability and provides guidelines for controller design.

**4. Gain and Phase Margins**: These measures quantify how much gain or phase shift a system can tolerate before becoming unstable. Classical frequency methods use these margins to assess the robustness of a control system.

These methods remain foundational in control engineering education due to their intuitive visual approach to system stability and design.

### 1.2.2 Optimal Methods (1960s)

Optimal methods involve mathematical optimization to achieve specific objectives in control systems:

**1. State-Space Representation**: An abstract mathematical model that represents the system's dynamics using vectors and matrices. It offers a comprehensive framework for analyzing and controlling MIMO systems.

**2.Linear Quadratic Regulator (LQR)**: A method that seeks to minimize a cost function representing a combination of control effort and system deviation from desired behavior. It leads to the derivation of optimal feedback gain matrices.

**3.Kalman Filter**: An optimal estimation algorithm that combines predictions from system models with actual measurements to produce accurate estimates of system states. This approach is used for optimal control in noisy environments. Optimal methods significantly advance control theory by providing rigorous mathematical frameworks for designing controllers that meet predefined performance criteria.

### 1.2.3 The Renaissance of Frequency Methods (1980s)

During this period, a resurgence in interest in frequency domain methods occurred due to their robustness to uncertainties:

1. **$H^\infty$ Control**: A robust control technique designed to minimize the maximum gain of the transfer function from disturbances to outputs. It provides guaranteed performance even in the worst-case scenario, making it ideal for systems with uncertain parameters.

2. **Norm-Based Analysis**: This involves the use of mathematical norms like the $\infty$-norm or the 2-norm to quantify the system's robustness to disturbances, allowing for the design of controllers that maintain stability despite uncertainties. The resurgence of these methods addressed the increasing need for robust control strategies capable of handling complex real-world disturbances.
   The second category includes methods tailored to nonlinear systems, which pose unique challenges due to their nonlinear dynamics:Input-output Feedback Linearization,Backstepping Integrator Techniques.

## 1.3 Input-output Feedback Linearization

This type of control (input-output feedback linearization) emerged in the 1980s [II95] with the works of Isidori and has benefited from the contributions of differential geometry. A large number of nonlinear systems can be partially or fully transformed into systems exhibiting linear input-output or state-space behavior through the appropriate choice of an endogenous nonlinear state feedback law.

When the zero dynamics are stable, it is possible to transform a nonlinear system into a linear chain of integrators. After linearization, classical linear techniques can be applied. This approach has often been employed to solve practical control problems, but this technique requires that the state vector be measured and necessitates an accurate model of the process to be controlled. Moreover, robustness properties are not guaranteed against model parametric uncertainties.

Indeed, this technique is based on the exact cancellation of nonlinear terms, and hence, the presence of modeling uncertainties on the nonlinear terms leads to an imperfect cancellation, resulting in an input-output equation that remains nonlinear.

### 1.3.1 Design of Feedback Linearization Controllers

The concept of input-output linearization is now very well-understood. We will demonstrate how to achieve a linear relationship between the output y and a new input v by making a good choice of the linearizing law. Since the equivalent model is linear, we can ensure stable dynamics by relying on classical linear methods.

First and foremost, we consider a nonlinear system with $p$ inputs and $p$ outputs of the following form:

$$\dot{x} = f(x) + \sum_{i=1}^{p} g_i(x)u_i \tag{1.1}$$

$$y_i = h_i(x), \quad i = 1, 2, \ldots, p \tag{1.2}$$

where $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$ is the state vector, $u = [u_1, u_2, \ldots, u_p]^T \in \mathbb{R}^p$ is the input vector, and $y = [y_1, y_2, \ldots, y_p]^T \in \mathbb{R}^p$ represents the output vector. The functions $f, g_i$ are smooth vector fields and $h_i, i = 1, 2, \ldots, p$, are smooth functions. The objective is to establish a linear relationship between the input and the output by differentiating the output until at least one input appears, utilizing the following formulation:

$$y_j^{(r_j)} = L_f^{r_j} h_j(x) + \sum_{i=1}^{p} L_{g_i}\left(L_f^{r_j-1} h_j(x)\right) u_i, \quad j = 1, 2, \ldots, p \tag{1.3}$$

Here, $L_f h_j$ and $L_{g_i} h_j$ represent the $i$-th Lie derivatives of $h_j$ along the directions of $f$ and $g$ respectively. The term $r_j$ denotes the number of derivatives needed for at least one of the inputs to appear in the expression (1.1) of the output's relative degree corresponding to $y_j$.

The total relative degree $r$ is defined as the sum of all the individual relative degrees obtained with the help of the above equation and must be less than or equal to the system's order: $r = \sum_{j=1}^{p} r_j \leq n$.

A system described by equation (1.1) is said to have a relative degree $r$ if it satisfies:

$$L_{g_i} L_f^k h_j = 0 \quad \text{for} \quad 0 < k < r_j - 1, \quad 1 \leq j \leq p, \quad 1 \leq i \leq p \tag{1.4}$$

And:

$$L_{g_i} L_f^k h_j \neq 0 \text{ for } k = r_j - 1 \tag{1.5}$$

In cases where the total relative degree is equal to the system's order, we have state-space linearization. However, if the total relative degree is strictly less than the system's order, the process is called input-output linearization.

To derive the expression for the linearizing control law $u$ that linearizes the relationship between the input and the output, we rewrite equation (1.1) in matrix form.

$$\begin{bmatrix} y_1^{r_1} & \cdots & y_p^{r_p} \end{bmatrix}^T = \xi(x) + D(x)u$$

where

$$\xi(x) = \begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_p} h_p(x) \end{bmatrix}$$

and

$$D(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & L_{g_2} L_f^{r_1-1} h_1(x) & \cdots & L_{g_p} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & L_{g_2} L_f^{r_2-1} h_2(x) & \cdots & L_{g_p} L_f^{r_2-1} h_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_p-1} h_p(x) & L_{g_2} L_f^{r_p-1} h_p(x) & \cdots & L_{g_p} L_f^{r_p-1} h_p(x) \end{bmatrix}$$

The matrix $D(x)$ is referred to as the system's decoupling matrix.

Assuming that $D(x)$ is non-singular, the form of the linearizing control law is given by:

$$u = D(x)^{-1} \left( -\xi(x) + v \right) \tag{1.6}$$

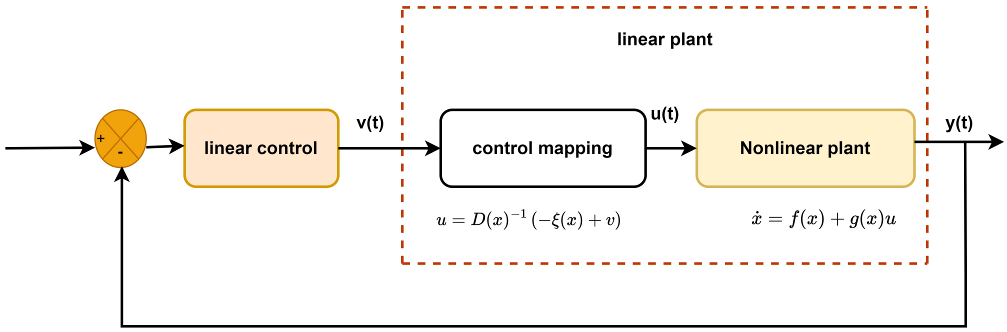



Fig 1.1: Structure of State Feedback Linearization

By substituting equation (1.6) into (1.1) , the equivalent system becomes linear and completely decoupled in the form:

$$y_j^{(r_j)} = v_i \tag{1.7}$$

Or more explicitly:

$$\begin{bmatrix} y_1^{r_1} \\ \vdots \\ y_p^{r_p} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} \tag{1.8}$$

This allows us to impose any stable dynamics through a well-designed new input vector $v = [v_1, \ldots, v_p]^T$.

Note that the expression (1.8) represents $p$ cascaded integrators, whose behavior may not always be desirable.

#### 1.3.1.1 Canonical form

Suppose the system (3.1) has relative degrees $\{r_1, r_2, \ldots, r_p\}$ such that the sum $r = \sum_{i=1}^{p} r_i$ equals the system's order $n$. Define $r$ functions $\{\phi_1, \phi_2, \ldots, \phi_p\}$ allowing us to express:

$$\begin{aligned} z &= (\phi_1, \phi_2, \ldots, \phi_p, \phi_{p+1}, \ldots, \phi_r) \\ &= ([h_1, L_f h_1, \ldots, L_f^{r_1-1} h_1, h_2, \ldots, L_f^{r_2-1} h_2, \ldots, h_p, \ldots, L_f^{r_p-1} h_p]) \end{aligned} \tag{1.9}$$

Depending on the values of $\{r_1, r_2, \ldots, r_p\}$, we can distinguish two possible cases:

<u>**Case N°1**</u>: $r = r_1 + r_2 + \ldots + r_p = n$

In this scenario, the set of functions $\phi^k = L_f^{k-1} h$, where $1 \leq k \leq r_i$ and $1 \leq i \leq p$, define a diffeomorphism, such that:

$$\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_r \end{bmatrix} = \begin{bmatrix} \left[h_1, L_f h_1, \ldots, L_f^{r_1-1} h_1\right]^T \\ \vdots \\ \left[h_p, L_f h_p, \ldots, L_f^{r_p-1} h_p\right]^T \end{bmatrix} \tag{1.10}$$

<u>**Case N°2**</u>: $(r = r_1 + r_2 + \ldots + r_p < n)$

In this case, it is possible to find $(n - r)$ other functions $\phi^k$, $r + 1 \leq k \leq n$ such that $\phi^k$, $1 \leq k \leq n$ is of rank $n$. We introduce a vector of complementary variables $\eta$ such that:

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-r} \end{pmatrix} = \begin{pmatrix} \phi_{r+1} \\ \phi_{r+2} \\ \dots \\ \phi_n \end{pmatrix}$$

In the new coordinates, the system (1.1) is written as:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_1 &= z_3 \\ &\vdots \\ \dot{z}_{r_1-1} &= z_{r_1} \\ \dot{z}_{r_1} &= L_f^{r_1} h_1 + \sum_{j=1}^{p} L_{g_j}^{r_1-1} h_1 u_j \\ \dot{z}_{r_1+1} &= z_{r_1+2} \\ &\vdots \\ \dot{z}_r &= L_f^{r_p} h_p + \sum_{j=1}^{p} L_{g_j}^{r_p-1} h_p u_j \end{aligned} \tag{1.11}$$

For the remaining $(n-r)$ functions, it is difficult to provide a detailed form of the new variables; however, we denote them generally by $\dot{\eta} = \psi(z,\eta) + \theta(z,\eta)u$.

Regarding the output, the vector $y = \begin{bmatrix} y_1 & y_2 & \dots & y_p \end{bmatrix}^T$ can be expressed in the new coordinates as:

$$\begin{aligned} y_1 &= z_1, \\ y_2 &= z_{r_1+1}, \\ &\vdots \\ y_p &= z_{r_i+\dots+r_{p-1}+1}. \end{aligned}$$

Applying the linearizing law (1.6) to the system(1.11) , we obtain:

$$\dot{z} = \begin{bmatrix} A_{r_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_{r_n} \end{bmatrix} z + \begin{bmatrix} B_{r_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_{r_n} \end{bmatrix} u \tag{1.12}$$

$$\eta = \psi(z,\eta) + \Theta(z,\eta)u$$

With:

$$A_{r_i} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{r_i \times r_i}, \quad B_{r_i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{r}_{i}, \quad C_{r_i} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

And for the output:

$$y = \begin{bmatrix} C_{r_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & C_{r_2} \end{bmatrix} z \quad (3.15)$$

#### 1.3.1.2 Design of The Virtual Input $v$

The vector $v$ is often designed according to the control objectives in order to ensure the tracking of the envisioned path and must satisfy the following conditions:

$$v_j = y_{d_j}^{(r_j)} + k_{r_j}\left(y_{d_j}^{(r_j-1)} - y_j^{(r_j-1)}\right) + \dots + k_1\left(y_{d_j} - y_j\right) \quad 1 \le j \le p \tag{1.13}$$

Where the vectors $\{y_{d_j}, y_{d_j}^{(1)}, \dots, y_{d_j}^{(r_j-1)}, y_{d_j}^{(r_j)}\}$ define the imposed reference trajectories for the different outputs. If the $k_i$ are chosen such that the polynomial: $s^{r_j} + k_{r_j-1}s^{r_j-1} + \dots + k_2 s + k_1 = 0$ is a Hurwitz polynomial (possesses roots with negative real parts), then we can show that the error $e_j(t) = y_{d_j}(t) - y_j(t)$ satisfies:

$$\lim_{t\to\infty} e_j(t) = 0.$$

The closed-loop linearized system is depicted in Figure (1.2) below:

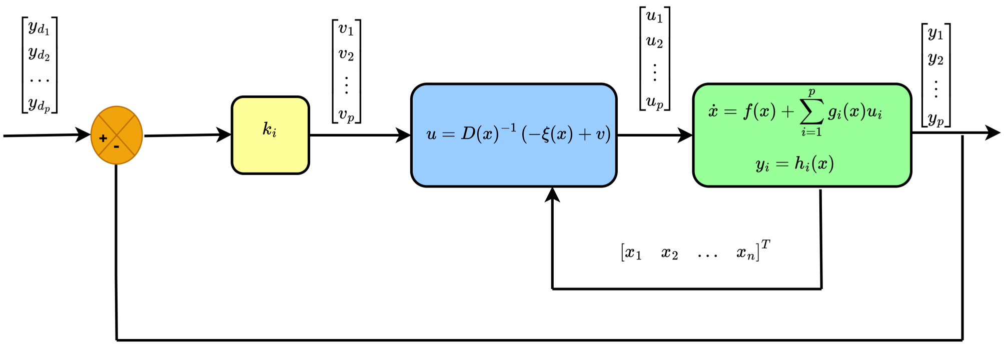



Fig 1.2: Structure of State Feedback Linearization

### 1.3.2 Feedback Linearization for Three-Phase PWM AC-DC Converter:

Let's recall that the dynamic equations of the three-phase PWM AC-DC converter 'Statcom' in the (d-q) reference frame are as follows:

$$\begin{cases} di_d = -\frac{R}{L}i_d + wi_q\frac{1}{L}(e_d - v_d) \\ di_q = -\frac{R}{L}i_q + wi_d\frac{1}{L}(e_q - v_q) \\ dv_{dc} = -\frac{2}{3Cv_{dc}}(e_d i_d - e_q i_q) - \frac{v_{dc}}{CR_{ch}} \end{cases} \tag{1.14}$$

If we define x as the state vector and u as the control vector, we can express the system of equations (1.14) in the suggested form to apply the theory of input-output feedback linearization as follows:

$$\dot{x} = f(x) + g(x)u \tag{1.15}$$

With

$$\begin{bmatrix} i_d & i_q & v_{dc} \end{bmatrix}^T, u = \begin{bmatrix} u_d \\ u_q \end{bmatrix} = \begin{bmatrix} (e_d - v_d) \\ (e_q - v_q) \end{bmatrix}$$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L}i_d + wi_q \\ -\frac{R}{L}i_q + wi_d \\ -\frac{2}{3Cv_{dc}}(e_d i_d - e_q i_q) - \frac{v_{dc}}{CR_{ch}} \end{bmatrix} \tag{1.16}$$

and

$$g(x) = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{L} \\ 0 & 0 \end{bmatrix} \tag{1.17}$$

Our goal is to ensure that the voltage across the capacitor $v_{dc}$ follows a reference $v_{dcref}$, maintain a unity power factor, and minimize harmonics in the lines. To achieve this, we apply input-output feedback linearization theory, ensuring complete decoupling between the controls and the outputs. In this context, we select the output variables as the current $i_d$ and the voltage across the capacitor $v_{dc}$.

The desired objective (unity power factor, harmonic minimization, and regulation of the voltage $v_{dc}$) leads us to impose $i_{dcref}$, while the voltage $v_{dc}$ must follow its reference, which can be a step or any trajectory defined by $v_{dcref}$.

**Remark:**

The current $i_q$ acts as a regulator for the capacitor voltage.

To derive the linearizing control law, we calculate the relative degree of the output $y$, which is the number of times we need to differentiate the output to reveal at least one input.

**a) The output $i_d$:**

Given$y_1 = i_d = h_1(x)$, then:

$$\nabla h_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{1.18}$$

When we differentiate it with respect to time, we get

$$\begin{aligned} \dot{y_1} &= \dot{h_1}(x) = \frac{\partial h_1(x)}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial h_1}{\partial x}\dot{x} = \frac{\partial h_1}{\partial x}(f(x) + g(x)u) \\ \dot{y} &= L_f h_1(x) L_g h_1(x).u \\ &= f_1 + g_1 u_d \end{aligned} \tag{1.19}$$

Since the input ud appears in expression (1.19), we stop. For this output, note that the relative degree $r_1 = 1$

**a) The output $v_{dc}$:**

Given $y_2 = v_{dc} = h_2(x)$, then:

$$\nabla h_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{1.20}$$

If we differentiate $y_2$ with respect to time, we find

$$\begin{aligned} \dot{y_2} &= L_f h_2(x) L_g h_2(x).u + f_3 + 0.u \\ &= L_f h_2(x) L_g h_2(x) = F_3 \end{aligned} \tag{1.21}$$

The Lie derivative of $h_2(x)$ with respect to g is zero ($L_g h_2(x) = 0$), which forces us to differentiate it a second time

$$\ddot{y_2}(x) = \ddot{h}_2(x) = \frac{d(\dot{y_2})}{\dot{\mathrm{d}}\mathrm{t}} = \frac{d(L_f h_2)}{\dot{\mathrm{d}}\mathrm{t}}$$
$$\ddot{y_2}(x) = L_f^2 h_2(x) L_g L_f h_2(x).u \tag{1.22}$$

with

$$L_f^2 h_2(x) = \frac{2}{3C\nu_{dc}}(e_d f_1 + e_q f_2) - \left\{\frac{2}{3C\nu_{dc}}(e_d i_d + e_q i_q) + \frac{1}{CR_{ch}}\right\} f3 \tag{1.23}$$

$$L_g L_f h_2(x) = \begin{bmatrix} \frac{2e_d}{3LC\nu_{dc}} & \frac{2e_q}{3LC\nu_{dc}} \end{bmatrix} \tag{1.24}$$

Observe that $f_1$, $f_2$, and $f_3$ are functions given by (1.16).

The relative degree with respect to $y_2$ is $r_2 = 2$. Thus, the total relative degree of the system is $r = r_1 + r_2 = 3$ . Since $r = n = 3$, the system is exactly linearizable, where $n$ is the order of the system.

From (1.21) to (1.22), we obtain the input-output relation of the system as follows:

$$\frac{d}{dt}[y_1, \dot{y}_2]^{\mathrm{T}} = \zeta(x) + D(x).u \tag{1.25}$$

Where

$$\zeta(x) = \left[\frac{2}{3C\nu_{dc}}(e_d f_1 + e_q f_2) - \left\{\frac{f_1}{3C\nu_{dc}}(e_d i_d + e_q i_q) + \frac{1}{CR_{ch}}\right\} f3\right] \tag{1.26}$$

and

$$D(x) = \begin{bmatrix} \frac{1}{L} & \frac{1}{L} \\ \frac{2e_d}{3LC\nu_{dc}} & \frac{2e_q}{3LC\nu_{dc}} \end{bmatrix} \tag{1.27}$$

D:represents the decoupling matrix

If $(3L^2 Cv_{dc})$, then the matrix $D$ is invertible. Hence, the linearizing control law that relates the new internal inputs $(v_1, v_2)$ to the real inputs $(u_d, u_q)$ is expressed as follows

$$\begin{bmatrix} u_d \\ u_q \end{bmatrix} = D^{-1} \left[-\zeta + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right] \tag{1.28}$$

By substituting expression (1.28) into (1.25), we obtain a completely decoupled linear system in the form

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{1.29}$$

To impose steady-state behavior on the error (desired objective), the new internal inputs $(v_1, v_2)$ must be designed to ensure

$$\begin{cases} \lim\limits_{t\to\infty} i_d = 0 \\ \lim\limits_{t\to\infty} \nu_{dc} = \nu_{dcref} \end{cases} \tag{1.30}$$

In the general case, for a trajectory tracking problem, we have

$$\begin{cases} v_1 = k_1(i_{dref} - i_d) + \frac{d(i_{dref})}{dt} \\ v_2 = k_2(\nu_{dcref} - \nu_{dc}) + k_3 \left\{ \frac{d(\nu_{dref})}{dt} - \frac{d(\nu_{dc})}{dt} \right\} + \frac{d^2(\nu_{dcref})}{dt^2} \end{cases} \tag{1.31}$$

Our goal is to keep the voltage $v_{dc}$ constant with a unity power factor, so we enforce[Yac04]

$$\dot{v}_{dcref} = \ddot{v}_{dcref} = \dot{i}_{dref} = 0 \tag{1.32}$$

Thus

$$\begin{cases} v_1 = k_1(i_{dref} - i_d) \\ v_2 = k_2(v_{dcref} - v_{dc}) - k_3 \frac{d(v_{dc})}{dt} \end{cases} \tag{1.33}$$

The control gains are calculated based on the desired poles, which are located at $-1200 \pm j1200$ for current control, and $-500$ and $-400 \pm j300$ for voltage control, respectively. The block diagram of the closed-loop linearized system is depicted in the following figure, from which we derive the gains [LLL98]

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 24 \times 10^2 \\ 125 \times 10^6 \\ 7 \times 10^5 \end{bmatrix} \tag{1.34}$$

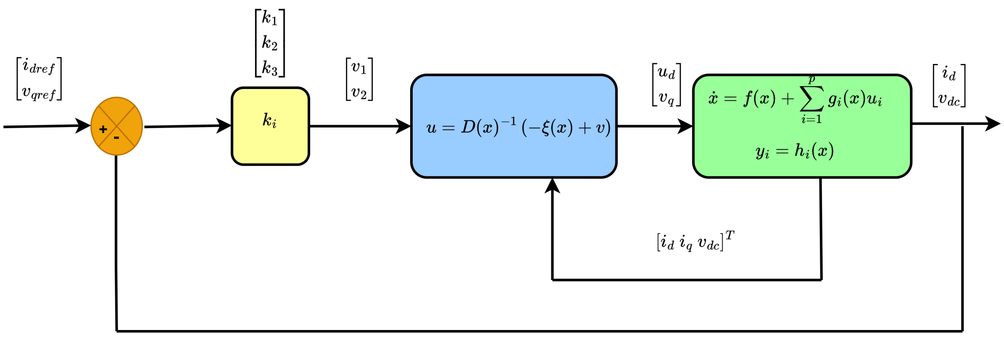



Fig 1.3: Structure of State Feedback Linearization

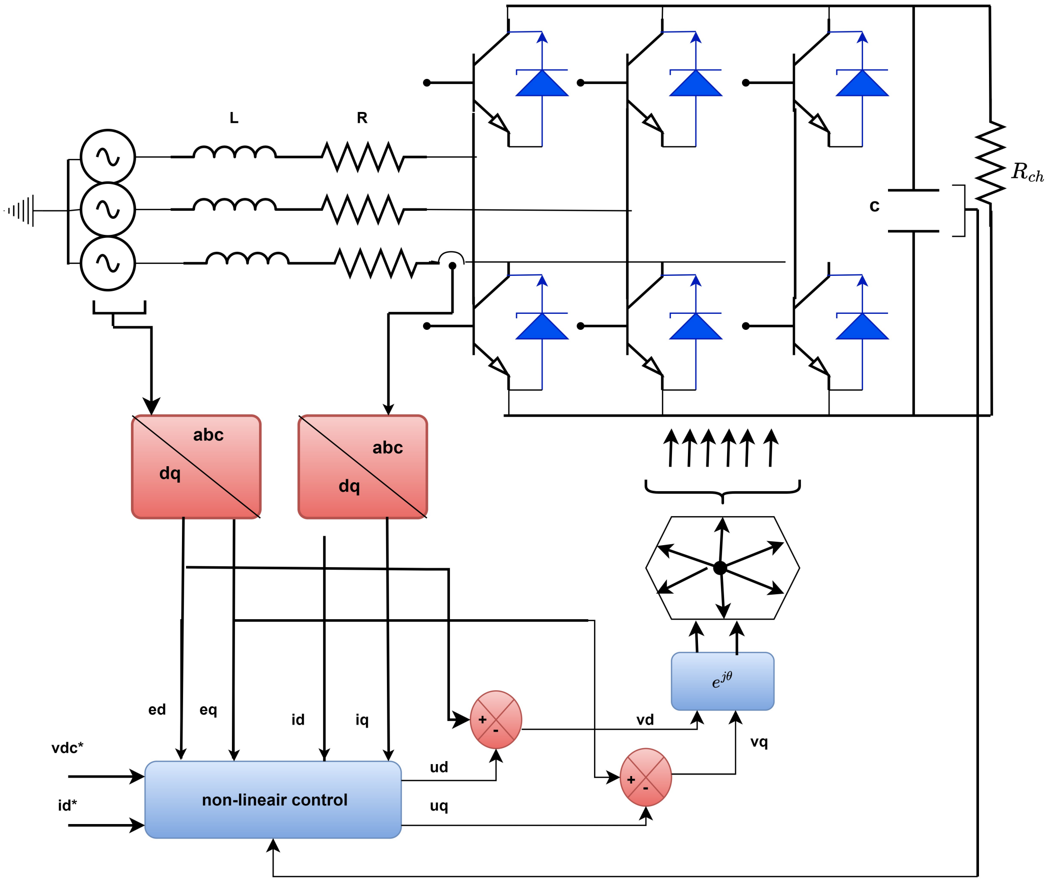



Fig 1.4: Structure of State Feedback Linearization

According to the figures (1.5)and (1.6) , it is observed that the voltage vac follows the reference, and the current of the line does not contain harmonics and is in phase with the voltage of the source. From what has been obtained, it can be seen that the relationship between the output $i_d$ and $i_{dref}$ is represented by an integrator, whereas the relationship between $v_{dc}$ and its reference is represented by a double integrator.

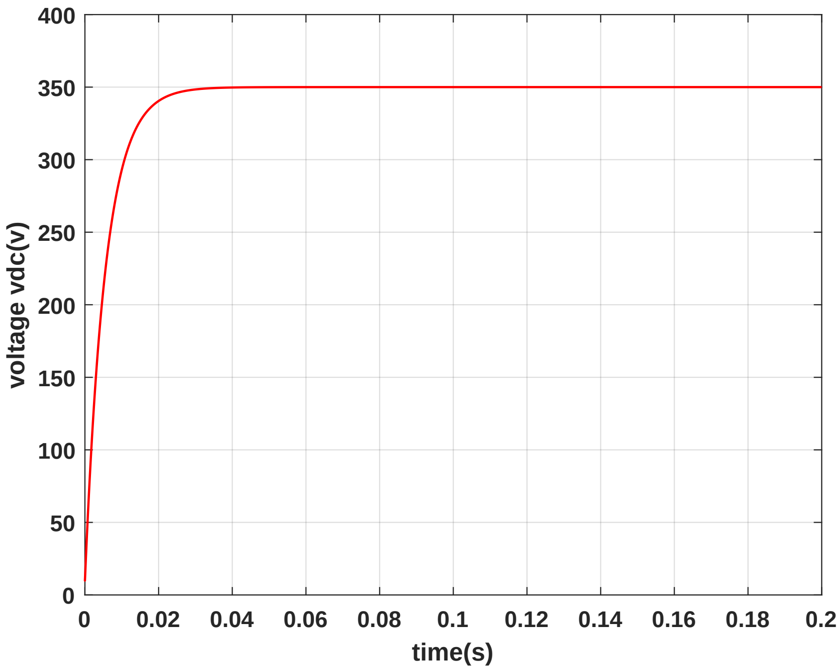



Fig 1.5: the voltage $v_{dc}$

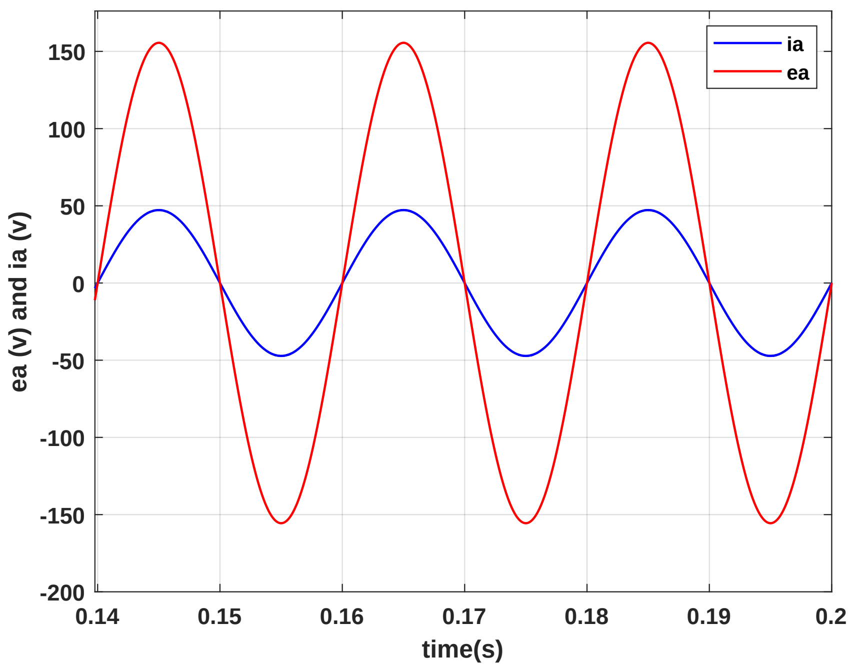



Fig 1.6: Line current and voltage

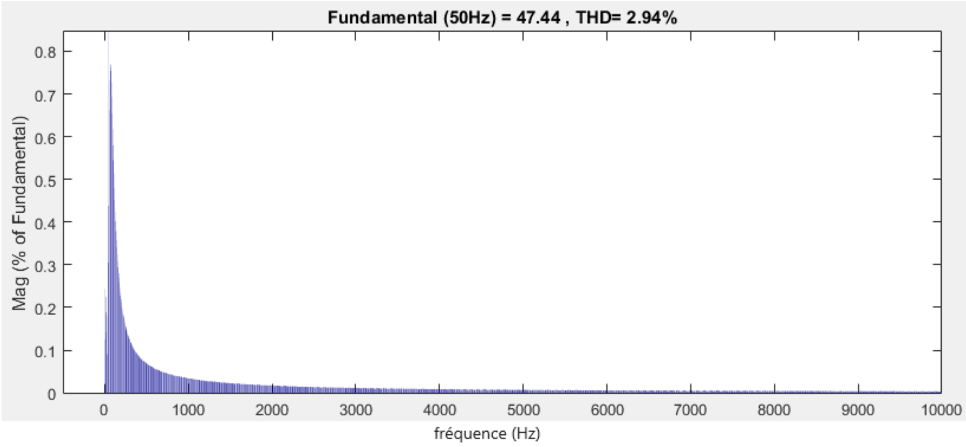



Fig 1.7: The harmonic spectrum of the line current

And according to the previous figures, it can be observed that the input-output linearization technique can eliminate the issue of ripple caused by the opening and closing of switches, and make the input-output system linear at all points, and that this system will be equivalent to an integrator

### 1.3.3 Feedback Linearization for Permanent Magnet Synchronous Motor

Let us recall that the dynamic equations of the PMSM in the d-q reference frame are:

$$\begin{aligned}\frac{di_d}{dt} &= -\frac{R}{L_d}i_d + \frac{L_q}{L_d}p\omega_r i_q + \frac{1}{L_d}u_d\\ \frac{di_q}{dt} &= -\frac{R}{L_q}i_q - \frac{L_d}{L_q}p\omega_r i_d - \frac{\Phi_v}{L_q}p\omega_r + \frac{1}{L_q}u_q\\ \frac{d\omega_r}{dt} &= \frac{3p}{2J}(\Phi_v i_q + (L_d - L_q)i_d i_q) - \frac{1}{J}T_L - \frac{B}{J}\omega_r\end{aligned} \tag{1.35}$$

The load torque $T_L$ does not appear in these equations as it is considered a disturbance. The system of equations is rewritten in the form suggested for the application of input-output linearization as follows

$$\dot{x} = f(x) + g_1(x) \cdot u_d + g_2(x) \cdot u_q \tag{1.36}$$

Where the state vector $x$and the input vector $u$ are

$$x = \begin{bmatrix} i_d & i_q & \omega_r \end{bmatrix}^T \text{et} u = \begin{bmatrix} u_d & u_q \end{bmatrix}^T \tag{1.37}$$

With

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L_d} i_d + \frac{L_q}{L_d} p\omega_r i_q \\ -\frac{R}{L_q} i_q - \frac{L_d}{L_q} p\omega_r i_d - \frac{\Phi_v}{L_q} p\omega_r \\ \frac{3p}{2J}(\Phi_v i_q + (L_d - L_q) i_d i_q) - \frac{B}{J}\omega_r \end{bmatrix} \tag{1.38}$$

and

$$g_1(x) = \begin{bmatrix} \frac{1}{L_d} \\ 0 \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ \frac{1}{L_q} \\ 0 \end{bmatrix} \tag{1.39}$$

The objective is to ensure speed regulation of the motor while maintaining maximum torque operation (where the d-component of the stator currents id is forced to remain zero at all times). To achieve this, input-output linearization is applied to the model, which ensures total decoupling between the inputs and outputs. In this case, the outputs must be the rotor speed($w_r$) and the current ($i_d$)

$$y_1 = i_d \quad et \quad y_2 = \omega_r \tag{1.40}$$

These two outputs must follow the trajectories imposed on them. The maximum torque operation strategy leads us to impose $i_{dref} = 0$, while the speed must follow its reference, which can be a step or any other trajectory defined by wref.
For the first output ($i_d$), we have

$$y_1 = i_d = h_1(x), \quad \nabla h_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{1.41}$$

By differentiating it with respect to time, we will have

$$\begin{aligned} \dot{y}_1 &= L_f h_1(x) + L_{g_1} h_1(x) u_d + L_{g_2} h_1(x) u_q \\ &= \frac{\partial h_1}{\partial x} \cdot f(x) + \frac{\partial h_1}{\partial x} \cdot g_1(x) \cdot u_d + \frac{\partial h_1}{\partial x} \cdot g_2(x) \cdot u_q \\ &= -\frac{R}{L_d} i_d + \frac{L_q}{L_d} p\omega_r i_q + \frac{1}{L_d} u_d \end{aligned} \tag{1.42}$$

Thus, the input $u_d$ appears in the expression (1.42). We stop here and note, for this output, a relative degree r = 1.
For the second output, we have

$$y_2 = \omega_r = h_2(x), \quad \nabla h_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{1.43}$$

By differentiating it once, we have

$$\begin{aligned} \ddot{y}_2 &= L^2{}_f h_2(x) + L_{g_1}(L_f h_2(x)) \cdot u_d + L_{g_2}(L_f h_2(x)) \cdot u_q \\ &= \Lambda(L_d - L_q) i_q f_1(x) + \Lambda(\Phi, +(L_d - L_q) i_d) f_2(x) - \frac{B}{J} f_3(x) + \\ &\frac{\Lambda(L_d - L_q)}{L_d} i_q u_d + \frac{\Lambda(\Phi_\nu + (L_d - L_q) i_d)}{L_q} u_q \end{aligned} \tag{1.44}$$

$$\Lambda = \frac{3p}{2J} \tag{1.45}$$

Where $f_2(x)$ and $f_3(x)$ are given by (1.38). The two inputs $u_d$ and $u_q$ appear in (1.42) and the relative degree is therefore $(r_2 = 2)$. The total relative degree is $r = r_1 + r_2 = n = 3$ and thus we have achieved an exact linearization.(No internal dynamics need to be considered).

By combining the expressions (1.42) and (1.44), we obtain the following form

$$\begin{bmatrix} \dot{y}_1 & \ddot{y}_2 \end{bmatrix}^T = \zeta(x) + D(x) \cdot u \tag{1.46}$$

Where

$$\zeta(x) = \begin{bmatrix} L_f h_1(x) \\ L_f^2 h_2(x) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L_d} i_d + \frac{L_q}{L_d} p \omega_r i_q \\ \Lambda(L_d - L_q) i_q f_1(x) + \Lambda(\Phi_\nu + (L_d - L_q) i_d) f_2(x) - \frac{B}{J} f_3(x) \end{bmatrix} \tag{1.47}$$

And

$$D(x) = \begin{bmatrix} \frac{1}{L_d} & 0 \\ \frac{\Lambda(L_d - L_q)}{L_d} i_q & \frac{\Lambda(\Phi_v + (L_d - L_q) i_d)}{L_q} \end{bmatrix} \tag{1.48}$$

The matrix $D(x)$ is invertible if the following condition is satisfied

$$det[D(x)] = \frac{\Lambda(\Phi_v + (L_d - L_q) i_d)}{L_\sigma L_d} \neq 0 \quad or \quad \Phi_v \neq (L_q - L_d) i_d \tag{1.49}$$

Therefore, the linearizing control law that ensures decoupling is expressed by

$$\begin{bmatrix} u_d \\ u_q \end{bmatrix} = D(x)^{-1} \left( \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \zeta(x) \right) \tag{1.50}$$

Where

$$D(x)^{-1} = \begin{bmatrix} L_d & 0 \\ \frac{-(L_d-L_q)i_q L_q}{(\Phi_v+(L_d-L_q)i_d)} & \frac{L_q}{\Lambda(\Phi_v+(L_d-L_q)i_d)} \end{bmatrix} \tag{1.51}$$

Substituting expression (1.50) into (1.46) yields a fully decoupled linear system of the form:

$$\begin{bmatrix} \dot{y}_1 & \ddot{y}_2 \end{bmatrix}^T = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T \tag{1.52}$$

The new inputs $(v_1, v_2)$ must be designed to ensure that

$$\lim_{t\to\infty} y_1 = i_{dref} \quad and \quad \lim_{t\to\infty} y_2 = \omega_{ref} \tag{1.53}$$

To achieve this, pole placement is employed. In the general case, for a trajectory tracking problem, we have:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} k_d \cdot (i_{dref} - i_d) \\ \ddot{\omega_{ref}} + k_{w_1}(\dot{\omega_{ref}} - \dot{\omega}_r) + k_{w_2}(\omega_{ref} - \omega_r) \end{bmatrix} \tag{1.54}$$

But if the imposed trajectory is a step, then we have $\ddot{w_{ref}} = \dot{w_{ref}} = 0$, and expression (1.54) becomes

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} k_d \cdot (i_{dref} - i_d) \\ -k_{w_1}\dot{\omega}_r + k_{w_2}(\omega_{ref} - \omega_r) \end{bmatrix} \tag{1.55}$$

The block diagram of the linearized closed-loop system is represented by the following figure 1.8

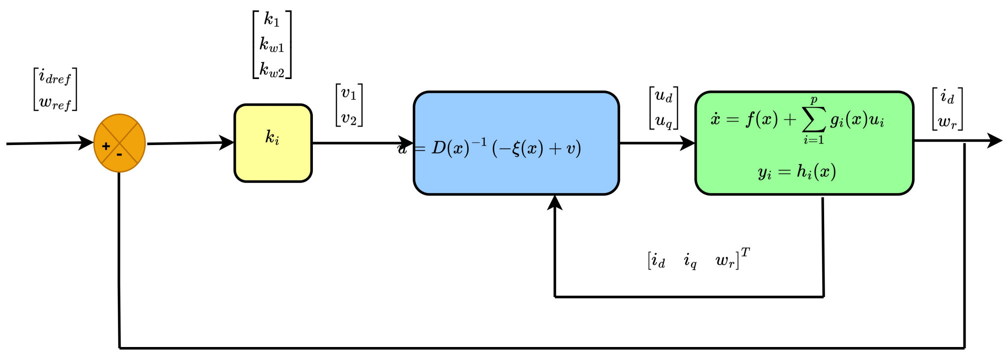



Fig 1.8: Structure of State Feedback Linearization

It has been decided to impose the following poles $(-10, -0.125, -4000)$so that the closed-loop system is stable and its response does not exhibit any overshoot. The controller gains are then [Kad00]:

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 500 \\ 4000 \end{bmatrix} \tag{1.56}$$

Figure 1.9 shows the speed profile, which perfectly follows its reference speed of 100 rad/s, achieved very quickly with an acceptable response time. After applying a load torque of 5 N.m at $t = 0.25$ s, a slight overshoot is observed, after which the speed returns to its reference value. A similar observation is made during the reversal of the reference speed, where it is noted that the motor speed follows the reference speed, albeit with the presence of oscillations during the transient phase.

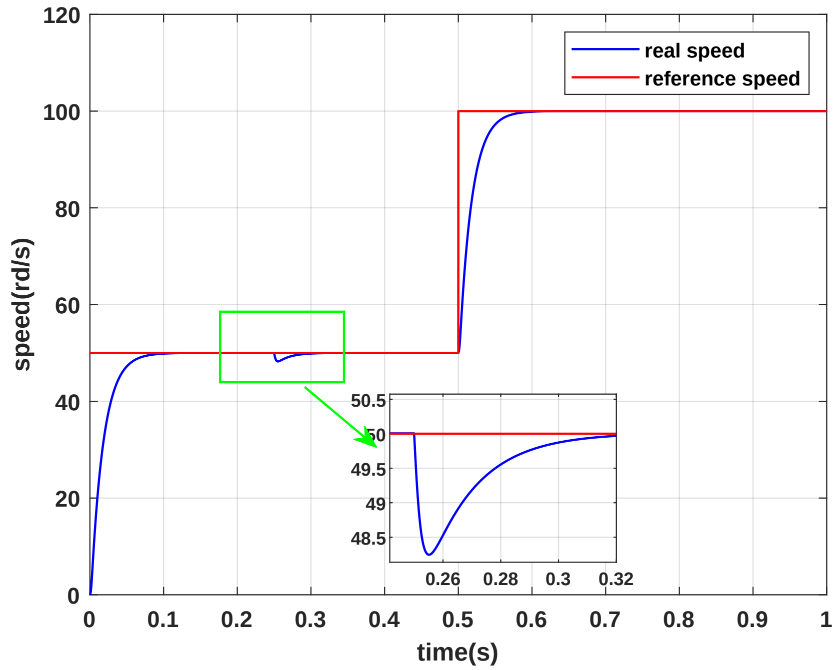



Fig 1.9: rotor speed $w_r$

## 1.4 Backstepping control

In this section, we provide a brief summary of the well-known backstepping control method in the control theory literature [Kha02], [KKK95]. The backstepping control approach is a recursive design technique that associates the selection of a control Lyapunov function with the development of a feedback controller, ensuring global asymptotic stability for strict feedback systems [Kok92], [KA01].

For the design of an integrating backstepping controller, we consider the following control system:

$$\dot{\xi} = F(\xi) + G(\xi)\eta \tag{1.57.a}$$

$$\dot{\eta} = v. \tag{1.57.b}$$

In the system defined by (1.57), the state vector is $X = (\xi, \eta) \in \mathbb{R}^{n+1}$, and the control input is $v \in \mathbb{R}$. The objective is to design a backstepping control law $v$ such that $X(t) \to 0$ as $t \to \infty$. We assume that both $F$ and $G$ are known functions, continuously differentiable on $\mathbb{R}^n$, with the conditions $F(0) = 0$ and $G(0) = 0$.
Figure 1.10 illustrates the block diagram of the control system (1.57). The system can be interpreted as a cascade of two components, where the first is an integrator.

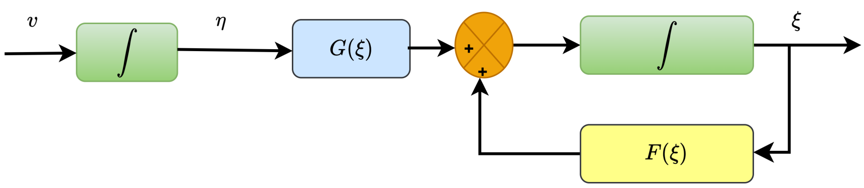



Fig 1.10: Backstepping control design for the control system (1.57)

In the subsystem (1.57.a), we treat $\eta$ as a virtual controller. We assume the existence of a smooth feedback control law $\eta = \varphi(\xi)$ with $\varphi(0) = 0$, such that $\xi = 0$ is an asymptotically stable equilibrium point for the first system.

$$\dot{\xi} = F(\xi) + G(\xi)\varphi(\xi). \tag{1.58}$$

We also assume the existence of a Lyapunov function $V_1(\xi)$ that satisfies the inequality

$$\frac{\partial V_1}{\partial \xi}[F(\xi) + G(\xi)\varphi(\xi)] \leq -W(\xi). \tag{1.59}$$

where $W(\xi)$ is positive definite on $\mathbf{R}^n$.
By adding and subtracting $G(\xi)\varphi(\xi)$ on the right-hand side of (1.57.a), we can express

the control system (1.57) in the following form:

$$\dot{\xi} = [F(\xi) + G(\xi)\varphi(\xi)] + G(\xi)[\eta - \varphi(\xi)] \tag{1.60a}$$

$$\dot{\eta} = v. \tag{1.60b}$$

We now define a change of variables

$$y = \eta - \varphi(\xi) \tag{1.61}$$

The output $y$ can also be interpreted as the error between the state $\eta$ and the pseudo-control $\varphi(\xi)$.

Therefore, one of the design objectives in the backstepping control procedure is to find $v$ such that $y(t) \to 0$ as $t \to \infty$.

If we express the initial system (1.60) in the $(\xi, y)$ coordinates, we obtain:

$$\dot{\xi} = [F(\xi) + G(\xi)\varphi(\xi)] + G(\xi)y \tag{1.62a}$$

$$\dot{y} = v - \dot{\varphi}(\xi). \tag{1.62b}$$

Since $F, G$, and $\varphi$ are known, we can express $\dot{\varphi}(\xi)$ as follows:

$$\dot{\varphi}(\xi) = \frac{\partial \varphi}{\partial \xi}[F(\xi) + G(\xi)\eta]. \tag{1.63}$$

We set

$$u = v - \dot{\varphi}(\xi). \tag{1.64}$$

The transformed system (1.62) can then be expressed as follows:

$$\dot{\xi} = [F(\xi) + G(\xi)\varphi(\xi)] + G(\xi)y, \tag{1.65a}$$

$$\dot{y} = u. \tag{1.65b}$$

The transformed system (1.65) has the same structure as the original control system (1.57). The key advantage of the transformed system (1.65) is the following important observation:

When the input is zero, the first subsystem (1.65a) is asymptotically stable at $\xi = 0$. In the backstepping control design, the control Lyapunov function $V_1(\xi)$ is used to stabilize the overall control system (1.65).

Next, we consider the total Lyapunov function for the original system (1.57), which is given by:

$$V(\xi,\eta) = V_1(\xi) + \frac{1}{2}\,y^2 = V_1(\xi) + \frac{1}{2}\,[\eta - \varphi(\xi)]^2. \tag{1.66}$$

Next, we show that $V(\xi,\eta)$ is a positive definite function on $\mathbb{R}^{n+1}$.

Since $V_1(\xi)$ is a positive definite function, it follows directly that

$$V_1(\xi) \geq 0 \text{ for all } \xi \in \mathbf{R}^n \text{ and } V_1(\xi) = 0 \Longleftrightarrow \xi = 0. \tag{1.67}$$

From (1.66) and(1.67) ;it is immediate that $V(\xi,\eta) \geq 0$ for all$(\xi,\eta) \in \mathbf{R}^{n+1}$.

Next, we will demonstrate that

$$V(\xi,\eta) = 0 \iff (\xi,\eta) = (0,0) \tag{1.68}$$

Let $(\xi,\eta) = (0,0)$. Then $\xi = 0$ and $\eta = 0$. This implies that $V_1(0) = 0$ and $\varphi(0) = 0$. Hence, $V(\xi,\eta) = 0$. Next, suppose that $V(\xi,\eta) = 0$. Since $V$ is a sum of two nonnegative numbers, it is immediate that

$$V_1(\xi) = 0 \text{ and } \eta - \varphi(\xi) = 0. \tag{1.69}$$

Since $V_1$ is a positive definite function, $\xi = 0$. Since $\varphi(0) = 0$, we must have $\eta = 0$. Therefore, $(\xi,\eta) = (0,0)$.

Thus, we have established that $V$ is a positive definite function on $\mathbb{R}^{n+1}$.

Next, we compute the derivative of the candidate Lyapunov function $V$ as follows:

$$\begin{aligned} \dot{V} = &\ \frac{\partial V_1}{\partial \xi}\,[F(\xi) + G(\xi)\varphi(\xi)] + \frac{\partial V_1}{\partial z}\,G(\xi)y + yv \\ \leq &\ -W(\xi) + \frac{\partial V_1}{\partial z}\,G(\xi)y + yu. \end{aligned} \tag{1.70}$$

We select the backstepping control law $u$ as

$$u = -\frac{\partial V_1}{\partial z}\,G(\xi) - ky \ (k > 0) \tag{1.71}$$

Substituting (1.71) into the inequality in(1.70), we obtain

$$\dot{V} \leq -W(\xi) - ky^2.$$

Hence, by Lyapunov stability theory [Kha02], we deduce that $(\xi,\eta) = (0{,}0)$ is an asymptotically stable equilibrium for the original system (1.57)

From (1.64),we know that $u = v - \dot{\varphi}(\xi)$.

Therefore, the required backstepping control law is expressed as:

$$v = u + \dot{\varphi}(\xi) = \frac{\partial \varphi}{\partial \xi}[F(\xi) + G(\xi)\eta] - \frac{\partial V_1}{\partial z} G(\xi) - k[\xi - \varphi(\xi)] \ (k > 0). \tag{1.72}$$

The above backstepping calculations are summarized in the following result:

**Theorem 1.4.1:**

Consider the control system (1.57) defined on $\mathbb{R}^{n+1}$ with smooth vector fields $F$ and $G$ with $F(0) = 0$ and $G(0) = 0$. Let $\eta = \phi(\xi)$ be a stabilizing state feedback law for the subsystem (1.57.a), where $\phi(0) = 0$. Suppose that $V_1(\xi)$ is a Lyapunov function such that

$$\frac{\partial V_1}{\partial \xi}[F(\xi) + G(\xi)\varphi(\xi)] \leq -W(\xi) \tag{1.73}$$

where $W(\xi)$ is positive definite on $R^n$.Then the backstepping control law

$$v = u + \dot{\varphi}(\xi) = \frac{\partial \varphi}{\partial \xi}[F(\xi) + G(\xi)\eta] - \frac{\partial V_1}{\partial z} G(\xi) - k[\xi - \varphi(\xi)] \ (k > 0) \tag{1.74}$$

Stabilizes the equilibrium $(\xi, \eta) = (0, 0)$ of the system (1.57) with the total Lyapunov function

$$V(\xi, \eta) = V_1(\xi) + \frac{1}{2}[\eta - \varphi(\xi)]^2. \tag{1.75}$$

Next, we examine the backstepping control design for a general system of the form

$$\dot{\xi} = F(\xi) + G(\xi)\eta, \tag{1.76a}$$

$$\dot{\eta} = \alpha(\xi, \eta) + \beta(\xi, \eta)u \tag{1.76b}$$

We define the control input $u$ as

$$u = \frac{1}{\beta(\xi, \eta)}[v - \alpha(\xi, \eta)]. \tag{1.77}$$

Substituting (1.77) into (1.76),we get

$$\begin{aligned} \dot{\xi} &= F(\xi) + G(\xi)\eta, \\ \dot{\eta} &= v. \end{aligned} \tag{1.78}$$

The design process of the backstepping controller, which is inherently recursive, depends on the system's equations having a particular triangular structure, often referred to as the pure-feedback form. This structural condition is crucial for the successful application of backstepping methods in control design [LK97].

$$\begin{aligned}
\dot{\eta_1} &= F_0(\xi) + G_0(\xi)\eta_1 \\
\dot{\eta_2} &= F_1(\xi, \eta_1) + G_1(\xi, \eta_1)\eta_2 \\
&\vdots \\
\dot{\eta_{k-1}} &= F_{k-1}(\xi, \eta_1, \ldots, \eta_{k-1}) + G_{k-1}(\xi, \eta_1, \ldots, \eta_{k-1})\eta_k \\
\dot{\eta_k} &= F_k(\xi, \eta_1, \ldots, \eta_k) + G_k(\xi, \eta_1, \ldots, \eta_k)u
\end{aligned} \tag{1.79}$$

where $\xi, \eta_1, \eta_2, \ldots, \eta_k \in \mathbf{R}$ are the states and $u \in \mathbf{R}$ is the control input. In (1.79)$F_0, F_1, \ldots, F_k, G_0, G_1, \ldots, G_k$ are known smooth functions. It is noted that in the strict feedback form (1.79), $\dot{\xi}_i$ depends only on the states $\xi, \eta_1, \ldots, \eta_i$.

In most cases, the feedback linearization method used to stabilize the strict feedback control system (1.79) may result in the cancellation of important nonlinear terms. However, the standard backstepping control design offers greater flexibility compared to the feedback linearization approach, as it does not require the final input-output dynamics to be a linear system. The backstepping control design is a recursive process, and by applying Lyapunov stability theory, a Lyapunov function is constructed for the entire system (1.79).

### 1.4.1 Numerical Application

$$\begin{cases} \dot{x_1} = x_1^2 - x_1^3 + x^2 \\ \dot{x_2} = x_3 \\ \dot{x_3} = u \end{cases} \qquad x_{eq}(0,0,0) \tag{1.80}$$

It is necessary to determine $u$ in a way that ensures $x_{eq}$ is asymptotically stable (AS)

1: consider $\phi_0(x_1) = -x_1^2 - x_1$

let $v = 1/2x_1^2 \Longrightarrow \dot{v} = x_1\dot{x_1} = -x_1^4 - x_1^2 < 0$

2: consider

$$\begin{cases} \dot{x_1} = x_1^2 - x_1^3 + x^2 \\ \dot{x_2} = x_3 \end{cases} \tag{1.81}$$

$$\phi_1(x_1, x_2) = -k\left(\zeta - \phi_0(x_1)\right) + \frac{\partial \phi_0}{\partial \eta}\left(f(\eta) + g(\eta)\zeta\right) - \frac{\partial v}{\partial \eta} \cdot g(\eta) \tag{1.82}$$

$\eta = x_1$ and $\zeta = x_1$

$f = x_1^2 - x_1^3$ and $g = 1$

$$\phi_1(x_1, x_2) = -k\left(x_2 + x_1^2 + x_1)\right) + (-2x_1 - 1)\left(x_1^2 - x_1^3 + x_2\right) - x_1 \tag{1.83}$$

$$v_1 = \frac{1}{2}x_1^2 + \frac{1}{2}[\zeta - \phi_0(x_1)]^2 \tag{1.84}$$

3: consider

$$\begin{cases} \dot{x_1} = x_1^2 - x_1^3 + x^2 \\ \dot{x_2} = x_3 \\ \dot{x_3} = u \end{cases} \tag{1.85}$$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_1^2 - x_1^3 + x^2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_3$$

$\dot{x_3} = u$ We define

$$\eta_1 = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix}, f_1 = \begin{bmatrix} x_1^2 - x_1^3 + x^2 \\ 0 \end{bmatrix}, g_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \zeta_1 = x_3$$

The control law

$$\phi_1(x_1, x_2) = -k_1\left(\zeta_1 - \phi_1(x_1, x_2)\right) + \frac{\partial \phi_1}{\partial \eta_1}\left(f_1 + g_1\zeta_1\right) - \frac{\partial v_1}{\partial \eta_1} \cdot g_1 \tag{1.86}$$

$$\frac{\partial v_1}{\partial \eta_1} g_1 = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial v_1}{\partial x_2}$$

$$u = -k\left(x_3 - \phi_1(x_1, x_2)\right) + \frac{\partial \phi_1}{\partial x_1}\left(x_1^2 - x_1^3 + x_2\right) + \frac{\partial \phi_1}{\partial x_2} x_3 - \frac{\partial v_1}{\partial x_2} \tag{1.87}$$

### 1.4.2 Backstepping for Three-Phase PWM AC-DC Converter

To apply the backstepping control technique to the static compensator, we need to put our system into a 'strict-feedback' form. To do this, we will apply Lie theory after a suitable choice of output variables.

The desired objective is to force the voltage $v_{dc}$ to track its reference value and maintain the current $i_d$ at zero.

To achieve this objective using the backstepping technique, we will choose the output variables as follows:

$$\begin{cases} y_1 = h_1(x) = v_{dc} \\ y_2 = L_f h_1(x) \\ y_3 = i_d \end{cases} \tag{1.88}$$

Applying Lie theory, we find :

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} L_f h_1 + L_{g1} h_1 u_d + L_{g2} h_1 u_q \\ L_f^2 h_1 + L_{g1} L_f h_1 u_d + L_{g2} L_f h_1 u_q \\ L_f h_2 + L_{g1} h_2 u_d + L_{g2} h_2 u_q \end{bmatrix} \tag{1.89}$$

with

$$L_f h_1(x) = \frac{2}{3C\nu_{dc}}(e_d i_d + e_q i_q) - \frac{\nu_{dc}}{CR_{ch}}$$

$$L_{g1} h_1 (x) = L_{g2} h_1 (x) = 0$$

$$L^2 f h_1(x) = \frac{2\left(e_d f_1(x) + e_q f_2(x)\right)}{3Cv_{dc}} - \left\{ \frac{2\left(e_d i_d + e_q i_q\right)}{3Cv_{dc}^2} + \frac{1}{CR_{ch}} \right\} f_3$$

$$L_{g1} L_f h_1(x) = \frac{2e_d}{3LC\nu_{dc}}$$

$$L_{g2} L_f h_1(x) = \frac{2e_q}{3LC\nu_{dc}}$$

$$L_f h_2(x) = -\frac{R}{L} i_d + w i_q$$

$$L_{g1} h_2 (x) = \frac{1}{L}$$

$$L_{g2} h_2(x) = 0$$

We can write the system (1.89) in the following form:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = L^2 f h_1 + L_{g1} L_f h_1 u_d + L_{g2} L_f h_1 u_q \\ \dot{y}_3 = L_f h_2 + L_{g1} h_2 u_d + L_{g2} h_2 u_q \end{cases} \tag{1.90}$$

If we choose new control inputs, the system (1.90) can be written as two subsystems, where the first subsystem is in strict-feedback form [TC99]

Subsystem 01:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = L^2 \bar{f} h_1 + \bar{u}_d \end{cases} \tag{1.91}$$

Subsystem 02:

$$\dot{y}_3 = L_{\hat{f}}h_2 + \overline{u}_q \tag{1.92}$$

Such that

$$\begin{bmatrix} \overline{u}_d \\ \overline{u}_q \end{bmatrix} = \begin{bmatrix} L_{g1}L_{\tilde{f}}h_1u_d + L_{g2}L_{\tilde{f}}h_1u_q \\ L_{g1}h_2u_d + L_{g2}h_2u_q \end{bmatrix} \tag{1.93}$$

The compact form of the model is

$$\dot{y} = \overline{A}\,(x) + B\,(x)\,\overline{U} \tag{1.94}$$

To achieve good transient performance [ZW02], a linear reference model is defined as:

$$\begin{gathered} \dot{y}_m = k_m y_m + B_m u_{ref} \\ \begin{bmatrix} \dot{y}_{m1} \\ \dot{y}_{m2} \\ \dot{y}_{m3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -k_{m1} & k_{m2} & 0 \\ 0 & 0 & k_{m1} \end{bmatrix} \begin{bmatrix} y_{m1} \\ y_{m2} \\ y_{m3} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_{m1} & 0 \\ 0 & k_{m3} \end{bmatrix} \begin{bmatrix} vdc^* \\ i_d^* \end{bmatrix} \end{gathered} \tag{1.95}$$

Now, we will define the error variables as follows:

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} y_1 - y_{m1} \\ y_2 - y_{m2} \\ y_3 - y_{m3} \end{bmatrix} \tag{1.96}$$

We use the following transformation

$$\widetilde{U} = \begin{bmatrix} \widetilde{u}_d \\ \widetilde{u}_q \end{bmatrix} = \begin{bmatrix} \overline{u}_d + k_{m1}y_{m1} + k_{m2}y_{m2} - k_{m1}vdc^* \\ \overline{u}_q + k_{m3}y_{m3} - k_{m3}i_d^* \end{bmatrix} \tag{1.97}$$

Then the error differential equations are:

$$\dot{e} = \overline{A}\,(x) + B(x)\widetilde{U} \tag{1.98}$$

With

$$\overline{A}\,(x) = \begin{bmatrix} e_2 \\ L^2\tilde{f}h_1 \\ L_{\tilde{f}}h_2 \end{bmatrix}, B(x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**The step 01:**

We choose the state $e_2$ as the virtual input for the first equation of system (1.98), hence, the first backstepping variable is chosen as:

$$z_1 = e_1 \tag{1.99}$$

The virtual input is defined by:

$$z_2 = \alpha_1 e_2 \tag{1.100}$$

With $\alpha$ is a stabilizing function. $z_2$ is a new state variable.

Then the $z_1$ system can be written in the form:

$$\dot{z}_1 = \alpha_1 + z_2 \tag{1.101}$$

The stabilizing function $\alpha(x)$

$$\alpha(x) = -k_1 z_1 \tag{1.102}$$

So $\dot{z}_1 = -k_1 z_1 + z_2$

The Lyapunov function of the $z_1$ system is

$$\begin{aligned} V_1 &= \frac{1}{2} z_1^2 \\ \dot{V}_1 &= z_1 \dot{z}_1 \\ &= -k_1 z_1^2 + z_1 z_2 \end{aligned} \tag{1.103}$$

With $k_1 > 0$, it is the feedback gain, hence the $z_1$ system is now stable.

**The step 02:**

The dynamics of $z_2$ are described by the following equation:

$$\begin{aligned} \dot{z}_2 &= -\dot{e}_2 + \dot{\alpha}(x) \\ &= L_f^2 h_1 + \tilde{u}_d + \dot{\alpha}(x) \end{aligned} \tag{1.104}$$

Then the Lyapunov function of the $z_2$ system is

$$V_2 = V_1 + \frac{1}{2} z_2^2$$

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + \frac{1}{2} {z_2}^2 \\ &= \dot{V}_1 + \dot{z}_2 z_2 \\ &= (-k_1 z_1^2 + z_1 z_2) + \dot{z}_2 z_2 \\ &= -k_1 {z_1}^2 + z_2(z_1 + \dot{z}_2) \end{aligned}$$

$$\begin{aligned}\dot{V}_2 &= -k_1 z_1^2 + z_2(z_1 + L_{\tilde{f}}^2 h_1 + \tilde{u}_d + \dot{\alpha}) \\ &= -k_1 z_1^2 + z_2(z_1 + L_{\tilde{f}}^2 h_1 + \tilde{u}_d + -k_1 \dot{z}_1) \\ &= -k_1 z_1^2 + z_2(z_1 + L_{\tilde{f}}^2 h_1 + \tilde{u}_d - k_1 z_1 + z_2)\end{aligned} \tag{1.105}$$

**The step 03:**

In the third step, a Lyapunov function must be generated to stabilize the third equation of (1.98). To do this, we take

$$\dot{z}_3 = L_{\tilde{f}} h_2(x) + \tilde{u}_q \tag{1.106}$$

Defining a Lyapunov function $v_3(z_3)$ such that:

$$V_3(z_3) = \frac{1}{2} z_3^2 \tag{1.107}$$

Finally, to design the backstepping controller for the system, we define an augmented Lyapunov function $v(z_1, z_2, z_3)$, such that

$$V(z_1, z_2, z_3) = V_2(z_1, z_2) + V_3(z_3)$$

So

$$\begin{aligned}\dot{V}(z_1, z_2, z_3) = &-k_1 z_1^2 + z_2(z_1 + L^2 \tilde{f} h_1 + \tilde{u}_d - k_1 z_1 + z_2) \\ &+ z_3(L_{\tilde{f}} h_2 + \tilde{u}_q)\end{aligned} \tag{1.108}$$

For $\dot{V} \leq 0$ to hold, the second and third terms must be equal to $-k_2 z_2$ and $-k_3 z_3$ respectively, hence

$$\begin{aligned}&z_1 + L_{\tilde{f}}^2 h_1 + \tilde{u}_d - k_1 z_1 + z_2 = -k_2 z_2 \\ &L_{\tilde{f}} h_2 + \tilde{u}_q = -k_3 z_3\end{aligned} \tag{1.109}$$

Then the control inputs are :

$$\begin{aligned}\tilde{u}_d &= -z_1 - L_{\tilde{f}}^2 h_1 + k_1 z_1^2 - k_1 z_2 - k_2 z_2 \\ \tilde{u}_q &= -L_{\tilde{f}} h_2 - k_3 z_3\end{aligned} \tag{1.110}$$

To obtain the actual control inputs $u_d$ and $u_q$, we use equations (1.97) and (1.93)

Figures (1.11),(1.12) represent the responses of the voltage $v_{dc}$, the current, and the line-to-line voltage of the backstepping control applied to the three-phase PWM AC-DC converter. From these figures, we can observe that the $v_{dc}$ follows its reference, and the power factor is nearly unity.

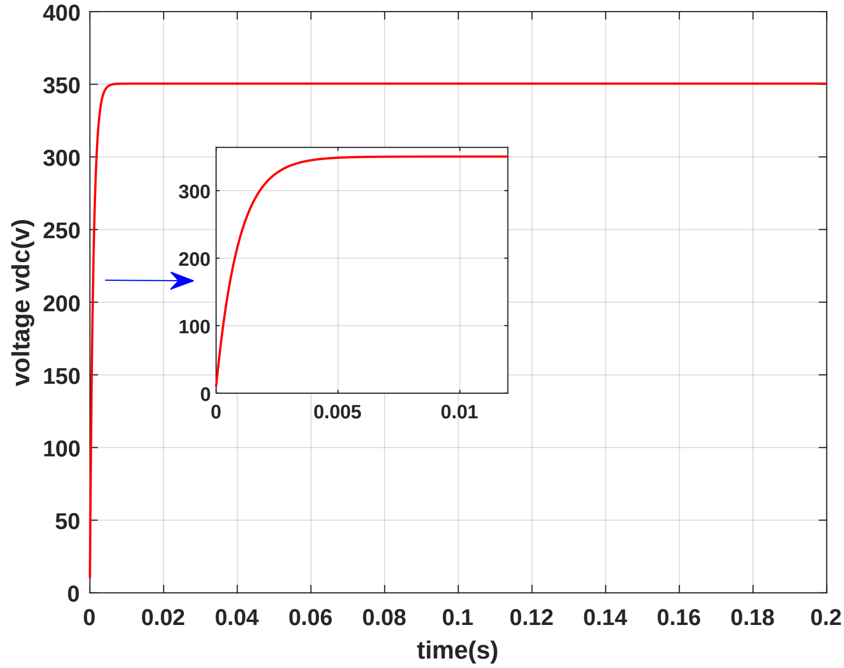



Fig 1.11: the voltage $v_{dc}$

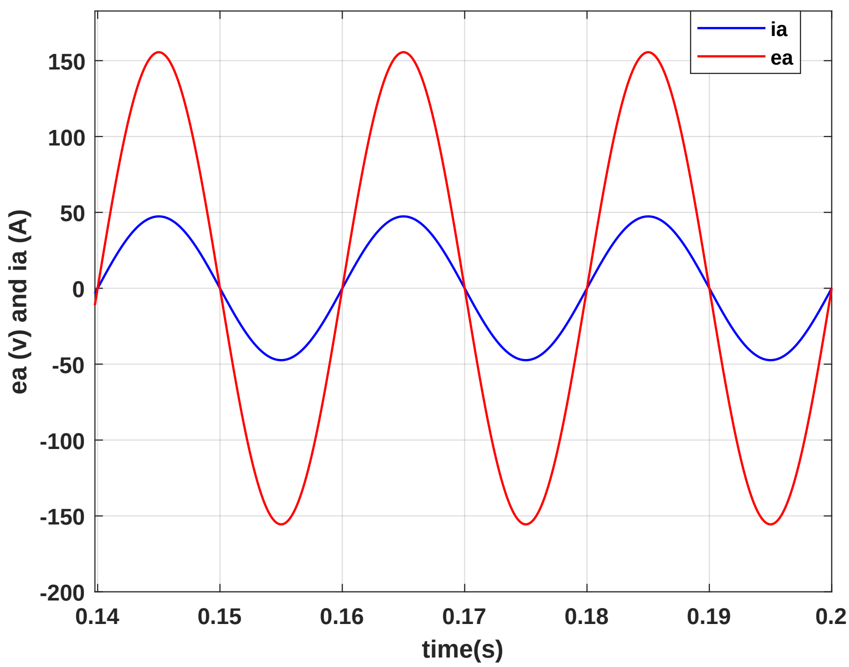



Fig 1.12: Line current and voltage

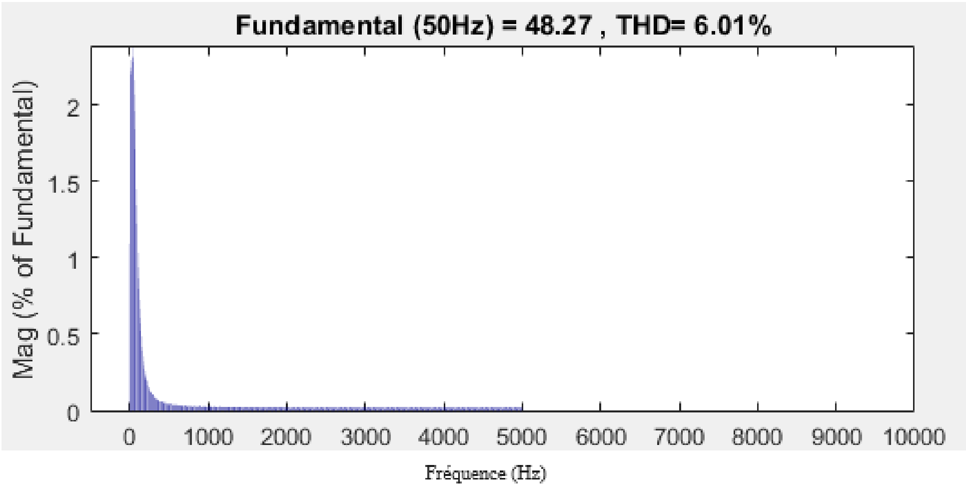



Fig 1.13: The harmonic spectrum of the line current

### 1.4.3 Backstepping for Permanent Magnet Synchronous Motor

In this section, we have demonstrated the implementation of backstepping control in the PMSM engine. This approach is designed to maintain the same overall structure of vector control, as depicted in Figure 1.14, while ensuring regulation and limitation of currents.[LK97]

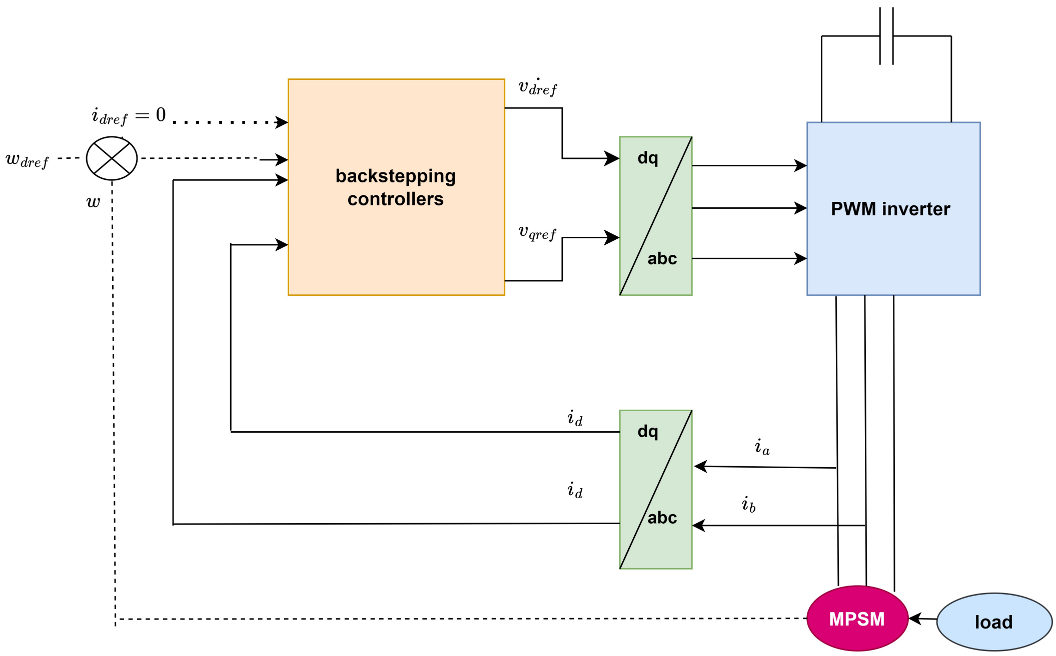



Fig 1.14: Overall structure of the speed control of the PMSM using backstepping

The model written in the following form:

$$\begin{bmatrix} \frac{di_d}{dt} \\ \frac{di_q}{dt} \\ \frac{d\omega}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-R_s}{L_d} i_d + \frac{\omega_r L_q}{L_d} i_q \\ \frac{-R_s}{L_q} i_q - \frac{\omega_r L_d}{L_q} i_d - \frac{\omega_r \phi_v}{L_q} \\ \frac{3}{2}\frac{p\phi_v}{J} i_q - \frac{p(L_q - L_d)}{J} i_d i_q - \frac{f}{J}\omega \end{bmatrix} + \begin{bmatrix} \frac{1}{L_d} & 0 & 0 \\ 0 & \frac{1}{L_q} & 0 \\ 0 & 0 & \frac{-1}{J} \end{bmatrix} \begin{bmatrix} V_d \\ V_q \\ T_L \end{bmatrix} \tag{1.111}$$

can be rewritten in the following form:

$$\begin{cases} \frac{di_d}{dt} &= -\frac{Rs}{L_d} i_d + \frac{L_d}{L_q}\omega_r i_q + \frac{1}{L_d} v_d \\ \frac{di_q}{dt} &= -\frac{Rs}{L_q} i_q - \frac{L_d}{L_q}\omega_r i_d - \frac{\phi_v}{L_q}\omega_r + \frac{1}{L_q} v_q \\ \frac{d\omega}{dt} &= -\frac{p\phi_v}{J} i_q - \frac{p(L_q - L_d)}{J} i_d i_q - \frac{1}{J} T_L \end{cases} \tag{1.112}$$

The core idea behind backstepping control is to render the overall system as a cascade of stable subsystems of order one, in the sense of Lyapunov stability, which confers qualities of robustness and global asymptotic stability. The objective is to control the speed by choosing the time derivatives $\frac{di_d}{dt}$ and $\frac{di_q}{dt}$ as the subsystem expressions and $i_d$ and $i_q$ as intermediate variables (the stator currents). Based on these variables ($i_d$ and $i_q$), the voltage control inputs ($v_d$ and $v_q$) required to achieve the desired speed control and overall system stability are calculated.

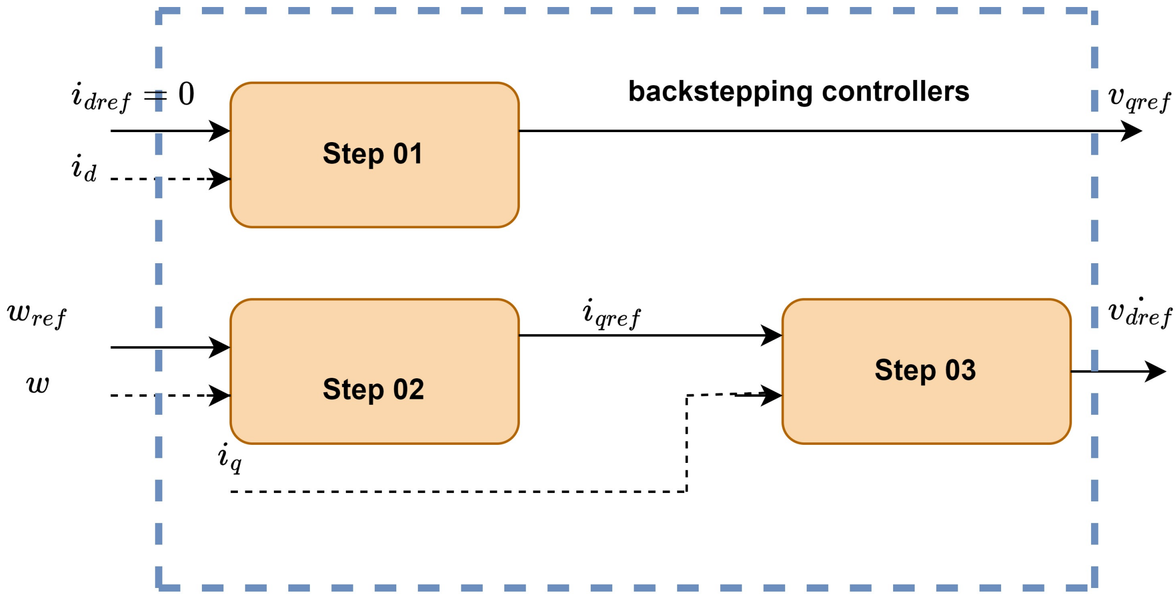



Fig 1.15: Internal structure of the backstepping control block

**The step 01:**

Since direct current is a control variable, its desired value and its regulation error are defined by

$$\begin{aligned} i_{dref} &= 0 \\ e_1 &= i_{dref} - i_d \\ \omega_r &= \omega . p \end{aligned} \tag{1.113}$$

Based on equations (1.112)and(1.113) , the error dynamics equations are:

$$\begin{aligned} \dot{e}_1 &= i_{\text{dref}} - i_d \\ \dot{e}_1 &= i_{\text{dref}} + \frac{R}{L_d} i_d - \frac{L_d}{L_q} \omega_r i_q - \frac{1}{L_d} v_d \end{aligned} \tag{1.114}$$

Since the objective requires error $e_1$ to converge to zero and also demands that the current be regulated and limited, we employ the Lyapunov function $V_1$, which in a sense represents the error energy

$$V_1 = \frac{1}{2} e_1^2 \tag{1.115}$$

The derivative of the function is expressed as follows, based on equations (1.113) and (1.114):

$$\begin{aligned} \dot{V}_1 &= e_1 \dot{e}_1 \\ \dot{V}_1 &= e_1 \left( i_{dref} + \frac{R}{L_d} i_d - \frac{L_d}{L_q} \omega_r i_q - \frac{1}{L_d} v_d \right) \end{aligned} \tag{1.116}$$

To ensure that the derivative of the criterion is always negative, the derivative of $V_1$ must take the form $\dot{V}_1 = -k_1 e_1^2$ introduced by the backstepping method, thus leading to:

$$\dot{V}_1 = e_1 \left( k_1 e_1 + \frac{R}{L_d} i_d - \frac{L_d}{L_q} \omega_r i_q - \frac{1}{L_d} v_d \right) = -k_1 e_1^2 \tag{1.117}$$

This equation allows us to define the voltage control signal $v_d$ of the subsystem to ensure Lyapunov stability and force the current $i_d$ to follow its reference $i_{dref} = 0$. The reference voltage $v_{dref}$ is obtained as follows:

$$v_{dref} = L_d \left( k_1 e_1 + \frac{R}{L_d} i_d - \frac{L_d}{L_q} \omega_r i_q \right) \tag{1.118}$$

**The step 02:**

Given that the rotor speed is the primary modulating variable, we define its specified trajectory using a reference value and regulation error

$$\begin{aligned} e_2 &= \omega_{\text{ref}} - \omega \\ \dot{e}_2 &= \dot{\omega}_{ref} - \dot{\omega} \\ \dot{e}_2 &= \dot{\omega}_{ref} - \left( \frac{p\left(L_d - L_q\right)}{J} i_d + \frac{p\phi_v}{J} \right) i_q - \frac{f}{J}\omega - \frac{1}{J}C_r \end{aligned} \tag{1.119}$$

Our objective demands that the error e converge to zero. This is achieved by selecting b as the virtual control in equation (1.119).

The extended Lyapunov function will be defined as:

$$\begin{aligned} V_2 &= V_1 + \frac{1}{2}e_2^2 \\ V_2 &= \frac{1}{2}[e_1^2 + e_2^2] \end{aligned} \tag{1.120}$$

By choosing $\dot{V}_2$ to be negative semi-definite such that

$$\dot{V}_2 = -k_1 e_1^2 - k_2 e_2^2 \leq 0 \tag{1.121}$$

We obtain

$$k_2 e_2 + \dot{\omega}_{ref} - \left( \frac{p\left(L_d - L_q\right)}{J} i_d + \frac{p\phi_v}{J} \right) i_q + \frac{f}{J}\omega + \frac{1}{J}T_L = 0 \tag{1.122}$$

Considering that the time derivative of the desired current $i_{dref} = 0$ , which leads to defining the required control action for the desired current derivative $i_{dref}$ in order to determine the desired voltage $v_{dref}$

$$i_{\text{qref}} = \left( k_2 e_2 + \frac{f}{J}\omega + \frac{1}{J}T_L + \dot{\omega}_{\text{ref}} \right) \left( \frac{J}{p\phi_v} \right) \tag{1.123}$$

**The step 03:**

This step allows determining the reference voltage for the overall system (1.112), with the new control objective being the current, considered as a virtual control input at this stage. A new regulation error is defined as

$$e_3 = i_{qref} - i_q \tag{1.124}$$

Thus, the dynamic equations for the error, based on the system (1.112), are

$$\begin{aligned} \dot{e}_3 &= i_{qref} - i_q \\ \dot{e}_3 &= i_{qref} + \frac{R}{L_q} i_q - \frac{L_d}{L_q}\omega_r i_d - \frac{\phi_f}{L_q}\omega_r - \frac{1}{L_q}v_q \end{aligned} \tag{1.125}$$

Since the objective requires this error to also converge to zero, and also requires that the current be regulated and limited, we use the following extension of the Lyapunov function

$$V_3 = V_1 + V_2 + \frac{1}{2}e_3^2 V_3 = \frac{1}{2}[e_1^2 + e_2^2 + e_3^2] \tag{1.126}$$

The derivative of the function is written as follows

$$\begin{aligned} \dot{V}_3 &= \dot{V}_1 + \dot{V}_2 + e_3\dot{e}_3 \\ \dot{V}_3 &= \dot{V}_1 + \dot{V}_2 + e_3\left[\dot{i}_{qref} + \frac{R}{L_q}i_q - \frac{L_d}{L_q}\omega_r i_d - \frac{\phi_f}{L_q}\omega_r - \frac{1}{L_d}v_q\right] \end{aligned} \tag{1.127}$$

By choosing the derivative of $\dot{V}_3$ to be negative semi-definite such that

$$\dot{V}_3 = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 \leq 0 \tag{1.128}$$

We obtain

$$\dot{V}_3 = -k_1e_1^2 - k_2e_2^2 + e_3\dot{e}_3 = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 e_3\dot{e}_3 = -k_3e_3e_3\dot{e}_3 = -k_3e_3 \tag{1.129}$$

and

$$\left[\dot{i}_{qref} + \frac{R}{L_q}i_q - \frac{L_q}{L_d}\omega_r i_d - \frac{\phi_f}{L_q}\omega_r - \frac{1}{L_d}v_q\right] = -k_3e_3 \tag{1.130}$$

From this, we deduce the final control law $v_{dref}$ :

The currents $i_{sd}$ and $i_{sq}$ are represented in Figures 1.17 and 1.18, respectively.The speed profile 1.16 perfectly follows its reference speed of 100 rad/s, which is reached very quickly with an acceptable response time. After applying a load torque of 5 N.m at t=0.25 s, a slight overshoot is observed, and then the speed returns to its reference value. A similar observation is made during the reversal of the reference speed, where it is noted that the motor speed follows the reference speed with the presence of oscillations during the transient phase.

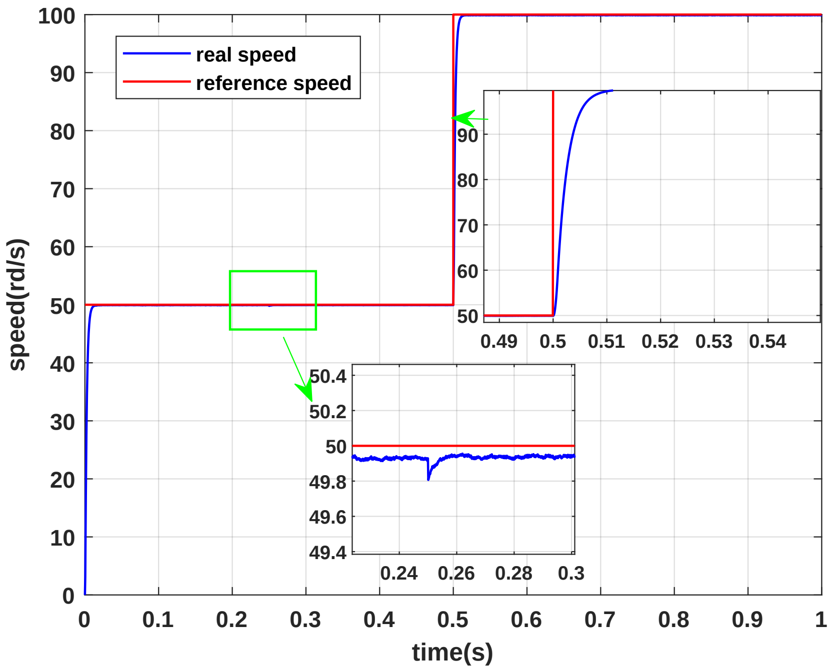



Fig 1.16: rotor speed $w_r$

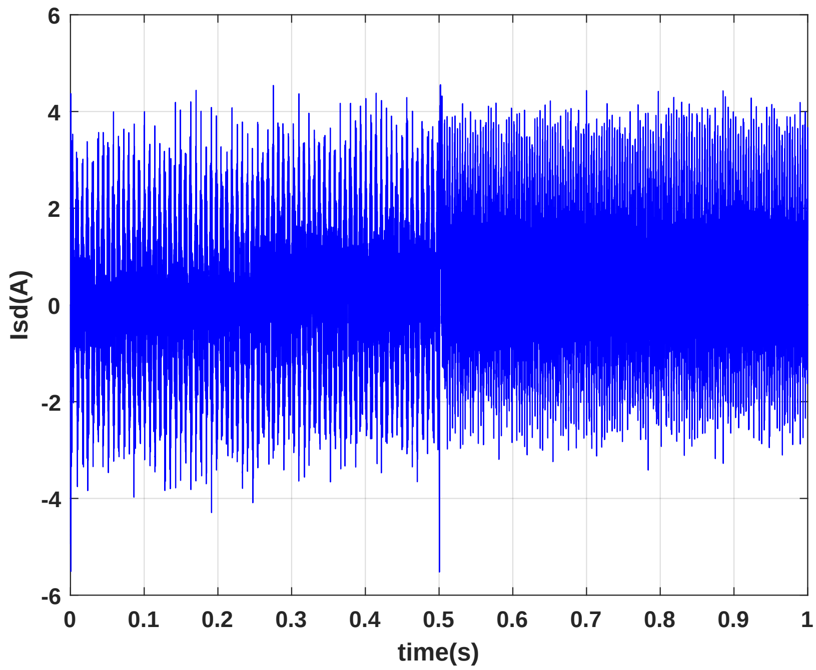



Fig 1.17: the current $isd$

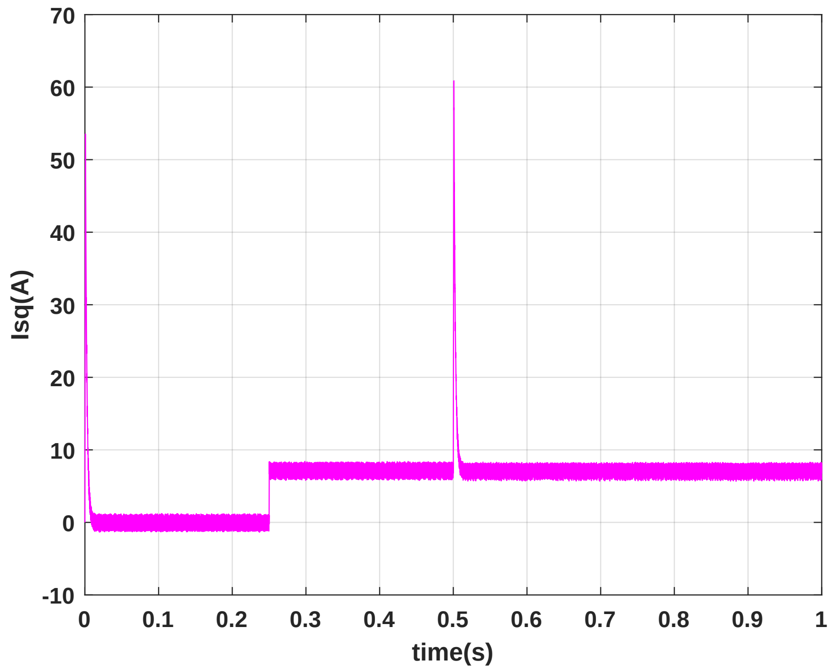



Fig 1.18: the current $isq$

## 1.5 Nonlinear Adaptive Control

### 1.5.1 Matching Conditions:

Consider the following ideal system:

$$\dot{x} = f(x) + \sum_{i=1}^{p} g_i(x)u_i \tag{1.131}$$

Where $f$, $g_i$ for $i = 1, 2, \ldots, p$ are smooth functions and $f(0) = 0$. This same system, subject to uncertainties in the parameters, is written as:

$$\dot{x} = f(\delta, x) + \sum_{i=1}^{p} g_i(\delta, x)u_i \tag{1.132}$$

Where $f(\delta, x)$, $g_i(\delta, x)$ for $i = 1, 2, \ldots, p$ are also smooth functions. Assuming the uncertain system has the same dimension as the ideal system and shares the same inputs (controls), we then have:

$$\Delta f = f(\delta, x) - f(x), \quad \Delta g_i = g_i(\delta, x) - g_i(x) \quad \text{for } i = 1, 2, \ldots, p \tag{1.133}$$

If the system (1.131) is linearizable, then there must exist a transformation that converts it into a controllable canonical form, thus facilitating pole placement. For the uncertain system to be linearized while remaining in a controllable canonical form, it is necessary for the error between the uncertain model and the ideal model to satisfy the triangularity condition.

### 1.5.2 Theorem (Triangularity Condition):

Consider the uncertain system (3.b) where the nominal model (3.a) is linearizable by state feedback with Kronecker indices $k_1, k_2, \ldots, k_p$. The uncertain system is also linearizable if the $\Delta f$, $\Delta g_1$, $\Delta g_2$, ...

### 1.5.3 Steps in the design of an adaptive controller:

The design of the adaptive controller takes place in two phases. In the first phase, we develop a non-adaptive controller using the nominal model ($\delta = 0$). In the second phase, we aim to estimate the vector of uncertain parameters by adopting an appropriate adaptation law [Kad00].

- **First step:** Consider a multivariable dynamic system described by:

$$\dot{x} = f(x, \delta) + \sum_{i=1}^{p} g_i(x, \delta) u_i \tag{1.134}$$

Where:

$$f(x, \delta) = f_0(x) + \Delta f(x, \delta), \quad g_i(x, \delta) = g_i(x) + \Delta g_i(x, \delta) \tag{1.135}$$

where $\Delta f(x, \delta)$ and $\Delta g_i(x, \delta)$ represent the parts produced by the uncertain parameters. $f_0(x)$, $g_i(x)$ are the functions of the nominal model:

$$\dot{x} = f_0(x) + \sum_{i=1}^{p} g_i(x) u_i \tag{1.136}$$

We start by considering the nominal case represented by the model (1.136). The design procedures for the corresponding controller are the same as those given in the previous chapter. The model (1.136) is linearizable if there exists a region $U$ around 0 such that for all $x \in U$ and for all $\delta \in B_\delta$, there exists a change of variables:

$$x_i = L_f^{i-1} \phi_j(x, \delta) \quad \text{for } j = 1, \ldots, p, \quad i = 1, \ldots, r_i \tag{1.137}$$

where $r_i$ is the relative degree corresponding to output $i$, and a linearizing law $u = D(x)^{-1}(-\zeta(x) + v)$ transforms the system (1.136) into canonical form. Consider the following dynamic expression

$$z = \begin{bmatrix} A_{r_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_{r_p} \end{bmatrix} z + \begin{bmatrix} B_{r_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_{r_p} \end{bmatrix} v \tag{1.138}$$

where each $A_{r_i}$ is a matrix of the form

$$A_{r_i} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{r_i \times r_i} \tag{1.139}$$

and each $B_{r_i}$is a column vector:

$$B_{r_i} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{r_i} \tag{1.140}$$

And the function $\zeta(x)$ is given by:

$$\zeta(x) = \begin{bmatrix} L_f^{r_i} h_1(x) \\ \vdots \\ L_f^{r_p} h_p(x) \end{bmatrix} \tag{1.141}$$

The matrix $D(x)$, defining the coupling between the inputs and outputs, is

$$D(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & L_{g_2} L_f^{r_1-1} h_1(x) & \dots & L_{g_p} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & L_{g_2} L_f^{r_2-1} h_2(x) & \dots & L_{g_p} L_f^{r_2-1} h_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_p-1} h_p(x) & L_{g_2} L_f^{r_p-1} h_p(x) & \dots & L_{g_p} L_f^{r_p-1} h_p(x) \end{bmatrix} \tag{1.142}$$

The matrix $D(x)$ is the decoupling matrix, which must be non-singular, and $r_i$ indicates all the associated relative degrees. Subsequently, we assign the new input $v$ a dynamics determined by the selected control strategy.

- **Second step**:

In the second step, it is necessary to determine how to apply the controller designed for the nominal case, taking into account the constraints associated with uncertain parameters. It is also crucial to assess how far this new controller can maintain its performance. We consider the nonlinear system (1.134) with $L_g h(x) \neq 0$ for any vector $x$, such that

$$f(x) = \sum_{i=1}^{n_1} \theta_i^{(1)}(t) f_i(x) \tag{1.143}$$

$$g(x) = \sum_{j=1}^{n_2} \theta_j^{(2)}(t) g_j(x) \tag{1.144}$$

where $\theta_i^{(1)}$ for $i = 1, \ldots, n_1$ and $\theta_j^{(2)}$ for $j = 1, \ldots, n_2$ are the unknown parameters, and $f_i(x)$, $g_j(x)$ are known functions over time. Our estimate for $f$ and $g$ is:

$$\tilde{f}(x) = \sum_{i=1}^{n_1} \tilde{\theta}_i^{(1)}(t) f_i(x) \tag{1.145}$$

$$\tilde{g}(x) = \sum_{j=1}^{n_2} \tilde{\theta}_j^{(2)}(t) g_j(x) \tag{1.146}$$

where $\tilde{\theta}_i^{(1)}(t)$ and $\tilde{\theta}_j^{(2)}(t)$ represent the estimates of the parameters $\theta_i^{(1)}(t)$ and $\theta_j^{(2)}(t)$, respectively, consequently, the linearization law (1.142) is replaced by

$$u = \tilde{D}(x)^{-1}(-\tilde{\zeta}(x) + v) \tag{1.147}$$

Where

$$\tilde{\zeta}(x) = \begin{bmatrix} (L_f^{r_1} h_1)\text{est}(x) \\ \vdots \\ (L_f^{r_p} h_p)\text{est}(x) \end{bmatrix} \tag{1.148}$$

$$\tilde{D}(x) = \begin{bmatrix} (L_{g_1} L_f^{r_1-1} h_1)\text{est}(x) & \cdots & (L_{g_p} L_f^{r_1-1} h_1)\text{est}(x) \\ \vdots & \ddots & \vdots \\ (L_{g_1} L_f^{r_p-1} h_p)\text{est}(x) & \cdots & (L_{g_p} L_f^{r_p-1} h_p)\text{est}(x) \end{bmatrix} \tag{1.149}$$

And

$$(L_f^{r_k} h_k)\text{est} = \sum_{i=1}^{n_1} \tilde{\theta}_i^{(1)} L_f^{r_k} h_k \quad k = 1, \ldots, p \tag{1.150}$$

$$(L_{g_k} L_f^{r_i-1})\text{est} = \sum_{j=1}^{n_2} \tilde{\theta}_j^{(2)} (L_{g_k j} L_f^{r_i-1} h_i) \quad i = 1, \ldots, p, \quad k = 1, \ldots, p \tag{1.151}$$

If we define $\theta_n \in \mathbb{R}^{n_1+n_2}$ as the nominal vector of the parameters $(\theta(1), \theta(2))$, $\theta(t) \in \mathbb{R}^{n_1+n_2}$, and $\psi$ as the error vector between the nominal value and the estimated value such that

$$\psi = [\psi(1)\ \psi(2)] = [\tilde{\theta} - \theta_n] = [\tilde{\theta}_i^{(1)} - \theta_{in}^{(1)}\ \tilde{\theta}_j^{(2)} - \theta_{jn}^{(2)}] \tag{1.152}$$

If we substitute equation (1.147) into (1.148) after a lengthy calculation, we obtain

$$y = v + \psi(1)^T w(1) + \psi(2)^T w(2) \tag{1.153}$$

with

$$w(1) = -\begin{bmatrix} L_{f1}^{r_1} h_1 \\ \vdots \\ L_{f_{n1}}^{r_1} h_1 \\ \vdots \\ L_{f1}^{r_p} h_p \\ \vdots \\ L_{f_{n1}}^{r_p} h_p \end{bmatrix} \tag{1.154}$$

and

$$w(2) = \begin{bmatrix} L_{g_{11}} L_f^{r_1-1} h_1 & \cdots & L_{g_{p1}} L_f^{r_1-1} h_1 \\ \vdots & \ddots & \vdots \\ L_{g_{1n2}} L_f^{r_1-1} h_1 & \cdots & L_{g_{pn2}} L_f^{r_1-1} h_1 \\ \vdots & \ddots & \vdots \\ L_{g_{11}} L_f^{r_p-1} h_p & \cdots & L_{g_{p1}} L_f^{r_p-1} h_p \\ \vdots & \ddots & \vdots \\ L_{g_{1n2}} L_f^{r_p-1} h_p & \cdots & L_{g_{pn2}} L_f^{r_p-1} h_p \end{bmatrix} \tag{1.155}$$

If $\tilde{\zeta}(x)$ defined by (1.148) is different from zero, we can define the parameter adaptation law using the gradient method and the control $u$ as follows:

$$u = \tilde{D}(x)\{-\tilde{\zeta}(x) + v\} \tag{1.156}$$

$$\psi = -\gamma W^T e \tag{1.157}$$

The adaptive controller ensures that the error between the outputs and their references tends toward zero, but it does not do so for the estimation error of the uncertain parameters [Yac04], [Kad00].

### 1.5.4 Nonlinear Adaptive Control for Three-Phase PWM AC-DC Converter:

In this section, we design a nonlinear adaptive controller that ensures the regulation of the voltage $\nu_{dc}$ across the capacitor terminals with respect to the variation in the load $R_{ch}$. To do this, we begin by designing a controller based on the input-output linearization technique applied to the nominal model, and then we calculate the adaptation law that will allow us to estimate the load resistance $R_{ch}$. Let us recall that the dynamic equations of the three-phase PWM AC-DC converter are:

$$\begin{cases} \frac{di_d}{dt} = -\frac{R}{L}i_d + wi_q + \frac{1}{L}(e_d - \nu_d) \\ \frac{di_q}{dt} = -\frac{R}{L}i_q - wi_q + \frac{1}{L}(e_q - \nu_q) \\ \frac{d\nu_{dc}}{dt} = \frac{2}{3C\nu_{dc}}(e_d i_d + e_q i_q) - \frac{\nu_{dc}}{CR_{ch}} \end{cases} \tag{1.158}$$

The system (1.158) is written in the form of state equations as follows

$$\dot{x} = f(x) + g.u \tag{1.159}$$

With

$$\begin{cases} x = \begin{bmatrix} i_d & i_q & \nu_{dc} \end{bmatrix}^T \\ u = [(e_d - v_d) \qquad (e_q - v_q)]^T \end{cases}$$

Our system satisfies the triangularity conditions, which can be written in the following form

$$\dot{x} = f_n(x) + \delta f_\delta(x) + g(x)u \tag{1.160}$$

Where

$$f_n(x) = \begin{bmatrix} -\frac{R}{L}i_d + wi_q \\ -\frac{R}{L}i_q - wi_d \\ \frac{2}{3C\nu_{dc}}(e_d i_d + e_q i_q) - \frac{\nu_{dc}}{CR_{chnom}} \end{bmatrix} \tag{1.161}$$

$$f_\delta(x) = \begin{bmatrix} 0 \\ 0 \\ -\frac{v_{dc}}{c} \end{bmatrix} \tag{1.162}$$

$$g(x) = \begin{bmatrix} \frac{1}{L}(e_d - v_d) \\ \frac{1}{L}(e_q - v_q) \\ 0 \end{bmatrix} \tag{1.163}$$

With $\delta$ representing the error between the inverse of the actual value $R_{ch}$ and the inverse of its nominal value such that:

$$\delta = \frac{1}{R_{ch}} - \frac{1}{R_{chnom}} \tag{1.164}$$

**a- The non-adaptive version:**

As we saw in the previous chapter, we can determine the linearizing control law without adaptation (where $\delta = 0$ is the nominal case). It is worth mentioning that our objective is to force the voltage $\nu_{dc}$ to follow its trajectory and to regulate $i_d$ in order to achieve a unit power factor with minimized harmonics in the lines. By applying the input-output linearization theory, we obtain:

$$\frac{d}{dt}[y_1 \dot{y}_2]^T = \zeta(x) + D(x).u \tag{1.165}$$

With

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} i_d \\ \\ v_{dc} \end{bmatrix} \tag{1.166}$$

And

$$\zeta(x) = \begin{bmatrix} f_1 \\ \frac{2}{3Cv_{dc}}(e_d f_1 + e_q f_2) - \left\{ \frac{2}{3Cv_{dc}^2}(e_d i_d + e_q i_q) + \frac{1}{CR_{CD}} \right\} f3 \end{bmatrix} \tag{1.167}$$

$$D(x) = \begin{bmatrix} \frac{1}{L} & 0 \\ \frac{2e_d}{3LCv_{dc}} & \frac{2e_q}{3LCv_{dc}} \end{bmatrix} \tag{1.168}$$

Where

$$\begin{cases} f_1 = -\frac{R}{L} i_d + w i_q \\ f_2 = -\frac{R}{L} i_q - w i_d \\ f_3 = \frac{2}{3C\nu_{dc}}(e_d i_d + e_q i_q) - \frac{\nu_{dc}}{CR_{ch}} \end{cases}$$

The linearizing control law is therefore

$$\begin{bmatrix} u_d \\ u_q \end{bmatrix} = D^{-1}\left[-\zeta + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right] \tag{1.169}$$

Where $(v_1, v_2)$ are the new control inputs designed to ensure the tracking of the voltage $\nu_{dc}$ and the regulation of the current $i_d$, such that:

$$\begin{cases} \nu_1 = k_1(i_{dref} - i_d) \\ \\ \nu_2 = k_2(\nu_{dcref} - \nu_{dc}) - k_3\frac{d(\nu_{dc})}{dt} \end{cases} \tag{1.170}$$

**b- The adaptive version:**

In this section, we will design a nonlinear adaptive controller to control the outputs $y$, taking into account the variation in the load $R_{ch}$, such that:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} i_d \\ \\ v_{dc} \end{bmatrix}$$

The system (1.158) can be written in the following form

$$\dot{x} = f(x) + g(x).u \tag{1.171}$$

such that

$$f(x) = \textstyle\sum_{i=1}^{n1} \theta_i^{(1)} f_i(x), g(x) = \sum_{j=1}^{n2} \theta_j^{(2)} g_j(x)$$

Where

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L}x_1 + wx_2 \\ -\frac{R}{L}x_2 - wx_1 \\ \frac{2}{3Cx_3}(e_d x_1 + e_q x_2) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{x_3}{C} \end{bmatrix}.\theta \tag{1.172}$$

$$g = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{L} \\ 0 & 0 \\ & \end{bmatrix} \tag{1.173}$$

And

$$\theta = \frac{1}{R_{ch}}$$

The variable $\theta$ is unknown, so it must be replaced by its estimate $\widetilde{\theta}$, while $f$ will be replaced by its estimate $\widetilde{f}$, such that:

$$\widetilde{f}(x) = \begin{bmatrix} -\frac{R}{L}x_1 + wx_2 \\ -\frac{R}{L}x_2 - wx_1 \\ \frac{2}{3Cx_3}(e_d x_1 + e_q x_2) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{x_3}{C} \end{bmatrix} .\widetilde{\theta} \tag{1.174}$$

Consequently, the control law (1.169) is replaced by:

$$\begin{bmatrix} u_d \\ u_q \end{bmatrix} = \tilde{D}^{-1} \left[ -\tilde{\zeta} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right] \tag{1.175}$$

With $\tilde{\zeta}$ representing the estimate of $\zeta$:

$$\tilde{\zeta}(x) = \begin{bmatrix} f_1 \\ \frac{2}{3C\nu_{dc}}(e_d f_1 + e_q f_2) - \left\{ \frac{2}{3C\nu_{dc}^2}(e_d i_d + e_q i_q) \right\} f_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{f_3}{C} \end{bmatrix} \tilde{\theta} \tag{1.176}$$

Now, to obtain the adaptation law of the parameter, we will define the parameter error vector $\psi$, such that:

$$\psi = \tilde{\theta} - \theta_n \tag{1.177}$$

With $\theta_n$ being the nominal value of the parameter. Using equation (1.154), we obtain:

$$\dot{y} = v + \psi W \tag{1.178}$$

With

$$W = \begin{bmatrix} 0 \\ \frac{f_3}{C} \end{bmatrix} \tag{1.179}$$

The vector $v$ is defined in the previous chapter such that:

$$\begin{cases} v_1 = k_1(i_{dref} - i_d) \\ v_2 = k_2(\nu_{dcref} - v_{dc}) - k_3\frac{d(v_{dc})}{dt} \end{cases} \tag{1.180}$$

The parameter adaptation law and the control law are obtained

$$\dot{\psi} = -\gamma W^T e \tag{1.181}$$

$$u = \tilde{D}^{-1}\left[ -\tilde{\zeta} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right] \tag{1.182}$$

$-\gamma$ :represents the adaptation gain

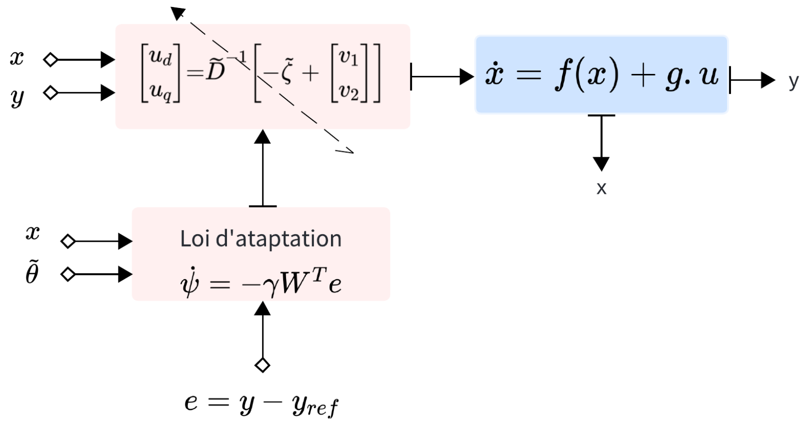



Fig 1.19: Block diagram of the nonlinear adaptive control

We implemented a nonlinear adaptive controller in the Matlab/Simulink software environment to evaluate the stability and tracking performance. Figure (1.20) illustrates the response of the voltage $\nu_{dc}$ with an ideal linearizing control, that is, without an adaptation law, applied to a three-phase AC-DC converter with PWM. Figures (1.20-1.23) show the responses of the voltage $v_{dc}$, the current, and the voltage of a phase line, as well as the estimation of the load resistance, when a nonlinear adaptive control is applied to the three-phase converter. It is observed that the load resistance changes abruptly from $25\Omega$ to $45\Omega$ after $25\Omega$ at times $t = 0.15s$ and $t = 0.45s$, respectively.

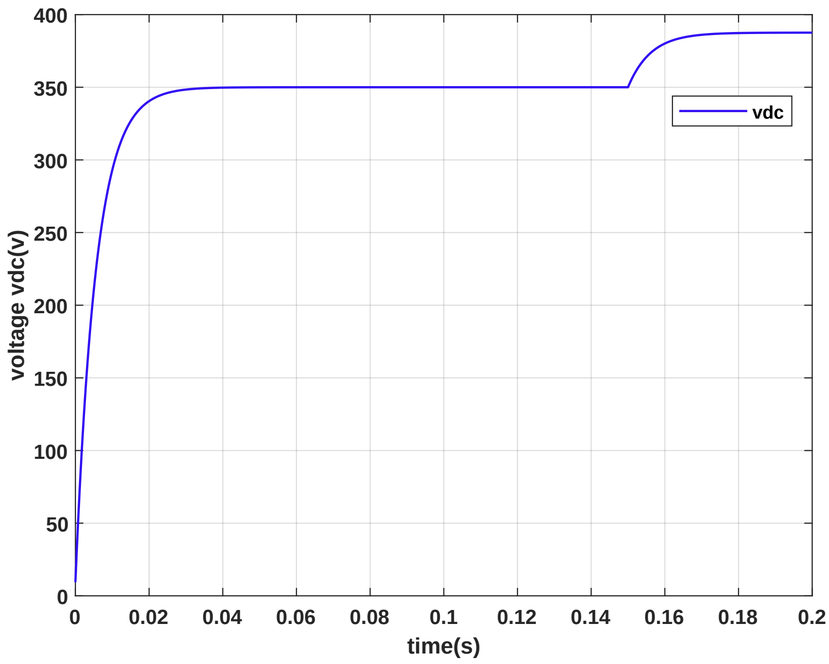



Fig 1.20: The response of the voltage without adaptation law (effect of the load)

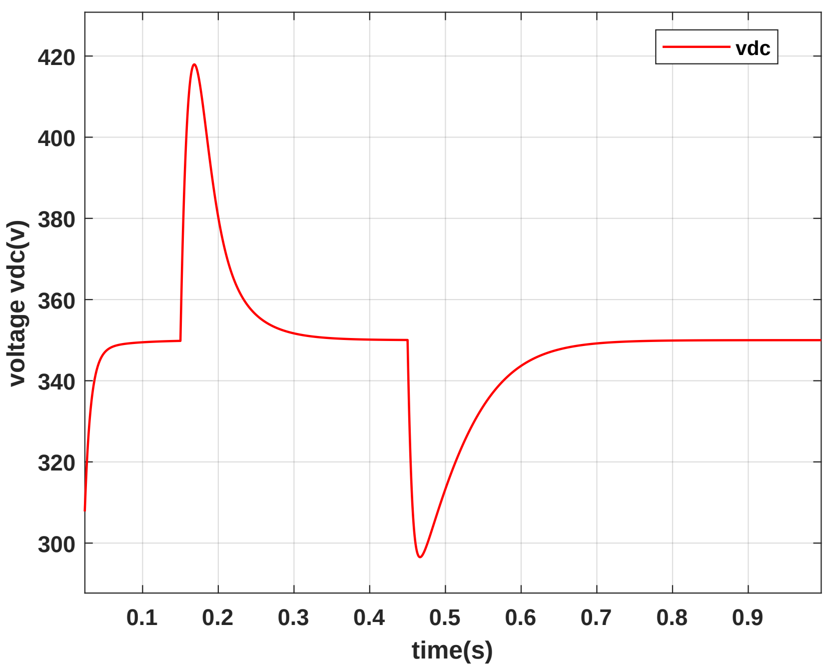



Fig 1.21: The response of the voltage with the adaptation law (Rch = 1.8 Rchnom)

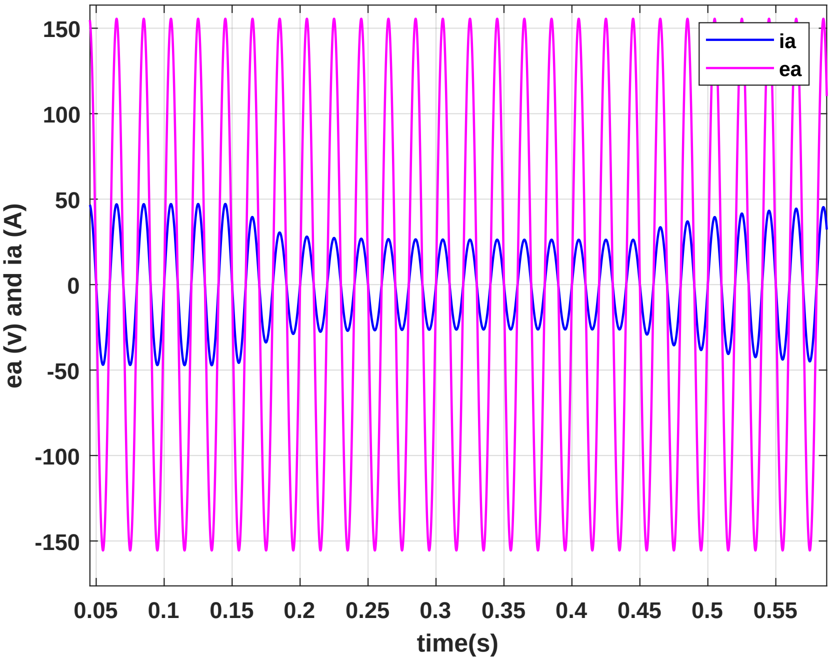



Fig 1.22: The line current and voltage.

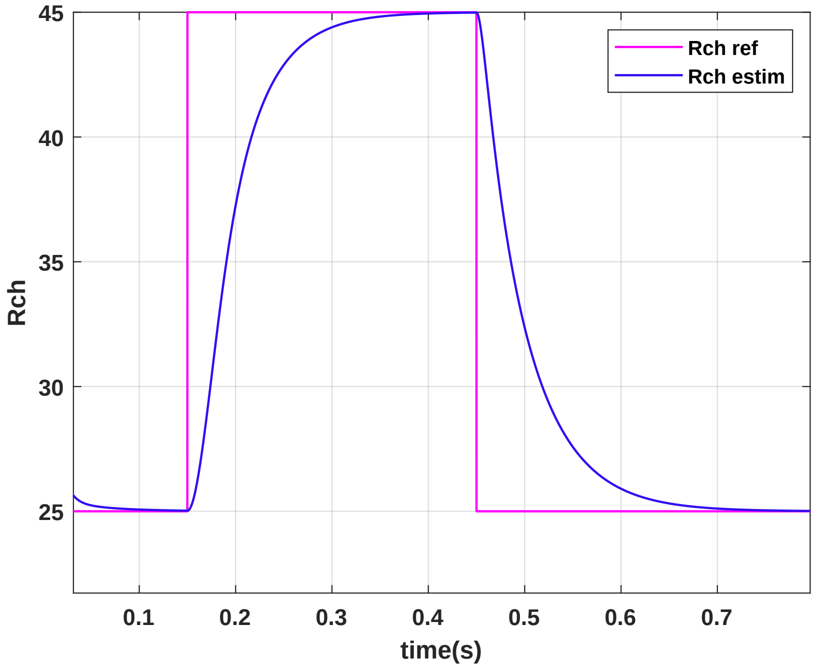



Fig 1.23: Estimation of the load resistance (Rch = 1.8Rchnom).

To evaluate the robustness of the controller against variations in the system parameters, we examined the variation of the load resistance $R_{ch} = R_{chnom}$ to $R_{ch} = 1.8R_{chnom}$ , as shown in Figures (1.21)and (1.23) . The results obtained demonstrate the effectiveness of this control technique, although a slight instantaneous variation in the voltage $\nu_{dc}$ is observed. However, the steady-state error is eliminated thanks to the effectiveness of the adaptation laws.

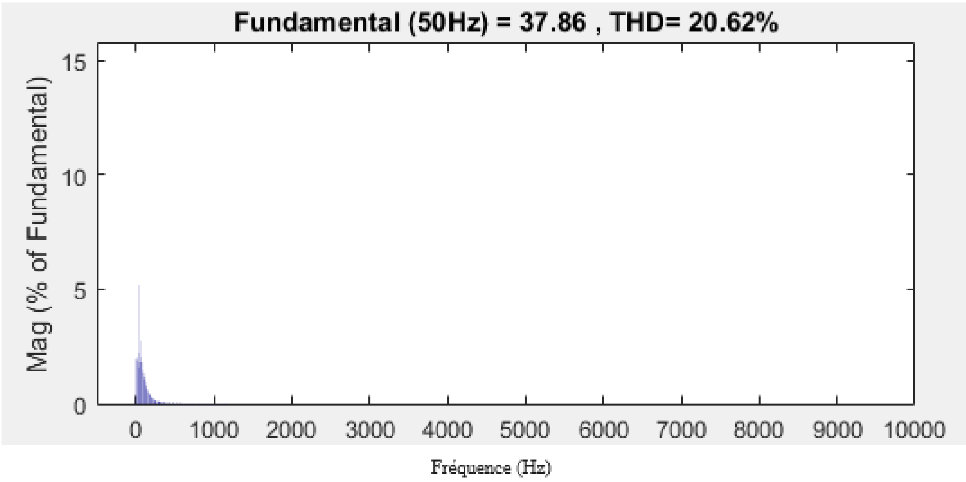



Fig 1.24: The harmonic spectrum of the line current

## 1.6 Conclusion

The chapter presents a detailed study of nonlinear system control strategies. It covers several techniques, including input-output feedback linearization and backstepping control design, applied to a three-phase AC-DC converter with pulse width modulation (PWM) and a permanent magnet synchronous motor (PMSM). The results of nonlinear control simulations show that these techniques are effective for tracking and regulation problems, achieving a unit power factor. However, their main disadvantage is their sensitivity to variations in system parameters.

In the second part of the chapter, an adaptive nonlinear control strategy is developed for the same type of converter. Simulation results indicate that these control strategies outperform other techniques. They stand out for their robustness to parameter uncertainties, achieving an almost unit power factor and significantly reducing ripple in the line current and the voltage across the capacitor

Chaptre 2

# State of the Art for Multi-models Approach

## 2.1 Introduction

This chapter centers on the multi-model approach, a powerful methodology for modeling complex nonlinear systems. This approach involves the creation of multiple models, each tailored to capture the system's behavior within specific operational zones. These individual models are then ingeniously woven together to form a comprehensive representation that excels in accurately characterizes the nonlinear system.

We explore various multi-model structures, such as coupled, decoupled, and hierarchical configurations, shedding light on their distinctive features and applications. Additionally, we navigate the methods used to obtain these multi-models, delving into techniques including identification, linearization, neural approaches, and the intriguing sector nonlinearity approach.

Furthermore, this chapter dedicates itself to the analysis of dynamic system stability, placing a particular emphasis on the rigorous examination of the stability of Takagi-Sugeno fuzzy systems. Our journey through this chapter will equip you with a profound understanding of the multi-model approach, its diverse structures, the methodologies employed in its implementation, and its role in ensuring the stability of complex dynamic systems, Particularly focusing on the stability analysis of Takagi-Sugeno fuzzy systems.

## 2.2 Modeling with Multi-Model Approaches

Multi-models excel at replicating complex dynamic behaviors in diverse systems. They are highly effective for modeling systems using empirical data, enabling precise state estimation and effective control. With intriguing mathematical features, multi-models simplify the extension of analytical techniques from linear to nonlinear systems, eliminating the need for in-depth nonlinearity analysis[Ham12] .

In the realm of literature, dynamic systems are commonly depicted using models in the following form:

$$\begin{cases} \dot{x} &= f(x(t), u(t)) \\ y &= h(x(t)) \end{cases} \tag{2.1}$$

In these equations, $f$ and $h$ represent differentiable functions that can be either linear or nonlinear.

Additionally, we have the following variables:

- $x(t) = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$, which is the state vector.
- $u(t) = [u_1, u_2, \dots, u_m]^T \in \mathbb{R}^m$, representing the input variable.
- $y \in \mathbb{R}^{ny}$ is the measurement vector.

Under certain conditions applied to the functions $f$ and $g$, the previous system can be reformulated in a quasi-linear form as shown below:

$$\begin{cases} \dot{x}(t) &= A(x(t))x(t) + B(x(t))u(t) \\ y(t) &= C(x(t))x(t) + D(x(t))u(t) \end{cases} \tag{2.2}$$

Here, $A(\cdot) \in \mathbb{R}^{nx \times nx}$, $B(\cdot) \in \mathbb{R}^{nx \times nu}$, $C(\cdot) \in \mathbb{R}^{ny \times nx}$, and $D(\cdot) \in \mathbb{R}^{ny \times nu}$ are matrices known as the dynamics, control, observation, and direct action matrices, respectively.

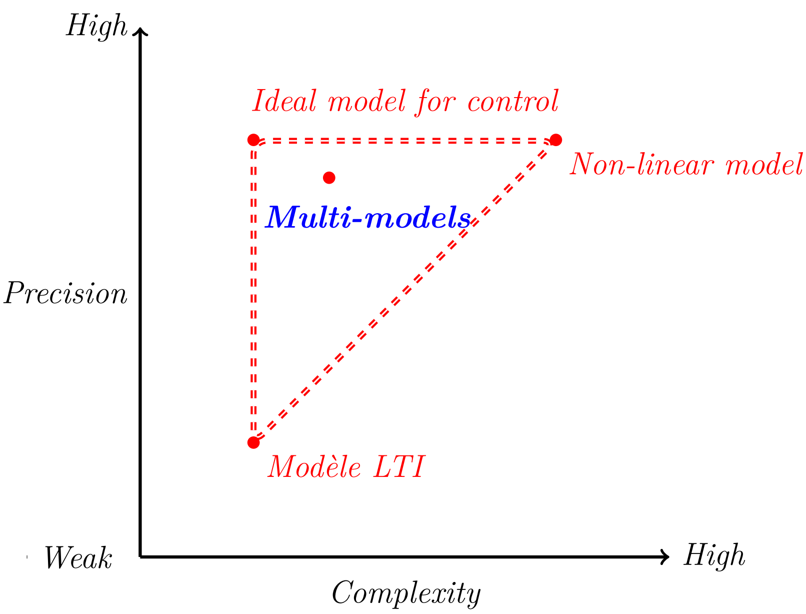



Fig 2.1: Complexity and precision of the representation of non-linear systems

The Multi-Model (MM) approach is extensively employed in modeling nonlinear systems. Various terminologies, essentially synonymous, are used to describe this model type, as illustrated in Figure (2.2) . These include the Multi-Model [MSJ], the Fuzzy

Takagi-Sugeno Model [TS85], and the Polytopic Linear Model [Ang01], among others. Central to this approach is the integration of sub-models into a comprehensive system model. This integration is facilitated through a weighting function, often referred to as the activation function. This function represents a convex combination of the subsystems, effectively contributing to the overarching system model[BOU23].

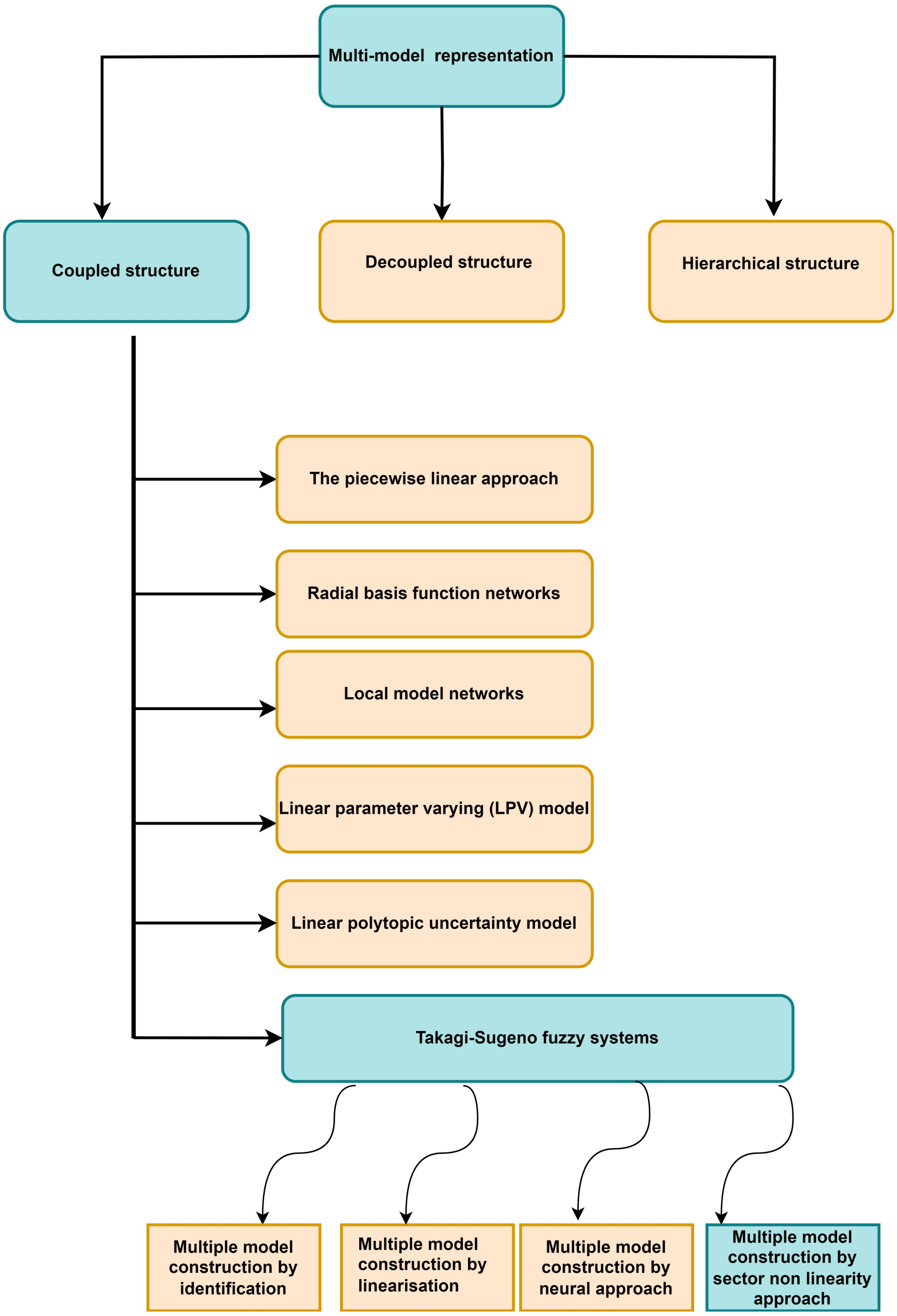



Fig 2.2: Multiple model structure

### 2.2.1 Operating space

This is a vector space in which the system's variables evolve.

### 2.2.2 Operating zone

The functioning areas represent the validity domains of local models, where each domain is defined around an operating point [Kso99]. These domains may be either disjoint in validity or overlapping, as shown in Figure (2.3)

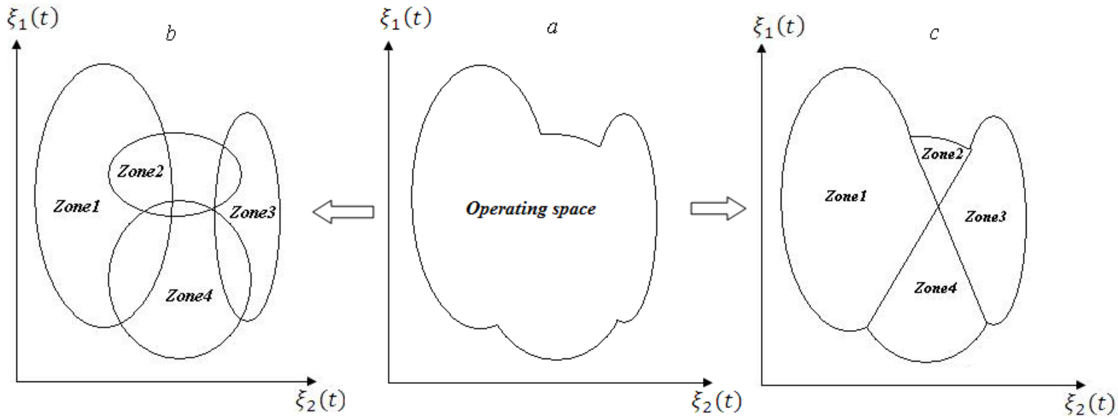



Fig 2.3: Schematic diagram of the multi-model approach
$a$)- Non-linear system , $b-c$)- Multi-models representation

In cases where the validity domain is disjoint, the activation functions can only take values of 0 or 1. At any given moment, only one model is valid, and the others are null. This type of partitioning is common in systems with multiple configurations or modes of operation, and the model obtained is called 'piecewise affine' [SSPP23, Ham12]. The other situation that can also be encountered in a multi-model description is when the validity domains overlap or have common areas. This overlap is due to the substitution of sharply defined activation functions with those having a softer slope. In this case, these functions become continuously differentiable functions, where the slope determines the speed of transitioning from one model to another[Ham15, Ham12]

### 2.2.3 Sub-model

It is the representation of the nonlinear system's behavior in a particular operating zone.

### 2.2.4 Premise variable

Also known as the decision variable $\xi(t)$ , it is a vector variable that is characteristic of the system and is involved in the weighting functions $\mu(t)$. This variable may include one or more internal or external variables of the system. These variables can either be measurable, such as measurable state variables or system input signals, or they can be unmeasurable

### 2.2.5 Activation function

This function determines the activation degree of the associated local sub-model. Depending on the area where the system operates, this function indicates the varying degree of contribution of the corresponding local model to the overall model. It facilitates a gradual transition from this model to the neighboring local models. These functions are dependent on the decision variables.

$$h_i(\xi(t)) = \frac{\mu_i(\xi(t))}{\sum_{i=1}^{r} \mu_i(\xi(t))} \tag{2.3}$$

Activation functions can be constructed either from functions with discontinuous derivatives, such as triangular or trapezoidal functions, or from functions with continuous derivatives, such as Gaussian functions. They are selected to satisfy the following convex sum properties:

$$\begin{cases} 0 \leq h_i(\xi(t)) \leq 1 \\ \sum_{i=1}^{r} h_i(\xi(t)) = 1 \end{cases}$$

Activation functions that are constructed based on an exponential law are frequently utilized in continuous cases [ACMR04]

### 2.2.6 Multi-model

The multi-model approach is based in the decomposition of a system's dynamic behavior into distinct operating zones, each of which is characterized by its corresponding subsystem. Depending on the region in which the system operates, each subsystem makes varying contributions to the approximation of the overall system behavior. Typically, within a specific operating zone, the system exhibits homogeneous dynamic behavior.Consequently, the contribution of each subsystem to the comprehensive model, which can be conceptualized as a convex combination of all subsystems, is determined by a weighting function [Nag10]

## 2.3 Multi-Model Structures

The representation of a non-linear system using a multi-model approach can take various forms. A state-based representation of these sub-models provides a concise and versatile means of highlighting them. This multi-model state representation offers the advantages of being compact and straightforward compared to presenting the system as an input/output regression equation. Furthermore, when synthesizing control laws or constructing non-linear observers, such a description of the system is often essential [Orj08].

The choice of multi-model representation for a non-linear system depends on factors such as whether segmentation occurs based on input or output variables (i.e., measurable state variables) and the nature of coupling between local models associated with different operating zones. Generally, three primary multi-model structures exist:

- coupled structure
- decoupled structure
- hierarchical structure

**1. Coupled structure** :also referred to as the Fuzzy TS fuzzy system, expresses the state vector as a weighted combination of local model states. This structure is widely adopted due to its common usage and can be readily derived without information loss using non-linear sector transformations[Ham15][Nag10].

**2. Decoupled structure**, also known as Local Multi-Models: In this structure, the system's representation assumes the presence of local models that operate independently, allowing for separate state vectors. This concept was introduced by Filev[Fil91] , who developed the idea of interpolating decoupled state sub-models. Each sub-model maintains its distinct state space, evolving independently of other sub-models [Mar16] .

$$\begin{cases} \dot{x}_i(t) = \sum_{i=1}^{r} h_i(\xi(t))(A_i x_i(t) + B_i u(t)) \\ y_i(t) = C_i x_i(t) + E_i u(t) \\ y(t) = \sum_{i=1}^{r} h_i(\xi(t)) y_i(t) \end{cases} \tag{2.4}$$

**3. Hierarchical structure:** While the multi-model approach has demonstrated success in diverse fields like control and diagnosis, it encounters limitations when applied to systems with a high number of variables or increased dimensionality. As the number of variables grows, the quantity of local models increases exponentially. To illustrate, consider a multi-model with a single output and n variables, each associated with m activation functions; such a system would involve $m^n$ local models [Akh04] .

## 2.4 Basic Concepts of T-S Fuzzy systems

Takagi-Sugeno (TS) models have been the subject of numerous studies since their introduction in 1985 by Takagi and Sugeno [TS85] . These models belong to the class of convex polytopic systems and allow for the extension of certain concepts from linear system control to the case of nonlinear affine control systems.

Historically rooted in fuzzy formalism, more recent methods for obtaining TS models, such as sector nonlinearity decomposition, enable the exact representation of a nonlinear system within a compact space of its state variables. As a result, a TS model can be equivalently expressed as a Quasi-Linear Parameter-Varying (Quasi-LPV) model, represented as a collection of linear dynamics (polytopes) interpolated by a set of nonlinear functions (satisfying convex sum properties). Numerous research efforts have been dedicated to this class of systems [Jab11] .

## 2.5 T-S Fuzzy model construction approaches

In the literature, there are three approaches for transforming an affine nonlinear control model into a T-S model. These approaches aim to represent complex nonlinear systems over a wide operating range. These different approaches are as follows:

### 2.5.1 The Approach by Identification

By representing a nonlinear system in a multi-model form, the problem of identifying nonlinear systems is reduced to identifying the subsystems defined by linear local models and activation functions. Numerical optimization methods are then employed to estimate these parameters.

For parameter estimation, several numerical optimization methods can be used, depending on the available a priori information. These methods typically involve minimizing the discrepancy function between the estimated multi-model output $\hat{y}$(t) and the measured system output $y_m$(t) [CB12],[Ham15].

### 2.5.2 The Approach by linearisation

The fundamental idea of this approach is to linearize the nonlinear system around a carefully chosen set of operating points, resulting in a defined number of Linear Time-Invariant (LTI) models. Creating a Takagi-Sugeno (T-S) representation in this context involves interconnecting these LTI models using carefully selected nonlinear membership functions, such as Gaussians, triangles, trapezoids, etc. This concept is discussed in Bouarar's work [Bou09].
Consider a nonlinear system characterized by the following equations:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = g(x(t), u(t)) \end{cases} \tag{2.5}$$

where $x(t) \in \Re^{n_x}, u(t) \in \Re^{n_u}$, and $y \in \Re^{n_y}$ denote the state, input, and output measurement vectors, respectively. The functions $f$ and $g$ are continuous and nonlinear, belonging to $\mathbb{R}^{2n}$. This nonlinear system can be represented by a multi-model, comprising several local linear or affine models. These models are derived by linearizing the nonlinear system around different operating points $(x_i, u_i) \in \mathbb{R}^n \times \mathbb{R}^m$ [Gas00], [Oud08]:

$$\begin{cases} \dot{x}_m(t) = \sum_{i=1}^{r} h_i(\xi(t))(A_i x_m(t) + B_i u(t) + D_i) \\ y_m(t) = \sum_{i=1}^{r} h_i(\xi(t))(C_i x_m(t) + E_i u(t) + N_i) \end{cases} \tag{2.6}$$

with the coefficients given by:

$$A_i = \frac{\partial f(x,u)}{\partial x}|_{(x,u)=(x_i,u_i)} \ , \ B_i = \frac{\partial f(x,u)}{\partial u}|_{(x,u)=(x_i,u_i)}$$
$$C_i = \frac{\partial h(x,u)}{\partial x}|_{(x,u)=(x_i,u_i)} \ , \ E_i = \frac{\partial h(x,u)}{\partial u}|_{(x,u)=(x_i,u_i)}$$
$$D_i = f(x_i,u_i) - A_i x - B_i u \ , \ N_i = h(x_i,u_i) - C_i x - E_i u$$

The number of local models $(r)$ is influenced by the desired accuracy of the model, the complexity of the nonlinear system, and the chosen structure of the activation function.

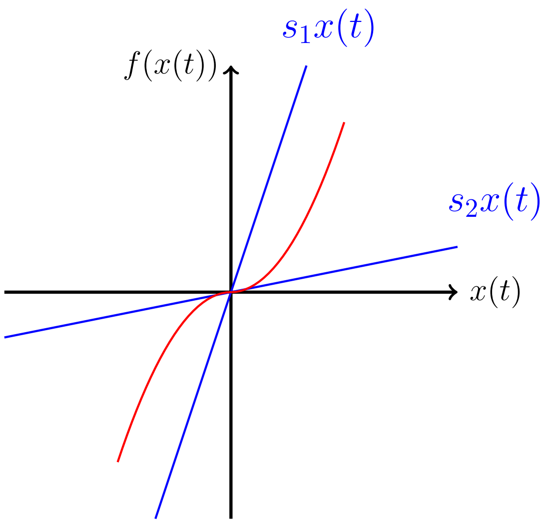
### 2.5.3 The Neural Approach

The neural approach to constructing multiple models offers a robust solution for representing complex systems, especially when an initial general model is not readily available. This methodology leverages input-output signals and neural classification techniques to partition data into distinct classes, effectively determining the required number of models. The classification results are further refined using Kohonen self-adaptive networks and the fuzzy K-means method, enhancing the granularity of data understanding.

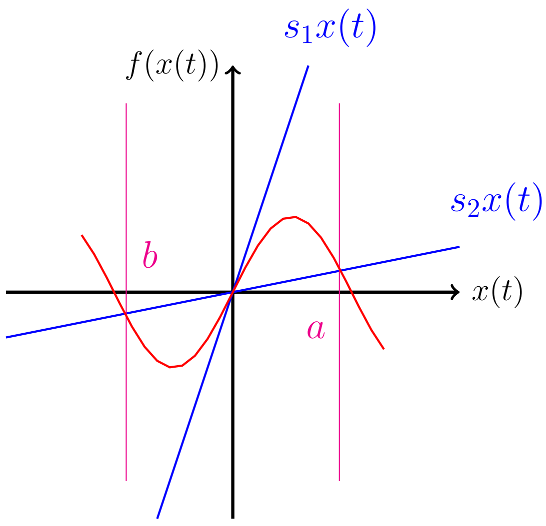
These refined classifications are then associated with models, often linear, corresponding to each dataset. Additionally, the approach calculates the validity or coefficient of each model at various operating points, resulting in a versatile and resilient multiple model representation. This process plays a pivotal role in the fields of machine learning and data analysis, enabling the effective handling of the complexity and variability found in large datasets.[EDBB10][CB12]

### 2.5.4 The Nonlinear Sector Approach.

In this subsection, we describe the three methods for obtaining a coupled multi-model structure from a non-linear model.




*Global sector non linearity.*




*Local sector non linearity.*

Fig 2.4: Non-linear sectors

We can represent the system stated in Equation (2.7) as an LPV form given by Equation (2.5):

$$\begin{cases} \dot{x}(t) = A(s(t))x(t) + B(s(t))u(t) \\ y(t) = C(s(t))x(t) + E(s(t))u(t) \end{cases} \tag{2.7}$$

Let us define $k$ as the count of nonlinear functions within the system of Equation (2.7). These functions are present in the state matrices $A(.)$, $B(.)$, $C(.)$, and $E(.)$, and they typically depend on the state $x$ and the input $u$, symbolized as $\xi_i(t)$ for $i = 1, ..., k$. If we consider a compact set $C$ for the variables $\xi(t)$, the nonlinearities are constrained as per the bounds provided in [Bez13], shown in Equation (3.22):

$$\xi_i(t) \in [\xi_{i,2}, \xi_{i,1}], \text{ for } i = 1, \dots, k. \tag{2.8}$$

The nonlinearity $\xi_i$ can be reformulated as in Equation (2.9):

$$\xi(t) = F_{i,1}(\xi_i(t))\xi_{i,1} + F_{i,2}(\xi_i(t))\xi_{i,2}, \tag{2.9}$$

where the terms are defined as in Equation (2.10):

$$\begin{cases} \xi_{i,1} = \max\{\xi_i(t)\} \\ \xi_{i,2} = \min\{\xi_i(t)\} \\ F_{i,1}(\xi_i(t)) = \frac{\xi_i(t)-\xi_{i,2}}{\xi_{i,1}-\xi_{i,2}} \\ F_{i,2}(\xi_i(t)) = \frac{\xi_{i,1}-\xi_i(t)}{\xi_{i,1}-\xi_{i,2}} \end{cases} \tag{2.10}$$

Activation functions $\mu_i(\xi(t))$ derive from $F_{i,1}(\xi(t))$ and $F_{i,2}(\xi_i(t))$, as illustrated in Equation (2.11):

$$\mu_r(\xi(t)) = \prod_{i=1}^{2^k} F_{i,\sigma_r^i}(\xi_i(t)). \tag{2.11}$$

Here, we have $r = 2^k$ sub-models. The indices $\sigma_r^i$ (with $r = 1, ..., 2^k$ and $i = 1, ..., k$) take values 1 or 2, indicating which partition of the sub-model $i$ (either $F_{i,1}$ or $F_{i,2}$) is used to define sub-model $r$. The connection between the sub-model number $i$ and the indices $\sigma_r^i$ is described by Equation (2.12):

$$i = 2^{n-1}\sigma_i^1 + 2^{n-2}\sigma_i^2 + \ldots + 2^0\sigma_i^n - (2^1 + 2^2 + \ldots + 2^{n-1}). \tag{2.12}$$

We obtain matrices $A_i, B_i$, and $C_i$ by substituting $\xi_i(t)$ with $\xi_{i,\sigma_i^r}$ in $A(\xi(t))$, $B(\xi(t))$, and $C(\xi(t))$, as shown in Equation(2.7).

This yields the T-S system in Equation (2.13):

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r=2^k} h_i(s(t))(A_i x(t) + B_i u(t)) \\ y(t) = \sum_{i=1}^{r=2^k} h_i(s(t)) C_i x(t) \end{cases} \tag{2.13}$$

This multi-model structure correlates with the number of nonlinear terms in the initial system. A potential downside of this approach is the proliferation of local models and the challenges in accessing the decision variables of the weighting functions. Nevertheless, from a structural perspective, all the sub-models within this multi-model maintain the same dimensionality, using a single state vector. Thus, the complexity of each sub-model remains constant, regardless of the system's complexity across different operational zones. However, there is a risk of over-parameterization with this multi-model, potentially leading to unnecessary complexity, as noted in [Ham12].

**Example 1.1**

Let's examine the following nonlinear system: Let's consider a system where the states $x_1(t)$ and $x_2(t)$ are each bounded within the interval $[-1, 1]$. The system's dynamics can then be expressed as:

$$\dot{x}(t) = \begin{bmatrix} -1 & x_1(t)x_2^2(t) \\ (3 + x_2(t))x_1^2(t) & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{2.14}$$

where $x(t) = [x_1(t), x_2(t)]^T$ and the elements $x_1(t)x_2^2(t)$ and $(3+x_2(t))x_1^2(t)$ introduce nonlinearity. To manage this, we define:

$$\xi_1(t) = x_1(t)x_2^2(t) \quad \text{and} \quad \xi_2(t) = (3 + x_2(t))x_1^2(t),$$

which allows us to rewrite the system as:

$$\dot{x}(t) = \begin{bmatrix} -1 & \xi_1(t) \\ \xi_2(t) & -1 \end{bmatrix} x(t).$$

Next, we determine the boundaries of $\xi_1(t)$ and $\xi_2(t)$, given the constraints on the states:

$$\xi_{1max} = 1, \quad \xi_{1min} = -1,$$
$$\xi_{2max} = 4, \quad \xi_{2min} = 0.$$

These nonlinear elements can be parameterized linearly as:

$$\xi_1(t) = M_1(\xi_1(t)) \cdot \xi_{1max} + M_2(\xi_1(t)) \cdot \xi_{1min},$$
$$\xi_2(t) = N_1(\xi_2(t)) \cdot \xi_{2max} + N_2(\xi_2(t)) \cdot \xi_{2min},$$

with the conditions that $M_1(\xi_1(t)) + M_2(\xi_1(t)) = 1$ and $N_1(\xi_2(t)) + N_2(\xi_2(t)) = 1$.

The membership functions are then defined as:

$$M_1(\xi_1(t)) = \frac{\xi_1(t) + 1}{2}, \quad M_2(\xi_1(t)) = \frac{1 - \xi_1(t)}{2},$$
$$N_1(\xi_2(t)) = \frac{\xi_2(t)}{4}, \qquad N_2(\xi_2(t)) = \frac{4 - \xi_2(t)}{4}.$$

These functions correspond to "Positive," "Negative," "Big," and "Small," respectively. Utilizing these, the original system is described by a Takagi-Sugeno fuzzy model with the following rules:

- If $\xi_1(t)$ is "Positive" and $\xi_2(t)$ is "Big," then $\dot{x}(t) = A_1 x(t)$.
- If $\xi_1(t)$ is "Positive" and $\xi_2(t)$ is "Small," then $\dot{x}(t) = A_2 x(t)$.
- If $\xi_1(t)$ is "Negative" and $\xi_2(t)$ is "Big," then $\dot{x}(t) = A_3 x(t)$.
- If $\xi_1(t)$ is "Negative" and $\xi_2(t)$ is "Small," then $\dot{x}(t) = A_4 x(t)$.

Each rule is associated with a system matrix $A_i$, and the overall dynamics are represented by the weighted sum of these individual linear systems:

$$\dot{x}(t) = \sum_{i=1}^{4} h_i(\xi(t)) A_i x(t),$$

where $h_i(\xi(t))$ are the weighting functions derived from the membership functions. This concise representation captures the essence of the nonlinear dynamics within the specified state space region.

Let's consider a system where the states $x_1(t)$ and $x_2(t)$ are each bounded within the interval [-1 1].
Avec

$$A_1 = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} - & -1 \\ 0 & -1 \end{bmatrix}, \tag{2.15}$$

The results of the fuzzy and real model simulation for $x_1(0) = [0.5 \quad 0.5]$ and $x_2(0) = [0.1 \quad 0.1]$ with $u = 0$ are presented in Figure (2.5) . It is evident that the two models are identical.This example clearly demonstrates that the nonlinear system (2.14) can be represented exactly by a Takagi-Sugeno type fuzzy model.

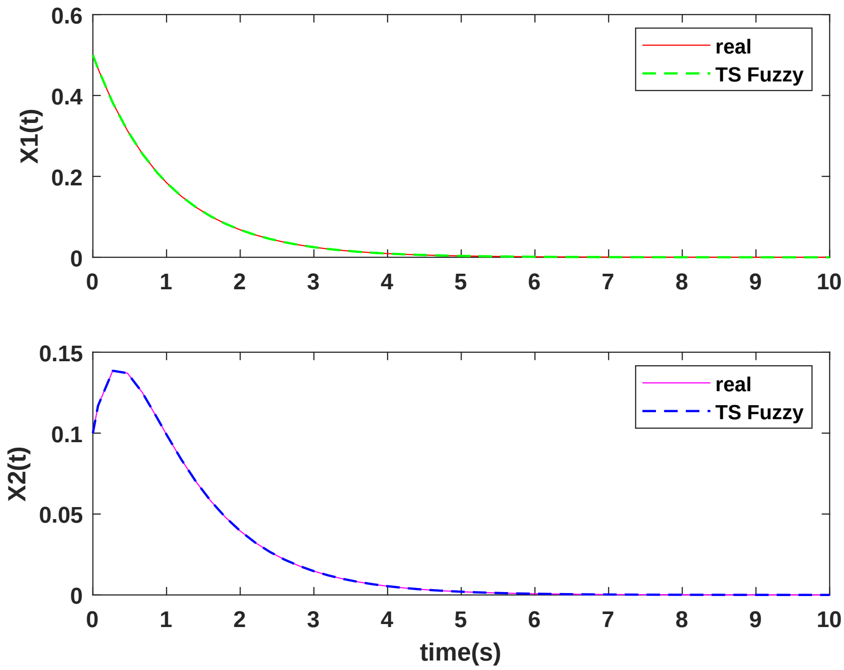



Fig 2.5: States evolution of T-S fuzzy model and real model

## 2.6 Stability analysis of nonlinear Systems

Lyapunov's method provides a lens through which to view the stability of dynamical systems by considering their energy-like properties. We define the system's energy through a positive-definite function, $V(x(t))$, as a function of the state $x$. This function, often representing a norm or metric, is crucial for deducing stability. The stability is inferred by analyzing the sign of the time derivative of $V(x(t))$ within a certain vicinity of the equilibrium point. The essence of Lyapunov's stability theorem is that for an autonomous continuous-time system $\dot{x} = f(x)$, if there exists a function that satisfies predefined conditions, then the system's equilibrium at the origin is globally asymptotically [Aou12].

**Definition 1** A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable is deemed a candidate Lyapunov function if it fulfills the following:

1. $V(x)$ is positive definite with $V(0) \neq 0$ for $x \neq 0$.

2. $\dot{V}(x) < 0$ for all $x \neq 0$.

3. $V(x) \to \infty$ as $\|x\| \to \infty$.

Lyapunov theory empowers us to determine the necessary conditions for stability. However, the level of conservatism in these conditions depends on the chosen form of $V(x)$. The selection, dictated by the system under study and the targeted stability objectives, falls into one of two categories: quadratic or non-quadratic Lyapunov functions [AGH07].

### 2.6.1 Stability in the Sense of Lyapunov

The quadratic Lyapunov function is pivotal in the domain of stability analysis, control mechanisms, and state estimation within Takagi-Sugeno (TS) fuzzy systems. Its prevalence in literature underscores its significance. Works ranging from the early studies by Takagi and Sugeno in 1992 to more recent research highlight the fundamental role that these quadratic functions play. They serve as a key tool in managing the intricacies of TS fuzzy models, enabling the construction of robust control strategies and reliable estimations of system states. The depth of research, as cited by numerous scholars [[TS92],[Kru06],[WTG96], reflects the ongoing advancement and refinement of these methods in the field.

#### 2.6.1.1 Lyapunov first method

The Indirect Method of Lyapunov, often termed the First Method, is instrumental in assessing the stability of equilibrium points within dynamical systems. This method hinges on the linearization of the system's equations around an equilibrium point, denoted as $\bar{x}$. Specifically, the Jacobian matrix $A$, derived as

$$A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}},$$

captures the system's local behavior. The eigenvalues $\lambda_i(A)$ of this matrix are then scrutinized. An equilibrium point $\bar{x}$ is deemed exponentially stable if all the eigenvalues exhibit negative real parts, formally stated as $Re(\lambda_i(A)) < 0$. Conversely, the presence of any eigenvalue with a positive real part, $Re(\lambda_i(A)) > 0$, signifies that the equilibrium is inherently unstable.

This approach, while straightforward in application, offers only a snapshot of the system's stability. It fails to provide insights into the extent of the system's basins of attraction or its behavior in the nonlinear regime beyond the vicinity of the equilibrium. As such, while the Indirect Method is a useful preliminary tool, it is often supplemented by other analyses for a comprehensive stability evaluation [Zer11].

#### 2.6.1.2 Lyapunov second method

Lyapunov's second method, known as the Direct Method, is pivotal in the realm of dynamical systems for assessing the stability of equilibrium points. This approach employs a specially crafted positive definite function, typically designated as $V(x(t))$, referred to as the Lyapunov function. For an equilibrium point to be considered stable, this function must exhibit a decrease along the trajectories of the system and remain positive definite within the system's basin of attraction. Despite being more universally applicable than the indirect method, finding a suitable Lyapunov function is often a more intricate task [Zer11, Cha02].

**Criteria for Local and Asymptotic Stability:** To ascertain the local stability of a system described by equation (2.5), one must identify a continuous and differentiable Lyapunov function $V(x(t))$, and define a neighborhood $V_0$. Within this vicinity, the following conditions should hold:

- For every state $x$ within $V_0$, the condition $V(x(t)) > 0$ must be satisfied, indicating positive definiteness.
- The time derivative of $V$, given by $\dot{V}(x(t)) = \dfrac{dV(x(t))}{dt} = \dfrac{\partial V(x(t))}{\partial x}\dot{x}(t)$, should not exceed zero, ensuring that $V$ is non-increasing along trajectories.

If the Lyapunov function not only remains positive but also strictly decreases over time ($\dot{V}(x(t)) < 0$), then it is considered a strict Lyapunov function, confirming the system's asymptotic stability at the origin.

**Defining Exponential Stability:** The origin is regarded as an exponentially stable equilibrium point for the system outlined in equation (2.5) under the presence of a Lyapunov function $V(x(t))$ that satisfies certain growth conditions. Precisely, for constants $\alpha, \beta, \gamma > 0; p \geq 0$, and within a neighborhood $V_0$, the function $V$ should comply with the following:

- It must be bound between two radial unbounded functions of $x$: for all $x$, $\alpha \parallel x \parallel^p \leq V(x(t)) < \beta \parallel x \parallel^p$.

- The time derivative of $V$ should satisfy $\dot{V}(x(t)) < -\gamma V(x(t))$, implying an exponential decay.

### 2.6.2 Choice of the Lyapunov Function

The cornerstone of the Lyapunov stability method is the appropriate selection of the Lyapunov function. The choice is guided by the characteristics of the system under study, such as whether it is a linear system, a piecewise continuous system, a system with delays, or an uncertain linear system. In this thesis, our focus is on the exploration of stability through both quadratic and non-quadratic Lyapunov functions.

#### 2.6.2.1 Quadratic Lyapunov Functions

Quadratic Lyapunov functions represent the quintessential choice for assessing the stability of linear systems and are defined as:

$$V(x(t)) = x(t)^T P x(t) \quad \text{with} \quad P = P^T > 0 \tag{2.16}$$

In this context, $P$ is a positive definite matrix, and identifying such a matrix is tantamount to discovering a quadratic Lyapunov function. When employing a multi-model approach, the convex nature of the problem formulation assists in deriving a suitable function, if one exists. Nevertheless, this method's primary limitation is that it often yields overly conservative stability conditions, as documented in the literature [TW04, Cha02].

#### 2.6.2.2 Polyquadratic Lyapunov approach

Polyquadratic Lyapunov functions offer a versatile approach to stability analysis by interpolating between multiple quadratic functions. These functions are particularly effective for systems that are not adequately described by a single quadratic Lyapunov function. Among various forms, the Fuzzy Lyapunov Function (FLF) stands out for its widespread use and robustness, as documented in the literature [MPA09, BGM13, HHH24].

The standard representation of an FLF is given by:

$$V(x(t)) = x^T(t) P(\xi(t)) x(t) \tag{2.17}$$

where the matrix $P(\xi(t))$ is constructed as a convex combination of several matrices:

$$P(\xi(t)) = \sum_{i=1}^{r} h_i(\xi(t)) P_i \tag{2.18}$$

Here, $P_i \in \mathbb{R}^{n\times n}$ are symmetric and positive-definite matrices, and $h_i(\xi(t))$ represent the membership functions (MFs) of the Takagi-Sugeno (TS) fuzzy system. These membership functions are designed to fulfill the convex sum property, ensuring that the sum of all $h_i(\xi(t))$ equals one at any given time $t$, which guarantees a consistent fusion of the individual quadratic matrices.

Importantly, if each $P_i$ is replaced by a singular matrix $P$, the function reverts to the classical quadratic Lyapunov function. This adaptability allows for a seamless transition from general to specific stability assessments, enabling a more comprehensive exploration of system dynamics that might be missed by a purely quadratic analysis.

#### 2.6.2.3 Parametric Affine Lyapunov Functions

For systems characterized by time-varying parameters, parametric affine Lyapunov functions are often employed:

$$V(x(t)) = x(t)^T P(\theta) x(t) \tag{2.19}$$

with the matrix $P(\theta)$ defined as:

$$P(\theta) = P_0 + \sum_{i=1}^{r} \theta_i P_i > 0$$

These systems can typically be described by:

$$\dot{x}(t) = A(\theta) x(t) \tag{2.20}$$

where $A(\theta)$ is a matrix composed similarly to $P(\theta)$, accommodating uncertain but bounded parameter variations. This class of Lyapunov functions, less conservative than their quadratic counterparts, takes into account parameter variations and provides results whose quality is contingent upon the compatibility of the chosen Lyapunov function with the nature of the system [Cha02].

#### 2.6.2.4 Piecewise Continuous Lyapunov Functions

These functions can be either piecewise continuous linear or quadratic and are suitable for analyzing the stability of fuzzy control systems. They have been successfully applied to various types of fuzzy systems, as shown in:

$$V(x(t)) = \max\left(V_1(x(t)), \ldots, V_i(x(t)), \ldots, V_n(x(t))\right) \tag{2.21}$$

with each $V_i(x(t))$ being a positive definite quadratic form:

$$V_i(x(t)) = x(t)^T P_i x(t), \quad P_i > 0 \tag{2.22}$$

This approach has demonstrated advantages over the quadratic functions in terms of conservativeness when applied to linear time-variant systems [BEGFB94, Cha02].

#### 2.6.2.5 Line Integral Lyapunov Functions

For the analysis and design of fuzzy control systems, the line integral Lyapunov function stands out due to its capacity to form stability conditions without the need for time derivative computations of the membership functions. Proposed as an efficient tool, this type of function is given by:

$$V(x(t)) = 2\int_{\Gamma(0,x)} g(\bar{\omega})d\bar{\omega} \tag{2.23}$$

where $g(x)$ represents a force field and the integral computes the work done along a path $\Gamma(0, x)$ from the origin to $x$. The conditions for this function to be a valid Lyapunov candidatesuch as being smooth, positive definite, and radially unbounded?are dependent on its path independence:

For equation(2.23) to be considered as a suitable Lyapunov candidate function (as defined in Definition 1), it is essential that the associated curvilinear integral demonstrates path independence, a prerequisite highlighted in [GGB$^+$09]. Let's denote $\mathfrak{J}(x)$ by the vector $[\mathfrak{J}_1(x), \ldots, \mathfrak{J}_n(x)]^T$. To establish the path independence of equation (2.23) , a particular (2.24) provides both necessary and sufficient conditions. This eduation is crucial in ensuring the integral maintains a consistent value across different paths in the state space, a fundamental criterion for a function's qualification as a valid Lyapunov candidate.

$$\frac{\partial g_i(x)}{\partial x_j} = \frac{\partial g_j(x)}{\partial x_i} \quad \text{for all } i, j \tag{2.24}$$

Relying on the lemma, Rhee and Won[RW06] demonstrated that the Lyapunov Integral-Like Function (LILF), as given in equation (2.23) , can be shown to be path-independent by imposing a specific structure on the function $\mathfrak{J}(x)$. This structure is represented as follows:

$$\mathfrak{J}(x) = \sum_{i=1}^{r} h_i(x) P_i \tag{2.25}$$

where each matrix $P_i$ is defined as:

$$P_i = \begin{bmatrix} d_{\alpha_{il}}^{1,1} & p_{1,2} & \cdots & p_{1,n} \\ p_{1,2} & d_{\alpha_{il}}^{2,2} & \cdots & p_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1,n} & p_{2,n} & \cdots & d_{\alpha_{il}}^{n,n} \end{bmatrix} \tag{2.26}$$

with $d_{\alpha_{il}}^{j,j}$ representing the diagonal elements and $p_{j,k}$ the off-diagonal elements in each $P_i$.

The distinction between Fuzzy Lyapunov Functions (FLF) and Lyapunov Integral-Like Functions (LILF) is evident in their matrix formulations. In FLF, the entire matrix $P_i$ can vary according to fuzzy rules, where as in LILF, only the diagonal elements are subject to change. This difference is highlighted when comparing equations(2.17) and (2.23).

For example, consider the fuzzy rules defined as:

$$\begin{aligned} &\text{IF } \xi_1(t) \text{ is } F_{\alpha_{i1}}^1 \text{ and } \ldots \xi_l(t) \text{ is } F_{\alpha_{il}}^l \ldots \text{ and } \xi_q(t) \text{ is } F_{\alpha_{iq}}^q \\ &\text{THEN } \dot{x}(t) = A_i x(t) + B_i u(t), \quad \forall i \in \{1, \ldots, r\} \end{aligned} \tag{2.27}$$

The modifications to $P_i$ are determined by the fuzzy sets in these IF-THEN rules. For instance, if a premise variable appears in the same fuzzy set across different rules, the corresponding diagonal elements of $P_i$ for that variable will be identical. However, if a variable $x_j(t)$ belongs to different sets in rules $k$ and $w$, the $j^{th}$ diagonal elements of $P_k$ and $P_w$ will differ.

For nonlinear systems where nonlinearities depend on several premise variables, the matrix $A(x)$ in the system dynamics might include terms like $\cos(x_1 x_3)$, complicating the application of the structure (2.25). For example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 & 2 & -3x_2 \\ \sin x_2 & -1 & -2\cos x_1 \\ \cos(x_1 x_2) & 0 & 2 \end{bmatrix}}_{A(x)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In contrast, for systems where nonlinearities depend on a single premise variable, the structure(2.25) can be achieved, as demonstrated in the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos x_2 & 2 \\ -1 & x_1 \end{bmatrix}}_{A(x)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In this case, the system's nonlinearities each depend on a single premise variable, allowing the structure (2.25) to be met by setting: The matrix $P_i$ is defined as:

$$P_i = \begin{bmatrix} d^i_{1,1} & p_{1,2} \\ p_{1,2} & d^i_{2,2} \end{bmatrix}$$

### 2.6.3 Sum relaxation

The Takagi-Sugeno Linear Matrix Inequality (TS-LMI) framework often yields inequalities involving convex sums with membership functions (MFs) that encapsulate nonlinearities. To transform these inequalities into a Linear Matrix Inequality (LMI) form, the nonlinear elements must be eliminated. This conversion is challenging, leading to the development of various sum relaxation techniques to facilitate the creation of LMIs.

We highlight several lemmas that provide conditions under which certain inequalities are satisfied.

***Lemma 2.6.1:***[TS94] For $i, j = 1, \ldots, r$ and $h_i, h_j > 0$, the inequality

$$\sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \Upsilon_{ij} < 0 \tag{2.28}$$

is satisfied if:

$$\Upsilon_{ii} < 0, \quad \forall i = 1, \ldots, r \tag{2.29}$$

and

$$\Upsilon_{ij} + \Upsilon_{ji} < 0, \quad \forall i, j = 1, \ldots, r, i \neq j \tag{2.30}$$

***Lemma 2.6.2:***[TANY01] For the same initial conditions, the inequality holds if:

$$2\sum_{i=1}^{r-1} \Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \quad \forall i, j = 1, \ldots, r, i \neq j \tag{2.31}$$

***Lemma 2.6.3:***[XQ03] If we have:

$$\Upsilon_{ii} > \Xi_{ii}, \quad \forall i = 1, \ldots, r \tag{2.32}$$

and

$$\Upsilon_{ij} + \Upsilon_{ji} > \Xi_{ij} + \Xi_{ji}, \quad \forall i, j = 1, \ldots, r, i < j \tag{2.33}$$

with the matrix $\Xi$ defined as:

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \ldots & \Xi_{1r} \\ \Xi_{21} & \Xi_{22} & \ldots & \Xi_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \Xi_{r1} & \Xi_{r2} & \ldots & \Xi_{rr} \end{bmatrix} > 0 \tag{2.34}$$

then the inequality is satisfied.

Notably, Xiaodong and Qingling's[XQ03] relaxation method is the least conservative, though it increases computational complexity due to additional decision variables. Tuan et al.'s[TANY01] method, in contrast, offers a compromise between conservatism and computational efficiency.

## 2.7 Stability analysis of T-S fuzzy systems

The stability of nonlinear systems has been extensively studied, with Lyapunov's theory serving as a cornerstone. The central premise of this theory posits that if there exists a function with an energy-like form that dissipates over time, it will converge towards an equilibrium point. Within this framework, the Lyapunov function acts as a metric for gauging the proximity of state variables to the equilibrium point.

The challenge of this approach lies in the determination of suitable Lyapunov functions. Yet, two predominant families of Lyapunov functions exist: quadratic and non-quadratic. This thesis investigates the stability of systems through both quadratic and non-quadratic Lyapunov functions, offering insights into their application and effectiveness.

### 2.7.1 Quadratic stability analysis

Quadratic stability analysis is a method used to assess the stability of a nonlinear system by verifying if a quadratic Lyapunov function can be found for the system. If such a function exists and satisfies certain conditions, then the system is said to be quadratically stable. To analyze the stability of autonomous Takagi-Sugeno (T-S) fuzzy models, we focus on the system described as follows:

$$\dot{x} = \sum_{i=1}^{r} h_i(\xi) A_i(t) x \tag{2.35}$$

Here, $\dot{x}$ represents the time derivative of the state vector $x$, while $h_i(\xi)$ are the fuzzy membership functions, and $A_i(t)$ are system matrices corresponding to each fuzzy rule.

The time derivative of the quadratic Lyapunov function, which satisfies the stability conditions, along the trajectories of our system (2.35), is presented by:

$$\dot{V}(x(t)) = \dot{x}^T(t) P x(t) + x^T P \dot{x}(t) < 0 \tag{2.36}$$

than

$$\dot{V}(x(t)) = x(t)^T \left( \sum_{i=1}^{r} h_i(\xi(t))(A_i(t)^T P + P A_i(t)) \right) x(t) < 0 \tag{2.37}$$

Here, $P$ is a symmetric positive-definite matrix that ensures the Lyapunov function's validity for our system. This expression is a critical component in establishing the stability criteria for the fuzzy model in question.

Here's a simplified version of a theorem related to quadratic stability[TS92]:

**Theorem 2.7.1:**

Let $A_i(t)$ be a time-varying matrix that is bounded for all $t$. If there exists a constant, symmetric, positive-definite matrix $P$ such that for all $t$, the following LMI holds:

$$A_i^T(t) P + P A_i(t) < 0 \quad i = 1, \ldots, r \tag{2.38}$$

Then the system described by $\dot{x}(t) = A_i(t)x(t)$ is quadratically stable.

### 2.7.2 Non-quadratic stability analysis

Our stability analysis employs nonquadratic Lyapunov functions, specifically piecewise continuous and polyquadratic types, to adapt to the intricate nature of nonlinear systems.

This allows for a nuanced approach to understanding the stability of complex nonlinear systems

**1:Non-quadratic stability independent of time derivatives membership functions** To address the complexities arising from the inclusion of time derivatives of membership functions in stabilization conditions, the strategies proposed in [RW06] and [Gue14] suggest utilizing a Lyapunov function that is defined through a curvilinear integral, indicated in equation (2.24) .

Remarkably, as elucidated in [RW06], by harnessing the gradient theorem (referenced as Lemma 5.3.1 in the Appendix) and adopting the structure presented in equation (2.24) for the LILF, it is feasible to extract a stability condition devoid of the membership functions' time derivatives, as shown in equation (2.39)

$$\dot{V}(x) = \dot{x}^\top \left( \sum_{i=1}^{r} h_i(x) P_i \right) x + x^\top \left( \sum_{i=1}^{r} h_i(x) P_i \right) \dot{x} < 0 \tag{2.39}$$

Utilizing approaches from [RW06] , we can now examine the global stability of Takagi-Sugeno (TS) fuzzy systems through a Lyapunov Integral-Like Function (LILF) that is independent of the time derivatives of membership functions. This advancement significantly simplifies the stability analysis within a non-quadratic framework.

To delineate the stability conditions for the TS fuzzy system as outlined in system (2.35), it is imperative to consider the following lemma.

***Lemma1:***:the membership functions for fuzzy rules are shown to comply with a specified inequality (2.40). This inequality sets a constraint on the maximum number of fuzzy rules that can be active simultaneously, which is denoted by $\eta$.

$$\sum_{i=1}^{r} h_i^2(x) - \frac{2}{(\eta - 1)} \sum_{i=1}^{r} \sum_{j>i} h_i(x) h_j(x) \geq 0 \tag{2.40}$$

where $1 < \eta \leq r$.

Additionally, the foundational stability conditions originating from a LILF are summarized in an ensuing theorem.

**Theorem 2.7.2:**

The autonomous TS fuzzy system, referenced as (2.35), attains global asymptotic stability when positive matrices $P_i$ (defined in(2.26) and a positive scalar $X$ exist. This is contingent upon the satisfaction of two inequalities, indexed as(2.41) and

(2.42), which constrain the matrices in relation to $X$ and $n$, the latter being the cap on the number of active fuzzy rules at once, where $1 < n \leq r$.

$$P_iA_i + A_i^TP_i + (n-1)X < 0, \quad \forall i \in \{1, ..., r\}, \quad 1 < s < r \tag{2.41}$$

$$P_iA_j + A_j^TP_i + P_jA_i + A_i^TP_j - 2X < 0, \quad i < j \tag{2.42}$$

where $n$ represents the maximum number of simultaneously active fuzzy rules $(1 < n \leq r)$.

In an effort to reduce the conservatism found in the stability conditions originally proposed by [RW06], [Gue14] has presented a set of new, more flexible conditions. These updated conditions are outlined in the theorem that follows.

**Theorem 2.7.3:**

For the autonomous T-S fuzzy system, defined in equation (2.35), to be considered globally asymptotically stable, it is necessary that there exist matrices $P_i$ that are positive (referenced in equation (2.26)), and matrices $U_z$, and $V_z$, which meet the following criteria:

$$S_{ii} < 0, \text{ for each } i \in \{1, \ldots, r\} \tag{2.43}$$

$$\frac{2}{r-1}S_{ii} + S_{ij} + S_{ji} < 0, \text{ for all distinct pairs } (i,j) \in \{1, \ldots, r\}^2 \tag{2.44}$$

where

$$S_{ij} = \begin{bmatrix} U_iA_i + A_i^TU_j^T & (*) \\ P_i + V_iA_i - U_j^T & -V_j - V_j^T \end{bmatrix}, \tag{2.45}$$

(*) denotes terms that are to be specified.

**2:Non-quadratic stability based on piecewise continuous functions**

When compared to the quadratic form (2.16), the stability conditions derived from piecewise continuous functions (2.21) are found to be less restrictive. The pioneering work in this area is credited to [JRA99], who delineated the stability conditions for affine TSF models, expressed as:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(\xi(t))(A_ix(t) + a_i), \tag{2.46}$$

articulated through the formalism of Linear Matrix Inequalities (LMIs). Utilizing the function described in Equation (2.22) yields the results that follow:

**Theorem 2.7.4:**

A Takagi-Sugeno fuzzy system is asymptotically stable if, for each $i \in \{1, \ldots, r\}$, there exist positive definite matrices $P_i$ with $P_i = P_i^T$, and non-negative scalars $\gamma_{ijk}$, such that for all $(i, j) \in I^2$, the following matrix inequality is satisfied:

$$A_i^T P_j + P_j A_i + \sum_{k=1}^{r} \gamma_{ijk}(P_j - P_k) < 0. \tag{2.47}$$

**Example 2**

This example focuses on analyzing the stability of an autonomous TS fuzzy system (2.35) by comparing two methods: the use of a quadratic Lyapunov function and a line integral Lyapunov function, which is the central topic of this thesis

$$A_1 = \begin{bmatrix} -0.7 & -a+2 \\ 1.4b+a & -10 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & -a+2 \\ -0.2b+a & -10 \end{bmatrix}, \tag{2.48}$$

$$w_i^1(x_i) = \begin{cases} \frac{1}{2}(1 - \sin(x_i)), & \text{for } |x_i| \leq \frac{\pi}{2} \\ 0, & \text{for } x_i > \frac{\pi}{2} \\ 1, & \text{for } x_i < -\frac{\pi}{2} \end{cases} \tag{2.49}$$

$$w_i^2(x_i) = 1 - w_i^1(x_i). \tag{2.50}$$

Where

$$\begin{aligned} h_1(x) &= w_1^1(x_1)w_2^1(x_2) \\ h_3(x) &= w_2^1(x_1)w_1^1(x_2) \end{aligned} \tag{2.51}$$

**Proof:**

The theorem (2.7.2) is proven by employing a line integral Lyapunov function, which is defined as follows:

The time derivative of (2.23) is:

$$\dot{V}(t) = x^T \left( \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(x)h_j(x)(P_i A_j + A_j^T P_i) \right) x \tag{2.52}$$

$$\dot{V}(t) = x^T \left( \sum_{i=1}^{r} h_i^2(x)(P_i A_i + A_i^T P_i) + 2 \sum_{i=1}^{r} \sum_{j>i} h_i(x)h_j(x)Q \right) x \tag{2.53}$$

Where

$$2Q = (P_iA_j + A_j^TP_i + P_jA_i + A_i^TP_j) \tag{2.54}$$

Informed by theorem (2.7.2), equation(2.53) is subsequently expressed as:

$$\dot{V}(x) \leq x^T \left( \sum_{i=1}^{r} h_i^2(t)(A_i^TP_i + P_iA_i + (n-1)Q) \right) x \tag{2.55}$$

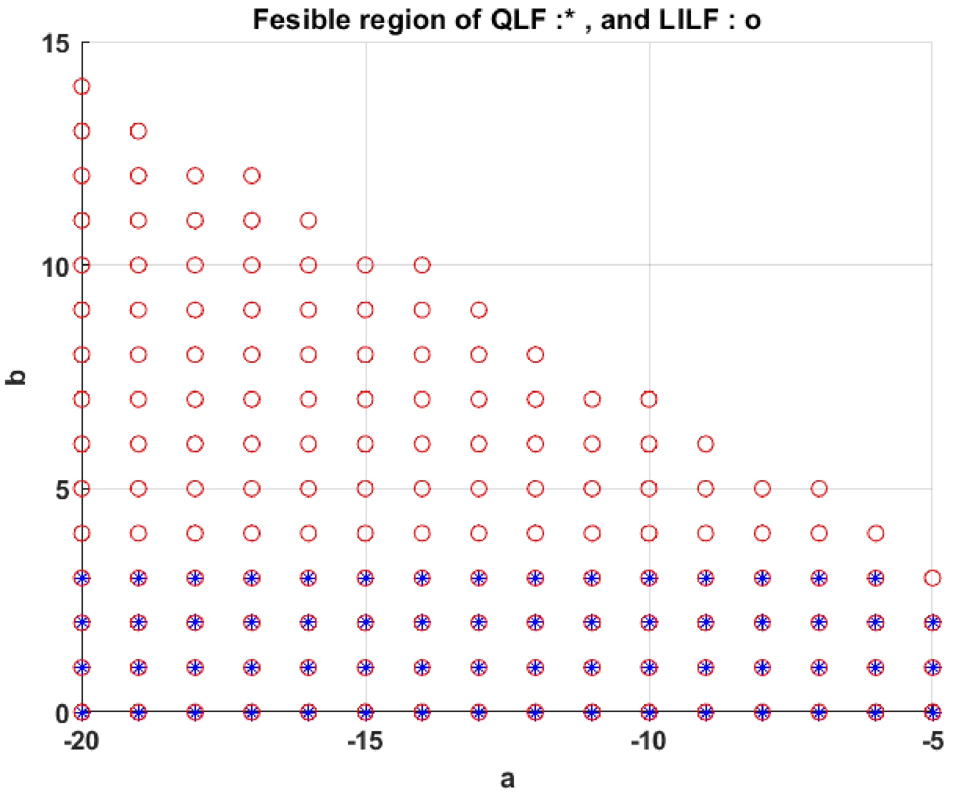



Fig 2.6: Stability region of theorems 2.7.1 and 4.2.1.

Employing the Yalmip toolbox along with the mosek solver enabled us to examine the stability regions for a range of values of a and b. Figure (2.6) displays the stability-assured region when applying the Line integral Lyapunov function technique as detailed in [RW06] and cited as theorem (2.7.2), denoted by (o). It also compares this with the regions corresponding to the conventional Lyapunov function approach of theorem (2.7.2), marked by (*). Upon analysis, it becomes clear that Theorem (2.7.2) offers a set of stability conditions that are more lenient than those of Theorem (2.7.1).

## 2.8 Conclusion

The multi-model approach effectively maps nonlinear system's behaviors, pinpointing their responses within distinct operational zones. This methodology, with its array of structures and derivation techniques, enriches the landscape of nonlinear system modeling. Stability analysis, a crucial aspect of this domain, ensures that models not only perform but persist under varied conditions. In this chapter, we explore Lyapunov's stability analysis, delving into both quadratic and non-quadratic stability within the realm of Takagi-Sugeno fuzzy models techniques imperative for the system's steadfastness. In sum, the multi-model methodology stands as a cornerstone in the modeling of nonlinear systems, shedding light on the intricacies of complex system operations.

Chapter 3

# Control of TS Fuzzy Systems via Quadratic Function



## 3.1 Introduction

The control of industrial systems typically assumes that all states are available at every instant. However, due to technological and economic constraints, measuring these states is often impractical in many applications. As a result, state reconstruction for unmeasured variables becomes essential. Traditional control laws, while effective for many linear systems, often fail to provide robust performance in nonlinear systems. These methods struggle to handle the complexities and unpredictability inherent in nonlinear dynamics. As a result, more advanced control strategies are needed to ensure stability and robustness in such systems

This chapter provides a comprehensive overview of the mathematical tools and concepts discussed in this thesis, which are vital for the design and analysis of advanced control systems. A primary focus is the challenge of developing controllers for nonlinear systems, particularly those characterized by Takagi-Sugeno type multi-models.After introducing the problem, we will outline the proposed approach for control design synthesis in Section (3.3.3). These algorithms are advantageous due to their cost-efficiency, adaptability, and ease of implementation. Various methodologies for state feedback control and PDC (Parallel Distributed Compensation) controllers for systems described by Takagi-Sugeno multi-models have been explored, as discussed in [BMMR16] and [Orj08]. Our approach utilizes the Mean Value Theorem (MVT) [Zem07] [Pha11] and applies transformations through nonlinear sectors. This method formulates the controller design as a convex combination of the derivatives of the nonlinear functions. Notably, the necessary gain to ensure control convergence can be efficiently determined through Linear Matrix Inequalities (LMIs), which are processed using optimization techniques.

## 3.2 Stabilization of T-S Fuzzy System

During the last few years, extensive research has been conducted to explore the stability and stabilization of multi-models in the Takagi-Sugeno type by controllers. These are based on the second Lyapunov method so as to lead, where possible, to a formulation in terms of linear matrix inequalities. The most widespread stabilization is based on control law types such as Parallel Distributed Compensation (PDC) [WTG96], [TIW98], and its derivatives like Proportional PDC (PPDC) [EL02], as well as the command law type known as division and Fusion Compensation (CDF) [GVDB99],[PKP01]. When the system state is not available, output feedback stabilization may be considered. In this context, three approaches are distinguished:

- **Static Output Feedback**

  This type of controller is particularly straightforward and helps minimize online computation costs [Cha02], [NTHB07], [Tah09].

- **Dynamic Output Feedback**

  Dynamic output feedback control improves controller performance by incorporating specifications related to the desired closed-loop dynamics. It is noted, however, that this approach can be conservative due to the emergence of cross terms within the LMIs [Yon09], [GBM09], [ZGM08].

- **Observer-Based Output Feedback**

  This type of controller involves the introduction of an observer to estimate unmeasured state variables, allowing a static state feedback control law to stabilize the system [GKVT06], [CEH07].

### 3.2.1 Stabilization using PDC Technique

The PDC (Parallel Distributed Compensation) approach is utilized to develop a control law for systems described by multi-models. The significant advantage of these controllers is their possession of the same interconnection structure as the multi-models from which they are derived. This feature allows for the extension of some linear system control theories to nonlinear systems. Figure (3.1)illustrates the concept of the PDC regulator.

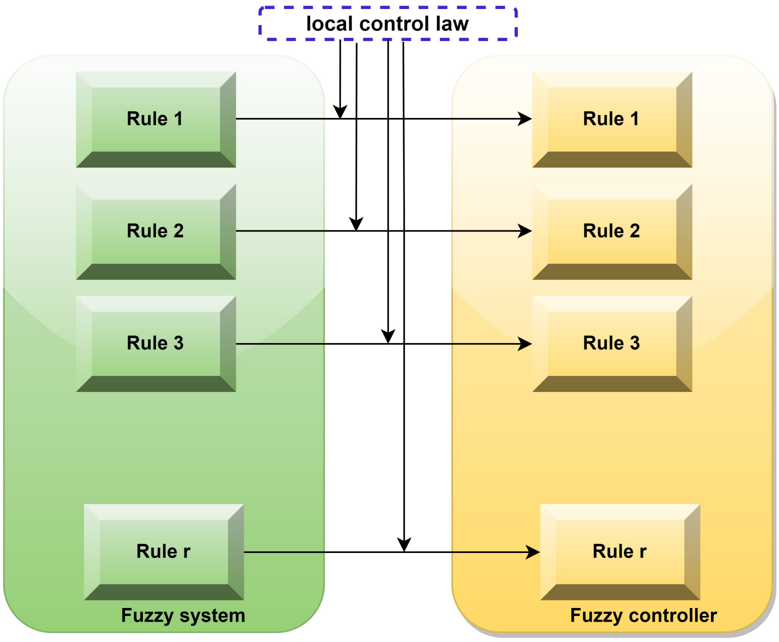



Fig 3.1: Principle of the PDC Controller.

For systems(2.4) , the implementation of the PDC controller is realized in the following manner: If $z_1$ is $M_1$, ..., $z_p$ is $M_p$, then:

$$u(t) = -F_i x(t) \tag{3.1}$$

Finally, the controller takes the form:

$$u(t) = -\sum_{i=1}^{n} h_i(\xi(t)) F_i x(t) \tag{3.2}$$

with $F_i \in \mathbb{R}^{m\times n}$ representing the controller gains. The synthesis of the PDC corrector then consists of determining the state feedback gains $F_i$. The system (2.4) controlled by a PDC controller (3.2) , is then written as:

$$\dot{x}(t) = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} h_i(\xi(t)) h_j(\xi(t)) (A_i - B_i F_j) \right) x(t) \tag{3.3}$$

Here we distinguish between the cross terms $i$ and $j$, and the non-cross terms (only $i$). The aim of the PDC command is found in the non-cross terms. The cross terms are consequently undesirable terms that we want to minimize as much as possible according to the norm [Bla01].

$$\dot{x}(t) = \left( \sum_{i=1}^{n} h_i^2(\xi(t)) G_{ii} + \sum_{i=1}^{n} \sum_{j\neq i} h_i(\xi(t)) h_j(\xi(t)) \frac{G_{ij} + G_{ji}}{2} \right) x(t) \tag{3.4}$$

With

$$G_{ij} = (A_i - B_i F_j), \quad \text{for } i < j, \quad i,j = 1,2,\ldots,n \tag{3.5}$$

Several relaxations of system (3.3) have been proposed in the literature. Hence, based solely on the relaxations of Tanaka [TW04].

**Theorem 3.2.1:**

The equilibrium of the continuous-time Takagi-Sugeno model (3.3) is asymptotically stable if there exists a positive definite matrix $P > 0$ satisfying:

$$G_{ii}^T P + P G_{ii} < 0 \tag{3.6}$$

$$\left( \frac{G_{ij} + G_{ji}}{2} \right)^T P + P \left( \frac{G_{ij} + G_{ji}}{2} \right) < 0, \quad \forall i < j \tag{3.7}$$

for all $i, j = 1, 2, \ldots, n$, except for pairs $(i, j)$ such that $\forall t, h_i(\xi(t)) h_j(\xi(t)) = 0$.

The development of a PDC controller involves determining the regulator gains that satisfy conditions (3.7), Utilizing conditions(3.7) allows for a reduction in the conservatism of the results since it is not mandatory for all the cross sub-models to be stable.

In this context, a change of the bijective variables is necessary in order to convert the problem into LMI's.

$$X = P^{-1} \quad \text{and} \quad M_i = F_i X$$

Equation (3.7) can be rewritten in the form:

$$\begin{aligned} X - (A_iX - B_iM_i)^T X^{-1}(A_iX - B_iM_i) > 0 \\ X - \frac{1}{2}(A_iX + A_jX - B_iM_j - B_jM_i)^T X^{-1} \\ (A_iX + A_jX - B_iM_j - B_jM_i) > 0 \end{aligned} \tag{3.8}$$

By using Schur's complement [BEGFB94], inequality (3.8) can be converted into LMI form:

$$\begin{bmatrix} X & (A_iX - B_iM_i)^T \\ A_iX - B_iM_i & X \end{bmatrix} \geq 0 \tag{3.9}$$

and

$$\begin{bmatrix} X & \frac{1}{2}(A_iX + A_jX - B_iM_j - B_jM_i)^T \\ \frac{1}{2}(A_iX + A_jX - B_iM_j - B_jM_i) & X \end{bmatrix} \geq 0 \tag{3.10}$$

The controller gains are given by:

$$F_i = M_i X^{-1} \tag{3.11}$$

**Numerical exemple:**

In this subsection, we demonstrate the proposed method using a Single-Link Flexible Joint Robot example.Let's consider a continuous-time T-S fuzzy system with $r = 2^n = 2$ given by [EHRBB22]

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 1.95 & 0 & -2.28 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 1.95 & 0 & -2.16 & 0 \end{bmatrix} \tag{3.12}$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}, \tag{3.13}$$

The weighting functions, which depend on $x_3$from the state vector $x$ are specified as follows:

$$\begin{cases} h_1(\xi(t)) = \dfrac{\xi(t) + 0.2172}{1.2172}, \\ h_2(\xi(t)) = 1 - h_1(\xi(t)) \end{cases} \tag{3.14}$$

To stabilize the Joint Robo at its equilibrium point, we use a control law given by:

$$u(t) = -\sum_{i=1}^{2} \mu_i(z(t)) k_i x(t)$$

The resolution of the LMIs in Theorem (3.2.1) has given us the following results:

$$P = \begin{bmatrix} 35.4163 & 6.2339 & -21.8199 & 4.1308 \\ 6.2339 & 1.5584 & -3.9757 & 0.8419 \\ -21.8199 & -3.9757 & 18.4953 & -2.0374 \\ 4.1308 & 0.8419 & -2.0374 & 1.0244 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 0.8129 & 0.4509 & 0.0245 & 0.4110 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 0.8520 & 0.4588 & 0.0194 & 0.1536 \end{bmatrix}$$

The results of the simulation are illustrated in the figures that follow

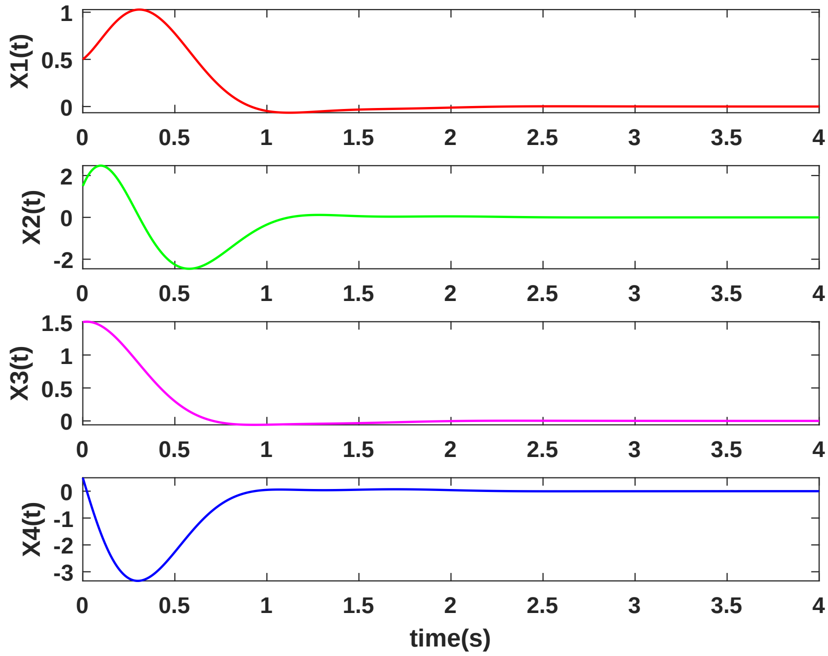



Fig 3.2: State evolution of T-S fuzzy model under PDC control law obtained via Theorem 3.2.1

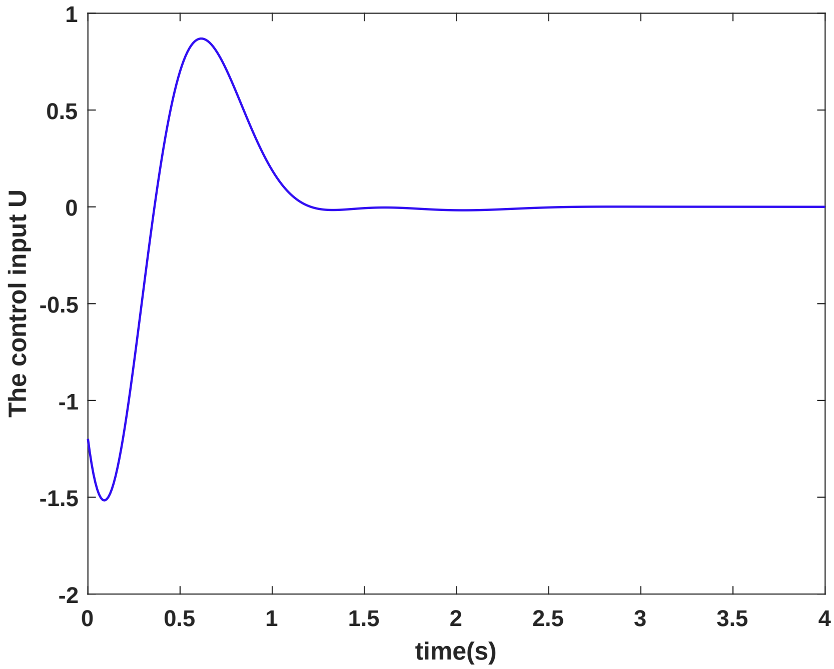



Fig 3.3: Control law evolution with gains obtained via Theorem 3.2.1

Figure (3.2), respectively illustrate the evolution of the state variables $x_1$ ,$x_2$, $x_3$,

$x_4$;these graphs demonstrate a rapid stabilization of the system: the augmented system reaches stability at the origin in approximately 2 seconds. The stability conditions, as outlined in LMIs (3.6) and(3.7), are fulfilled. This implies that the fuzzy control system, comprising the fuzzy model and the Parallel Distributed Compensation (PDC) controller, achieves global asymptotic stability. Figure (3.3) demonstrates the performance of the fuzzy control system under the same initial conditions depicted in Figure (3.2)

## 3.3 Controller Synthesis via DMVT Approach

In tackling the challenge of controller design within nonlinear systems, the application of the Mean Value Theorem (MVT) stands out as a pivotal technique, offering a strategic path through the complexities inherent in such systems. This approach is deeply rooted in the fundamental calculus principle of the MVT, providing a profound framework for understanding and regulating system dynamics across varied conditions. Unlike traditional controller design methodologies, which often presume that all system states can be directly controlled or influenced, real-world scenarios frequently reveal that only a subset of these states is accessible for manipulation. In this context, the MVT approach shines by enabling precise and effective control strategies, even for systems with partially inaccessible states. This method sidesteps the limitations of linearization and oversimplification, which can degrade the efficacy of control efforts. The subsequent sections will delve into the detailed mechanics, practical implementations, and the strengths and weaknesses of employing this strategy in the realm of nonlinear systems.

### 3.3.1 Problem statement

A controller acts as a dynamic mechanism that adjusts a system's state, either asymptotically or exponentially, by utilizing the system's inputs, outputs, and understanding of its dynamic model. This process aims to influence the behavior of the system to achieve desired outcomes, ensuring stability and optimal performance based on predefined criteria.

Let's examine a nonlinear system characterized by the ensuing format:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(x(t))(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases} \tag{3.15}$$

Where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, and $y(t) \in \mathbb{R}^m$ respectively denote the state vector, input vector, and output vector. $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{m\times n}$ are the appropriate matrices. $h_i(x(t))$ represent the activation functions.

The proposed control law structure is in the form of

$$u(t) = -K(x - x_c) \tag{3.16}$$

The state error is defined as $e(t) = x(t) - x_c(t)$, with $K$ signifying the controller gain. Considering system (3.15) and the control law (3.16),The dynamics of the state error are given by:

$$\dot{e}(t) = \sum_{i=1}^{r} h_i(x(t))(A_i x(t) - B_i K x(t)) \tag{3.17}$$

To simplify the dynamics of the state error (3.17) into two components, it is necessary to apply a transformation to the system (3.15):

$$\begin{cases} \dot{x}(t) = A_0 x(t) + B_0 u(t) + \sum_{i=1}^{r} h_i(x(t))(\bar{A}_i x(t) + \bar{B}_i u(t)) \\ y(t) = Cx(t) \end{cases} \tag{3.18}$$

where

$$\begin{cases} A_0 = \frac{1}{r}\sum_{i=1}^{r} A_i \\ B_0 = \frac{1}{r}\sum_{i=1}^{r} B_i \\ \bar{A}_i = A_i - A_0 \\ \bar{B}_i = B_i - B_0 \end{cases} \tag{3.19}$$

Here, $A_0$ and $B_0$ denote the average matrices of $A_i$ and $B_i$, respectively. Consequently, the state error dynamics (3.17) can be reformulated as:

$$\dot{e}(t) = (A_0 - B_0 K)e(t) + \sum_{i=1}^{r} h_i(x(t))(\bar{A}_i x(t) + \bar{B}_i u(t)) \tag{3.20}$$

Our objective is to guarantee the asymptotic convergence of the state error to zero. For this purpose, formulating stability conditions as Linear Matrix Inequalities (LMIs) is essential to determine the static gain $K$, as shown in equation (3.17)

### 3.3.2 Differential Mean value theorem

In this segment, we explore the application of the finite difference method, commonly referred to as the Differential Mean Value Theorem ($DMVT$), applicable to both scalar and vectorial functions. This methodology enables the representation of the nonlinear dynamics associated with estimation errors in the form of a Linear Parameter-Varying ($LPV$) system. This approach will be pivotal in the next stage of our discussion.

It is essential to first establish the following definitions

***Lemma 3.4.1:(Scalar DMVT)***[Pha11]

Consider a smooth function $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ on the closed interval $[a, b]$. Then, there exists some $C$ in the open interval $(a, b)$ such that:

$$f(a) - f(b) = \frac{df(x)}{dt}\Big|_C \times (a - b) \tag{3.21}$$

***Lemma 3.4.2:(Canonical basis).)***[ZBB05]

Consider the vector function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $f(x) = [f_1(x), f_2(x), \ldots, f_i(x), \ldots, f_m(x)]^T$, where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is the $i$-th component of $f(x)$. Define the set $E_s$ as:

$$\begin{cases} e_s(i) = [0, .., 0, 1, 0, \ldots., 0], \; pour \; i = 1, 2, \ldots, s \\ E_s = e_s(i) \end{cases} \tag{3.22}$$

By using the definition of$E_s$, the function $f(x)$ can be rewritten in the form of:

$$f(x) = \sum_{i=1}^{q} e_q(i) f_i(x) \tag{3.23}$$

***Lemma 3.4.3(DMVT for Vector Functions.)***[Ham15]

Let $f(x) \in \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a differentiable function. Consider the interval $[a, b] \in \mathbb{R}^n$, with in which there exists a constant $c$ in the open interval $(a, b)$. We define the convex set $Co(x, y)$ such that:

$$f(a) - f(b) = \nabla f(c)(a - b) \tag{3.24}$$

By applying Lemma 3.3.2 to equation (3.23), we arrive at a modified form of the Differentiable Mean Value Theorem (DMVT).

**Theorem 3.3.1:**

***Modified DMVT for Vector Functions*** [Ham15]

Let $f(x) \in \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function on the interval $[a, b] \in \mathbb{R}^n$ and differentiable in $Co(a, b)$. There exists a vector $c \in Co(a, b)$ with $c_i \neq a$ and $c_i \neq b$ for $i = 1, \ldots, n$, such that [Pha11]:

$$f(a) - f(b) = \left[ \sum_{i,j,k=1}^{n,n} H_{ijk}(c_j)\delta_{ij} \right] (a - b) \tag{3.25}$$

Such that

**Hypothesis** : Assuming that the function $f(x)$ is Lipschitz continuous, its derivatives are bounded within specified limits. This boundedness allows for the application of equations (2.10)and (2.11)

$$\underline{\delta}_{ij} \leq \delta_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{c_i} < \overline{\delta}_{ij} \tag{3.26}$$

where

$$\underline{\delta}_{ij} = \min \left.\frac{\partial f_i}{\partial x_j}\right|_{c_i}, \text{and} \quad \overline{\delta}_{ij} = \max \left.\frac{\partial f_i}{\partial x_j}\right|_{c_i} \tag{3.27}$$

Any manifestation of nonlinearity can be characterized by the following expression:

$$\underline{\delta}_{ij} = \sum_{k=1}^{2} \Lambda_{ij}^{k} \sigma_{ijk} \tag{3.28}$$

where $\sigma_{ij1} = \underline{\delta}_{ij}$ and $\sigma_{ij2} = \overline{\delta}_{ij}$

$$\begin{cases} A_{ij}^1 = \dfrac{\delta_{ij} - \underline{\delta}_{ij}}{\overline{\delta}_{ij} - \underline{\delta}_{ij}}, \\ A_{ij}^2 = \dfrac{\overline{\delta}_{ij} - \delta_{ij}}{\overline{\delta}_{ij} - \underline{\delta}_{ij}} \end{cases} \tag{3.29}$$

$$\sum_{k=1}^{2} \Lambda_{ij}^{k}(c_i) = 1; \quad 0 \leq \Lambda_{ij}^{k}(c_i) \leq 1 \quad \text{for} \quad k = 1, 2 \tag{3.30}$$

By applying Lemma 3.3.2, Equation 4.44 can be reformulated as follows:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} e_n(i) e_n^T(j) \frac{\partial f_i}{\partial x_j}(c_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{2} \lambda_{ij}^{k}(c_i) H_{ij} \sigma_{ijk} = \sum_{i=1}^{n} \sum_{j=1}^{n} H_{ij} \delta_{ij} \tag{3.31}$$

In the above, $H_{ij}$ are zero matrices where the $(i, j)$ element is equal to 1, with $H_{ij} = e_n(i)^T e_n(j)$.

### 3.3.3 Controller Design

In the present discussion, our main emphasis is on designing a static controller specifically for a designated class of systems, as described in equation (3.15). We have modified the dynamics of the state error, originally detailed in equation (3.20), to enhance its clarity and applicability. The revised formulation is provided below

$$\dot{e}(t) = (A_0 - BK)e(t) + \Phi(x(t), u(t)) - \Phi(x_c(t), u(t)) \tag{3.32}$$

Where

$$\begin{cases} \Phi(x(t), u(t)) = \sum_{i=1}^{r} h_i(x(t))(\bar{A}_i x(t) + \bar{B}_i u(t)) \\ \Phi(x_c(t), u(t)) = 0 \end{cases} \tag{3.33}$$

To solve this problem, we use the modified mean value theorem (3.3.2) and the approach of transformation by nonlinear sectors, which allows us to express the dynamics of the state error in the form of an autonomous T-S system

In this context, we have a vector $c(t)$ in the set $(Ç(x, x_c))$ such that:

$$\Phi(x(t), x_c(t), u(t)) = \Phi(x(t), u(t)) \tag{3.34}$$

$$\Phi(x(t), u(t)) = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} e_n(i) e_n^T(j) \frac{\partial \Phi_i(c_i(t))}{\partial x_j} |_{\hat{x}_i < c_i < x_i} \right) (x(t) - x_c(t)(t)) \tag{3.35}$$

By exploiting the approach of transformation by nonlinear sectors to rewrite the quantities $\frac{\partial \Phi_i(c_i(t))}{\partial x_j}$ in the form of sums as follows:

$$\frac{\partial \Phi_i(c_i(t))}{\partial x_j} = \sum_{i=1}^{n} \sum_{j=1}^{n} H_{ij} \delta_{ij}(c_i) \tag{3.36}$$

Thus, the dynamics of the state error (3.32) can be represented by:

$$\dot{e}(t) = \left( A_0 - B_0 K + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{2} H_{ij} \Lambda_{ij}^k \delta_{ijk}(c_i) \right) e(t) \tag{3.37}$$

For simplicity, we define:

$$\sum_{i=1}^{q} h_i(x) \mathcal{A}_i = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{2} H_{ij} \Lambda_{ij}^k \delta_{ijk}(c_i) \tag{3.38}$$

$$X_i = A_0 - B_0 K + \mathcal{A}_i \tag{3.39}$$

The dynamics of the state estimation error can then be represented as follows:

$$\dot{e}(t) = \left( \sum_{i=1}^{q} h_i(x(t)) X_i(t) \right) e(t) \quad with \quad q = 2^{(n)^2} \tag{3.40}$$

The weighting functions $h_i(x)$ depend on the products of $\Lambda_{ij}^1$ and $\Lambda_{ij}^2$, ensuring that they uphold the subsequent convex property:

$$\sum_{i=1}^{q} h_i(x) = 1, \quad 0 \leq h(x) \leq 1, \quad \forall i = 1, \ldots, q \tag{3.41}$$

The weighting functions $h_i(x)$ depend on the products of $\Lambda_{ij}^1$ and $\Lambda_{ij}^2$, ensuring that they uphold the subsequent convex property:

$$\sum_{i=1}^{q} h_i(x) = 1, \quad 0 \leq h_i(x) \leq 1, \quad \forall i = 1, \ldots, q \tag{3.42}$$

The matrices $\mathcal{A}_i$ are constants derived from the parameters $\sigma_{ijk}$. To compute the sub-model matrices $\mathcal{A}_i$, we begin by approximating the final summation in equation(3.38) with a Jacobian matrix. This is achieved by taking the partial derivative of the nonlinear function $f$ presented in equationeq (3.18). Thus, we have [NKMS21] :

$$\sum_{i=1}^{q} h_i(x)\mathcal{A}_i = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \delta_{11} & \cdots & \delta_{1n} \\ \vdots & \ddots & \vdots \\ \delta_{n1} & \cdots & \delta_{nn} \end{bmatrix} \tag{3.43}$$

Following this, we replace each element of the Jacobian matrix of $f(x)$ with the respective upper or lower limit as prescribed by equation(4.42). This process is crucial for determining the components of the sub-model matrices.

In line with Hypothesis(3.3.2) , the parameters $\delta_{ij}$ are constrained within a specified convex set $\mathcal{F}_{n,n}$, which is characterized by a set of $2n^2$ vertices, as defined by [MHS+23]:

$$\nu_{\mathcal{F}_{n,n}} = \left\{ \Omega = (\Omega_{11}, \ldots, \Omega_{1n}, \ldots, \Omega_{nn}) : \Omega_{ij} \in \{\underline{\Omega}_{ij}, \overline{\Omega}_{ij}\} \right\}$$

In a broader context, it is crucial to note that the number of sub-model matrices $A_i$ matches $q = 2^{n^2}$ precisely when each element $\Omega_{ij}$ within the Jacobian matrix of $f$ is neither zero nor constant. Conversely, when certain elements are presumed to be either constant or zero, the quantity of sub-models is reduced to below $2^{n^2}$ [NKMS21].

Equation (3.40) describes an autonomous system within the framework of LPV (Linear Parameter Varying) theory. Given this, established stability results for multi-models that incorporate both measurable and non-measurable decision variables are applicable [THW03][Ich09].

The primary objective is to ascertain the gain $K$ that guarantees the asymptotic convergence of the state error to nil.

This is predicated on a quadratic Lyapunov function of the form:

$$V(e(t)) = e^T(t)Pe(t) \quad / \quad P = P^T > 0 \tag{3.44}$$

**Theorem 3.3.2:**

The state error converges asymptotically to zero if there exists a symmetric and positive definite matrix $P$ in $\mathbb{R}^{n\times n}$ and a matrix $M$ in $\mathbb{R}^{n\times m}$, such that the following linear matrix inequalities are satisfied:

$$PA_0 + P\mathcal{A}_i - BN + A_0^T P + \mathcal{A}_i^T P - N^T B^T < 0 \tag{3.45}$$

The convergence of the controller, as indicated in equation (3.17), hinges on the presence of a symmetric matrix P and a gain matrix M that conform to the requirements of equation (3.40). Once these matrices are established, the observer's gain can be accurately calculated using the specified equation.

$$K = NP^{-1}. \tag{3.46}$$

*Proof of Theorem:*

The proof of the theorem utilizes a quadratic Lyapunov function, defined as:

$$V(e(t)) = e(t)^\top Pe(t), \quad \text{where} \quad P = P^\top > 0. \tag{3.47}$$

This establishes that the estimation error asymptotically converges to zero under two key conditions:

$$\begin{cases} V(e(t)) > 0, \\ \dot{V}(e(t)) < 0. \end{cases} \tag{3.48}$$

Given that the matrix $P$ is positive definite, the first condition naturally holds for all $e(t) \neq 0$. However, it is still necessary to confirm the second condition, namely $\dot{V}(e(t)) < 0$.

To do this, we consider the derivative of $V(e(t))$ over time:

$$\dot{V}(e(t)) = e(t)^\top \left(\sum_{i=1}^{q} h_i(x(t))(A_0 - BK + \mathcal{A}_i)\right)^\top P + P\left(\sum_{i=1}^{q} h_i(x(t))(A_0 - BK + \mathcal{A}_i)\right) e(t). \tag{3.49}$$

Since the term $\sum_{i=1}^{q} h_i(x(t))X_i(t)$ represents an affine function, we can apply the convexity principle as outlined in [BV97]

$$(A_0 - BK + \mathcal{A}_i)^T P + P(A_0 - BK + \mathcal{A}_i) \implies \dot{V}(e(t)) < 0 \tag{3.50}$$

We can simplify the controller's gain equation by introducing a variable substitution $N = KP$. Doing so reveals that equation(3.50) is effectively the same as(3.45) . Consequently, we express the controller's gain as $K = NP^{-1}$.

## 3.4 Improvement of dynamic performances

The approach of placing poles in a chosen region of the plane is called D-stability.

**Definition 1.** *(Région LMI).*

*A region of the complex plane $\mathcal{D}$ is an LMI region when it can be expressed in the following form:*

$$\mathcal{D} = \{z \in \mathbb{C} : \alpha + z\beta + \bar{z}\beta^T < 0\} \tag{3.51}$$

The exponential convergence of the observer (2.3) will be established under the principle of pole placement in a desired LMI region. This latter must be defined as follows:

- ***Limitation of the imaginary parts of the eigenvalues***: $|\Re_e(z)| > \alpha$

To minimize the exponential convergence rate, we define the characteristic function of the LMI region $\mathcal{D}$ as follows:

$$\mathcal{D} = \{z \in \mathbb{C} : f_{\mathcal{D}}(z) = \alpha + z + \bar{z} < 0\} \tag{3.52}$$

- ***Limitation of gain amplitude***: Radius r

The setting of the natural frequency allows for the limitation of the amplitude of the observation gain

$$\mathcal{D} = \{z \in \mathbb{C} : f_{\mathcal{D}}(z) = \begin{bmatrix} -r & z \\ \bar{z} & -r \end{bmatrix} < 0\} \tag{3.53}$$

- ***Minimization of damping***: Angle $\theta$

To ensure minimal damping,$f_{\mathcal{D}}$ must be defined as follows:

$$\mathcal{D} = \{z \in \mathbb{C} : f_{\mathcal{D}}(z) = \begin{bmatrix} (z+\bar{z})sin(\theta) & (z-\bar{z})cos(\theta) \\ -(z-\bar{z})cos(\theta) & (z+\bar{z})sin(\theta) \end{bmatrix} < 0\} \tag{3.54}$$

### 3.4.1 Pole Placement

The placement of the eigenvalues of the state observer in a region ($\mathcal{D}$) of the complex plane figure (3.4) must be considered in order to take into account the dynamic performance of the observer.

**Theorem 3.4.1:**

The error asymptotically approaches zero with pole placement in a region $\mathcal{D}$ of the complex plane, provided there exists a symmetric and positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a matrix $M \in \mathbb{R}^{n\times m}$ such that the following matrix linear inequalities are satisfied:

$$PA_0 + P\mathcal{A}_i - BN + A_0^T P + \mathcal{A}_i^T P - N^T B^T + \alpha P < 0 \tag{3.55}$$

$$\begin{bmatrix} -rP & PA_0 + P\mathcal{A}_i - BN \\ A_0^T P + \mathcal{A}_i^T P - N^T B^T & -rp \end{bmatrix} < 0 \tag{3.56}$$

$$\begin{bmatrix} \tilde{\mathcal{G}}_1 cos(\theta) & \tilde{\mathcal{G}}_2 sin(\theta) \\ -\tilde{\mathcal{G}}_2 sin(\theta) & \tilde{\mathcal{G}}_1 cos(\theta) \end{bmatrix} < 0 \tag{3.57}$$

with

$$\begin{cases} \tilde{\mathcal{G}}_1 = PA_0 + P\mathcal{A}_i - BN + A_0^T P + \mathcal{A}_i^T P - N^T B^T \\ \tilde{\mathcal{G}}_2 = PA_0 + P\mathcal{A}_i - BN - A_0^T P - \mathcal{A}_i^T P + N^T B^T \end{cases}$$

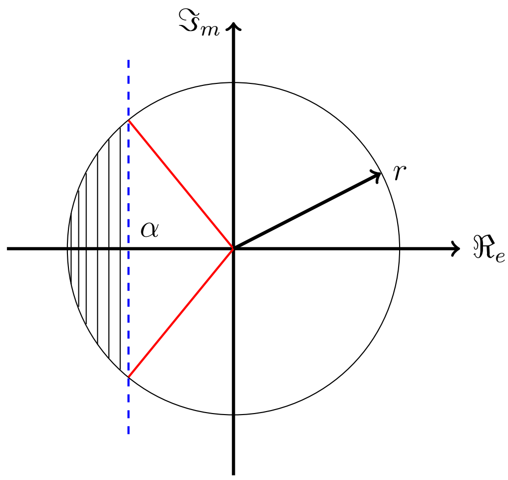



Fig 3.4: LMI Region

#### 3.4.1.1 $H^\infty$ Performance Criterion:

Up to now, we have considered the case of a proportional control law for nonlinear systems. The major drawback of this control structure is its inability to guarantee zero static errors when the process is subject to nonzero external disturbances.
Our objective in this section is to extend the stabilization conditions outlined in theorem 3.4.1 to the class of nonlinear systems that are affected by unknown inputs.
The Proportional-Integral (PI) controller based on the DMVT approach differs from the traditionally used PI controller. This distinction arises from how the integral action is utilized.

**a/Problem Formulation**

Considering the class of T-S multi-model systems subject to disturbances, which are represented as follows:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{8} h_i((z(t))(A_i x(t) + B_i u(t)) + Dw(t)) \\ y(t) = C_i x(t) \end{cases} \tag{3.58}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^p$ represents the input vector, and $y(t) \in \mathbb{R}^m$ is the output vector. The matrices $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{m\times n}$ are known appropriate matrices, and $w(t) \in \mathbb{R}^l$ is the vector of unknown inputs.

The error state vector is expressed as follows:

$$e(t) = x(t) - x_r(t) \tag{3.59}$$

Where $x_r$represents the stepwise reference state, with the dynamics of the state error(if $x_r(t) = 0$) described by the following expression:

$$\dot{e}(t) = \dot{x}(t) \tag{3.60}$$

Thus, the dynamics of the error for the multi-model system (3.58) can be represented in the following form:

$$\dot{e}(t) = \sum_{i=1}^{8} h_i((z(t))(A_i x(t) + B_i u(t)) + Dw(t)) \tag{3.61}$$

**b/$H^{\infty}$ Control Synthesis:**

In this portion, we will develop an $H^{\infty}$ controller that ensures both the closed-loop stability of system (3.58) and the convergence of the tracking error (3.59) , while also mitigating the impact of external disturbances.

**Definition 2.** *An $H^{\infty}$ criterion is written as follows:*

$$\int_0^{\infty} \bar{e}^T(t)\bar{e}(t)dt \leq \gamma^2 \int_0^{\infty} \bar{w}^T(t)\bar{w}(t)dt \tag{3.62}$$

*where $\gamma$ is the desired rate of attenuation for external disturbances.*

The new control law $u(t)$, based on observer performance under the $H^{\infty}$ criterion, is given by:

$$u(t) = -Ke_c(t) - K' \int e_c(t)\ dt \tag{3.63}$$

such that $K$ and $K'$ are the gains of the PI controller that need to be designated.

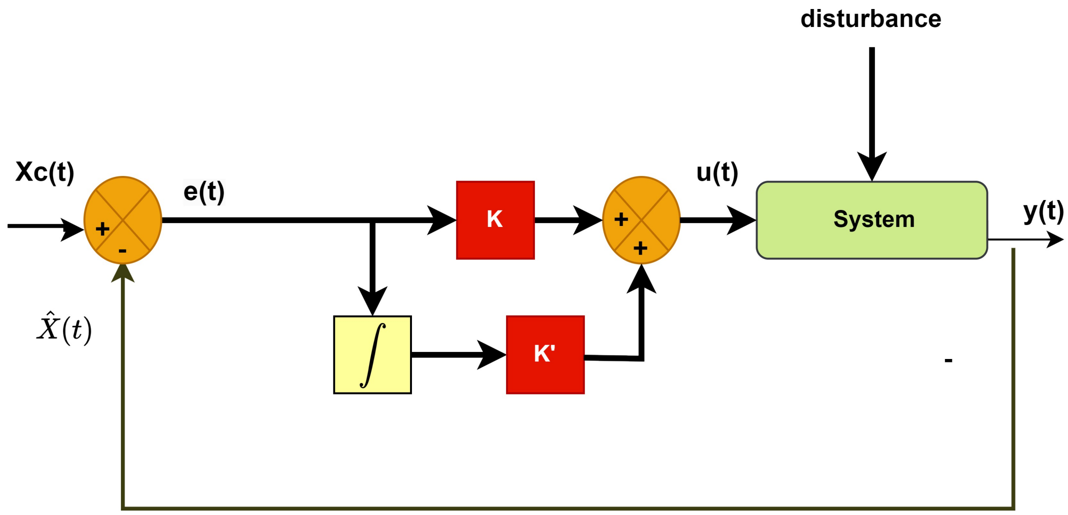



Fig 3.5: State feedback control structure with integral action

Taking into account the T-S fuzzy framework detailed in (3.64) and the proportional-integral controller introduced in (3.65) , they can be articulated as augmented configurations in the forms below:

$$\begin{cases} \dot{x}_a(t) = \sum_{i=1}^{r} h_i(x_a(t)) \left(A_i^a x_a(t) + B_i^a u(t) + D_i^a w^a(t)\right) \\ y(t) = C^a x_a(t) \end{cases} \quad (3.64)$$

where

$$A_i^a = \begin{bmatrix} A_0 & 0 \\ I & 0 \end{bmatrix}, B_i^a = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad et \quad D_i^a = \begin{bmatrix} A_i & D \\ 0 & 0 \end{bmatrix}$$

and

$$K_i^a = \begin{bmatrix} K & K' \end{bmatrix} \quad (3.65)$$

Let's consider the dynamics of the state error in the augmented system, which can be defined as:

$$\dot{e^a}(t) = \dot{x^a}(t)$$

We can describe the dynamics of this error, $\dot{e^a}(t)$ , using the following expression:

$$\dot{e^a}(t) = \sum_{i=1}^{r} h_i(x_a(t)) \left(A_i^a - B_i^a K_i^a) e^a(t) + D_i^a w^a(t)\right) \quad (3.66)$$

In this context, we have previously assumed certain conditions. Based on these, we determine the gain matrices $K_i^a$ for the proportional-integral controller to ensure the system's stability, even when $w^a(t)$ is not zero.

**Theorem 3.4.2:**

If there exists a positive scalar $\gamma > 0$, and symmetric and positive definite matrices $X, N$ such that the following matrix inequalities are sufficient:

$$\begin{bmatrix} A_i^a X + X A_i^{aT} - B_i^a N_i - N_i^T B_i^{aT} & D_i^a & X \\ D_i^{aT} & -\gamma^2 I & 0 \\ X & 0 & -I \end{bmatrix} < 0 \tag{3.67}$$

then, the closed-loop system (3.58) is asymptotically stable and the $H^\infty$ performance (3.62) is guaranteed via the state feedback control law (3.63).
The controller gains are given by:

$$K_i^a = N_i X^{-1} \tag{3.68}$$

*Proof of Theorem:*

The analysis of this controller's convergence centers on its asymptotic stability, with a particular focus on the Proportional-Integral (PI) controller. The goal is to ascertain the appropriate gain $K_i^a$ which ensures that the state error, , achieves asymptotic stability

$$\lim_{t\to\infty} e^a(t) = 0 \tag{3.69}$$

To assess the convergence of the state estimation error, we apply a quadratic Lyapunov function, as detailed in Equation (2.16). Over time, as we approach infinity, this error is expected to diminish to zero. Convergence criteria, encapsulated by Linear Matrix Inequalities (LMIs), are established using the same quadratic Lyapunov function:

$$V(e^a(t)) = e^{aT}(t) P e^a(t), \text{ where } P = P^T > 0. \tag{3.70}$$

We verify asymptotic stability when:

$$\dot{V}(e^a(t)) < 0. \tag{3.71}$$

The Lyapunov function's derivative is expressed as:

$$\dot{V}(e^a(t)) = \dot{e}^{aT}(t) P e^a(t) + e^{aT}(t) P \dot{e}^a(t). \tag{3.72}$$

Upon integrating the insights from equation (3.66) with equation (3.72), we deduce the following results

$$\dot{V}(e^a(t)) = e^{aT}(t)\left(A_i^{aT} - K_i^{aT}B_i^{aT}\right)Pe^a(t) + w^{aT}(t)D_i^{aT}Pe^a(t) + e^{aT}(t)P\left((A_i^a - B_i^aK_i^a)e^a(t)\right)$$
$$+ e^{aT}(t)PD_i^a(t)w_i^a(t) \tag{3.73}$$

$$\dot{V}(e^a(t)) = e^{aT}(t)\left((A_i^{aT} - K_i^{aT}B_i^{aT})P + P(A_i^a - B_i^aK_i^a)\right)e^a(t) + w^{aT}(t)D_i^{aT}Pe^a(t)$$
$$+ e^{aT}(t)PD_i^a(t)w_i^a \tag{3.74}$$

Considering the assumptions previously mentioned, the function$w^a(t)$ is bounded. This is supported by Lemma 1:

$$\|e^a(t)\|_2 < \gamma\|w^a(t)\|_2 \tag{3.75}$$

$$\dot{V}(t) - \gamma^2 w^{aT}w^a = \begin{bmatrix} e^a \\ w^a \end{bmatrix}^T \begin{bmatrix} A_i^{aT}P + PA_i^a - PB_i^aK_i^a - K_i^{aT}B_i^{aT}P + I & PD_i^a \\ -D_i^{aT}P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} e^a \\ w^a \end{bmatrix} < 0 \tag{3.76}$$

Based on the Schur complement, equation(3.76) can be reformulated as follows:

$$\begin{bmatrix} P^{-1}[PA_i^a + A_i^{aT}P]P^{-1} - PB_i^aN - N^TB_i^{aT}P & P^{-1}PD_i^a & P^{-1} \\ D_i^{aT}PP^{-1} & -\gamma^2 I & 0 \\ I & 0 & -I \end{bmatrix} < 0 \tag{3.77}$$

The preceding equation can be expanded as follows:

$$\begin{bmatrix} A_i^aX + XA_i^{aT} - B_i^aN_i - N_i^TB_i^{aT} & D_i^a & X \\ D_i^{aT} & -\gamma^2 I & 0 \\ X & 0 & -I \end{bmatrix} < 0 \tag{3.78}$$

The controller gain has been determined as follows:

$$K_i^a = N_iX^{-1} \tag{3.79}$$

## 3.5 Conclusion

In this chapter, several complementary control methods are proposed, including the integration of the Mean Value Theorem (MVT) with nonlinear sector transformations. Furthermore, the MVT is employed to express the error dynamics in a manner that minimizes the conservatism inherent in the assumptions about bounded terms.These methods are based on Lyapunov's quadratic theory and are formulated as linear matrix inequalities (LMIs), which can be solved using convex optimization tools.The second part of the chapter emphasizes stabilizing nonlinear systems using Takagi-Sugeno multi-models. It introduces two main stabilization techniques. The first approach uses PDC controllers, developed through the MVT technique. The second approach involves designing robust control based on $H^{\infty}$ performance. This section clearly demonstrates the effectiveness of the robust $H^{\infty}$ with MVT controller in minimizing the impact of disturbances.

Chaptre 4

# Control of TS Multi-Model via non-quadratic Function



## 4.1 Introduction

The preceding chapters established a basic framework for analyzing stability and formulating stabilisation in nonlinear systems, particularly in the context of multi-model frameworks. The main approach revolved around employing Lyapunov's quadratic functions. Although these functions provide useful insights and techniques, their results tend to be on the conservative side, which may restrict their use in some cases. To address this constraint, the current chapter shifts the focus to a new approach by exploring a different category of functions non-quadratic Lyapunov functions, especially those based on curvilinear integrals (LILF).

At the outset of our discussion in the seconde chapter, we highlighted the comparative benefits of adopting LILF over the standard non-quadratic Lyapunov functions (NQLF). The primary advantage of LILF lies in its ability to circumvent the appearance of derivatives of membership functions, thereby facilitating a global analysis of the entire state space of the considered model. However, it has been noted that existing research often presents BMI (Bilinear Matrix Inequalities) conditions for stabilization in more general cases [RW06].

In this chapter, our objective is to broaden the scope of the existing findings by eliminating certain constraints, leading to more relaxed LMI (Linear Matrix Inequality) conditions. Our approach, which is inspired by the study [IMRM12], introduces an innovative strategy for achieving LMI conditions. This strategy is situated within the realm of non-quadratic controller law synthesis using a LILF (Lyapunov functions involving curvilinear integrals) framework, as detailed in[HHBT23] . Central to this approach is the belief that the stability of a T-S (Takagi-Sugeno) system can be consistently guaranteed and upheld.

This chapter further delves into the practical application of these theoretical concepts in its subsequent sections, focusing on numerical examples. Initially, we will thoroughly investigate the realm of innovative control design tailored for nonlinear systems. These designs are specifically structured within the Takagi-Sugeno fuzzy system framework and give particular attention to non-measurable decision variables. Building on this foundation, we then shift to establishing a set of relaxed yet comprehensive conditions, based on LMIs, for implementing state feedback controller conditions. Our methodology not only presents promising prospects but also furnishes examples that address challenges previously highlighted in [RW06]. In the latter half of this chapter, we delve deeply into the practical application of the methodologies discussed, with a focus on control systems designed for state feedback conditions. Our discussion encompasses systems characterized by the Takagi-Sugeno framework, covering both systems that use non-measurable decision variables.

## 4.2 Stabilization of TS Fuzzy Systems via LILF

In the next section, we will explore in detail the design aspects of our proposed controller.A key element of this design is the utilization of the Mean Value Theorem. Additionally, our approach distinctively features a non-quadratic Lyapunov function, with a specific focus on the Line Integral Lyapunov Function (LILF).The combination of these essential mathematical principles is fundamental to our objective of developing an effective controller.

### 4.2.1 Design of the Proposed Control System

To enhance the stability conditions in control design, specifically for state feedback controllers, without the need for bilinear matrix inequalities (BMI), a novel Lyapunov function candidate is introduced. This function, in the form of a line-integral along a trajectory from the origin to the current state, is explored for its effectiveness in the stability and stabilization of TSF systems. This proposed function, as a specific instance of the classical Quadratic Lyapunov Function (QLF), offers more flexible stability conditions. A notable implementation of this concept is presented by the authors in[HHBT23], where they propose a state feedback controller gain for nonlinear systems . This approach employs convex structures paired with a non-quadratic Lyapunov function [HHT24], enhancing robustness against sensor noise. Further expanding on this, Maalej et al. [MKB17]introduced an innovative method for designing state feedback control in nonlinear systems, particularly those characterized by Takagi-Sugeno fuzzy systems with non-measurable premise variables.

We have developed a methodical strategy for controller design, specifically for the nonlinear system detailed by (3.1). Here is a synopsis of the structure we suggest:

$$u = -\left(\sum_{j=1}^{q} h_j(c_j)P_j\right)^{-1} k(x - x_c) \tag{4.1}$$

$$\begin{cases} \dot{x} = \sum_{i=1}^{r} h_i(x)\left(A_i x - B_i \left(\sum_{j=1}^{q} h_j(c_j)P_j\right)^{-1} k(x - x_c)\right) \\ y = Cx \end{cases} \tag{4.2}$$

For the essential transformation as indicated in subsection 3.3.1 and equation (3.19) , we define the following matrices:

$$A_0 = \frac{1}{r}\sum_{i=1}^{r} A_i, \quad B_0 = \frac{1}{r}\sum_{i=1}^{r} B_i \tag{4.2}$$

As a result, the TSF system, which is delineated by equations (3.15 ) and(4.2) , can be reformulated as follows

$$\begin{cases} \dot{x} = A_0 x + B_0 u + \sum_{i=1}^{r} h_i(x)(\bar{A}_i x + \bar{B}_i u) \\ y = Cx \end{cases} \tag{4.3}$$

Additionally, the observer's structure is outlined by the equations:

$$\begin{cases} \dot{x} = A_0 x - B_0 \left( \sum_{j=1}^{q} h_j(c_j) P_j \right)^{-1} K(x - x_c) + \sum_{i=1}^{r} h_i(x) \left( \bar{A}_i x - \bar{B}_i \left( \sum_{j=1}^{q} h_j(c_j) P_j \right)^{-1} K(x - x_c) \right) \\ y = Cx \end{cases} \tag{4.4}$$

We aim to find the appropriate controller gain, denoted as $K$, that ensures the dynamics state of the system stabilize over time, ideally reaching zero as time approaches infinity.

When we integrate equations (4.2) and(4.3) into equation (4.4), the resulting equation is:

$$\dot{x} = \left( A_0 - B_0 \left( \sum_{j=1}^{q} h_j(c_j) P_j \right)^{-1} K \right) x + \phi(x, x_c) \tag{4.5}$$

Here, the term $\phi(x, x_c, u)$ is defined as:

$$\phi(x, x_c) = B_0 \left( \sum_{j=1}^{q} h_j(c_j) P_j \right)^{-1} K x_c + \sum_{i=1}^{r} h_i(x) \left( \bar{A}_i x - \bar{B}_i \left( \sum_{j=1}^{q} h_j(c_j) P_j \right)^{-1} K(x - x_c) \right) \tag{4.6}$$

By invoking Theorem 3.3.2 , it leads us to conclude that there is a function $c$ within the range $[x, x_c]$, which meets the following condition (for more detailssee see Assumption in annexe (4.3))

$$\phi(x, x_c) = \frac{\partial \phi(c)}{\partial x} x \tag{4.7}$$

and

$$\phi(x, x_r) = \sum_{i=1}^{n} \sum_{j=1}^{n} e_i(n) e_j(T) \left. \frac{\partial \phi_j}{\partial x_i} \right|_{c_i} \times x \tag{4.8}$$

Consequently, the dynamics of the state can be succinctly expressed as follows:

$$\dot{x} = \left( A_0 - B_0 \left( \sum_{j=1}^{q} h_j(c_j) P_j \right)^{-1} K + \sum_{i=1}^{n} \sum_{j=1}^{n} e_n(i) e_n^T(j) \frac{\partial \phi_j}{\partial x_i} \bigg|_{c_i} \right) x \tag{4.9}$$

$$\dot{x} = \left( -B_0 \left( \sum_{j=1}^{q} h_j(c_j) P_j \right)^{-1} K + \sum_{i=1}^{q} h_i(c_i) \mathcal{A}_i \right) x \tag{4.10}$$

To comprehensively describe the dynamics of the state , we define the matrix $\delta_{ij}$ as: $\delta_{ij} = \mathcal{A}_i - B_0 P_j^{-1} K$.

$$\dot{x} = \sum_{i=1}^{q} \sum_{j=1}^{q} h_i(c_i) h_j(c_j) \delta_{ij} x \tag{4.11}$$

***Lemma 4.2:*** [Tanaka et al. [TIW98]]

Let $s$ represent the maximum number of fuzzy rules that can be activated at the same time. It is given that $s$ satisfies the constraint $1 \leq s \leq r$. Under these conditions, the membership functions of the fuzzy rules adhere to the inequality:

$$(s-1) \sum_{i=1}^{r} h_i^2(x) - 2 \sum_{i=1}^{r} \sum_{j>i} h_i(x) h_j(x) \geq 0. \tag{4.12}$$

This inequality is valid for any value of $x$

The conditions for stabilization convergence are now presented in the following theorem.

**Theorem 4.2.1:**

The TS fuzzy system, detailed in (4.3), with fuzzy controller (4.1) attains asymptotic stability provided there are positive definite matrices $P_i$ and $X > 0$ that satisfy the Linear Matrix Inequalities (LMIs) for each $i, j$ ranging from 1 to $q$.

$$P_i > 0 \tag{4.13}$$

$$\mathcal{A}_i^T P_i + P_i \mathcal{A}_i - K^T B^T - BK + (S-1)X < 0 \quad i = j \tag{4.14}$$

$$\mathcal{A}_i^T P_j + P_j \mathcal{A}_i - K^T B^T - BK \quad + \mathcal{A}_j^T P_i + P_i \mathcal{A}_j - K^T B^T - BK \leq 2X \quad i < j \tag{4.15}$$

**Proof:**

Take into account the LILF

$$V(x) = 2 \int_{l(0,x)} f^T(\theta) d\theta \tag{4.16}$$

The time derivative of $V(x)$ is described by:

$$\dot{V}(x) = x^T \sum_{j=1}^{q} h_j(c_j) P_j x + x^T \sum_{j=1}^{q} h_j(c_j) P_j \dot{x} \tag{4.17}$$

then

$$\dot{V}(x) = \sum_{i=1}^{q} \sum_{j=1}^{q} h_i(c_i) h_j(c_j) x^T M_{ij} x \tag{4.18}$$

$$\dot{V}(x) = x^T \left( \sum_{i=1}^{q} h_i(c) M_{ii} + 2 \sum_{i=1}^{q} \sum_{j>i}^{q} h_i(c) h_j(c) X \right) x \tag{4.19}$$

with

$$M_{ii} = \mathcal{A}_i^T P_i + P_i \mathcal{A}_i - K^T B^T - BK \tag{4.20}$$

and

$$2X = \mathcal{A}_i^T P_j + P_j A_i - K^T B^T - BK + \mathcal{A}_j^T P_i + P_i \mathcal{A}_j - K^T B^T - BK \tag{4.21}$$

Therefore, applying Lemma (4.2.1) allows us to reformulate equation (4.22) in the following way:

$$\dot{V}(x) = x^T \left( \sum_{i=1}^{q} h_i^2(c_i) M_{ii} + (S-1) X \right) x \tag{4.22}$$

For all $x \neq 0 \Rightarrow V(x) < 0$

### 4.2.2 Enhanced of The Controller Performance

In this subsection, we delve into the initial D-stability conditions, originally introduced for linear systems by Chilali and Gahinet in 1996 [Gah96]. Building upon this foundation, Peaucelle et al. further developed these concepts in 2000 [PABB00], extending their application to uncertain linear systems, particularly those represented in convex polytopic forms. Their primary goal was to reduce the conservatism associated with Linear Matrix Inequality (LMI) conditions in linear systems. Further advancements in this area, especially pertaining to Takagi-Sugeno (T-S) fuzzy systems, have been documented in various studies, including those by [Ass14], [SBAT22], and [BLL+15].

Moreover, the research by Toulotte et al. in 2008 [TDGB08] introduced specialized LMI constraints. These constraints are specifically associated with certain regions, encompassing decay rate, conical sector, and circle, and were integrated into the conventional stabilization criteria for uncertain systems, as noted in [Che17]. Building on the D-stability concept from [Gah96], we describe the requirement to position the poles of the

$i$th subsystem within the shaded region shown in Fig(3.4). This positioning is articulated through Linear Matrix Inequalities (LMIs) in the following Lemma.

***Lemma 4.2:***[RW06] A condition for the $\mathcal{D}$-stability of the T-S model, as indicated in (4.28), is the existence of matrices $P_j$ satisfying $P_j = P_j^T > 0$, in addition to matrices $Q$ and $K$, which together must comply with the established Linear Matrix Inequality (LMI) criteria:

$$\Gamma_i^k + (\eta - 1)Q < 0, \qquad \forall i \in \{1, \dots, r\} \text{ and } k \in \{1, 2, 3\} \tag{4.23}$$

$$\Gamma_{ij} + \Gamma_{ji} - 2Q < 0, \qquad \forall (i, j) \in \{1, \dots, r\}^2 \text{ and } i < j \tag{4.24}$$

with

$$\Gamma_{ii}^1 = G_{ii} + G_{ii}^T + \alpha_i P_i < 0,$$

$$\Gamma_{ii}^2 = \begin{bmatrix} \sin\theta & (G_{ii} + G_{ii}^T) \\ \cos\theta & (G_{ii} - G_{ii}^T) \end{bmatrix} < 0,$$

$$\Gamma_{ii}^3 = \begin{bmatrix} -rP_j & G_{ii} \\ G_{ii}^T & -rP_j \end{bmatrix} < 0,$$

where

$$G_{ii} = (A_i + A_0)P_i - BK$$

### 4.2.3 Simulation Examples

In this subsection, we explore two illustrative examples that highlight the potential and robustness of our proposed observer design for continuous TS fuzzy systems. The performance of the various techniques was evaluated using the MATLAB/Simulink environment.The first example facilitates a direct comparison between the conservativeness of our proposed line integral Lyapunov function (2.23) and the traditional quadratic function (2.16). This comparison is designed to illuminate the subtleties and benefits of each approach. Following this, our second example adopts a more practical perspective. Here, we demonstrate the numerical simulation applicability of our methodology by outlining its implementation in the design of an observer for a significant engineering challenge: a flexible joint robot. This not only showcases the merits of our design but also underscores its utility in complex systems.

**Numerical Example 1**

Let's explore the following non-linear mode.

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - 5 & -4 \\ 0.2(x_1(t) * b - x_2(t)) + x_2(t) * a & -x_2(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{4.25}$$

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The T-S fuzzy model is given as the following form:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(x)(A_i x(t) + B_i u(t)) \tag{4.26}$$

Utilizing the technique of sector nonlinearity, the T-S fuzzy model can be depicted in the subsequent manner:

$$\begin{cases} \dot{x} = A_0 x + B_0 u + \sum_{i=1}^{r} h_i(x)(A_i x + B_i u) \\ y = Cx \end{cases} \tag{4.27}$$

where the $A_i$ matrices are given as follows :

$$A_1 = \begin{bmatrix} -12 & -4 \\ 0.2(7b - 6) + 6a & -6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -12 & -4 \\ 0.2(7b - 1) + a & -1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -6 & -4 \\ 0.2(b - 6) + 6a & -6 \end{bmatrix} \quad A_4 = \begin{bmatrix} -6 & -4 \\ 0.2(b - 1) + a & -1 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The dynamics of state error can be characterized as follows:

$$\dot{x} = \sum_{i=1}^{4} \sum_{j=1}^{4} h_i(c_i) h_j(c_j) \left( \mathcal{A}_i - B P_j^{-1} K \right) x \tag{4.28}$$

$$\mathcal{A}_i = \begin{bmatrix} -2x_1(t) - 5 & 0 \\ 0.4 b_1 x_1(t) - (0.2 - a_2) x_2(t) & -2x_2(t) - (0.2 - a_1) x_1(t) \end{bmatrix}$$

$$h_1(t) = \frac{x_1(t) - 1}{6}$$
$$h_2(t) = \frac{7 - x_1(t)}{6}$$
$$h_3(t) = \frac{x_2(t) - 1}{5}$$
$$h_4(t) = \frac{6 - x_2(t)}{5}$$

The intervals $a \in [0, 50]$ and $b \in [0, 80]$ define the range of values utilized to assess the feasible regions for the problems based on Linear Matrix Inequalities (LMIs), as explored in Theorems 2.7.1 and 4.2.1.

Figure 4.1 presents a comparative analysis of the stability regions obtained from the Linear Matrix Inequalities (LMIs) associated with the conventional Lyapunov function described in Theorem 2.7.1, against the non-quadratic constraints delineated in Theorem 4.2.1. The comparative results evidently confirm that the non-quadratic Lyapunov functions introduced herein yield less conservative outcomes than their classical quadratic counterparts, thereby signifying an advancement in reducing the conservativeness of the stabilization problem through the adoption of non-quadratic Lyapunov functions, as advocated by the current study.

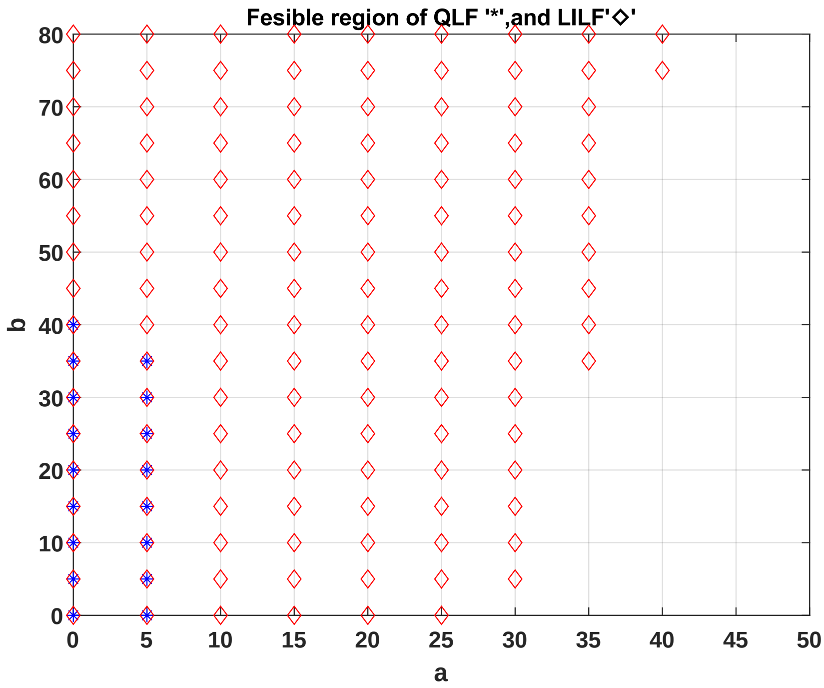



Fig 4.1: Comparison of the feasibility fields on parameter space $a \times b$ obtained via Theorems 2.7.1 "$*$" and 4.2.1 "$\diamond$".

**Remark:** When the parameters $a$ and $b$ are assigned values of 5 and 40 respectively, the stabilization criteria outlined in Theorem 2.7.1 are not met. This indicates that achieving a stable controller with these values is not feasible using conventional methods. However, employing Theorem 4.2.1 offers a resolution to the stabilization issues, aligning them within the constraints of linear matrix inequalities.

By employing the LMI control toolbox in MatLab, we conducted a stability analysis for various combinations of parameters $a$ and $b$. Figure 4.1 offers a comparative view of the feasibility fields derived from Theorems 2.7.1 and 4.2.1. The findings are revealing. Theorem 2.7.1 yields 17 feasible solutions, which represent 9.09% of the total. In contrast, Theorem 4.2.1 presents a remarkable 130 solutions, or 69.51%. This stark difference underlines the broader feasibility scope of Theorem 4.2.1 compared to Theorem 2.7.1. Such a difference highlights the significant decrease in conservatism brought about by the LMI conditions introduced in our study, especially when compared to the traditional quadratic function approach.

To gain a deeper understanding of the complexities involved in these LMI-based conditions, we have detailed their computational aspects in Table (4.1). Our examination focuses on three critical factors in conditions based on inequalities: the total number of constraints (C), the sum of decision variables (V), and, importantly, the ratio between these two elements. This ratio is crucial as it indicates the computational intensity of the process. It highlights a key trade-off: higher complexity often leads to results that are less conservative, striking a balance between computational demand and the precision of outcomes.

Table 4.1: Performance evaluation indicators.

| **Method** | **Feasibility %** | **Nb of dec.var.(V)** | **Nb of LMIs (C)** | $n = V/C$ |
|---|---|---|---|---|
| **Theorem 2** | 9.09 % | 1 | 5 | 0.2 |
| **Theorem 3** | 69.51 % | 5 | 15 | 0.33 |

For the specific case where $a = -10$ and $b = -60$, Theorem 4.2.1 yields a single viable solution. This solution is encapsulated in the state-feedback control law detailed in equation (4.1), with the gain matrices structured in accordance with linear matrix inequality constraints. Furthermore, implementing Theorem 3.3.2 with this model successfully produces a robust solution, complete with extended controller and observer gains.

The control gain is presented as follows:

$$K = \begin{bmatrix} -0.0669 & 0.3807 \end{bmatrix}$$

The observer gain can be given as follows:

$$L = \begin{bmatrix} 0.4295 & -0.3219 \\ -0.3219 & 0.5488 \end{bmatrix}$$

For detailed insights into the observer structure, refer to the annex (for more details see Annex (4.3)).

The positive definite matrices can be obtained as :

$$P_1 = \begin{bmatrix} 0.0003 & 0.0009 \\ 0.0009 & 0.0045 \end{bmatrix}, P_2 = \begin{bmatrix} 0.0003 & 0.0009 \\ 0.0009 & 0.0045 \end{bmatrix}, P_3 = \begin{bmatrix} 0.0005 & 0.0009 \\ 0.0009 & 0.0032 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 0.0005 & 0.0009 \\ 0.0009 & 0.0045 \end{bmatrix}, X = \begin{bmatrix} 0.0002 & -0.0006 \\ -0.0006 & 0.0210 \end{bmatrix}$$

Initially, we set the conditions as $x_0(t) = [0\ 0]$ and $\hat{x}_0(t) = [0\ 1]$. The simulation's results, which illustrate the closed-loop state trajectories, are shown in Figure 4.2.

Figure 4.6 suggests that system stability can be achieved by increasing the values of the monitoring system and controller gains.

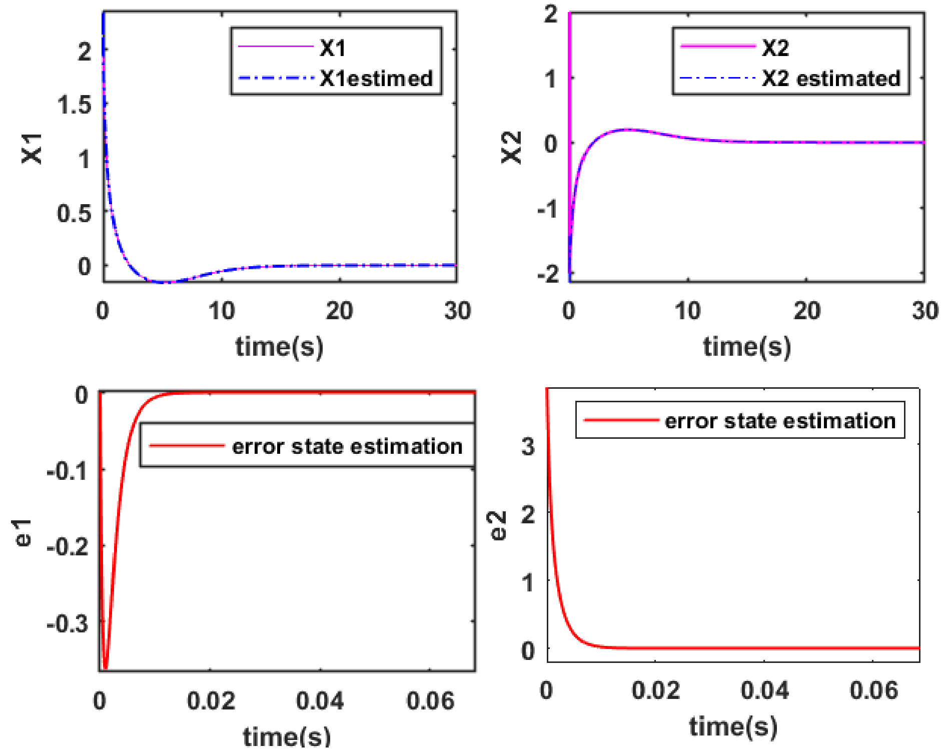



Fig 4.2: State evolution of T-S fuzzy model under control law obtained via Theorem 4.2.1.

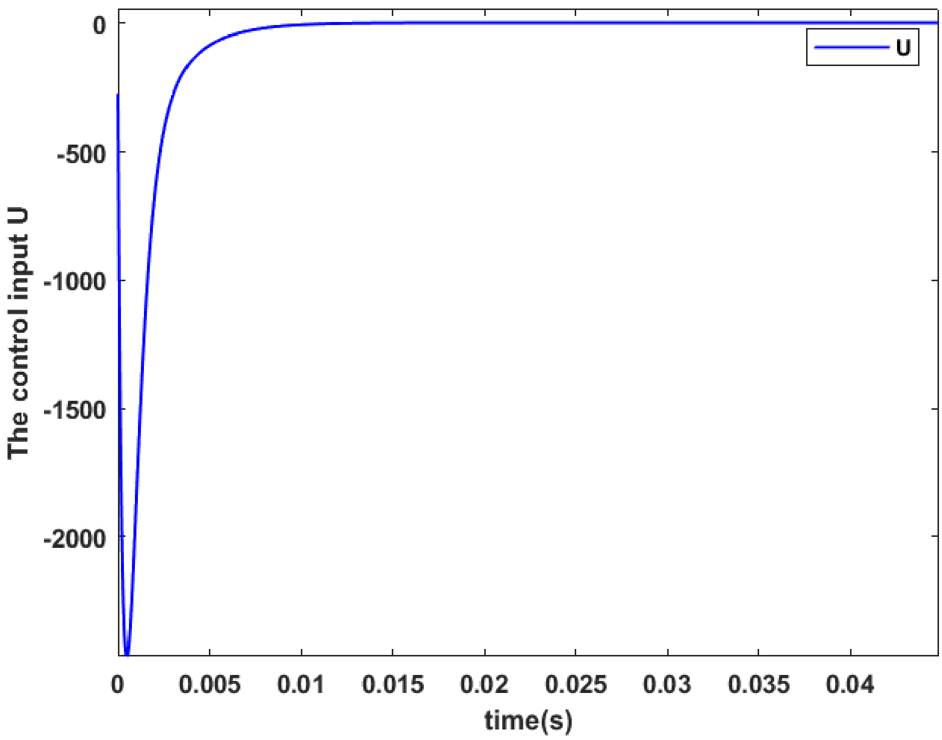



Fig 4.3: Control law evolution with gains obtained via Theorem 4.2.1

## Example 2:Application to Single-Link Flexible Joint Robot

In this part , we exhibit the application of our formulated strategy, tailored for T-S fuzzy systems moderated by a state feedback controller. Our objective is to demonstrate the practicality and efficiency of this strategy. For this purpose, a comprehensive simulation is conducted. This simulation involves a single-link flexible joint robot, details of which are graphically presented in Figure 4.4.

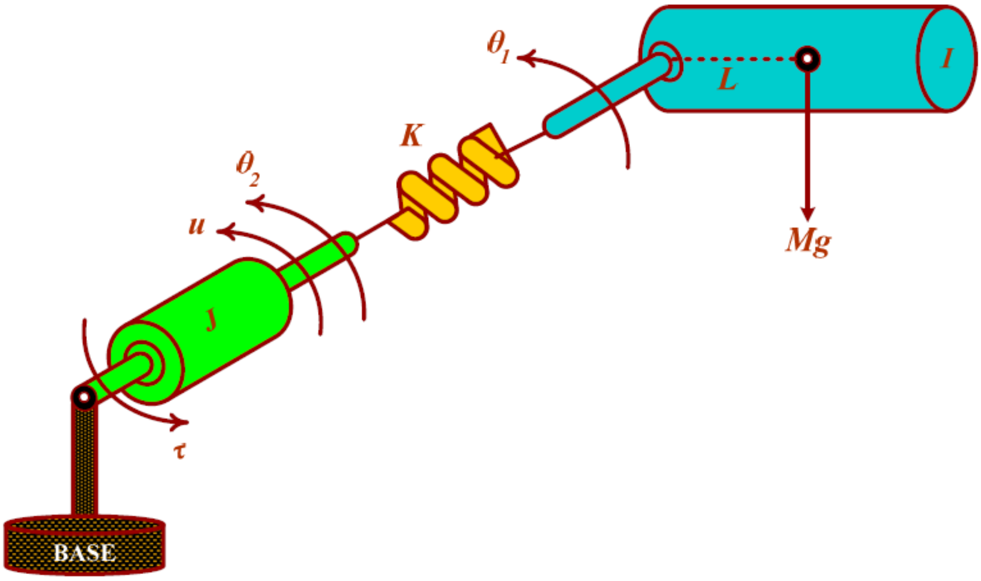



Fig 4.4: Schematic Representation of a Single-Link Flexible Joint Robotic Manipulator

Figure 4.4 depicts the foundational schematic of the single-link manipulator with a flexible joint. The corresponding dynamic model is delineated in Equations as follows [UMR+20]:

$$I\ddot{q}_1 + MgL\sin(q_1) + K(q_1 - q_2) = 0 \tag{4.29}$$

$$J\ddot{q}_2 - K(q_1 - q_2) = \tau \tag{4.30}$$

Here, $q_1$ represents the angular position of the link, and $q_2$ denotes the angular position of the motor. The parameters $I$ and $J$ are the inertias of the link and the motor, respectively. The term $M$ signifies the mass of the link, $g$ is the acceleration due to gravity, $L$ is the distance from the mass to the pivot point, $K$ is the spring stiffness constant, and $\tau$ is the applied torque on the motor shaft.

To facilitate analysis, the system is often expressed in the state space form. Hence, the nonlinear dynamic model of the single-link robotic manipulator with a flexible joint, encapsulated by equations (4.29 , 4.30) is transposed into the state-space representation.

**a/Takagi-Sugeno Model Design:**

The nonlinear model for a single-link flexible joint robot arm is encapsulated by the equations below [KWP07]:

$$\begin{cases} \dot{\theta}_m(t) = \omega_m(t) \\ \dot{\omega}_m(t) = \frac{k}{J_m}(\theta_l(t) - \theta_m(t)) - \frac{B_v}{J_m}\omega_m(t) + \frac{K_t}{J_m}u(t) \\ \dot{\theta}_l(t) = \omega_l(t) \\ \dot{\omega}_l(t) = -\frac{k}{J_l}(\theta_l(t) - \theta_m(t)) - \frac{mgh}{J_l}\sin(\theta_l(t)) \end{cases} \tag{4.31}$$

In these equations, $J_m$ denotes the motor's inertia, $J_l$ represents the controlled link's inertia, $m$ stands for the mass of the link, $h$ indicates the center of mass, $g$ is the gravitational acceleration, $k$ is the spring constant, $B_v$ is the coefficient of viscous friction, and $K_t$ is the gain of the amplifier. Table 4.2 [EHRBB22] provides the values for these parameters.

Table 4.2: System parameters used in simulations

| **Parameter** | **Meaning** | **Value** |
|---|---|---|
| $J_m$ | Motor inertia | $3.7 \times 10^{-3}\, Kg \cdot m^2$ |
| $J_l$ | Link inertia | $9.3 \times 10^{-2}\, Kg \cdot m^2$ |
| $h$ | Link length | $1.55 \times 10^{-2}\, m$ |
| $m$ | Pointer mass | $2.04 \times 10^{-1}\, Kg$ |
| $k$ | Torsional spring constant | $1.8 \times 10^{-1}\, N \cdot m \cdot rad^{-1}$ |
| $B_v$ | Viscous friction coefficient | $4.6 \times 10^{-3}\, N \cdot m \cdot s^{-1}$ |
| $K_t$ | Amplifier gain | $8 \times 10^{-2}\, N \cdot m \cdot V^{-1}$ |

The state vector $x(t)$ and the output vector $y(t)$ are respectively defined as $x(t) = [\theta_m(t) \quad \omega_m(t) \quad \theta_l(t) \quad \omega_l(t)]^{\mathrm{T}}$ and$y(t) = [\theta_m(t) \quad \omega_m(t)]^{\mathrm{T}}$, where $x_1(t)$ denotes the motor's angular rotation, $x_2(t)$ is the motor's angular velocity, $x_3(t)$ represents the link's angular position, and $x_4(t)$ signifies the link's angular velocity. The dynamics described by system (4.31) are thus formulated.

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{4.32}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_m} & -\frac{B_v}{J_m} & \frac{k}{J_m} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{J_l} & 0 & -\frac{k}{J_l} - \frac{mgh}{J_l}\sin(x_3) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{K_t}{J_m} \\ 0 \\ 0 \end{bmatrix}$$

In developing the T-S fuzzy model for a single-link flexible joint robot, we introduce the premise variable defined as:

$$\xi(t) = \frac{\sin(x_3)}{x_3} \tag{4.33}$$

where $x_3$ is constrained to the interval $\left[-\frac{\pi}{2}, 0\right]$. Given that there is one nonlinearity ($n = 1$), the overall model is composed of $r = 2^n = 2$ sub-models.

Subsequently, the single-link flexible joint robot system, designated as (4.31), is characterized using the Takagi-Sugeno (T-S) fuzzy model:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases} \tag{4.34}$$

The matrices $A_i$ and $B_i$for each sub-model have been computed and are presented below:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 1.95 & 0 & -2.28 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 1.95 & 0 & -2.16 & 0 \end{bmatrix}, \tag{4.35}$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}, \tag{4.36}$$

$$\begin{cases} \mu_1(\xi(t)) = \dfrac{\xi(t) + 0.2172}{1.2172}, \\ \mu_2(\xi(t)) = \dfrac{1 - \xi(t)}{1.2172} \end{cases} \tag{4.37}$$

By solving the Linear Matrix Inequality (LMI) presented in equation of theorem 4.2.1, we obtain a feasible solution. This solution yields the subsequent matrices for the positive definite conditions and the control law, which are specified as follows:

$$P_1 = \begin{bmatrix} 0.1010 & -0.1546 & 0.0664 & -0.1400 \\ -0.1546 & 2.0511 & 0.0478 & -0.3945 \\ 0.0664 & 0.0478 & 0.0951 & -0.1071 \\ -0.1400 & -0.3945 & -0.1071 & 0.7661 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.0966 & -0.1546 & 0.0664 & -0.1400 \\ -0.1546 & 1.9427 & 0.0478 & -0.3945 \\ 0.0664 & 0.0478 & 0.1029 & -0.1071 \\ -0.1400 & -0.3945 & -0.1071 & 0.7661 \end{bmatrix}$$

$$X = \begin{bmatrix} 0.1093 & 0.1895 & 0.0205 & -0.0370 \\ 0.1895 & 1.5220 & 0.0854 & 0.0073 \\ 0.0205 & 0.0854 & 0.0464 & -0.0039 \\ -0.0370 & 0.0073 & -0.0039 & 0.1748 \end{bmatrix}, \quad K = \begin{bmatrix} 0.0419 & 0.4082 & 0.0605 & -0.0791 \end{bmatrix}$$

**b/Simulation Validation and Discussion:**

To validate the controller design, the simulations have been achieved by using MATLAB/Simulink. In Table 4.2, all of the parameters are given which are used in simulations. The initial conditions set for the simulations specify that $x(1) = 0.5, x(2) = 0.8, x(3) = 0.8, x(4) = 0.5$

These features emphasize the reliability and robustness of the proposed fuzzy controller, showcasing its capability to handle the FJR's complex dynamics with precision. The controller's in figure accuracy in navigating intricate behaviors demonstrates its effectiveness. This underscores the controller's potential in ensuring precise and stable operation within such dynamic environments.

In control systems, convergence to zeros signifies that the system's output methodically decreases and stabilizes at zero over time

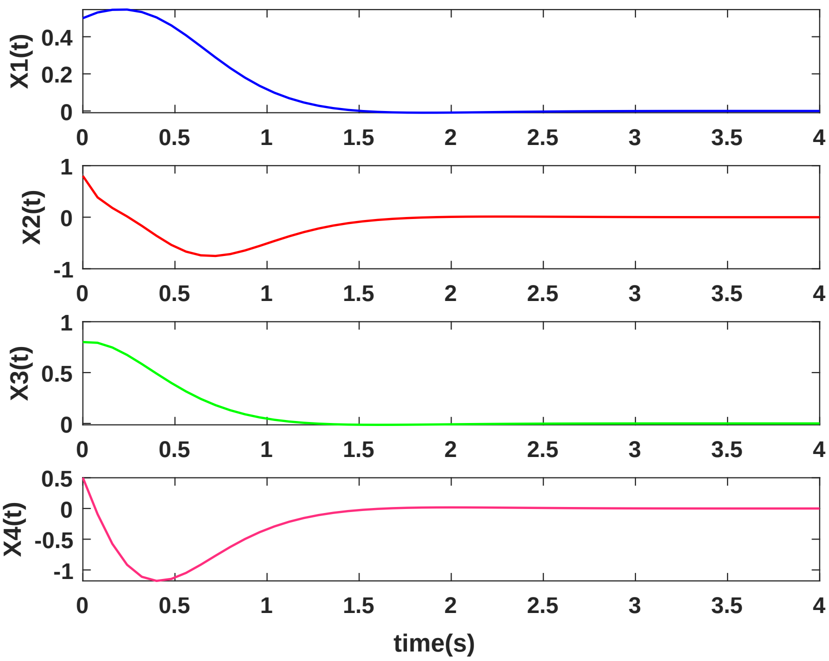



Fig 4.5: State evolution of T-S fuzzy model under control law obtained via Theorem 4.2.1.

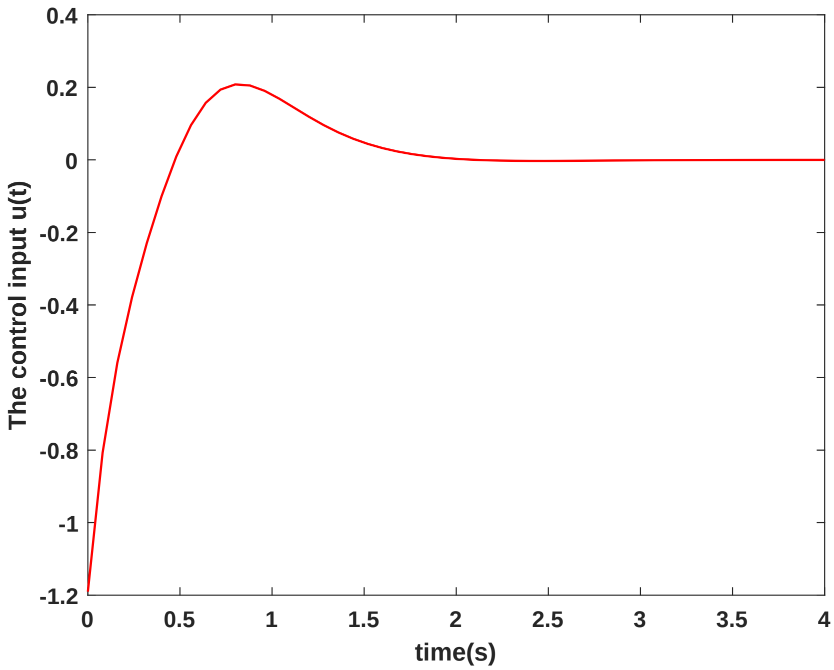



Fig 4.6: Control law evolution with gains obtained via Theorem 4.2.1

## 4.3 Conclusions

This chapter delves into advanced strategies for designing controllers for continuous-time nonlinear models, surpassing current standards in modern research. A significant advancement discussed is the controller design based on non-measurable premises. This innovative approach adeptly applies a convex reinterpretation of the model, specifically focusing on its Takagi-Sugeno Fuzzy (TSF) framework.

A subsequent set of solutions explores scenarios involving unmeasurable premises. Here, the challenge of state feedback control is transformed into a more manageable convex problem. This transformation is facilitated by the use of a line integral Lyapunov function combined with the differential mean value theorem.

In conclusion, a critical aspect emphasized in this chapter is the stability criteria for these controllers. The use of a line integral Lyapunov function has significantly reduced conservatism, enhancing the reliability of these controllers and highlighting their exceptional relevance. To support this argument, the chapter presents simulations and comparative results, offering a compelling demonstration of the effectiveness and efficiency of these innovative approaches

# General Conclusion

This thesis addresses the challenge of modeling and controlling nonlinear systems through the application of fuzzy multi-model representations. In our exploration of modeling, we have chosen to focus on the Takagi-Sugeno (T-S) method for its straightforwardness, especially in terms of analyzing stability, developing observers and controllers. Multi-model systems can be categorized into two primary structures based on the uniformity of the state vector across sub-models. The first type is the decoupled multi-model, and the second is the Takagi-Sugeno fuzzy multi-model. This latter approach has spurred significant advancements in several areas of automation, including identification, state estimation, and control. The opening chapter iexplores control strategies for nonlinear systems, focusing on techniques like input-output feedback linearization and backstepping control design. It addresses the challenges of obtaining accurate models due to parametric uncertainties and neglected dynamics. Nonlinear adaptive control methods are discussed as solutions to compensate for model inaccuracies by incorporating nominal models and adjusting for uncertainties. Following this, the second chapter delves into the multi-model approach, elucidating the underlying concepts and principles it rests on. The heart of the thesis, embodied in the third and fourth chapters, tackles the challenge of stabilizing nonlinear systems through their multi-model representations. The focus of the third chapter is on the quadratic stabilization of nonlinear systems based MVT approches, using Takagi-Sugeno (TS) multi-models that originate from nonlinear sectors. The interest of recent research has pivoted to crafting robust controllers for nonlinear systems depicted by TS fuzzy models.In the pivotal fourth chapter, we ventured into uncharted territories, marking the zenith of our research contributions. Here, the spotlight was on the innovative design of controllers, engineered for robust stabilization in the face of external disturbances or unidentified inputs. A significant hurdle was the inherent conservatism of previous models within the TS-LMI framework, often leading to overly cautious designs. Our approach was to confront this conservatism directly, aiming to diminish its impact and pave the way for the creation of more efficient and less restrictive controller designs for TS representations.

We explored two main avenues: one involving the use of standard quadratic Lyapunov functions, and the other delving into non-quadratic alternatives. The former approach relies on leveraging multiple convex combinations to achieve stabilization, whereas the latter explores the innovative use of line-integral Lyapunov functions. These methodologies collectively offered more flexible and less stringent conditions for controller design, providing a marked improvement over previous methodologies.

Further enriching our investigation, we ventured into the realm of $H^{\infty}$ performance design. This addition not only expanded our research scope but also added depth to our contributions, presenting a comprehensive exploration of controller design in various scenarios.

To connect theoretical insights with dynamical applications, our thesis is enriched with illustrative examples. The examples demonstrate the effectiveness of our methods, guiding readers from abstract concepts to concrete implementations in future work. The core takeaway from our research is unequivocal: advanced automation techniques are crucial in the field of control and diagnosis, hinting that we are merely beginning to uncover the full spectrum of possibilities.

An intriguing aspect of using the line integral Lyapunov function (LILF) is the requirement for Lyapunov matrices to have off-diagonal elements. This need arises from the path-independent nature of LILF, presenting both a challenge and an opportunity for innovation. Investigating ways to mitigate the constraints of path independence could be fruitful, potentially facilitating the development of advanced higher-order systems.

Exploring fault detection and diagnosis methods based presents another fertile ground for research.

Additionally, the development of fault-tolerant control strategies tailored for nonlinear systems, especially those characterized by Takagi-Sugeno (TS) multi-models, stands out as a vital area for future inquiry

- An intriguing aspect of using the line integral Lyapunov function (LILF) is the requirement for Lyapunov matrices to have off-diagonal elements. This need arises from the path-independent nature of LILF, presenting both a challenge and an opportunity for innovation. Investigating ways to mitigate the constraints of path independence could be fruitful, potentially facilitati.ng the development of advanced higher-order systems.

- Additionally, the development of fault-tolerant control strategies tailored for nonlinear systems, especially those characterized by Takagi-Sugeno (TS) multi-models, stands out as a vital area for future inquiry

# Annex

Table 4.3: Different simulation Converter Parameters

| Parameter | Value | Description |
|---|---|---|
| $V_s$ | $220V$ | Supply voltage |
| $f_s$ | $50HZ$ | Supply frequency |
| $F_{\text{PWM}}$ | $50HZ$ | PWM carrier frequency |
| $R$ | $2m\Omega$ | Line resistance |
| $L$ | $1mH$ | Line inductor |
| $C$ | $660\ \mu\text{F}$ | Output capacitor |
| $R_L$ | $25\Omega$ | load resistance |

Table 4.4: PMSM parameters

| Parameter | Value | Description |
|---|---|---|
| $R$ | $1.4\Omega$ | stator resistance |
| $L_d$ | $0.0066H$ | direct axe Inductance d |
| $L_q$ | $0.0058H$ | quadrature axe Inductance q |
| $f$ | $0.0003881\ N.m.s/rad$ | friction coefficient |
| $J$ | $0.00176\ Kg.m^2$ | Moment of inertia |
| $\phi$ | $0.1564\ Wb$ | lux linkage |
| $P$ | 3 | Number of pole pairs |
| $N$ | $1000\ tr/min$ | rated speed |

**LMI Tools**

LMI methods are based on formulating a given problem as an optimization problem with a linear objective and constraints in the form of Matrix Linear Inequalities (LMI). An LMI constraint in a vector $x \in \mathbb{R}^m$ is of the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \geq 0 \tag{4.38}$$

where the symmetric matrices $F_i = F_i^T \in \mathbb{R}^{N\times N}, i = 1, \cdots, m$, are given.

**Schur's complement**

Consider three matrices $R(x) = R^T(x), Q(x) = Q^T(x)$ and $S(x)$ affine with respect to the variable $x$. The following LMIs are equivalent:

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0,$$

$$R(x) > 0, Q(x) - S(x)R^{-1}(x)S^T(x) > 0,$$

$$Q(x) > 0, R(x) - S^T(x)Q^{-1}(x)S(x) > 0.$$

**Observer structure** [MIM23]

For the nonlinear system described by Equation 1.90, we have formulated a systematic approach to designing the observer. Below is a summary of the proposed framework

$$\begin{cases} \dot{\hat{x}} = \sum_{i=1}^{r} h_i(\hat{x})(A_i\hat{x} + B_i u) + \left(\sum_{j=1}^{q} h_j(\hat{x}; C_j)P_j\right)^{-1} L_0(y - \hat{y}) \\ y = C\hat{x} \end{cases} \tag{4.39}$$

In accordance with the crucial transformation outlined in Subsection 3.3.1 and equation 3.19, we specify the following matrices

$$A_0 = \frac{1}{r}\sum_{i=1}^{r} A_i, \quad B_0 = \frac{1}{r}\sum_{i=1}^{r} B_i \tag{4.2}$$

Thus, the Time-Scale Formulation (TSF) system, as depicted by equations 1.90 and 4.39, is articulated as follows:

$$\begin{cases} \dot{x} = A_0 x + B_0 u + \sum_{i=1}^{r} h_i(x)(\bar{A}_i x + \bar{B}_i u) \\ y = Cx \end{cases} \tag{4.3}$$

Additionally, the observer's structure is outlined by the equations:

$$\begin{cases} \dot{\hat{x}} = A_0\hat{x} + B_0 u + \sum_{i=1}^{r} h_i(\hat{x})(\bar{A}_i\hat{x} + \bar{B}_i u) + \left(\sum_{j=1}^{q} h_j(\hat{x}; C_j)P_j\right)^{-1} L_0(y - \hat{y}) \\ y = C\hat{x} \end{cases} \tag{4.4}$$

To describe the dynamics of the estimation error, denoted by $e = x - \hat{x}$, we refer to the equations that follow:

We aim to find the appropriate observer gain, denoted as $L_0$, that ensures the dynamics of the estimation error stabilize over time, ideally reaching zero as time approaches infinity.

When we integrate equations (4.3) and (4.4) into equation (4.5), the resulting equation is:

$$\dot{e} = \left( A_0 - \left( \sum_{j=1}^{q} h_j(c_j; P_j) \right)^{-1} L_0 C \right) e + \phi(x, \hat{x}, u) \tag{4.6}$$

Here, the term $\phi(x, \hat{x}, u)$ is defined as:

$$\phi(x, \hat{x}, u) = \sum_{i=1}^{r} h_i(x)(\bar{A}_i x + \bar{B}_i u) + \sum_{i=1}^{r} h_i(\hat{x})(\bar{A}_i \hat{x} + \bar{B}_i u) \tag{4.7}$$

Applying Theorem 3.3.2, we deduce the existence of a function $c$ within the range $[x, \hat{x}]$, satisfying:

$$\phi(x) - \phi(\hat{x}) = \frac{\partial \phi(c)}{\partial x}(x - \hat{x}) \tag{4.8}$$

and

$$\phi(x) - \phi(\hat{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} e_i(n) e_j(T) \left. \frac{\partial \phi_j}{\partial x_i} \right|_{c_i} \times (x - \hat{x}) \tag{4.9}$$

Consequently, the dynamics of the state estimation error can be expressed as:

$$\dot{e} = \left( A_0 - \left( \sum_{j=1}^{q} h_j(c_j; P_j) \right)^{-1} L_0 C + \sum_{i=1}^{n} \sum_{j=1}^{n} e_n(i) e_n^T(j) \left. \frac{\partial \phi_j}{\partial x_i} \right|_{c_i} \right) e \tag{4.10}$$

$$\dot{e} = \left( A_0 - \left( \sum_{j=1}^{q} h_j(c_j; P_j) \right)^{-1} L_0 C + \sum_{i=1}^{q} h_i(c_i) \mathcal{A}_i \right) e \tag{4.40}$$

To comprehensively describe the dynamics of the state estimation error, we define the matrix $\delta_{ij}$ as: $\delta_{ij} = \mathcal{A}_i + A_0 - P_j^{-1} L_0 C$.

$$\dot{e} = \sum_{i=1}^{q} \sum_{j=1}^{q} h_i(c_i) h_j(c_j) \delta_{ij} e \tag{4.41}$$

**Assumption**

Considering that the function $\phi(x, u)$is Lipschitz continuous, it ensures that its derivatives are bounded. Hence, this allows us to use equation (2.9 )

$$\underline{\delta}_{ij} \leq \delta_{ij} = \left.\frac{\partial \phi_i}{\partial x_j}\right|_{c_i} < \overline{\delta}_{ij} \tag{4.42}$$

where

$$\underline{\delta}_{ij} = \min \left.\frac{\partial \phi_i}{\partial x_j}\right|_{c_i}, \text{and} \ \ \overline{\delta}_{ij} = \max \left.\frac{\partial \phi_i}{\partial x_j}\right|_{c_i} \tag{4.43}$$

Any manifestation of nonlinearity can be characterized by the following expression:

$$\underline{\delta}_{ij} = \sum_{k=1}^{2} \Lambda_{ij}^{k} \sigma_{ijk} \tag{4.44}$$

where $\sigma_{ij1} = \underline{\delta}_{ij}$ and $\sigma_{ij2} = \overline{\delta}_{ij}$

$$\begin{cases} A_{ij}^{1} = \dfrac{\delta_{ij} - \underline{\delta}_{ij}}{\overline{\delta}_{ij} - \underline{\delta}_{ij}}, \\ A_{ij}^{2} = \dfrac{\overline{\delta}_{ij} - \delta_{ij}}{\overline{\delta}_{ij} - \underline{\delta}_{ij}} \end{cases} \tag{4.45}$$

$$\sum_{k=1}^{2} \Lambda_{ij}^{k}(c_i) = 1; \quad 0 \leq \Lambda_{ij}^{k}(c_i) \leq 1 \quad \text{for} \quad k = 1, 2 \tag{4.46}$$

# *Bibliography*


[ACMR04] Abdelkader Akhenak, Mohammed Chadli, Didier Maquin, and José Ragot. State estimation of uncertain multiple model with unknown inputs. In *43rd IEEE Conference on Decision and Control, CDC'04*, page CDROM. IEEE, 2004. (page 64).

[AGH07] Ibtissem Abdelmalek, Noureddine Goléa, and Mohamed Hadjili. A new fuzzy lyapunov approach to non-quadratic stabilization of takagi-sugeno fuzzy models. *International Journal of Applied Mathematics and Computer Science*, 17(1):39–51, 2007. (page 74).

[Akh04] Abdelkader Akhenak. *Conception d'observateurs non linéaires par approche multimodèle: application au diagnostic*. PhD thesis, éditeur inconnu, 2004. (page 66).

[Ang01] Georgo Z Angelis. System analysis, modelling and control with polytopic linear models. 2001. (page 62).

[Aou12] Nedia Nedia Aouani. *Commande Robuste des systèmes Linéaires continus à Paramètres Variant dans le temps*. PhD thesis, Université du 7 Novembre à Carthage, 2012. (page 73).

[Ass14] Wudhichai Assawinchaichote. Further results on robust fuzzy dynamic systems with lmi d-stability constraints. *International Journal of Applied Mathematics and Computer Science*, 24(4):785–794, 2014. (page 115).

[BEGFB94] Stephen P Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15. SIAM, 1994. (pages 78, 92).

[Bez13] Souad Bezzaoucha. *Commande tolérante aux défauts de systemes non linéaires représentés par des modeles de Takagi-Sugeno*. PhD thesis, Université de Lorraine, 2013. (pages 3, 69).

[BGM13] Tahar Bouarar, Kevin Guelton, and Noureddine Manamanni. Robust non-quadratic static output feedback controller design for takagi–sugeno systems using descriptor redundancy. *Engineering Applications of Artificial Intelligence*, 26(2):739–756, 2013. (page 76).

[Bla01] Yann Blanco. *Stabilisation des modèles Takagi-Sugeno et leur usage pour la commande des systèmes non linéaires*. PhD thesis, 2001. (page 91).

[BLL+15] Jianjun Bai, Renquan Lu, Xia Liu, Anke Xue, and Zhonghua Shi. Fuzzy regional pole placement based on fuzzy lyapunov functions. *Neurocomputing*, 167:467–473, 2015. (page 115).

[BMMR16] Souad Bezzaoucha, Benoît Marx, Didier Maquin, and José Ragot. State and output feedback control for takagi–sugeno systems with saturated actuators. *International Journal of Adaptive Control and Signal Processing*, 30(6):888–905, 2016. (page 89).

[BMR99] Anass Boukhris, Gilles Mourot, and Jose Ragot. Non-linear dynamic system identification: a multi-model approach. *International Journal of Control*, 72(7-8):591–604, 1999. (page 3).

[Bou09] Tahar Bouarar. *Contribution à la synthèse de lois de commande pour les descripteurs de type Takagi-Sugeno incertains et perturbés*. PhD thesis, Reims, 2009. (page 67).

[BOU23] Anouar BOUKHLOUF. *Contribution à la commande des systèmes non linéaires: application à la machine synchrone à réluctance variable*. PhD thesis, Université Mohamed Khider Biskra, 2023. (pages 4, 62).

[BV97] Stephen Boyd and Lieven Vandenberghe. Introduction to convex optimization with engineering applications. *Course notes*, 1997. (page 102).

[CB12] Mohammed Chadli and Pierre Borne. *Multiple models approach in automation: Takagi-Sugeno fuzzy systems*. John Wiley & Sons, 2012. (pages 3, 67, 68).

[CEH07] M Chadli and A El Hajjaji. Moment robust fuzzy observer-based control for improving driving stability. *International Journal of Vehicle Autonomous Systems*, 5(3):326–344, 2007. (page 90).

[Cha02] Mohammed Chadli. *Stabilité et commande de systèmes décrits par des multimodèles*. PhD thesis, Vandoeuvre-les-Nancy, INPL, 2002. (pages 75, 76, 77, 78, 89).

[Che17] Abdelmadjid Cherifi. *Contribution à la commande des modèles Takagi-Sugeno: approche non-quadratique et synthèse D-stable*. PhD thesis, Reims, 2017. (page 115).

[EDBB10] Nesrine Elfelly, Jean-Yves Dieulot, Mohamed Benrejeb, and Pierre Borne. A new approach for multimodel identification of complex systems based on both neural and fuzzy clustering algorithms. *Engineering Applications of Artificial Intelligence*, 23(7):1064–1071, 2010. (pages 3, 68).

[EHRBB22] Mohamed Elouni, Habib Hamdi, Bouali Rabaoui, and Naceur Ben-Hadj Braiek. Adaptive pid fault-tolerant tracking controller for takagi-sugeno fuzzy systems with actuator faults: Application to single-link flexible joint robot. *International Journal of Robotics & Control Systems*, 2(3), 2022. (pages 92, 124).

[EL02] MJ Er and DH Lin. A new approach for stabilizing nonlinear systems with time delays. *International Journal of Intelligent Systems*, 17(3):289–302, 2002. (page 89).

[Fil91] Dimiter Filev. Fuzzy modeling of complex systems. *International Journal of Approximate Reasoning*, 5(3):281–290, 1991. (pages 4, 65).

[Gah96] P Gahinet. $H^\infty$ design with pole placement constraints: An lmi approach. *IEEE Trans. on Auto. Cont.*, 45(3):358–367, 1996. (page 115).

[Gas00] Komi Gasso. *Identification des systèmes dynamiques non-linéaires: approche multi-modèle*. PhD thesis, 2000. (pages 3, 67).

[GBM09] Kevin Guelton, Tahar Bouarar, and Noureddine Manamanni. Robust dynamic output feedback fuzzy lyapunov stabilization of takagi-sugeno systems a descriptor redundancy approach. *Fuzzy sets and systems*, 160(19):2796–2811, 2009. (page 90).

[GGB+09] Kevin Guelton, Thierry-Marie Guerra, Miguel Bernal, Tahar Bouarar, and Noureddine Manamanni. Comments on fuzzy control systems design via fuzzy lyapunov functions. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 40(3):970–972, 2009. (page 78).

[GKVT06] Thierry-Marie Guerra, Alexandre Kruszewski, Laurent Vermeiren, and Helene Tirmant. Conditions of output stabilization for nonlinear models in the takagi–sugeno's form. *Fuzzy Sets and Systems*, 157(9):1248–1259, 2006. (page 90).

[Gue14] Kevin Guelton. Some refinements on stability analysis and stabilization of second order ts models using line-integral lyapunov functions. *IFAC Proceedings Volumes*, 47(3):7988–7993, 2014. (pages 83, 84).

[GVDB99] Thierry Marie Guerra, Laurent Vermeiren, François Delmotte, and Pierre Borne. Lois de commande pour systèmes flous continus. *Journal européen des systèmes automatisés*, 33(4):489–527, 1999. (page 89).

[Ham12] Habib Hamdi. *Approche Multi-Modèle pour l'Observation d'état et le Diagnostic des Systèmes Singuliers Non Linéaires*. PhD thesis, Ecole Polytechnique de Tunis, 2012. (pages 60, 63, 70).

[Ham15] Mohamed Yacine Hammoudi. *Contribution à la commande et à l'observation dans l'association convertisseurs machine*. PhD thesis, Université Mohamed Khider-Biskra, 2015. (pages 4, 63, 65, 67, 97, 98).

[HHBT23] Rabiaa Houili, Mohamed Yacine Hammoudi, Mohamed Benbouzid, and Abdennacer Titaouine. Observer-based controller using line integral lyapunov fuzzy function for ts fuzzy systems: Application to induction motors. *Machines*, 11(3):374, 2023. (pages 111, 112).

[HHH24] Wail Hamdi, Mohamed Yacine Hammoudi, and Madina Hamiane. Proportional multi-integral observer design for takagi–sugeno systems with unmeasurable premise variables: Conservatism reduction via polyquadratic lyapunov function. *European Journal of Control*, 75:100915, 2024. (page 76).

[HHT24] Rabiaa Houili, Mohamed Yacine Hammoudi, and Abdenacer Titaouine. Line integral fuzzy lyapunov function based control design of takagi-sugeno fuzzy systems using mean value theorem. *AIJR Abstracts*, pages 22–23, 2024. (page 112).

[Ich09] Dalil Ichalal. *Estimation et diagnostic de systèmes non linéaires décrits par un modèle de Takagi-Sugeno*. PhD thesis, Institut National Polytechnique de Lorraine-INPL, 2009. (page 101).

[II95] Alberto Isidori and Alberto Isidori. Local decompositions of control systems. *Nonlinear control systems*, pages 1–76, 1995. (page 11).

[Jab11] Dalel Jabri. *Contribution à la synthèse de lois de commande pour les systèmes de type Takagi-Sugeno et/ou hybrides interconnectés*. PhD thesis, Reims, 2011. (page 66).

[JRA99] Mikael Johansson, Anders Rantzer, and K-E Arzen. Piecewise quadratic stability of fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 7(6):713–722, 1999. (page 84).

[KA01] Petar Kokotović and Murat Arcak. Constructive nonlinear control: a historical perspective. *Automatica*, 37(5):637–662, 2001. (page 28).

[Kad00] Azeddine Kaddouri. *Etude d'une commande non-linéaire adaptative d'une machine synchrone à aimants permanents*. Université Laval, 2000. (pages 27, 47, 50).

[Kha02] Hassan K Khalil. *Control of nonlinear systems*. Prentice Hall, New York, NY, 2002. (pages 28, 31).

[KKK95] Miroslav Krstic, Petar V Kokotovic, and Ioannis Kanellakopoulos. *Nonlinear and adaptive control design*. John Wiley & Sons, Inc., 1995. (page 28).

[Kok92] Petar V Kokotovic. The joy of feedback: nonlinear and adaptive. *IEEE control systems magazine*, 12(3):7–17, 1992. (page 28).

[Kru06] Alexandre Kruszewski. *Lois de commande pour une classe de modèles non linéaires sous la forme Takagi-Sugeno: Mise sous forme LMI.* PhD thesis, Université de Valenciennes et du Hainaut-Cambresis, 2006. (page 74).

[Kso99] M Ksouri. *Contribution à la commande multimodèles des processus complexes.* PhD thesis, Thèse de Doctorat, Université des Sciences et Technologies de Lille, Lille, 1999. (page 63).

[KTIT92] Shunji Kawamoto, Kensho Tada, Atsushi Ishigame, and Tsuneo Taniguchi. An approach to stability analysis of second order fuzzy systems. In *Fuzzy Systems, 1992., IEEE International Conference on*, pages 1427–1434. IEEE, 1992. (page 3).

[KWP07] Józef Korbicz, Marcin Witczak, and Vicenc Puig. Lmi-based strategies for designing observers and unknown input observers for non-linear discrete-time systems. *Bulletin of the Polish Academy of Sciences: Technical Sciences*, pages 31–42, 2007. (page 123).

[LK97] Zhong-Hua Li and Miroslav Krstić. Optimal design of adaptive tracking controllers for non-linear systems. *Automatica*, 33(8):1459–1473, 1997. (pages 32, 40).

[LLL98] Dong-Choon Lee, Ki-Do Lee, and G-Myoung Lee. Voltage control of pwm converters using feedback linearization. In *Conference Record of 1998 IEEE Industry Applications Conference. Thirty-Third IAS Annual Meeting (Cat. No. 98CH36242)*, volume 2, pages 1491–1496. IEEE, 1998. (page 20).

[Mar16] Benoît Marx. *Estimation, diagnostic et commande tolérante de systèmes décrits par des multimodèles.* PhD thesis, Université de Lorraine, 2016. (page 66).

[MHS+23] Khalida Mimoune, Mohamed Yacine Hammoudi, Ramzi Saadi, Mohamed Benbouzid, and Anouar Boukhlouf. Real-time implementation of non linear observer based state feedback controller for induction motor using mean value theorem. *Journal of Electrical Engineering & Technology*, 18(1):615–628, 2023. (page 100).

[MIM23] Khalida MIMOUNE. *Contribution à l'estimation et à la commande des systèmes non linéaires*. PhD thesis, Faculté des Sciences et de la technologie, 2023. (page 133).

[MKB17] Sonia Maalej, Alexandre Kruszewski, and Lotfi Belkoura. Stabilization of takagi–sugeno models with non-measured premises: Input-to-state stability approach. *Fuzzy Sets and Systems*, 329:108–126, 2017. (page 112).

[MPA09] Leonardo A Mozelli, Reinaldo M Palhares, and Gustavo SC Avellar. A systematic approach to improve multiple lyapunov function stability and stabilization conditions for fuzzy systems. *Information Sciences*, 179(8):1149–1162, 2009. (page 76).

[MSH98] Xiao-Jun Ma, Zeng-Qi Sun, and Yan-Yan He. Analysis and design of fuzzy controller and fuzzy observer. *Fuzzy Systems, IEEE Transactions on*, 6(1):41–51, 1998. (page 3).

[MSJ] R Murray-Smith and TA Johansen. Multiple model approaches to modelling and control, 1997. (page 61).

[Nag10] Anca Maria Nagy. *Analyse et synthèse de multimodèles pour le diagnostic. Application à une station d'épuration*. PhD thesis, Institut National Polytechnique de Lorraine-INPL, 2010. (page 65).

[NKMS21] Mohamed Lamine Nacer, Hamid Kherfane, Sandrine Moreau, and Salah Saad. Robust observer design for uncertain lipschitz nonlinear systems based on differential mean value theorem: Application to induction motors. *Journal of Control, Automation and Electrical Systems*, 32:132–144, 2021. (page 100).

[NTHB07] Meriem Nachidi, Fernando Tadeo, Abdelaziz Hmamed, and Abdellah Benzaouia. Static output-feedback stabilization for time-delay takagi-sugeno fuzzy systems. In *Decision and Control, 2007 46th IEEE Conference on*, pages 1634–1639. IEEE, 2007. (page 89).

[Orj08] Rodolfo Orjuela. *A contribution to state estimation and diagnosis of systems modelled by multiple models*. PhD thesis, Institut National Polytechnique de Lorraine-INPL, 2008. (pages 65, 89).

[Oud08] Mohammed Oudghiri. *Commande multi-modèles tolérante aux défauts: Application au contrôle de la dynamique d'un véhicule automobile*. PhD thesis, Université de Picardie Jules Verne, 2008. (page 67).

[PABB00] Dimitri Peaucelle, Denis Arzelier, Olivier Bachelier, and Jacques Bernussou. A new robust d-stability condition for real convex polytopic uncertainty. *Systems & control letters*, 40(1):21–30, 2000. (page 115).

[Pha11] Gridsada Phanomchoeng. *State, Parameter, and Unknown Input Estimation Problems in Active Automotive Safety Applications*. PhD thesis, UNIVERSITY OF MINNESOTA, 2011. (pages 89, 97, 98).

[PKP01] Jooyoung Park, Jinsung Kim, and Daihee Park. Lmi-based design of stabilizing fuzzy controllers for nonlinear systems described by takagi–sugeno fuzzy model. *Fuzzy Sets and Systems*, 122(1):73–82, 2001. (page 89).

[RW06] Bong-Jae Rhee and Sangchul Won. A new fuzzy lyapunov function approach for a takagi–sugeno fuzzy control system design. *Fuzzy sets and systems*, 157(9):1211–1228, 2006. (pages 79, 83, 84, 86, 111, 116).

[SBAT22] B Sereni, Marco Antonio Leite Beteto, Edvaldo Assunção, and Marcelo CM Teixeira. Pole placement lmi constraints for stability and transient performance of lpv systems with incomplete state measurement. *Journal of the Franklin Institute*, 359(2):837–858, 2022. (page 115).

[SSPP23] Bernard Steyaert, Ethan Swint, W Wesley Pennington, and Matthias Preindl. Piecewise affine maximum torque per ampere for the wound rotor synchronous machine. *IEEE Transactions on Transportation Electrification*, 2023. (page 63).

[Tah09] Bouarar Tahar. *Contribution à la synthèse de lois de commande pour les descripteurs de type Takagi-Sugeno incertains et perturbés*. PhD thesis, Université de Reims-Champagne Ardenne, 2009. (page 89).

[TANY01] Hoang Duong Tuan, Pierre Apkarian, Tatsuo Narikiyo, and Yasuhiro Yamamoto. Parameterized linear matrix inequality techniques in fuzzy control system design. *IEEE Transactions on fuzzy systems*, 9(2):324–332, 2001. (page 81).

[TC99] Hualin Tan and Jie Chang. Adaptive backstepping control of induction motor with uncertainties. In *Proceedings of the 1999 American Control Conference (Cat. No. 99CH36251)*, volume 1, pages 1–5. IEEE, 1999. (page 35).

[TDGB08] P-F Toulotte, Sebastien Delprat, T-M Guerra, and Jacques Boonaert. Vehicle spacing control using robust fuzzy control with pole placement in lmi region. *Engineering Applications of Artificial Intelligence*, 21(5):756–768, 2008. (page 115).

[THW03] Kazuo Tanaka, Tsuyoshi Hori, and Hua O Wang. A multiple lyapunov function approach to stabilization of fuzzy control systems. *Fuzzy Systems, IEEE Transactions on*, 11(4):582–589, 2003. (page 101).

[TIW98] Kazuo Tanaka, Takayuki Ikeda, and Hua O Wang. Fuzzy regulators and fuzzy observers: relaxed stability conditions and lmi-based designs. *Fuzzy Systems, IEEE Transactions on*, 6(2):250–265, 1998. (pages 89, 114).

[TS85] Tomohiro Takagi and Michio Sugeno. Fuzzy identification of systems and its applications to modeling and control. *IEEE transactions on systems, man, and cybernetics*, (1):116–132, 1985. (pages 62, 66).

[TS92] Kazuo Tanaka and Michio Sugeno. Stability analysis and design of fuzzy control systems. *Fuzzy sets and systems*, 45(2):135–156, 1992. (pages 74, 82).

[TS94] Kazuo Tanaka and Manabu Sano. A robust stabilization problem of fuzzy control systems and its application to backing up control of a truck-trailer. *IEEE Transactions on Fuzzy systems*, 2(2):119–134, 1994. (page 80).

[TW04] Kazuo Tanaka and Hua O Wang. *Fuzzy control systems design and analysis: a linear matrix inequality approach*. John Wiley & Sons, 2004. (pages 3, 76, 91).

[UMR+20] Hameed Ullah, Fahad Mumtaz Malik, Abid Raza, Anjum Saeed, Naveed Mazhar, and Rameez Khan. Robust nonlinear output feedback tracking control of single-link flexible joint robotic manipulator system. In *2020 14th International Conference on Open Source Systems and Technologies (ICOSST)*, pages 1–6. IEEE, 2020. (page 123).

[WTG96] Hua O Wang, Kazuo Tanaka, and Michael F Griffin. An approach to fuzzy control of nonlinear systems: stability and design issues. *Fuzzy Systems, IEEE Transactions on*, 4(1):14–23, 1996. (pages 74, 89).

[XQ03] Liu Xiaodong and Zhang Qingling. New approaches to $h^\infty$ controller designs based on fuzzy observers for ts fuzzy systems via lmi. *Automatica*, 39(9):1571–1582, 2003. (page 81).

[Yac04] Hammoudi Mohamed Yacine. Commande nonlinéaire au filtre actif et principe de compensation, 2004. (pages 20, 50).

[Yon09] Jun Yoneyama. Output feedback control for fuzzy systems with immeasurable premise variables. In *Fuzzy Systems, 2009. FUZZ-IEEE 2009. IEEE International Conference on*, pages 802–807. IEEE, 2009. (page 90).

[ZBB05] Ali Zemouche, Mohamed Boutayeb, and G Iulia Bara. Observer design for nonlinear systems: An approach based on the differential mean value theorem. In *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 6353–6358. IEEE, 2005. (page 97).

[Zem07] Ali Zemouche. *Sur l'observation de l'état des systèmes dynamiques non linéaires*. PhD thesis, Université Louis Pasteur-Strasbourg I, 2007. (page 89).

[Zer11] Mohamed Zerrougui. *Observation et commande des systemes singuliers non linéaires*. PhD thesis, Université Henri Poincaré-Nancy I, 2011. (page 75).

[ZGM08] Madjid Zerar, Kevin Guelton, and Noureddine Manamanni. Linear fractional transformation based h-infinity output stabilization for takagi–sugeno fuzzy models. *Mediterranean Journal of Measurement and Control*, 4(3):111–121, 2008. (page 90).

[ZW02] J Zhou and Y Wang. Adaptive backstepping speed controller design for a permanent magnet synchronous motor. *IEE Proceedings-Electric Power Applications*, 149(2):165–172, 2002. (page 36).