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On partially observed optimal stochastic control of McKean-Vlasov systems in Wasserstein space of probability measures with applications

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Résumé

Cette thèse présente deux sujets de recherche sur les problèmes de contrôle stochastique des équations de type McKean–Vlasov, dans lesquelles les coefficients du système et la fonctionnelle de coût dépendent de l'état du processus de solution ainsi que sa loi de probabilité et la variable de contrôle. Dans le premier sujet, nous établissons des conditions nécessaires d'optimalité vérifiées par un contrôle partiellement observé pour des équations différentielles stochastiques progressivement rétrogrades (EDSPRs) gouvernées à la fois par une famille de martingales de Teugels et un mouvement brownien indépendant sous l'hypothèse que le domaine de contrôle est supposé convexe. En tant qu'application de la théorie générale, un problème de contrôle linéaire-quadratique est étudié en termes de filtrage stochastique. Le deuxième sujet consiste à étudier le principe du maximum pour le problème de contrôle optimal sensible au risque partiellement observé des EDSPRs, et la fonctionnelle de coût est une exponentielle de type intégral. De plus, sous certaines hypothèses de concavité, nous obtenons les conditions suffisantes d'optimalité. En tant qu'application, un problème de contrôle optimal sensible au risque linéaire-quadratique sous des informations partiellement observées et des informations entièrement observées est résolu en utilisant les principaux résultats.

Mots Clés. Principe du maximum stochastique, Equations différentielles stochastiques progressivement rétrograde, Contrôle optimal partiellement observé, Equations différentielles de tupe McKean-Vlasov, Martingales de Teugels, Contrôle optimal sensible au risque.

Abstract

This thesis presents two research topics about stochastic control problems of the general McKean–Vlasov equations, in which the coefficients depend nonlinearly on both the state process as well as its law. In the first topic, we establish partially observed necessary conditions of optimality for forward-backward stochastic differential equations driven by both a family of Teugels martingales and an independent Brownian motion under the assumption that the control domain is supposed to be convex. As an application of the general theory, a partially observed linear-quadratic control problem is studied in terms of stochastic filtering. The second topic is to study the maximum principle for the partially observed risk-sensitive optimal control problem of FBSDEs, and the cost functional is a McKean–Vlasov exponential of integral type. Moreover, under certain concavity assumptions, we obtain the sufficient conditions of optimality. As an application, a linear-quadratic risk-sensitive optimal control problem under partially observed information and fully observed information is solved by using main results.

Key words. Stochastic maximum principle, Forward-backward stochastic differential equations, Partially observed optimal Control, McKean–Vlasov differential equations, Teugels martingales, Risk-sensitive optimal control.

Symbols and acronyms

- $(\Omega, \mathcal{F}, \mathbb{P})$: Probability space.
- $\{\mathcal{F}_t\}_{t\geq 0}$: Filtration.
- $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$: Filtered probability space.
- $\mathbb{R}$: Real numbers.
- $\mathbb{N}$: Natural numbers.
- l^2 is the Hilbert space of real-valued sequences $x = (x_n)_{n\geq 0}$ such that

$$\|x\| = \left(\sum_{i=1}^{\infty} x_i\right)^{\frac{1}{2}} < \infty.$$

- $l^2(\mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued sequences $(x_i)_{i\geq 1}$ such that

$$\left(\sum_{i=1}^{\infty} \|x_i\|^2_{\mathbb{R}^n}\right)^{\frac{1}{2}} < \infty.$$

- $l^2_{\mathcal{F}}(0, T, \mathbb{R}^n)$ is the Banach space of $l^2(\mathbb{R}^n)$-valued $\mathcal{F}_t$-predictable processes such that

$$\left(\mathbb{E}\int_0^T \sum_{i=1}^{\infty} \left\|g_t^i\right\|^2_{\mathbb{R}^n} dt\right)^{\frac{1}{2}} < \infty.$$

- $\mathbb{S}^2_{\mathcal{F}}(0, T, \mathbb{R}^n)$ is the Banach space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted and càdlàg processes such that

$$\left(\mathbb{E}\sup_{0\leq t\leq T} |f_t|^2\right)^{\frac{1}{2}} < \infty.$$

- $\mathbb{L}^2_{\mathcal{F}}(0, T, \mathbb{R}^n)$ is the Banach space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes such that
$$\left(\mathbb{E}\int_0^T |f_t|^2_{\mathbb{R}^n}\, dt\right)^{\frac{1}{2}} < \infty.$$
- $\mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ is the Hilbert space with inner product $(x, y)_2 = \mathbb{E}[x.y]$, $x, y \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ and the norm $\|x\|_2 = \sqrt{(x, x)_2}$.
- $\mathbb{L}^2(\Omega, \mathcal{F}, P, \mathbb{R}^n)$ is the Banach space of all $\mathbb{R}^n$-valued square-integrable $\mathcal{F}_T$-measurable random variables on $(\Omega, \mathcal{F}, P)$.
- $Q_2\left(\mathbb{R}^d\right)$ the space of all probability measures μ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$.
- *a.e.,:* Almost everywhere.
- *a.s.,:* Almost surely.
- *e.g.*: For example (abbreviation of Latin exempli gratia).
- *i.e,.*: that is (abbreviation of Latin id est).
- *SDE*: Stochastic differential equations.
- *BSDE:* Backward stochastic differential equation.
- *PDE*: Partial differential equation.
- *ODE:* Ordinary differential equation.
- $\dfrac{\partial f}{\partial x}, f_x$: The derivatives with respect to x.
- $\mathbb{P}\otimes dt$: The product measure of $\mathbb{P}$ with the Lebesgue measure dt on $[0, T]$.
- $\mathbb{P}_X$ the law of the random variable $X(\cdot)$.
- $\mathbb{E}(\cdot)$: Expectation.
- $\mathbb{E}(\cdot | F_t)$: Conditional expectation.
- $\sigma(A)$: $\sigma-$algebra generated by A.
- $\mathbb{E}^v$ denotes expectation on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$.
- $\mathcal{F}^X$: The filtration generated by the process X.

- $W(\cdot)$: Brownian motions.
- $\mathcal{F}_t^W$: the natural filtration generated by the brownian motion $W(\cdot)$.
- $\mathcal{F}_1 \vee \mathcal{F}_2$ denotes the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$.
- $\partial_\mu f$: the derivatives with respect to measure μ.

Contents

Acknowledgements **i**

Résumé **ii**

Abstract **iii**

Symbols and acronyms **iv**

Introduction **11**

1 *Stochastic maximum principle for partially observed optimal control problem of FBSDEs with Teugels martingales* **19**

1.1 Introduction 19

1.2 Preliminaries 20

1.3 Stochastic maximum principle 24

1.4 Linear-quadratic control problem 31

2 *Stochastic maximum principle for partially observed optimal control problem of McKean–Vlasov FBSDEs with Teugels martingales* **33**

2.1 Introduction 33

2.2 Preliminaries 35

2.3 Necessary conditions of optimality 42

2.4 Partially observed linear-quadratic control problem of McKean-Vlasov FB-SDEs 50

3 *Maximum principle for partially observed Risk-sensitive optimal control problem of McKean–Vlasov FBSDEs involving impulse controls* **53**

3.1 Introduction 53
3.2 Statement of the problem 55
3.3 Necessary conditions of optimality 59
3.4 Sufficient conditions of optimality: 66
3.5 Application to LQ risk-sensitive control problem 71
3.5.1 Partially observed LQ risk-sensitive optimal control problem 71
3.5.2 Fully observed LQ risk-sensitive optimal control problem 73

Conclusion **75**

Bibliographie **76**

Introduction

The main objective of this thesis is to study two research topics about stochastic control problems. For the first topic, we are interested in a class of partially observed optimal control problems with McKean–Vlasov type:

$$\begin{cases} dx_t^v = b\left(t, x_t^v, P_{x_t^v}, v_t\right) dt + \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) dW_t \\ \qquad\quad + \tilde{\sigma}\left(t, x_t^v, P_{x_t^v}, v_t\right) d\widetilde{W}_t^v + \sum_{i=1}^{\infty} g^i\left(t, x_{t\_}^v, P_{x_{t\_}^v}, v_t\right) dH_t^i, \\ -dy_t^v = f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, q_t^v, P_{q_t^v}, v_t\right) dt - z_t^v dW_t \\ \qquad\quad -\bar{z}_t^v dY_t - \sum_{i=1}^{\infty} q_{t\_}^{v,i} dH_t^i, \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v, P_{x_T^v}\right), \end{cases}$$

where W_t is a one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, $H_t = \left(H_t^i\right)_{i\geq 1}$ is a family of pairwise orthogonal martingales associated with some Lévy process which is independent from W_t . These martingales are called Teugels martingales. $P_{x_t}, P_{y_t}, P_{z_t}, P_{\bar{z}_t}$ and P_{q_t} denotes the law of the random variable $x, y, z, \bar{z}$ and q respectively.

It is worth noting that the above forward-backward stochastic differential equation of type McKean–Vlasov is very general, in that the dependence of the coefficients on the probability law of the solution $P_{x_t^v}, P_{y_t^v}, P_{z_t^v}, P_{\bar{z}_t^v}$ and $P_{q_t^v}$ could be genuinely nonlinear as an element of the space of probability measures.

We assume that the state processes $(x^v, y^v, z^v, \bar{z}^v, q^v)$ cannot be observed directly, but the controllers can observe a related noisy process Y, which is the solution of the following equation

$$\begin{cases} dY_t = \xi\left(t, x_t^v, P_{x_t^v}\right) dt + d\widetilde{W}_t^v, \\ Y_0 = 0, \end{cases}$$

where $\xi : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\widetilde{W}_t^v$ is stochastic processes depending on the control v. The associated cost functional to be minimized defined as

$$\begin{aligned} J(v) &= \mathbb{E}^v\left[\int_0^T l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, q_t^v, P_{q_t^v}, v_t\right) dt\right] \\ &\quad + \mathbb{E}^v\left[M\left(x_T^v, P_{x_T^v}\right) + h\left(y_0^v, P_{y_0^v}\right)\right], \end{aligned}$$

where $\mathbb{E}^v$ denotes the expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P^v)$.

Our partially observed optimal control problem is to minimize the cost functional over $v \in \mathcal{U}_{ad}$, *i.e.*,

$$\min_{v \in \mathcal{U}_{ad}} J(v).$$

Stochastic optimal control problems related to Teugels martingales associated with some Lévy processes have been investigated by many authors through several papers, see, e.g. [7, 8, 34, 38]. Maximum principle for partially observed optimal control of FBSDEs driven by Teugels martingales and independent Brownian motion has been proved by Bougherara and Khelfallah [8]. Nualart and Schoutens [34] offered the incredibly helpful representation theory for researching SDEs and BSDEs driven by a brownian motion and Teugels martingales. The stochastic linear quadratic control problem associated with Lévy processes has been derived by Tang and Wu [38].

The stochastic differential equations (SDE) of McKean–Vlasov type play an important role in different fields of finance, economics, physics, and the game theory. Stochastic maximum principle of McKean–Vlasov systems has been studied by many authors, see, e.g. [9, 11, 30]. The stochastic maximum principle for general mean-field systems by using the tool of the second-order derivatives with respect to probability measures has been established by Buckdahn et al. [9]. Carmona and Delarue [11] proved a new version of the stochastic maximum principle of nonlinear stochastic dynamical systems of McKean–Vlasov type and gave sufficient conditions for existence of an optimal control. Stochastic maximum principle for optimal control of McKean–Vlasov FBSDEs with Lévy process has been studied by Meherrem and Hafayed [30].

However, in the above mentioned works all assume that the overall information is available to controllers. This assumption is not always satisfied in reality. Generally speaking, controllers can only get partial information in most cases. Then it is natural to study this kind of optimal control problems under partial observation. There is a rich literature on partially observed optimal control problem, see, e.g. [2, 6, 16, 17] and references therein. Stochastic maximum principle for partially observed optimal control problems of general McKean–Vlasov equations has been proved by Lakhdari et al. [21]. Ma and Liu [26] studied the maximum principle for partially observed risk-sensitive optimal control problems of mean-field type. Miloudi et al. [29] established the necessary conditions of partially observed optimal control of general McKean–Vlasov stochastic differential equations with jumps.

Partially observed stochastic optimal control of forward-backward stochastic differential equations has been studied by Wu [42]. Shi and Wu [44] established the maximum principle for partially observed optimal control of fully coupled forward-backward stochastic systems. Li and Fu [23] proved a general maximum principle for partially observed optimal control problems of mean-field FBSDEs under general control domains, with the help of Ekeland's variational principle and reduction method. Nie and Yan [33] studied an extended mean-field control problem with partial observation, where the state and the observation all depend on the joint distribution of the state and the control process. Abba and Lakhdari [1] etablished the necessary and sufficient conditions of optimality for partially observed optimal control problem of forward–backward stochastic differential equations of McKean–Vlasov type driven by a Poisson random measure and an independent Brownian motion. Wang et al. [40] studied three versions of stochastic maximum principle for partially observed optimal control problem for FBSDEs in the sense of weak solution by utilizing a direct method, an approximation method and a Malliavin derivative method. Partially observed optimal control problem of forward-backward stochastic jump diffusion differential system has been discussed by [41, 45].

Our main goal in this topic is to establish necessary conditions of partially observed optimal control problem of McKean–Vlasov FBSDEs driven by Teugels martingales, associated with some Lévy process. The coefficients of our system and the cost functional depend on the state of the solution process $(x_t^v, y_t^v, z_t^v, \bar{z}_t^v, q_t^v)$ as well as of its probability measures $\left(P_{x_t^v}, P_{y_t^v}, P_{z_t^v}, P_{\bar{z}_t^v}, P_{q_t^v}\right)$. The control domain must be convex. Our main result is supported by variational techniques and delicate estimates of SDE. As an application, a partially observed linear-quadratic control problem is provided.

For the second topic, we study risk-sensitive optimal control problems under partial observation, modeled by forward-backward stochastic differential equations (FBSDE) of general McKean-Vlasov form. The control variable consists of two components: a continuous control and an impulse control and the cost functional is an exponential of integral type based on the McKean-Vlasov framework. We consider the following stochastic controlled system

$$\begin{cases} dx_t^{v,\eta} = b(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, v_t)dt + \sigma(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, v_t)dW_t + \mathcal{C}_t d\eta_t, \\ -dy_t^{v,\eta} = f(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, y_t^{v,\eta}, P_{y_t^{v,\eta}}, z_t^{v,\eta}, P_{z_t^{v,\eta}}, v_t)dt - z_t^{v,\eta}dW_t + \mathcal{D}_t d\eta_t, \\ x_0^{v,\eta} = a, \quad y_T^{v,\eta} = \varphi(x_T^{v,\eta}, P_{x_T^{v,\eta}}), \end{cases}$$

where W_t is a one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and $\eta(\cdot) = \sum_{i\geq 1} \eta_i \mathbf{1}_{[\tau_i, T]}$ such that each $\eta_i \in \mathbb{R}^n$. $P_{x_t^{v,\eta}}, P_{y_t^{v,\eta}}$ and $P_{z_t^{v,\eta}}$ denotes the law of the random variable $x_t^{v,\eta}, y_t^{v,\eta}$ and $z_t^{v,\eta}$ respectively.

Consider state processes $(x_t^{v,\eta}, y_t^{v,\eta}, z_t^{v,\eta})$ are not fully observable. Rather, they are only partially observed through a noisy process Y, which is described by the following equation:

$$\begin{cases} dY_t = \xi(t, x_t^{v,\eta}, P_{x_t^{v,\eta}})dt + d\widetilde{W}_t^v, \\ Y_0 = 0, \end{cases}$$

here $\xi : [0, T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \to \mathbb{R}^n$ and $\widetilde{W}_t^v$ represents stochastic processes that rely on the control variable v.

The corresponding cost functional to be maximized is of the McKean-Vlasov type and is defined as follows

$$J^\theta(v, \eta) = \mathbb{E}^v \Bigg[\exp\theta \Bigg(\int_0^T l(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, y_t^{v,\eta}, P_{y_t^{v,\eta}}, z_t^{v,\eta}, P_{z_t^{v,\eta}}, v_t)dt + M(x_T^{v,\eta}, P_{x_T^{v,\eta}}) + h(y_0^{v,\eta}, P_{y_0^{v,\eta}}) + \sum_{i\geq 1} c(\tau_i, \eta_i) \Bigg) \Bigg],$$

here $\mathbb{E}^v$ represents expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P^v)$ and y_0^v is deterministic. θ represents risk-sensitive index for $\theta \in (0, 1]$.

The objective of our partially observed risk-sensitive optimal control problem is to maximize the above cost functional over $(v_\cdot, \eta_\cdot) \in \mathcal{A}$. A control $(u_\cdot, \zeta_\cdot) \in \mathcal{A}$ that satisfies

$$J^{\theta}(u, \zeta) = \max_{(v_\cdot, \eta_\cdot) \in \mathcal{A}} J^{\theta}(v, \eta),$$

is called a risk-sensitive optimal control.

Stochastic impulse control problems have attracted considerable research attention because of their wide applicability in numerous fields. For example, they are useful in portfolio optimization problems that account for transaction costs (refer to [15, 35]), in optimal impulse-type consumption issues (refer to [43]), in managing investment funds for financial institutions (see [13]), and in the optimal control of currency exchange rates (see [10]). For a thorough overview of impulse control theory and its applications, see [28] and the sources cited within. Given their broad range of applications across different domains, it is both significant and valuable to investigate problems involving FBSDEs coupled with impulse controls. Wu and Zhang [43] studied the stochastic maximum principle for optimal control problems involving forward-backward systems with impulse controls. Moreover Xu and Zhou [46] formulated the risk-sensitive stochastic maximum principle for forward-backward systems incorporating impulse controls. Maximum principle for progressive optimal control in mean-field forward-backward stochastic systems with random jumps and impulse controls has been demonstrated by [14].

McKean-Vlasov stochastic differential equations (SDEs) hold considerable importance in disciplines like physics, finance, economics, and game theory. Kac [18] and McKean [25] first introduced these equations to study physical systems characterized by the interaction of numerous particles. It is important to note that McKean-Vlasov type stochastic differential equations are quite general, as the coefficients can exhibit a genuinely nonlinear dependence on the probability law of the solution, viewed as an element of the probability measures space. Numerous researchers have explored stochastic optimal control problems associated with McKean-Vlasov SDEs in various papers; for instance, see [9, 11, 30].

However, the results for stochastic optimal control problems involving McKean-Vlasov SDEs discussed above are derived under the assumption of full information. In practice, controllers often have access to only partial information. Thus, it is natural to explore these optimal control problems within the context of partial observation. Extensive literature exists on optimal control problems involving McKean-Vlasov SDEs and McKean-Vlasov forward-backward SDEs under partial observation. As an example, Kaouache et al. [19] investigated a stochastic maximum principle for a partially observed optimal control problem in McKean-Vlasov type FBSDEs, which are driven by a mixture of independent Brownian motion and Teugels martingales. Nie and Yan [33] explored an expanded mean-field stochastic control problem under partial observations, where the state of the FBSDEs and the observations are influenced by the joint distribution of the state and control process. Abba and Lakhdari [1] developed a stochastic maximum principle for optimal control problems with partial observations in the framework of general McKean-Vlasov FBSDEs with random jumps.

It is important to note that the stochastic optimal control problems discussed in references [1, 3, 4, 5, 9, 11, 18, 21, 25, 29, 30, 33] can be considered as risk-neutral. In these cases, the cost functional is assessed solely based on the expected values of terminal, integral, and initial costs. This method can be extended to the risk-sensitive framework, where the cost function includes the expected value of the exponentiated terminal and integral costs. As a result, the risk-neutral case becomes a specific example within the broader risk-sensitive framework. The risk-sensitive control problem has garnered significant research interest and has been extensively explored under complete information by various authors, such as [12, 22, 20, 24, 32, 37, 39]. The foundational work on the stochastic maximum principle for risk-sensitive stochastic optimal control problems was introduced by Whittle [39], who applied large deviation theory. Subsequently, Lim and Zhou [24] introduced an innovative risk-sensitive stochastic maximum principle by employing logarithmic transformation and examining the relationship between the dynamic programming principle and the maximum principle, under the condition that the value function is smooth. Shi and Wu [37] formulated a risk-sensitive stochastic maximum principle aimed at optimal control of jump diffusions. Meanwhile, Khallout and Chala [20] investigated a risk-sensitive maximum principle tailored for fully coupled FBSDEs. Moon [32] investigated two distinct risk-sensitive maximum principles for FBSDEs, each with different configurations of the FBSDEs and the cost functional, by employing nonlinear transformations of the equivalent risk-neutral problems. Chala and Hafayed [12] presented a stochastic maximum principle for mean-field type fully coupled FBSDEs within a risk-sensitive performance framework.

A variety of studies have investigated risk-sensitive optimal control problems with partial observations. For example, Ma and Liu [26] analyzed the stochastic maximum principle in the context of risk-sensitive optimal control problems of the mean-field variety with partial observations. Similarly, Ma and Wang [27] established a stochastic maximum principle for partially observed risk-sensitive optimal control problems that involve mean-field FBSDEs. Meanwhile, Moon [31] formulated the risk-sensitive maximum principle for stochastic optimal control within mean-field type Markov regime-switching jump-diffusion systems. Additionally, Saldi et al. [36] utilized the risk-sensitive optimality criterion to establish results for discrete-time partially observed mean-field games.

This topic mainly concentrates on formulating the risk-sensitive maximum principle for McKean-Vlasov forward-backward stochastic differential equations (FBSDEs) with impulse control. In this framework, impulse control is modeled as a piecewise process that does not necessarily need to be monotonic. Additionally, the work introduces further concavity conditions under which the partial necessary risk-sensitive conditions of optimality are sufficient. As an example, our work examines a linear quadratic (LQ) risk-sensitive optimal control problem of the McKean-Vlasov type. It is noteworthy that the results offered in this study build upon the research conducted by Ma and Wang [27].

This thesis is structured around three chapters:

Chapter 1: In this chapter, we present the stochastic maximum principle for a partially observed optimal control problem of forward-backward stochastic differential equations (FBSDEs for short) driven by both a family of Teugels martingales and an independent Brownian motion in which the control domain is convex.

Chapter 2: This chapter contains the first main result of this thesis, which is the stochastic maximum principle for a partially observed optimal control problem of FBSDEs of the general McKean–Vlasov type driven by both a family of Teugels martingales and an independent Brownian motion. The coefficients of the system and the cost functional depending on the state of the solution process as well as its probability law and the control variable. Our main result is based on Girsavov's theorem and the derivatives with respect to probability law. As an application of the general theory, a partially observed linear-quadratic control problem of McKean–Vlasov type is studied in terms of stochastic filtering.

Chapter 3: This chapter contains the second result of this thesis, which is the maximum principle pertaining to risk-sensitive optimal control problems under partial observation, modeled by FBSDEs of the general McKean–Vlasov equations. The control variable consists of two components: a continuous control and an impulse control. The cost functional is an exponential of integral type based on the regularity McKean–Vlasov framework. Moreover, the sufficient conditions of optimality are obtained under certain concavity assumptions. As an application, the main outcomes are used to solve a linear-quadratic risk-sensitive optimal control problem of the regularity McKean–Vlasov type, both under partial and full observation conditions.

Published Author Papers

The content of this thesis was the subject of the following papers:

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2. Lakhdari, I. E., Djenaihi, Y., Kaouache, R., Boulaaras, S., Jan, R.: Maximum principle for partially observed risk-sensitive optimal control problem of McKean–Vlasov FBSDEs involving impulse controls. Journal of Pseudo-Differential Operators and Applications, 15(4), 1-28 (2024).

Chapter 1

Stochastic maximum principle for partially observed optimal control problem of FBSDEs with Teugels martingales

1.1 Introduction

In this chapter, we are interested in the stochastic maximum principle for a class of partially observed optimal control problem with dynamics:

$$\begin{cases} dx_t^v = b\left(t, x_t^v, v_t\right) dt + \sigma\left(t, x_t^v, v_t\right) dW_t \\ \qquad\quad + \widetilde{\sigma}\left(t, x_t^v, v_t\right) d\widetilde{W}_t^v + \sum\limits_{i=1}^{\infty} g^i\left(t, x_{t_-}^v, v_t\right) dH_t^i, \\ -dy_t^v = f\left(t, x_t^v, y_t^v, z_t^v, \bar{z}_t^v, q_t^v, v_t\right) dt - z_t^v dW_t \\ \qquad\quad - \bar{z}_t^v dY_t - \sum\limits_{i=1}^{\infty} q_{t_-}^{v,i} dH_t^i, \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v\right), \end{cases} \tag{1.1}$$

where W_t is a one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, $H_t = \left(H_t^i\right)_{i\geq 1}$ is a family of pairwise orthogonal martingales associated with some Lévy process which is independent from W_t . These martingales are called Teugels martingales. The coefficients $b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}, \widetilde{\sigma} : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}, g : [0,T] \times \mathbb{R}^n \times U \to l^2\left(\mathbb{R}^n\right), f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d} \times \mathbb{R}^n \times U \to l^2\left(\mathbb{R}^n\right), \varphi : \mathbb{R}^n \to \mathbb{R}^m$, are given deterministic functions.

We assume that the state processes $(x^v, y^v, z^v, \bar{z}^v, q^v)$ cannot be observed directly, but the controllers can observe a related noisy process Y, which is the solution of the following equation

$$\begin{cases} dY_t = \xi\left(t, x_t^v\right) dt + d\widetilde{W}_t^v, \\ Y_0 = 0, \end{cases} \tag{1.2}$$

where $\xi : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\widetilde{W}_t^v$ is stochastic processes depending on the control v.

The associated cost functional to be minimized defined as

$$J(v) = \mathbb{E}^v \left[\int_0^T l(t, x_t^v, y_t^v, z_t^v, \bar{z}_t^v, q_t^v, v_t) \, dt + M(x_T^v) + h(y_0^v) \right] \tag{1.3}$$

where $\mathbb{E}^v$ denotes the expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P^v)$ and $M : \mathbb{R}^n \to \mathbb{R}, h : \mathbb{R}^m \to \mathbb{R}, l : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^{m \times d} \times \mathbb{R}^n \times U \to \mathbb{R}$ are deterministic functions.

Our partially observed optimal control problem is to minimize the cost functional (1.3) over $v \in \mathcal{U}_{ad}$ subject to (1.1) and (1.2), *i.e.*,

$$\min_{v \in \mathcal{U}_{ad}} J(v).$$

1.2 Preliminaries

Let T be a fixed strictly positive real number and $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space equipped with two independent standard one-dimensional Brownian motions W and Y. Let $L = \{L_t : t \in [0, T]\}$ be a $\mathbb{R}$-valued Lévy process, independent of W and Y of the form $L_t = \lambda_t + bt$, where λ_t is a pure jump process. We assume that $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ and $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^Y \vee \mathcal{F}_t^L \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of P-null set and $\mathcal{F}_t^W, \mathcal{F}_t^Y$ and $\mathcal{F}_t^L$ denotes the P-completed natural filtration generated by W, Y and L respectively. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, and by $(\cdot, \cdot)$ (resp. $|\cdot|$) the inner product (resp. norm). The set of the admissible control variables is denoted by $\mathcal{U}_{ad}$. We also assume that the Lévy measure $\nu(dx)$ corresponding to the Lévy process λ_t satisfies the following.

1. For every $\delta > 0$, there exist $\gamma > 0$ such that $\int_{(-\delta,\delta)} \exp(\gamma |x|) \nu(dx) < \infty$.
2. $\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu(dx) < \infty$.

The above conditions settings imply that the random variable L_t have moments in all orders. Notice that the jump of the state x_t^v caused by the Lévy process is the power jump processes defined by

$$\begin{cases} L_t^i = \sum_{0<s\leq t} (\Delta L_s)^i, \text{ for } i>1, \\ L_t^1 = L_t, \end{cases}$$

where $\Delta L_t = L_t - L_{t_-}$.

The continuous part of L_t^i obtained by removing the jumps of L_t defined by

$$L_{t,c}^i = L_t^i - \sum_{0<s\leq t} (\Delta L_s)^i, \text{ for } i>1.$$

We distinguish between the jumps of state x_t^v and y_t^v caused by the Lévy martingales are defined by

$$\begin{cases} \Delta_L x_t^v = g\left(t, x_{t_-}^v, P_{x_{t_-}^v}, v_t\right) \Delta L_t, \\ \Delta_L y_t^v = \sum_{i=1}^{\infty} q_{t-}^{v,i} \Delta L_t^i. \end{cases}$$

Now, let $N_t^i = L_t^i - \mathbb{E}\left[L_t^i\right]$, for $i \geq 1$. Then, the family of Teugels martingales $\left(H_t^i\right)_{i\geq 1}$ is defined by $H_t^i = \sum_{j=1}^{j=i} \alpha_{ij} N_t^j$, where the coefficients α_{ij} associated with the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $m(dx) = x^2 \nu(dx)$. The Teugels martingales $\left(H_t^i\right)_{i\geq 1}$ are pathwise strongly orthogonal and their predictable quadratic variation processes are given by $\left\langle H_t^i, H_t^j \right\rangle_t = \delta_{ij} t$. For more information about Teugels martingales, Lévy processes and their practical examples, we refer to the work of Nualart and Schoutens [34].

Definition 1.1

Let U be a nonempty convex subset of $\mathbb{R}^k$. A control $v : \Omega \times [0,T] \to U$ is called admissible if it is $\mathcal{F}_t^Y$-adapted and satisfies $\sup_{0\leq t\leq T} \mathbb{E}\left|v_t\right|^2 < \infty$.

Now, inserting (1.2) into (1.1), we get

$$\begin{cases} dx_t^v = \left[b\left(t, x_t^v, v_t\right) dt - \widetilde{\sigma}\left(t, x_t^v, v_t\right) \xi\left(t, x_t^v\right)\right] dt \\ \qquad +\sigma\left(t, x_t^v, v_t\right) dW_t + \widetilde{\sigma}\left(t, x_t^v, v_t\right) dY_t + \sum_{i=1}^{\infty} g^i\left(t, x_{t_-}^v, v_t\right) dH_t^i, \\ -dy_t^v = f\left(t, x_t^v, y_t^v, z_t^v, \bar{z}_t^v, q_t^v, v_t\right) dt - z_t^v dW_t \\ \qquad -\bar{z}_t^v dY_t - \sum_{i=1}^{\infty} q_{t-}^{v,i} dH_t^i, \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v\right). \end{cases} \tag{1.4}$$

Define $dP^v = \rho_t^v dP$ with

$$\rho_t^v = \exp\left\{\int_0^t \xi\left(s, x_s^v\right) dY_s - \frac{1}{2}\int_0^t \left|\xi\left(s, x_s^v\right)\right|^2 ds\right\},$$

where ρ^v is the unique $\mathcal{F}_t^Y$-adapted solution of the SDE of McKean–Vlasov type

$$\begin{cases} d\rho_t^v = \rho_t^v \xi\left(t, x_t^v\right) dY_t, \\ \rho_0^v = 1. \end{cases} \tag{1.5}$$

Obviously, cost functional (1.3) can be rewritten as

$$J(v) = \mathbb{E}\left[\int_0^T \rho_t^v l\left(t, x_t^v, y_t^v, z_t^v, \bar{z}_t^v, q_t^v, v_t\right) dt + \rho_T^v M\left(x_T^v\right) + h\left(y_0^v\right)\right] \tag{1.6}$$

Then the original problem (1.3) is equivalent to minimize (1.6) over $v \in \mathcal{U}_{ad}$ subject to (1.1) and (1.5).

Condition (**H1**)

1. The function $\beta(\cdot, 0, 0) \in \mathbb{L}_{\mathcal{F}}^2(0, T, \mathbb{R}^n)$ for $\beta = b, \sigma, \widetilde{\sigma}$ and $g(\cdot, 0, 0) \in l_{\mathcal{F}}^2(0, T, \mathbb{R}^n)$, $\xi(\cdot, 0) \in \mathbb{L}_{\mathcal{F}}^2(0, T, \mathbb{R}^n)$, $f(\cdot, 0, 0, 0, 0, 0, 0) \in \mathbb{L}_{\mathcal{F}}^2(0, T, \mathbb{R}^n)$ and $\varphi(0) \in \mathbb{L}^2(\Omega, \mathcal{F}, P, \mathbb{R}^n)$.
2. The functions $b, \sigma, \widetilde{\sigma}$ and g are continuously differentiable in (x, v) and they are bounded by $C(1 + |x| + |v|)$, and the function ξ is continuously differentiable in x.
3. The functions f and l are continuously differentiable in $(x, y, z, \bar{z}, q, v)$, and they are bounded by $C(1+|x|+|y|+|z|+|\bar{z}|+|q|+|v|)$ and $C\left(1 + |x|^2 + |y|^2 + |z|^2 + |\bar{z}|^2 + |q|^2 + |v|^2\right)$. The derivatives of f and l with respect to $(x, y, z, \bar{z}, q, v)$ are uniformly bounded.
4. The functions φ and M are continuously differentiable in x, and the function h is continuously differentiable in y. The derivatives M_x, h_y are bounded by $C(1 + |x|)$ and $C(1 + |y|)$ respectively.
5. The derivatives $b_x, b_v, \sigma_x, \sigma_v, \widetilde{\sigma}_x, \widetilde{\sigma}_v, \xi_x$ are continuous and uniformly bounded.

Under condition (**H1**), and with the help of Theorem 3.1 in [9], and Lemma 2 in [41], for each $v \in \mathcal{U}_{ad}$, there is a unique solution $(x, y, z, \bar{z}, q) \in \mathbb{S}_{\mathcal{F}}^2(0, T, \mathbb{R}^n) \times \mathbb{S}_{\mathcal{F}}^2(0, T, \mathbb{R}^n) \times \mathbb{L}_{\mathcal{F}}^2\left(0, T, \mathbb{R}^{n\times d}\right) \times \mathbb{L}_{\mathcal{F}}^2\left(0, T, \mathbb{R}^{n\times d}\right) \times l_{\mathcal{F}}^2(0, T, \mathbb{R}^n)$ which solves

$$\begin{cases} x_t^v = x_0 + \int_0^t \left[b\left(s, x_s^v, v_s\right) - \widetilde{\sigma}\left(s, x_s^v, v_s\right) \xi\left(s, x_s^v\right)\right] ds + \int_0^t \sigma\left(s, x_s^v, v_s\right) dW_s \\ \qquad + \int_0^t \widetilde{\sigma}\left(s, x_s^v, v_s\right) dY_s + \sum_{i=1}^{\infty} \int_0^t g^i\left(s, x_{s_-}^v, v_s\right) dH_s^i, \\ y_t^v = y_T^v - \int_t^T f\left(s, x_s^v, y_s^v, z_s^v, q_s^v, v_s\right) dt + \int_t^T z_s^v dW_s + \int_t^T \bar{z}_s^v dY_s + \sum_{i=1}^{\infty} \int_t^T q_{s_-}^{v,i} dH_s^i. \end{cases}$$

We denote for ξ and $\phi = b, \sigma, \widetilde{\sigma}, g$

$$\begin{aligned} \xi(t) &= \xi\left(t, x_t\right), \qquad \phi(t) = \phi\left(t, x_t, u_t\right), \\ \xi_x(t) &= \xi_x\left(t, x_t\right), \quad \phi_\rho(t) = \phi_\rho\left(t, x_t, u_t\right), \text{for } \rho = x, v, \end{aligned}$$

Similarly, we denote for $\Lambda = f, l$ and $\rho = x, y, z, \bar{z}, q, v$

$$\begin{aligned} \Lambda(t) &= \Lambda\left(t, x_t, y_t, z_t, \bar{z}_t, q_t, u_t\right), \\ \Lambda_\rho(t) &= \Lambda_\rho\left(t, x_t, y_t, z_t, \bar{z}_t, q_t, u_t\right). \end{aligned}$$

Now, we introduce the following variational equations which is a linear FBSDEs

$$\begin{cases} dx_t^1 = \Big[\left(b_x(t) - \widetilde{\sigma}_x(t)\xi(t) - \widetilde{\sigma}(t)\xi_x(t)\right)x_t^1 + \left(b_v(t) - \widetilde{\sigma}_v(t)\xi(t)\right)v_t\Big]dt \\ \qquad + \Big[\sigma_x(t)x_t^1 + \sigma_v(t)v_t\Big]dW_t + \Big[\widetilde{\sigma}_x(t)x_t^1 + \widetilde{\sigma}_v(t)v_t\Big]dY_t \\ \qquad + \sum_{i=1}^{\infty}\Big[g_x^i(t)x_t^1 + g_v^i(t)v_t\Big]dH_t^i, \\ -dy_t^1 = \Big[f_x(t)x_t^1 + f_y(t)y_t^1 + f_z(t)z_t^1 + f_{\bar{z}}(t)\bar{z}_t^1 + f_q(t)q_t^1 + +f_v(t)v_t\Big]dt \\ \qquad - z_t^1 dW_t - \bar{z}_t^1 dY_t + \sum_{i=1}^{\infty} q_{t\_}^1 dH_t^i, \\ x_0^1 = 0, \quad y_T^1 = \varphi_x(x_T)x_T^1, \end{cases} \tag{1.7}$$

and a linear SDE

$$\begin{cases} d\rho_t^1 = \Big[\rho_t^1\xi(t) + \rho_t\xi_x(t)x_t^1\Big]dY_t, \\ \rho_0^1 = 0. \end{cases} \tag{1.8}$$

Set $\vartheta = \rho^{-1}\rho^1$, using Itô's formula, we have

$$\begin{cases} d\vartheta_t = \xi_x(t)x_t^1 d\widetilde{W}_t, \\ \vartheta_0 = 0. \end{cases} \tag{1.9}$$

Next, we introduce the following adjoint equations of McKean–Vlasov type

$$\begin{cases} -d\Psi_t = \Big[b_x(t)\Psi_t - \widetilde{\sigma}(t)\xi_x(t)\Psi_t - \widetilde{\sigma}_x(t)\xi(t)\Psi_t + \sigma_x(t)k_t \\ \qquad + \widetilde{\sigma}_x(t)\bar{k}_t + \sum_{i=1}^{\infty} g_x^i(t)n_t^i + \xi_x(t)Q_t - f_x(t)q_t + l_x(t)\Big]dt \\ \qquad - k_t dW_t - \bar{k}_t d\widetilde{W}_t - \sum_{i=1}^{\infty} n_t^i dH_t^i, \\ d\Phi_t = \left[f_y(t)\Phi_t - l_y(t)\right]dt + \left[f_z(t)\Phi_t - l_z(t)\right]dW_t \\ \qquad + \left[f_{\bar{z}}(t)\Phi_t - \xi(t)\Phi_t - l_{\bar{z}}(t)\right]d\widetilde{W}_t, \\ \qquad + \sum_{i=1}^{\infty}\Big[f_q^i(t)\Phi_t^i - l_q^i(t)\Big]dH_t^i, \\ \Psi_T = M_x(x_T) - \varphi_x(x_T)\Phi_t, \\ \Phi_0 = -h_y(y_0). \end{cases} \tag{1.10}$$

It is clear that, under (**H1**), there exists a unique $\left(\Psi, k, \bar{k}, n, \Phi\right) \in \mathbb{S}^2_{\mathcal{F}}(0,T,\mathbb{R}^n) \times \mathbb{L}^2_{\mathcal{F}}\left(0,T,\mathbb{R}^{n\times d}\right) \times \mathbb{L}^2_{\mathcal{F}}\left(0,T,\mathbb{R}^{m\times d}\right) \times l^2_{\mathcal{F}}(0,T,\mathbb{R}^n) \times \mathbb{S}^2_{\mathcal{F}}(0,T,\mathbb{R}^n)$ satisfying the FBSDE (1.10).

Now, we introduce the following BSDE involved in the stochastic maximum principle

$$\begin{aligned} -d\Gamma_t &= l\left(t, x_t, y_t, z_t, \bar{z}_t, q_t, u_t\right) dt - \bar{Q}_t dW_t - Q_t d\widetilde{W}_t, \\ \Gamma_T &= M(x_T). \end{aligned} \tag{1.11}$$

Under condition (**H1**), it is easy to prove that equation (1.11) admits a unique strong solution.

We define the Hamiltonian function

$$\begin{aligned} H &: [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d} \times \mathbb{R}^{n\times d} \times l^2(\mathbb{R}^n) \times l^2(\mathbb{R}^n) \\ &\times U \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d} \times \mathbb{R}^{n\times d} \times l^2(\mathbb{R}^n) \times \mathbb{R}^{n\times d} \to \mathbb{R}, \end{aligned}$$

associated with the stochastic control problem (1.1)-(1.6) by

$$\begin{aligned} &H(t,x,y,z,\bar{z},q,v,\Psi,\Phi,k,\bar{k},n,Q) \\ &= \Psi\left(b(t,x,v) - \tilde{\sigma}(t,x,v)\,\xi(t,x)\right) - \Phi f(t,x,y,z,\bar{z},q,v) \\ &\quad + k\sigma(t,x,v) + \bar{k}\tilde{\sigma}(t,x,v) + \sum_{i=1}^{\infty} n_t^i g^i(t,x,v) + Q\xi(t,x) \\ &\quad + l(t,x,y,z,\bar{z},q,v)\,. \end{aligned} \tag{1.12}$$

1.3 Stochastic maximum principle

In this section, we prove the necessary conditions of optimality for our system, satisfied by a partially observed optimal control, assuming that the solution exists. The proof is based on convex perturbation and on some estimates of the state processes of system and observed process.

Suppose that u is an optimal control with the optimal trajectory $(x, y, z, \bar{z}, q)$ of FBSDE (1.1). For any $0 \le \theta \le 1$ and $v + u \in \mathcal{U}_{ad}$, we define a perturbed control $u_t^\theta = u_t + \theta v_t$.

Lemma 1.1

*Under condition (**H1**), the following estimations holds*

$$\lim_{\theta\to 0} \mathbb{E}\left[\sup_{0\le t\le T} \left|\tilde{x}_t^\theta\right|^2\right] = 0, \tag{1.13}$$

$$\lim_{\theta\to 0} \mathbb{E}\left[\sup_{0\le t\le T} \left|\tilde{y}_t^\theta\right|^2 + \int_0^T \left(\left|\tilde{z}_t^\theta\right|^2 + \left|\tilde{\bar{z}}_t^\theta\right|^2 + \left\|\tilde{q}_t^\theta\right\|^2_{l^2(\mathbb{R}^n)}\right) ds\right] = 0, \tag{1.14}$$

$$\mathbb{E}\int_0^T \left|\tilde{\rho}_t^\theta\right|^2 dt = 0. \tag{1.15}$$

Proof. For notational ease, we introduce the following notations.

For $t \in [0,T]$, $\theta > 0$, we set

$$\begin{aligned}
\widetilde{x}_t^{\theta} &= \theta^{-1}\left(x_t^{\theta} - x_t\right) - x_t^1, & \widetilde{\bar{z}}_t^{\theta} &= \theta^{-1}\left(\bar{z}_t^{\theta} - \bar{z}_t\right) - \bar{z}_t^1,\\
\widetilde{y}_t^{\theta} &= \theta^{-1}\left(y_t^{\theta} - y_t\right) - y_t^1, & \widetilde{q}_t^{\theta} &= \theta^{-1}\left(q_t^{\theta} - q_t\right) - q_t^1,\\
\widetilde{z}_t^{\theta} &= \theta^{-1}\left(z_t^{\theta} - z_t\right) - z_t^1, & \widetilde{\rho}_t^{\theta} &= \theta^{-1}\left(\rho_t^{\theta} - \rho_t\right) - \rho_t^1,
\end{aligned}$$

and we denote by

$$\begin{aligned}
\widetilde{x}_t^{\lambda,\theta} &= x_t + \lambda\theta\left(\widetilde{x}_t^{\theta} + x_t^1\right), & \widetilde{z}_t^{\lambda,\theta} &= z_t + \lambda\theta\left(\widetilde{z}_t^{\theta} + z_t^1\right),\\
\widetilde{y}_t^{\lambda,\theta} &= y_t + \lambda\theta\left(\widetilde{y}_t^{\theta} + y_t^1\right), & \widetilde{\bar{z}}_t^{\lambda,\theta} &= \bar{z}_t + \lambda\theta\left(\widetilde{\bar{z}}_t^{\theta} + \bar{z}_t^1\right),\\
\gamma_t^{\lambda,\theta} &= \left(\widetilde{x}_t^{\lambda,\theta}, u_t^{\theta}\right), & \widetilde{q}_t^{\lambda,\theta} &= q_t + \lambda\theta\left(\widetilde{q}_t^{\theta} + q_t^1\right).
\end{aligned}$$

First, we have

$$\begin{cases}
d\tilde{x}_t^{\theta} = \left(\left[b_t^x - \widetilde{\sigma}_t^x \xi_t - \widetilde{\sigma}_t \xi_t^x\right]\widetilde{x}_t^{\theta} + \beta_1^{\theta}\right)dt\\
\qquad + \left(\sigma_t^x \widetilde{x}_t^{\theta} + \beta_2^{\theta}\right)dW_t + \left(\widetilde{\sigma}_t^x \widetilde{x}_t^{\theta} dt + \beta_3^{\theta}\right)dY_t\\
\qquad + \sum_{i=1}^{\infty}\left(g_t^{i,x}\widetilde{x}_t^{\theta} + \beta_4^{i,\theta}\right)dH_t^i,\\
\tilde{x}_0^{\theta} = 0,
\end{cases} \tag{1.16}$$

where

$$\begin{aligned}
b_t^x &= \int_0^1 b_x\left(t, \gamma_t^{\lambda,\theta}\right)d\lambda, & \widetilde{\sigma}_t^x &= \int_0^1 \widetilde{\sigma}_x\left(t, \gamma_t^{\lambda,\theta}\right)d\lambda,\\
\xi_t^x &= \int_0^1 \xi_x\left(t, \gamma_t^{\lambda,\theta}\right)d\lambda, & \sigma_t^x &= \int_0^1 \sigma_x\left(t, \gamma_t^{\lambda,\theta}\right)d\lambda,\\
g_t^{i,x} &= \int_0^1 g_x^i\left(t, \gamma_t^{\lambda,\theta}\right)d\lambda,
\end{aligned}$$

and

$$\begin{aligned}
\beta_1^{\theta} = {} & \int_0^1 \left[b_x\left(t, \gamma_t^{\lambda,\theta}\right) - b_x(t)\right]d\lambda x_t^1\\
& - \xi_t \int_0^1 \left[\widetilde{\sigma}_x\left(t, \gamma_t^{\lambda,\theta}\right) - \widetilde{\sigma}_x(t)\right]d\lambda x_t^1 - \widetilde{\sigma}_t \int_0^1 \left[\xi_x\left(t, \gamma_t^{\lambda,\theta}\right) - \xi_x(t)\right]d\lambda x_t^1\\
& + \int_0^1 \left[b_v\left(t, \gamma_t^{\lambda,\theta}\right) - b_v(t)\right]d\lambda v_t - \xi_t \int_0^1 \left[\widetilde{\sigma}_v\left(t, \gamma_t^{\lambda,\theta}\right) - \widetilde{\sigma}_v(t)\right]d\lambda v_t,
\end{aligned}$$

$$\beta_2^{\theta} = \int_0^1 \left[\sigma_x\left(t, \gamma_t^{\lambda,\theta}\right) - \sigma_x(t)\right]d\lambda x_t^1 + \int_0^1 \left[\sigma_v\left(t, \gamma_t^{\lambda,\theta}\right) - \sigma_v(t)\right]d\lambda v_t,$$

$$\beta_3^{\theta} = \int_0^1 \left[\widetilde{\sigma}_x\left(t, \gamma_t^{\lambda,\theta}\right) - \widetilde{\sigma}_x(t)\right]d\lambda x_t^1 + \int_0^1 \left[\widetilde{\sigma}_v\left(t, \gamma_t^{\lambda,\theta}\right) - \widetilde{\sigma}_v(t)\right]d\lambda v_t,$$

$$\beta_4^{i,\theta} = \int_0^1 \left[g_x^i\left(t, \gamma_t^{\lambda,\theta}\right) - g_x^i(t)\right]d\lambda x_t^1 + \int_0^1 \left[g_v^i\left(t, \gamma_t^{\lambda,\theta}\right) - g_v^i(t)\right]d\lambda v_t.$$

Under condition (**H1**), it is not difficult to see that

$$\lim_{\theta\to 0}\mathbb{E}\left[\left|\beta_1^\theta\right|^2+\left|\beta_2^\theta\right|^2+\left|\beta_3^\theta\right|^2+\left|\beta_4^{i,\theta}\right|^2\right]=0.$$

Applying Itô's formula to $\left|\widetilde{x}_t^\theta\right|^2$, we have

$$\begin{aligned}\mathbb{E}\left|\widetilde{x}_t^\theta\right|^2 &= 2\mathbb{E}\int_0^T \widetilde{x}_t^\theta\left(\left[b_t^x-\widetilde{\sigma}_t^x\xi_t-\widetilde{\sigma}_t\xi_t^x\right]\widetilde{x}_t^\theta+\beta_1^\theta\right)dt\\ &\quad+\mathbb{E}\int_0^T\left|\sigma_t^x\widetilde{x}_t^\theta+\beta_2^\theta\right|^2dt+\mathbb{E}\int_0^T\left|\widetilde{\sigma}_t^x\widetilde{x}_t^\theta+\beta_3^\theta\right|^2dt\\ &\quad+\sum_{i=1}^\infty\mathbb{E}\int_0^T\left|g_t^{i,x}\widetilde{x}_t^\theta+\beta_4^{i,\theta}\right|^2dt\\ &\leq C\mathbb{E}\int_0^T\left|\widetilde{x}_t^\theta\right|^2dt+\int_0^T\mathbb{E}\left[\left|\beta_1^\theta\right|^2+\left|\beta_2^\theta\right|^2+\left|\beta_3^\theta\right|^2+\left|\beta_4^{i,\theta}\right|^2\right]dt.\end{aligned}$$

Finally, estimate (1.13) now follows easily from the Gronwall inequality.

Let $\left(\widetilde{y}_t^\theta,\widetilde{z}_t^\theta,\widetilde{\bar{z}}_t^\theta,\widetilde{q}_t^\theta\right)$ be the solution of the following BSDE

$$\begin{cases} d\widetilde{y}_t^\theta=\left[f_t^x\widetilde{x}_t^\theta+f_t^y\widetilde{y}_t^\theta+f_t^z\widetilde{z}_t^\theta+f_t^{\bar{z}}\widetilde{\bar{z}}_t^\theta+f_t^q\widetilde{q}_t^\theta+\Upsilon_t^\theta\right]dt\\ \qquad\qquad+\widetilde{z}_t^\theta dW_t+\widetilde{\bar{z}}_t^\theta dY_t+\sum_{i=1}^\infty\widetilde{q}_t^\theta dH_t^i,\\ \widetilde{y}_T^\theta=\theta^{-1}\left[\varphi\left(x_T^\theta\right)-\varphi\left(x_T\right)\right]-\varphi_x\left(x_T\right)x_T^1,\end{cases}$$

where $\tilde{x}_t^\theta$ satisfies Eq. (1.16), and

$$f_t^\alpha=-\int_0^1 f_\alpha\left(t,\chi_t^{\lambda,\theta}\right)d\lambda,\ \text{ for }\alpha=x,y,z,\bar{z},q,$$

where

$$\chi_t^{\lambda,\theta}=\left(\widetilde{x}_t^{\lambda,\theta},\widetilde{y}_t^{\lambda,\theta},\widetilde{z}_t^{\lambda,\theta},\widetilde{\bar{z}}_t^{\lambda,\theta},\widetilde{q}_t^{\lambda,\theta},u_t^{\lambda,\theta}\right),$$

and Υ_t^θ is given by

$$\begin{aligned}\Upsilon_t^\theta &= \int_0^1\left[f_x\left(t,\chi_t^{\lambda,\theta}\right)-f_x\left(t\right)\right]d\lambda x_t^1+\int_0^1\left[f_y\left(t,\chi_t^{\lambda,\theta}\right)-f_y\left(t\right)\right]d\lambda y_t^1\\ &\quad+\int_0^1\left[f_z\left(t,\chi_t^{\lambda,\theta}\right)-f_z\left(t\right)\right]d\lambda z_t^1+\int_0^1\left[f_{\bar{z}}\left(t,\chi_t^{\lambda,\theta}\right)-f_{\bar{z}}\left(t\right)\right]d\lambda\bar{z}_t^1\\ &\quad+\int_0^1\left[f_q\left(t,\chi_t^{\lambda,\theta}\right)-f_q\left(t\right)\right]d\lambda q_t^1+\int_0^1\left[f_v\left(t,\chi_t^{\lambda,\theta}\right)-f_v\left(t\right)\right]d\lambda v_t.\end{aligned}$$

Due the fact that $f_t^x,f_t^y,f_t^z,f_t^{\bar{z}}$ and f_t^q are continuous, we have

$$\lim_{\theta\to 0}\mathbb{E}\left|\Upsilon_t^\theta\right|^2=0. \tag{1.17}$$

Appying Itô's formula to $\left|\widetilde{y}_t^\theta\right|^2$, we have

$$\begin{aligned}&\mathbb{E}\left|\widetilde{y}_t^\theta\right|^2+\mathbb{E}\int_t^T\left|\widetilde{z}_s^\theta\right|^2ds+\mathbb{E}\int_t^T\left|\widetilde{\bar{z}}_s^\theta\right|^2ds+\mathbb{E}\int_t^T\left\|\widetilde{q}_s^\theta\right\|^2_{l^2(\mathbb{R}^n)}ds\\ =\ &\mathbb{E}\left|\widetilde{y}_T^\theta\right|^2+2\mathbb{E}\int_t^T\widetilde{y}_s^\theta\left(f_s^x\widetilde{x}_s^\theta+f_s^y\widetilde{y}_s^\theta+f_s^z\widetilde{z}_s^\theta+f_s^{\bar{z}}\widetilde{\bar{z}}_s^\theta+f_s^q\widetilde{q}_s^\theta+\Upsilon_s^\theta\right)ds.\end{aligned}$$

By Young's inequality, for each $\varepsilon>0$, we get

$$\begin{aligned}&\mathbb{E}\left|\widetilde{y}_t^\theta\right|^2+\mathbb{E}\int_t^T\left|\widetilde{z}_s^\theta\right|^2ds+\mathbb{E}\int_t^T\left|\widetilde{\bar{z}}_s^\theta\right|^2ds+\mathbb{E}\int_t^T\left\|\widetilde{q}_s^\theta\right\|^2_{l^2(\mathbb{R}^n)}ds\\ \leq\ &\mathbb{E}\left|\widetilde{y}_T^\theta\right|^2+\frac{1}{\varepsilon}\mathbb{E}\int_t^T\left|\widetilde{y}_s^\theta\right|^2ds\\ &+\varepsilon\mathbb{E}\int_t^T\left|\left(f_s^x\widetilde{x}_s^\theta+f_s^y\widetilde{y}_s^\theta+f_s^z\widetilde{z}_s^\theta+f_s^{\bar{z}}\widetilde{\bar{z}}_s^\theta+f_s^q\widetilde{q}_s^\theta+\Upsilon_s^\theta\right)\right|^2ds\end{aligned}$$

$$\begin{aligned}\leq\ &\mathbb{E}\left|\widetilde{y}_T^\theta\right|^2+\frac{1}{\varepsilon}\mathbb{E}\int_t^T\left|\widetilde{y}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^x\widetilde{x}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^y\widetilde{y}_s^\theta\right|^2ds\\ &+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^z\widetilde{z}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^{\bar{z}}\widetilde{\bar{z}}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^q\widetilde{q}_s^\theta\right|^2ds.\end{aligned}$$

By the boundedness of $f_t^x, f_t^y, f_t^z, f_t^{\bar{z}}$, and f_t^q, we obtain

$$\begin{aligned}&\mathbb{E}\left|\widetilde{y}_t^\theta\right|^2+\mathbb{E}\int_t^T\left|\widetilde{z}_s^\theta\right|^2ds+\mathbb{E}\int_t^T\left|\widetilde{\bar{z}}_s^\theta\right|^2ds+\mathbb{E}\int_t^T\left\|\widetilde{q}_s^\theta\right\|^2_{l^2(\mathbb{R}^n)}ds\\ \leq\ &\left(\frac{1}{\varepsilon}+C_\varepsilon\right)\mathbb{E}\int_t^T\left|\widetilde{y}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|\widetilde{z}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|\widetilde{\bar{z}}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left\|\widetilde{q}_s^\theta\right\|^2_{l^2(\mathbb{R}^n)}ds\\ &+\mathbb{E}\left|\widetilde{y}_T^\theta\right|^2+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^x\widetilde{x}_s^\theta\right|^2ds+C_\varepsilon\mathbb{E}\int_t^T\left|\Upsilon_s^\theta\right|^2ds.\end{aligned}$$

Hence, in view of Eqs. (1.13), (1.17), the fact that f_t^x is continuous and bounded, by Gronwall's inequality, we obtain (1.14).

Now, we proceed to prove (1.15). It is plain to check that $\widetilde{\rho}_t^\theta$ satisfies the following equality

$$d\widetilde{\rho}_t^\theta=\left[\widetilde{\rho}_t^\theta\xi\left(t,x_t^\theta\right)+\bar{\Upsilon}_t^\theta\right]dY_t+\rho_t\xi_t^x\widetilde{x}_t^\theta dY_t,$$

where

$$\xi_t^x=\int_0^1\xi_x\left(t,\widetilde{x}_t^{\lambda,\theta}\right)d\lambda,$$

and $\bar{\Upsilon}_t^\theta$ is given by

$$\bar{\Upsilon}_t^\theta=\rho_t\int_0^1\left[\xi_x\left(t,\widetilde{x}_t^{\lambda,\theta}\right)-\xi_x\left(t\right)\right]d\lambda x_t^1+\rho_t^1\left[\xi\left(t,x_t^\theta\right)-\xi\left(t\right)\right].$$

Taking into account the fact that ξ_t^x is continuous, we deduce

$$\lim_{\theta\to 0}\mathbb{E}\left|\bar{\Upsilon}_t^\theta\right|^2=0. \tag{1.18}$$

Then, applying Itô's formula to $\left|\widetilde{\rho}_t^\theta\right|^2$ and taking expectation, we have

$$\mathbb{E}\left|\widetilde{\rho}_t^\theta\right|^2\leq C\mathbb{E}\int_0^T\left|\widetilde{\rho}_t^\theta\right|^2dt+C\mathbb{E}\int_0^T\left|\widetilde{x}_t^\theta\right|^2dt+C\mathbb{E}\int_0^T\left|\bar{\Upsilon}_t^\theta\right|^2dt.$$

Finally, by Gronwall's inequality, estimates (1.13) and recall to the Wasserstein metric, the above convergence result (1.15) holds. □

Since u is an optimal control, then, we have the following lemma.

Lemma 1.2

*Let condition (**H1**) hold. Then, we have*

$$\begin{aligned}0\ \leq\ & \mathbb{E}\left[\rho_TM_x\left(x_T\right)x_T^1\right]+\mathbb{E}\left[\rho_T^1M\left(x_T\right)\right]+\mathbb{E}\left[h_y\left(y_0\right)y_0^1\right]\\ &+\mathbb{E}\int_0^T\rho_t^1l(t)dt+\mathbb{E}\int_0^T\rho_tl_v(t)v_tdt+\mathbb{E}\int_0^T\rho_tl_x(t)x_t^1dt\\ &+\mathbb{E}\int_0^T\rho_tl_y(t)y_t^1dt+\mathbb{E}\int_0^T\rho_tl_z(t)z_t^1dt\\ &+\mathbb{E}\int_0^T\rho_tl_{\bar{z}}(t)\bar{z}_t^1dt+\mathbb{E}\int_0^T\rho_tl_q(t)q_t^1dt.\end{aligned} \tag{1.19}$$

Proof. Using Lemmas 1.1 and Taylor expansion, we get

$$\begin{aligned}0\ \leq\ & \theta^{-1}\left[J\left(u_t^\theta\right)-J\left(u_t\right)\right]\\ =\ & \theta^{-1}\mathbb{E}\left[\rho_T^\theta M\left(x_T^\theta\right)-\rho_TM\left(x_T\right)\right]\\ &+\theta^{-1}\mathbb{E}\left[h\left(y_0^\theta\right)-h\left(y_0\right)\right]\\ &+\theta^{-1}\mathbb{E}\int_0^T\left[\rho_t^\theta l^\theta\left(t\right)-\rho_tl\left(t\right)\right]dt\\ =\ & J_1+J_2+J_3,\end{aligned}$$

where $l^\theta\left(t\right)=l\left(t,x_t^\theta,,y_t^\theta,z_t^\theta,\bar{z}_t^\theta,q_t^\theta,u_t^\theta\right).$

Then, from the results of (1.13), (1.14) and (1.15), we derive

$$\begin{aligned}J_1\ =\ & \theta^{-1}\mathbb{E}\left[\rho_T^\theta M\left(x_T^\theta\right)-\rho_TM\left(x_T\right)\right]\\ =\ & \theta^{-1}\mathbb{E}\left[\left(\rho_T^\theta-\rho_T\right)M\left(x_T^\theta\right)\right]\\ &+\theta^{-1}\mathbb{E}\left[\rho_T\int_0^1M_x\left(x_T+\lambda\left(x_T^\theta-x_T\right)\right)\left(x_T^\theta-x_T\right)d\lambda\right]\\ \longrightarrow\ & \mathbb{E}^u\left[\vartheta_TM\left(x_T\right)\right]+\mathbb{E}^u\left[M_x\left(x_T\right)x_T^1\right].\end{aligned}$$

Similarly, we have

$$\begin{aligned}
J_2 &= \theta^{-1}\mathbb{E}\left[h\left(y_0^\theta\right) - h\left(y_0\right)\right] \\
&= \theta^{-1}\mathbb{E}\left[\int_0^1 h_y\left(y_0 + \lambda\left(y_0^\theta - y_0\right)\right)\left(y_0^\theta - y_0\right)d\lambda\right] \\
&\longrightarrow \mathbb{E}^u\left[h_y\left(y_0\right)y_0^1\right],
\end{aligned}$$

and

$$\begin{aligned}
J_3 &= \theta^{-1}\mathbb{E}\left[\int_0^T\left(\rho_t^\theta l^\theta\left(t\right) - \rho_t l\left(t\right)\right)dt\right] \\
&\longrightarrow \mathbb{E}^u\int_0^T\left[\vartheta_t l(t) + l_x(t)x_t^1 + l_y(t)y_t^1 + l_z(t)z_t^1\right. \\
&\quad + + l_{\bar{z}}(t)\bar{z}_t^1 + l_q(t)q_t^1 + l_v(t)v_t dt.
\end{aligned}$$

Then, the variational inequality (1.19) can be rewritten as

$$\begin{aligned}
0 \leq\ & \mathbb{E}^u\left[M_x\left(x_T\right)x^1\left(T\right)\right] \\
&+\mathbb{E}^u\left[\vartheta_T M\left(x_T\right)\right] + \mathbb{E}^u\left[h_y\left(y_0\right)y^1\left(0\right)\right] + \mathbb{E}^u\int_0^T\vartheta_t l(t)dt \\
&+\mathbb{E}^u\int_0^T l_v(t)v_t dt + \mathbb{E}^u\int_0^T\left[l_x(t)x_t^1\right]dt + \mathbb{E}^u\int_0^T\left[l_y(t)y_t^1\right]dt \\
&+\mathbb{E}^u\int_0^T\left[l_z(t)z_t^1\right]dt + \mathbb{E}^u\int_0^T\left[l_{\bar{z}}(t)\bar{z}_t^1\right]dt + \mathbb{E}^u\int_0^T\left[l_q(t)q_t^1\right]dt.
\end{aligned} \tag{1.20}$$

□

Theorem 1.1

*(Partial necessary conditions of optimality) Let condition (**H1**) hold and let* $(x, y, z, \bar{z}, q, u)$ *be an optimal solution of our partially observed optimal control problem. Then, there are* $(\Psi, \Phi, k, \bar{k}, n)$ *and* $(\Gamma, \bar{Q}, Q)$ *of* $\mathbb{F}$*-adapted processes that satisfy Eqs. (1.10), (1.11) respectively, and that for all* $v \in \mathcal{U}_{ad}$*, we have*

$$\mathbb{E}^u\left[H_v\left(t\right)\left(v_t - u_t\right)/\mathcal{F}_t^Y\right] \geq 0, a.\mathbb{E}, a.s, \tag{1.21}$$

where the Hamiltonian function

$$H\left(t\right) = H\left(t, x_t, y_t, z_t, \bar{z}_t, q_t, u_t, \Psi_t, \Phi_t, k_t, \bar{k}_t, n_t, Q_t\right),$$

is defined by (1.12).

Proof. Applying Itô's formula to $\Psi_t x_t^1$ and $\Phi_t y_t^1$ such that,

$$\Phi_0 = -h_y(y_0),$$
$$\Psi_T = M_x(x_T) - \varphi_x(x_T)\Phi_T,$$

we get

$$\begin{aligned}\mathbb{E}^u\left[\Psi_T x_T^1\right] &= \mathbb{E}^u \int_0^T \left[\Psi_t \left(b_v(t) - \widetilde{\sigma}_v(t)\xi(t)\right) v_t + \bar{k}_t \widetilde{\sigma}_v(t) v_t + k_t g_v(t) v_t + \sum_{i=1}^{\infty} n_t^i g_v^i(t) v_t\right] dt \\ &+ \mathbb{E}^u \int_0^T x_t^1 \left[f_x(t)\Phi_t - l_x(t)\right] dt - \mathbb{E}^u \int_0^T x_t^1 \left[\xi_x(t) Q_t\right] dt, \end{aligned} \tag{1.22}$$

and

$$\begin{aligned} & \mathbb{E}^u\left[\Phi_T y_T^1\right] + \mathbb{E}^u\left[h_y(y_0)\right] \\ = & -\mathbb{E}^u \int_0^T \Phi_t \left[f_v(t) v_t + f_x(t) x_t^1\right] dt \\ & -\mathbb{E}^u \int_0^T y_t^1 l_y(t)\, dt - \mathbb{E}^u \int_0^T z_t^1 l_z(t)\, dt \\ & -\mathbb{E}^u \int_0^T \bar{z}_t^1 l_{\bar{z}}(t)\, dt - \mathbb{E}^u \int_0^T q_t^1 l_q(t)\, dt. \end{aligned} \tag{1.23}$$

Now, applying Itô's formula to $\vartheta_t \Gamma_t$, we have

$$\mathbb{E}^u\left[\vartheta_T M(x_T)\right] = -\mathbb{E}^u \int_0^T \vartheta_t l(t)\, dt + \mathbb{E}^u \int_0^T Q_t \xi_x(t) x_t^1 dt. \tag{1.24}$$

From Eqs. (1.22), (1.23), and (1.24), we obtain

$$\begin{aligned} & \mathbb{E}^u\left[M_x(x_T)\right] + \mathbb{E}^u\left[h_y(y_0) + \vartheta_T M(x_T)\right] \\ = & \mathbb{E}^u \int_0^T \left[\Psi_t \left[b_v(t) - \widetilde{\sigma}_v \xi(t)\right] v_t + \bar{k}_t \widetilde{\sigma}_v(t) v_t + k_t g_v(t) v_t + \sum_{i=1}^{\infty} n_t^i g_v^i(t) v_t - \Phi_t f_v(t) v_t\right] dt \\ & -\mathbb{E}^u \int_0^T \vartheta_t l(t)\, dt - \mathbb{E}^u \int_0^T x_t^1 l_x(t) dt - \mathbb{E}^u \int_0^T y_t^1 l_y(t)\, dt - \mathbb{E}^u \int_0^T z_t^1 l_z(t)\, dt \\ & -\mathbb{E}^u \int_0^T \bar{z}_t^1 l_{\bar{z}}(t)\, dt - \mathbb{E}^u \int_0^T q_t^1 l_q(t)\, dt, \end{aligned} \tag{1.25}$$

thus

$$\begin{aligned} & \mathbb{E}^u\left[M_x(x_T)\right] + \mathbb{E}^u\left[h_y(y_0) + \vartheta_T M(x_T)\right] \\ = & \mathbb{E}^u \int_0^T H_v(t) v_t - \mathbb{E}^u \int_0^T l_v(t) v_t dt - \mathbb{E}^u \int_0^T \vartheta_t l(t)\, dt - \mathbb{E}^u \int_0^T x_t^1 \left[l_x(t)\right] dt \\ & -\mathbb{E}^u \int_0^T y_t^1 l_y(t)\, dt - \mathbb{E}^u \int_0^T z_t^1 l_z(t)\, dt - \mathbb{E}^u \int_0^T \bar{z}_t^1 l_{\bar{z}}(t)\, dt - \mathbb{E}^u \int_0^T q_t^1 l_q(t)\, dt. \end{aligned}$$

This together with the variational inequality (1.20) imply (1.21), the proof is then completed. □

1.4 Linear-quadratic control problem

In this section, we are going to consider a partially observed linear-quadratic control problem. We find an explicit expression of the corresponding optimal control by applying our partial necessary conditions established in the previous section.

We consider the following forward-backward system:

$$\begin{cases} dx_t = \left(b_t^1 x_t + b_t^2 v_t - \sigma_t^2 \gamma_t\right) dt + \sigma_t^1 dW_t + \sigma_t^2 dY_t + \sum_{i=1}^{\infty} g_t^i dH_t^i, \\ -dy_t = \left(f_t^1 x_t + f_t^2 y_t + f_t^3 z_t + f_t^4 \overline{z}_t + f_t^5 q_t + f_t^6 v_t\right) dt - z_t dW_t - \overline{z} dY_t - \sum_{i=1}^{\infty} q_t^i dH_t^i, \\ x(0) = x_0,\ y_T = \phi_1 x_T, \end{cases} \tag{1.26}$$

and the SDE

$$\begin{cases} dY_t = \gamma_t dt + d\widetilde{W}_t \\ Y_0 = 0, \end{cases} \tag{1.27}$$

where

$$\begin{aligned} & b_t^1 x_t + b_t^2 v_t = b\left(t, x_t^v, v_t\right), \\ & \sigma_t^1 = \sigma\left(t, x_t^v, v_t\right), \\ & \sigma_t^2 = \widetilde{\sigma}\left(t, x_t^v, v_t\right), \\ & g_t^i = g^i\left(t, x_{t_-}^v, v_t\right), \\ & \gamma_t = \xi\left(t, x_t^v\right), \end{aligned}$$

and

$$f\left(t, x_t^v, y_t^v, z_t^v, \overline{z}_t^v, q_t^v, v_t\right) = f_t^1 x_t + f_t^2 y_t + f_t^3 z_t + f_t^4 \overline{z}_t + f_t^5 q_t + f_t^6 v_t.$$

The quadratic cost function to be minimized

$$\begin{aligned} J(v(\cdot)) \;=\; & \mathbb{E}^u \int_0^T \left[L_t^1 x_t^2 + L_t^2 y_t^2 + L_t^3 v_t^2\right] dt \\ & + \mathbb{E}^u \left[M_1 x_T^2 + h_t y_0^2\right]. \end{aligned} \tag{1.28}$$

Here, all the coefficients $b^1(\cdot), b^2(\cdot), \sigma^1(\cdot), \sigma^2(\cdot), g(\cdot), \gamma(\cdot), f^{j_1}(\cdot)$ are bounded and deterministic functions for $j_1 = 1, ..., 6$, $L^{j_2}(\cdot)$ is positive function and bounded for $j_2 = 1, 2, 3$, and $M_1(\cdot), M_2(\cdot), h(\cdot)$ are positive constants. Then for any $v \in \mathcal{U}_{ad}$, Eqs. (1.26) and (1.27) have unique solutions, respectively. Now, we introduce

$$\rho_t = \exp\left\{\int_0^t \gamma_s dY_s - \frac{1}{2}\int_0^t |\gamma_s|^2 ds\right\},$$

which is the unique $\mathcal{F}_t^Y$-adapted solution of the SDE:

$$\begin{cases} d\rho_t = \rho_t \gamma_t dY_t, \\ \rho_0 = 1, \end{cases}$$

and we define the probability measure P^v by $dP^v = \rho_t^v dP$.

In this setting, the Hamiltonian function is defined as

$$\begin{aligned} & H(t, x, y, z, \bar{z}, q, v, \Psi, \Phi, k, \bar{k}, n, Q) \\ & = \Psi \left(b_t^1 x_t + b_t^2 v_t - \sigma_t^2 \gamma_t \right) - \Phi \left(f_t^1 x_t + f_t^2 y_t \right. \\ & + f_t^3 z_t + f_t^4 \bar{z}_t + f_t^5 q_t + f_t^6 v_t + k \sigma_t^1 + \bar{k} \sigma_t^2 \\ & \left. + \sum_{i=1}^{\infty} n_t^i g_t^i \right) + Q \gamma_t + L_t^1 x_t^2 + L_t^2 y_t^2 + L_t^3 v_t^2. \end{aligned} \tag{1.29}$$

Further due to Eqs. (1.10) and (1.11), the corresponding adjoint equations will be given by

$$\begin{cases} -d\Gamma_t = \left(L_t^1 x_t^2 + L_t^2 y_t^2 + L_t^3 v_t^2 \right) dt - \bar{Q}_t dW_t - Q_t d\widetilde{W}_t, \\ \Gamma_T = M(x_T, P_{x_T}), \end{cases} \tag{1.30}$$

and

$$\begin{cases} -d\Psi_t = \left[b_t^1 \Psi_t - f_t^1 \Phi_t + 2L_t^1 x_t \right] dt \\ \qquad -k_t dW_t - \bar{k}_t d\widetilde{W}_t - \sum_{i=1}^{\infty} n_t^i dH_t^i, \\ d\Phi_t = \left(f_t^2 \Phi_t - 2L_t^2 y_t \right) dt + \left(f_t^3 \Phi_t \right) dW_t \\ \qquad + \left[f_t^4 q_t \right] d\widetilde{W}_t + \sum_{i=1}^{\infty} \left(f_t^{i,5} \Phi_t \right) dH_t^i \\ \Psi_T = 2M_1 x_T - \phi_1 x_T, \\ \Phi_0 = -2h_t y_0. \end{cases} \tag{1.31}$$

According to Theorem 1.1, the necessary condition for optimality (1.21) will be

$$\mathbb{E}^u \left[\Psi_t b_t^2 - \Phi_t f_t^6 + 2L_t^3 u_t / \mathcal{F}_t^Y \right] = 0. \ \ a.s.a.e.$$

If $u(\cdot)$ is partial observed optimal control, then

$$u_t = -\frac{1}{2L_t^3} \left(b_t^2 \mathbb{E}^u \left[\Psi_t / \mathcal{F}_t^Y \right] - f_t^6 \mathbb{E}^u \left[\Phi_t / \mathcal{F}_t^Y \right] \right). \tag{1.32}$$

Chapter 2

Stochastic maximum principle for partially observed optimal control problem of McKean–Vlasov FBSDEs with Teugels martingales

2.1 Introduction

Our main goal in this chapter is to establish necessary conditions of partially observed optimal control problem of McKean–Vlasov FBSDEs driven by Teugels martingales, associated with some Lévy process under the assumption that the control domain is supposed to be convex. Our main result is supported by variational techniques and delicate estimates of SDE.

The stochastic system under consideration takes the following form:

$$\begin{cases} dx_t^v = b\left(t, x_t^v, P_{x_t^v}, v_t\right) dt + \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) dW_t \\ \qquad +\widetilde{\sigma}\left(t, x_t^v, P_{x_t^v}, v_t\right) d\widetilde{W}_t^v + \sum_{i=1}^{\infty} g^i\left(t, x_{t_-}^v, P_{x_{t_-}^v}, v_t\right) dH_t^i, \\ -dy_t^v = f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, q_t^v, P_{q_t^v}, v_t\right) dt - z_t^v dW_t \\ \qquad -\bar{z}_t^v dY_t - \sum_{i=1}^{\infty} q_{t_-}^{v,i} dH_t^i, \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v, P_{x_T^v}\right), \end{cases} \tag{2.1}$$

where W_t is a one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, $H_t = \left(H_t^i\right)_{i\geq 1}$ is a family of pairwise orthogonal martingales associated with some Lévy process which is independent from W_t . These martingales are called Teugels martingales. $P_{x_t}, P_{y_t}, P_{z_t}, P_{\bar{z}_t}$ and P_{q_t} denotes the law of the random variable

$x, y, z, \bar{z}$ and q respectively. The coefficients $b : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times U \to \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times U \to \mathbb{R}^{n\times d}, \widetilde{\sigma} : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times U \to \mathbb{R}^{n\times d}, g : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times U \to l^2(\mathbb{R}^n), f : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times \mathbb{R}^m \times Q_2(\mathbb{R}^m) \times \mathbb{R}^{m\times d} \times Q_2\left(\mathbb{R}^{m\times d}\right) \times \mathbb{R}^{m\times d} \times Q_2\left(\mathbb{R}^{m\times d}\right) \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times U \to l^2(\mathbb{R}^n), \varphi : \mathbb{R}^n \times Q_2(\mathbb{R}^n) \to \mathbb{R}^m$, are given deterministic functions, $Q_2\left(\mathbb{R}^d\right)$ the space of all probability measures μ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$, endowed with the following 2-Wasserstein metric (see Section 2.2 for more details).

It is worth noting that the above forward-backward stochastic differential equation (2.1) of type McKean–Vlasov is very general, in that the dependence of the coefficients on the probability law of the solution $P_{x_t^v}, P_{y_t^v}, P_{z_t^v}, P_{\bar{z}_t^v}$ and $P_{q_t^v}$ could be genuinely nonlinear as an element of the space of probability measures.

We assume that the state processes $(x^v, y^v, z^v, \bar{z}^v, q^v)$ cannot be observed directly, but the controllers can observe a related noisy process Y, which is the solution of the following equation

$$\begin{cases} dY_t = \xi\left(t, x_t^v, P_{x_t^v}\right) dt + d\widetilde{W}_t^v, \\ Y_0 = 0, \end{cases} \tag{2.2}$$

where $\xi : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \to \mathbb{R}^n$ and $\widetilde{W}_t^v$ is stochastic processes depending on the control v. The associated cost functional to be minimized is also of McKean–Vlasov type, defined as

$$\begin{aligned} J(v) = \mathbb{E}^v &\left[\int_0^T l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, q_t^v, P_{q_t^v}, v_t\right) dt\right] \\ &+ \mathbb{E}^v\left[M\left(x_T^v, P_{x_T^v}\right) + h\left(y_0^v, P_{y_0^v}\right)\right], \end{aligned} \tag{2.3}$$

where $\mathbb{E}^v$ denotes the expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P^v)$ and $M : \mathbb{R}^n \times Q_2(\mathbb{R}^n) \to \mathbb{R}, h : \mathbb{R}^m \times Q_2(\mathbb{R}^m) \to \mathbb{R}, l : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times \mathbb{R}^m \times Q_2(\mathbb{R}^m) \times \mathbb{R}^{m\times d} \times Q_2\left(\mathbb{R}^{m\times d}\right) \times \mathbb{R}^{m\times d} \times Q_2\left(\mathbb{R}^{m\times d}\right) \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times U \to \mathbb{R}$ are deterministic functions.

Our partially observed optimal control problem of general McKean–Vlasov FBSDE is to minimize the cost functional (2.3) over $v \in \mathcal{U}_{ad}$ subject to (2.1) and (2.2), *i.e.*,

$$\min_{v \in \mathcal{U}_{ad}} J(v).$$

2.2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space equipped with two independent standard one-dimensional Brownian motions W and Y. Let $L = \{L_t : t \in [0,T]\}$ be a $\mathbb{R}$-valued Lévy process, independent of W and Y of the form $L_t = \lambda_t + bt$, where λ_t is a pure jump process. We assume that $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ and $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^Y \vee \mathcal{F}_t^L \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of P-null set and $\mathcal{F}_t^W, \mathcal{F}_t^Y$ and $\mathcal{F}_t^L$ denotes the P-completed natural filtration generated by W, Y and L respectively. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, and by $(\cdot,\cdot)$ (resp. $|\cdot|$) the inner product (resp. norm). The set of the admissible control variables is denoted by $\mathcal{U}_{ad}$. We also assume that the Lévy measure $\nu(dx)$ corresponding to the Lévy process λ_t satisfies the following.

1. For every $\delta > 0$, there exist $\gamma > 0$ such that $\int_{(-\delta,\delta)} \exp(\gamma |x|)\, \nu(dx) < \infty$.
2. $\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu(dx) < \infty$.

The above conditions settings imply that the random variable L_t have moments in all orders. Notice that the jump of the state x_t^v caused by the Lévy process is the power jump processes defined by

$$\begin{cases} L_t^i = \sum\limits_{0<s\leq t} (\Delta L_s)^i, \text{ for } i > 1, \\ L_t^1 = L_t, \end{cases}$$

where $\Delta L_t = L_t - L_{t_-}$.

The continuous part of L_t^i obtained by removing the jumps of L_t defined by

$$L_{t,c}^i = L_t^i - \sum_{0<s\leq t} (\Delta L_s)^i, \text{ for } i > 1.$$

We distinguish between the jumps of state x_t^v and y_t^v caused by the Lévy martingales are defined by

$$\begin{cases} \Delta_L x_t^v = g\left(t, x_{t_-}^v, P_{x_{t_-}^v}, v_t\right) \Delta L_t, \\ \Delta_L y_t^v = \sum\limits_{i=1}^{\infty} q_{t-}^{v,i} \Delta L_t^i. \end{cases}$$

Now, let $N_t^i = L_t^i - \mathbb{E}\left[L_t^i\right]$, for $i \geq 1$. Then, the family of Teugels martingales $\left(H_t^i\right)_{i\geq 1}$ is defined by $H_t^i = \sum\limits_{j=1}^{j=i} \alpha_{ij} N_t^j$, where the coefficients α_{ij} associated with the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $m(dx) = x^2 \nu(dx)$.

The Teugels martingales $\left(H_t^i\right)_{i\geq 1}$ are pathwise strongly orthogonal and their predictable quadratic variation processes are given by $\left\langle H_t^i, H_t^j \right\rangle_t = \delta_{ij} t$. Let $Q_2\left(\mathbb{R}^d\right)$ be a space of all probability measures μ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$ with finite second moment, i.e, $\int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty$, endowed with the following 2-Wasserstein metric: for $\mu, \nu \in Q_2\left(\mathbb{R}^d\right)$,

$$\mathbb{D}_2(\mu_1, \mu_2) = \inf\left\{ \left[\int_{\mathbb{R}^d} |x-y|^2 \kappa(dx, dy)\right]^{\frac{1}{2}} : \kappa \in Q_2\left(\mathbb{R}^{2d}\right), \kappa\left(\cdot, \mathbb{R}^d\right) = \mu_1, \kappa\left(\mathbb{R}^d, \cdot\right) = \mu_2 \right\}.$$

Let $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{P}\right)$ be a copy of the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. For any random variable $(\vartheta, \alpha) \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right) \times \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$, we let $\left(\widehat{\vartheta}, \widehat{\alpha}\right)$ be an independent copy of the random variable (ϑ, α) defined on $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{P}\right)$. Let $\left(\widehat{u}_t, \widehat{x}_t, \widehat{y}_t, \widehat{z}_t, \widehat{\bar{z}}_t, \widehat{q}_t\right)$ be an independent copy of $(u_t, x_t, y_t, z_t, \bar{z}_t, q_t)$ so that $P_{x_t} = \widehat{P}_{\widehat{x}_t}$, $P_{y_t} = \widehat{P}_{\widehat{y}_t}$, $P_{z_t} = \widehat{P}_{\widehat{z}_t}$, $P_{\bar{z}_t} = \widehat{P}_{\widehat{\bar{z}}_t}$ and $P_{q_t} = \widehat{P}_{\widehat{q}_t}$. We denote by $\widehat{\mathbb{E}}[\cdot]$ the expectation under probability measure $\widehat{P}$ and $P_X = P \circ X^{-1}$ denotes the law of the random variable X.

In the following, we introduce the basic notations of the differentiability with respect to probability measures. The principal idea is to identify a distribution $\mu \in Q_2\left(\mathbb{R}^d\right)$ with a random variables $\vartheta \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ so that $\mu = P_\vartheta$. To be more precise, we assume that probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is rich enough in the sense that for every $\mu \in Q_2\left(\mathbb{R}^d\right)$, there is a random variable $\vartheta \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ such that $\mu = P_\vartheta$. It is well-known that the probability space $([0,1], \mathcal{B}[0,1], dx)$, where dx is the Borel measure, has this property.

Next, for any function $f : Q_2\left(\mathbb{R}^d\right) \to \mathbb{R}$, we induce a function $\widetilde{f} : \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right) \to \mathbb{R}$ such that $\widetilde{f}(\vartheta) := f(P_\vartheta)$, $\vartheta \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$. Clearly, the function $\widetilde{f}$ called the lift of f in the literature, depends only on the law of $\vartheta \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ and is independent of the choice of the representative ϑ.

Definition 2.1

A function $f : Q_2\left(\mathbb{R}^d\right) \to \mathbb{R}$ is said to be differentiable at $\mu_0 \in Q_2\left(\mathbb{R}^d\right)$ if there exists $\vartheta_0 \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ with $\mu_0 = P_{\vartheta_0}$ such that its lift $\widetilde{f}$ is Fréchet differentiable at ϑ_0. More precisely, there exists a continuous linear functional $D\widetilde{f}(\vartheta_0) : \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right) \to \mathbb{R}$ such that

$$\widetilde{f}(\vartheta_0 + \alpha) - \widetilde{f}(\vartheta_0) = \left\langle D\widetilde{f}(\vartheta_0), \alpha \right\rangle + O\left(\|\alpha\|_2\right) = D_\alpha f(\mu_0) + O\left(\|\alpha\|_2\right), \qquad (2.4)$$

where $\langle \cdot, \cdot \rangle$ is the dual product on $\mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$, and we will refer to $D_\alpha f(\mu_0)$ as the

Fréchet derivative of f at μ_0 in the direction α. In this case, we have

$$D_\alpha f(\mu_0) = \left\langle D\widetilde{f}(\vartheta_0), \alpha \right\rangle = \frac{d}{dt}\widetilde{f}(\vartheta_0 + t\alpha)\Big|_{t=0}, \text{ with } \mu_0 = P_{\vartheta_0}.$$

Note that by Riesz's representation theorem, there is a unique random variable $\Lambda_0 \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ such that $\left\langle D\widetilde{f}(\vartheta_0), \alpha\right\rangle = (\Lambda_0, \alpha)_2 = \mathbb{E}\left[(\Lambda_0, \alpha)_2\right]$, where $\alpha \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$. Then there exists a Boral function $h[\mu_0] : \mathbb{R}^d \to \mathbb{R}^d$, depending only on the law $\mu_0 = P_{\vartheta_0}$ but not on the particular choice of the representative ϑ_0 such that $\Lambda_0 = h[\mu_0](\vartheta_0)$.
So, we can write equation (2.4) as

$$f(P_\vartheta) - f(P_{\vartheta_0}) = (h[\mu_0](\vartheta_0), \vartheta - \vartheta_0)_2 + O(\|\vartheta - \vartheta_0\|_2), \ \forall \vartheta \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right).$$

We shall denote $\partial_\mu f(P_{\vartheta_0}, x) = h[\mu_0](x)$, $x \in \mathbb{R}^d$. Moreover, we have the following identities:

$$\begin{aligned} D\widetilde{f}(\vartheta_0) &= \Lambda_0 = h[\mu_0](\vartheta_0) = \partial_\mu f(P_{\vartheta_0}, \vartheta_0), \\ D_\alpha f(P_{\vartheta_0}) &= \langle \partial_\mu f(P_{\vartheta_0}, \vartheta_0), \alpha\rangle, \end{aligned}$$

where $\alpha = \vartheta - \vartheta_0$, and for each $\mu \in Q_2\left(\mathbb{R}^d\right)$, $\partial_\mu f(P_\vartheta, \cdot) = h[P_\vartheta](\cdot)$ is only defined in a $P_\vartheta(dx) - a.e$ sense, where $\mu = P_\vartheta$.

We say that the function $f \in C_b^{1,1}\left(Q_2\left(\mathbb{R}^d\right)\right)$ if for all $\vartheta \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$, there exists a P_ϑ-modification of $\partial_\mu f(P_\vartheta, \cdot)$ such that $\partial_\mu f : Q_2\left(\mathbb{R}^d\right) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipchitz continuous. That is for some $C > 0$, it holds that

1. $|\partial_\mu f(\mu, x)| \le C, \forall \mu \in Q_2\left(\mathbb{R}^d\right), \forall x \in \mathbb{R}^d$;
2. $|\partial_\mu f(\mu, x) - \partial_\mu f(\mu', x')| \le C\left(\mathbb{D}_2(\mu, \mu') + |x - x'|\right), \forall \mu, \mu' \in Q_2\left(\mathbb{R}^d\right), \forall x, x' \in \mathbb{R}^d$.

If $f \in C_b^{1,1}\left(Q_2\left(\mathbb{R}^d\right)\right)$, the derivative $\partial_\mu f(P_\vartheta, \cdot), \vartheta \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ indicated in the Definition 2.1 is unique.

Definition 2.2

Let U be a nonempty convex subset of $\mathbb{R}^k$. A control $v : \Omega \times [0,T] \to U$ is called admissible if it is $\mathcal{F}_t^Y$-adapted and satisfies $\sup_{0 \le t \le T} \mathbb{E}|v_t|^2 < \infty$.

Now, inserting (2.2) into (2.1), we get

$$\begin{cases} dx_t^v = \left[b\left(t, x_t^v, P_{x_t^v}, v_t\right)dt - \widetilde{\sigma}\left(t, x_t^v, P_{x_t^v}, v_t\right)\xi\left(t, x_t^v, P_{x_t^v}\right)\right]dt \\ \qquad +\sigma\left(t, x_t^v, P_{x_t^v}, v_t\right)dW_t + \widetilde{\sigma}\left(t, x_t^v, P_{x_t^v}, v_t\right)dY_t + \sum_{i=1}^{\infty} g^i\left(t, x_{t\_}^v, P_{x_{t\_}^v}, v_t\right)dH_t^i, \\ -dy_t^v = f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, q_t^v, P_{q_t^v}, v_t\right)dt - z_t^v dW_t \\ \qquad -\bar{z}_t^v dY_t - \sum_{i=1}^{\infty} q_{t\_}^{v,i} dH_t^i, \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v, P_{x_T^v}\right). \end{cases} \tag{2.5}$$

Define $dP^v = \rho_t^v dP$ with

$$\rho_t^v = \exp\left\{\int_0^t \xi\left(s, x_s^v, P_{x_s^v}\right)dY_s - \frac{1}{2}\int_0^t \left|\xi\left(s, x_s^v, P_{x_s^v}\right)\right|^2 ds\right\},$$

where ρ^v is the unique $\mathcal{F}_t^Y$-adapted solution of the SDE of McKean–Vlasov type

$$\begin{cases} d\rho_t^v = \rho_t^v \xi\left(t, x_t^v, P_{x_t^v}\right)dY_t, \\ \rho_0^v = 1. \end{cases} \tag{2.6}$$

Obviously, cost functional (2.3) can be rewritten as

$$J(v) = \mathbb{E}\left[\int_0^T \rho_t^v l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, q_t^v, P_{q_t^v}, v_t\right)dt\right] \\ + \mathbb{E}\left[\rho_T^v M\left(x_T^v, P_{x_T^v}\right) + h\left(y_0^v, P_{y_0^v}\right)\right]. \tag{2.7}$$

Then the original problem (2.3) is equivalent to minimize (2.7) over $v \in \mathcal{U}_{ad}$ subject to (2.1) and (2.6).

Let us impose some conditions on the coefficients of the state and the performance cost functional.

Condition (H1)

1. For all $t \in [0,T]$, the function $\beta(\cdot,0,0,0) \in \mathbb{L}_{\mathcal{F}}^2(0,T,\mathbb{R}^n)$ for $\beta = b, \sigma, \widetilde{\sigma}$ and $g(\cdot,0,0,0) \in l_{\mathcal{F}}^2(0,T,\mathbb{R}^n)$, $\xi(\cdot,0,0) \in \mathbb{L}_{\mathcal{F}}^2(0,T,\mathbb{R}^n)$, $f(\cdot,0,0,0,0,0,0,0,0,0,0,0) \in \mathbb{L}_{\mathcal{F}}^2(0,T,\mathbb{R}^n)$ and $\varphi(0,0) \in \mathbb{L}^2(\Omega,\mathcal{F},P,\mathbb{R}^n)$.
2. The functions $b, \sigma, \widetilde{\sigma}$ and g are continuously differentiable in (x,v) and they are bounded by $C(1+|x|+|v|)$, and the function ξ is continuously differentiable in x.

3. The functions f and l are continuously differentiable in $(x,y,z,\bar{z},q,v)$, and they are bounded by $C(1+|x|+|y|+|z|+|\bar{z}|+|q|+|v|)$ and $C\left(1+|x|^2+|y|^2+|z^{|2}+|\bar{z}|^2+|q^{|2}+|v|^2\right)$. The derivatives of f and l with respect to $(x,y,z,\bar{z},q,v)$ are uniformly bounded.

4. The functions φ and M are continuously differentiable in x, and the function h is continuously differentiable in y. The derivatives M_x, h_y are bounded by $C(1+|x|)$ and $C(1+|y|)$ respectively.

5. The derivatives $b_x, b_v, \sigma_x, \sigma_v, \widetilde{\sigma}_x, \widetilde{\sigma}_v, \xi_x$ are continuous and uniformly bounded.

Condition (H2)

1. The functions $b, \sigma, \widetilde{\sigma}, g, f, l, \xi, M, h, \varphi \in \mathbb{C}_b^{1,1}\left(Q_2\left(\mathbb{R}^n\right)\right)$.

2. The derivatives $\partial_\mu^{P_x} b, \partial_\mu^{P_x}\sigma, \partial_\mu^{P_x}\widetilde{\sigma}, \partial_\mu^{P_x} g, \partial_\mu^{P_x}\xi, \left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}, \partial_\mu^{P_q}\right)(f,l)$ are bounded and Lipchitz continuous, such that, for some $C>0$, it holds that

(i) For $\beta = b, \sigma, \widetilde{\sigma}, \xi, g$ and $\forall \mu, \mu' \in Q_2(\mathbb{R}), \forall x, x' \in \mathbb{R}$,

$$\begin{aligned}
&\left|\partial_\mu^{P_x}\beta(t,x,\mu)\right| \leq C,\\
&\left|\partial_\mu^{P_x}\beta(t,x,\mu) - \partial_\mu^{P_x}\beta(t,x',\mu')\right| \leq C\left(\mathbb{D}_2(\mu,\mu') + |x-x'|\right),
\end{aligned}$$

(ii) For $\beta = M, \varphi$, and $\forall \mu, \mu' \in Q_2(\mathbb{R}), \forall x, x' \in \mathbb{R}$,

$$\begin{aligned}
&\left|\partial_\mu^{P_x}\beta(x,\mu)\right| \leq C,\\
&\left|\partial_\mu^{P_x}\beta(x,\mu) - \partial_\mu^{P_x}\beta(x',\mu')\right| \leq C\left(\mathbb{D}_2(\mu,\mu') + |x-x'|\right);
\end{aligned}$$

(iii) For $\beta = f, l$, and $\forall \mu_1, \mu_1', \mu_2, \mu_2', \mu_3, \mu_3', \mu_4, \mu_4', \mu_5, \mu_5' \in Q_2(\mathbb{R})$ and $\forall x, x', y, y', z, z', \bar{z}, \bar{z}', q, q' \in \mathbb{R}$,

$$\begin{aligned}
&\left|\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}, \partial_\mu^{P_q}\right)\beta(t,x,\mu_1,y,\mu_2,z,\mu_3,\bar{z},\mu_4,q,\mu_5)\right| \leq C,\\
&\Big|\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}, \partial_\mu^{P_q}\right)\beta(t,x,\mu_1,y,\mu_2,z,\mu_3,\bar{z},\mu_4,q,\mu_5)\\
&-\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}, \partial_\mu^{P_q}\right)\beta(t,x',\mu_1',y',\mu_2',z',\mu_3',\bar{z}',\mu_4',q',\mu_5')\Big|\\
&\leq C\left(|x-x'| + |y-y'| + |z-z'| + |\bar{z}-\bar{z}'| + |q-q'| + \mathbb{D}_2(\mu_1,\mu_1')\right.\\
&\left.+\mathbb{D}_2(\mu_2,\mu_2') + \mathbb{D}_2(\mu_3,\mu_3') + \mathbb{D}_2(\mu_4,\mu_4') + +\mathbb{D}_2(\mu_5,\mu_5')\right).
\end{aligned}$$

Under conditions (**H1**) and (**H2**), with the help of Theorem 3.1 in [9], and Lemma 2 in [41], for each $v \in \mathcal{U}_{ad}$, there is a unique solution $(x,y,z,\bar{z},q) \in \mathbb{S}_{\mathcal{F}}^2(0,T,\mathbb{R}^n) \times$

$\mathbb{S}^2_{\mathcal{F}}(0,T,\mathbb{R}^n)\times\mathbb{L}^2_{\mathcal{F}}\left(0,T,\mathbb{R}^{n\times d}\right)\times\mathbb{L}^2_{\mathcal{F}}\left(0,T,\mathbb{R}^{n\times d}\right)\times l^2_{\mathcal{F}}(0,T,\mathbb{R}^n)$ which solves

$$\begin{cases} x_t^v = x_0 + \int_0^t \left[b\left(s,x_s^v,P_{x_s^v},v_s\right) - \widetilde{\sigma}\left(s,x_s^v,P_{x_s^v},v_s\right)\xi\left(s,x_s^v,P_{x_s^v}\right)\right]ds + \int_0^t \sigma\left(s,x_s^v,P_{x_s^v},v_s\right)dW_s \\ \qquad +\int_0^t \widetilde{\sigma}\left(s,x_s^v,P_{x_s^v},v_s\right)dY_s + \sum_{i=1}^{\infty}\int_0^t g^i\left(s,x_{s_-}^v,P_{x_{s_-}^v},v_s\right)dH_s^i, \\ y_t^v = y_T^v - \int_t^T f\left(s,x_s^v,P_{x_s^v},y_s^v,P_{y_s^v},z_s^v,P_{z_s^v},q_s^v,P_{q_s^v},v_s\right)dt + \int_t^T z_s^v dW_s \\ \qquad +\int_t^T \bar{z}_s^v dY_s + \sum_{i=1}^{\infty}\int_t^T q_{s_-}^{v,i}dH_s^i. \end{cases}$$

We denote for ξ and $\phi = b,\sigma,\widetilde{\sigma},g$

$$\begin{aligned} \xi(t) &= \xi\left(t,x_t,P_{x_t}\right), \qquad \phi(t) = \phi\left(t,x_t,P_{x_t},u_t\right), \\ \xi_x(t) &= \xi_x\left(t,x_t,P_{x_t}\right), \qquad \phi_\rho(t) = \phi_\rho\left(t,x_t,P_{x_t},u_t\right), \text{for } \rho = x,v, \end{aligned}$$

and the derivative processes

$$\begin{aligned} \partial_\mu^{P_x}\xi(t) &= \partial_\mu^{P_x}\xi\left(t,\widehat{x}_t,P_{x_t};x_t\right), \qquad \partial_\mu^{P_x}\phi(t) = \partial_\mu^{P_x}\phi\left(t,\widehat{x}_t,P_{x_t},\widehat{u}_t;x_t\right), \\ \partial_\mu^{P_x}\xi\left(t,\widehat{x}_t\right) &= \partial_\mu^{P_x}\xi\left(t,x_t,P_{x_t};\widehat{x}_t\right), \quad \partial_\mu^{P_x}\phi\left(t,\widehat{x}_t\right) = \partial_\mu^{P_x}\phi\left(t,x_t,P_{x_t},u_t;\widehat{x}_t\right), \end{aligned}$$

Similarly, we denote for $\Lambda = f,l$ and $\rho = x,y,z,\bar{z},q,v$

$$\begin{aligned} \Lambda(t) &= \Lambda\left(t,x_t,P_{x_t},y_t,P_{y_t},z_t,P_{z_t},\bar{z}_t,P_{\bar{z}_t},q_t,P_{q_t},u_t\right), \\ \Lambda_\rho(t) &= \Lambda_\rho\left(t,x_t,P_{x_t},y_t,P_{y_t},z_t,P_{z_t},\bar{z}_t,P_{\bar{z}_t},q_t,P_{q_t},u_t\right). \end{aligned}$$

Finally, we denote for $\zeta = x,y,z,\bar{z},q$

$$\begin{aligned} \partial_\mu^{P_\zeta}\Lambda(t) &= \partial_\mu^{P_\zeta}\Lambda\left(t,\widehat{x}_t,P_{x_t},\widehat{y}_t,P_{y_t},\widehat{z}_t,P_{z_t},\widehat{\bar{z}}_t,P_{\widehat{\bar{z}}_t},\widehat{q}_t,P_{q_t},\widehat{u}_t;\zeta\right), \\ \partial_\mu^{P_\zeta}\Lambda\left(t,\widehat{\zeta}_t\right) &= \partial_\mu^{P_\zeta}\Lambda\left(t,x_t,P_{x_t},y_t,P_{y_t},z_t,P_{z_t},\bar{z}_t,P_{\bar{z}_t},q_t,P_{q_t},u_t;\widehat{\zeta}_t\right). \end{aligned}$$

Now, we introduce the following variational equations which is a linear FBSDEs

$$\begin{cases} dx_t^1 = \Big[\left(b_x(t) - \widetilde{\sigma}_x(t)\xi(t) - \widetilde{\sigma}(t)\xi_x(t)\right)x_t^1 + \left(b_v(t) - \widetilde{\sigma}_v(t)\xi(t)\right)v_t \\ \qquad + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}b\left(t,\widehat{x}_t\right)\widehat{x}_t^1\right] - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\widetilde{\sigma}\left(t,\widehat{x}_t\right)\widehat{x}_t^1\right]\xi(t) - \widetilde{\sigma}(t)\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t,\widehat{x}_t\right)\widehat{x}_t^1\right]\Big]dt \\ \qquad + \left[\sigma_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma\left(t,\widehat{x}_t\right)\widehat{x}_t^1\right] + \sigma_v(t)v_t\right]dW_t \\ \qquad + \left[\widetilde{\sigma}_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\widetilde{\sigma}\left(t,\widehat{x}_t\right)\widehat{x}_t^1\right] + \widetilde{\sigma}_v(t)v_t\right]dY_t \\ \qquad + \sum_{i=1}^{\infty}\left[g_x^i(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}g^i\left(t,\widehat{x}_t\right)\widehat{x}_t^1\right] + g_v^i(t)v_t\right]dH_t^i, \\ -dy_t^1 = \Big[f_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}f\left(t,\widehat{x}_t\right)\widehat{x}_t^1\right] + f_y(t)y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y}f\left(t,\widehat{y}_t\right)\widehat{y}_t^1\right] \\ \qquad + f_z(t)z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z}f\left(t,\widehat{z}_t\right)\widehat{z}_t^1\right] + f_{\bar{z}}(t)\bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}}f\left(t,\widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right] \\ \qquad + f_q(t)q_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_r}f\left(t,\widehat{q}_t\right)\widehat{q}_t^1\right] + f_v(t)v_t\Big]dt - z_t^1dW_t - \bar{z}_t^1dY_t + \sum_{i=1}^{\infty}q_{t_-}^1dH_t^i, \\ x_0^1 = 0, \quad y_T^1 = \varphi_x\left(x_T,P_{x_T}\right)x_T^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(x_T,P_{x_T},\widehat{x}_t\right)\widehat{x}_T^1\right], \end{cases} \tag{2.8}$$

and a linear SDE

$$\begin{cases} d\rho_t^1 = \left[\rho_t^1 \xi(t) + \rho_t \left(\xi_x(t) x_t^1\right) + \rho_t \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \xi(t, \widehat{x}_t) \widehat{x}_t^1\right]\right] dY_t, \\ \rho_0^1 = 0. \end{cases} \tag{2.9}$$

Set $\vartheta = \rho^{-1}\rho^1$, using Itô's formula, we have

$$\begin{cases} d\vartheta_t = \left[\xi_x(t) x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \xi(t, \widehat{x}_t) \widehat{x}_t^1\right]\right] d\widetilde{W}_t, \\ \vartheta_0 = 0. \end{cases} \tag{2.10}$$

Next, we introduce the following adjoint equations of McKean–Vlasov type

$$\begin{cases} -d\Psi_t = \Big[b_x(t)\Psi_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b(t) \widehat{\Psi}_t\right] - \widetilde{\sigma}(t)\xi_x(t)\Psi_t - \widetilde{\sigma}(t)\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t)\widehat{\Psi}_t\right] \\ \qquad -\widetilde{\sigma}_x(t)\xi(t)\Psi_t - \xi(t)\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\widetilde{\sigma}(t)\widehat{\Psi}_t\right] + \sigma_x(t)k_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma(t)\widehat{k}_t\right] \\ \qquad +\widetilde{\sigma}_x(t)\bar{k}_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\widetilde{\sigma}(t)\widehat{\bar{k}}_t\right] + \sum_{i=1}^{\infty} g_x^i(t) n_t^i + \widehat{\mathbb{E}}\left[\sum_{i=1}^{\infty}\partial_\mu^{P_x} g^i(t)\widehat{n}_t^i\right] \\ \qquad +\xi_x(t)Q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t)\widehat{Q}_t\right] - f_x(t)\Phi_t - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f(t)\widehat{\Phi}_t\right] + l_x(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l(t)\right]\Big] dt \\ \qquad -k_t dW_t - \bar{k}_t d\widetilde{W}_t - \sum_{i=1}^{\infty} n_t^i dH_t^i, \\ d\Phi_t = \left[f_y(t)\Phi_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} f(t)\widehat{\Phi}_t\right] - l_y(t) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l(t)\right]\right] dt \\ \qquad + \left[f_z(t)\Phi_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} f(t)\widehat{\Phi}_t\right] - l_z(t) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l(t)\right]\right] dW_t \\ \qquad + \left[f_{\bar{z}}(t)\Phi_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} f(t)\widehat{\Phi}_t\right] - \xi(t)\Phi_t - l_{\bar{z}}(t) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l(t)\right]\right] d\widetilde{W}_t, \\ \qquad + \sum_{i=1}^{\infty}\left[f_q^i(t)\Phi_t^i + \widehat{\mathbb{E}}\left[\partial_\mu^{P_q} f^i(t)\widehat{\Phi}_t^i\right] - l_q^i(t) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_q} l^i(t)\right]\right] dH_t^i, \\ \Psi_T = M_x(x_T, P_{x_T}) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M(\widehat{x}_T, P_{x_T}, x_T)\right] \\ \qquad -\varphi_x(x_T, P_{x_T})\Phi_T - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi(\widehat{x}_T, P_{x_T}, x_T)\widehat{\Phi}_T\right], \\ \Phi_0 = -h_y(y_0, P_{y_0}) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right]. \end{cases} \tag{2.11}$$

It is clear that, under (**H1**) and (**H2**), there exists a unique $\left(\Psi, k, \bar{k}, n, \Phi\right) \in \mathbb{S}_{\mathcal{F}}^2(0, T, \mathbb{R}^n) \times \mathbb{L}_{\mathcal{F}}^2\left(0, T, \mathbb{R}^{n\times d}\right) \times \mathbb{L}_{\mathcal{F}}^2\left(0, T, \mathbb{R}^{m\times d}\right) \times l_{\mathcal{F}}^2(0, T, \mathbb{R}^n) \times \mathbb{S}_{\mathcal{F}}^2(0, T, \mathbb{R}^n)$ satisfying the FBSDE (2.11) of McKean–Vlasov type.

Note that the mean-field nature of FBSDE (2.11) comes from the terms involving Fréchet derivatives $\partial_\mu^{P_x} b(t), \partial_\mu^{P_x} g(t), \partial_\mu^{P_x}\widetilde{\sigma}(t), \partial_\mu^{P_x}\xi(t)$ and $\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}, \partial_\mu^{P_q}\right)(f, l)$, which will reduce to a standard BSDE if the coefficients do not explicitly depend on law of the solution.

Now, we introduce the following BSDE involved in the stochastic maximum principle

$$\begin{cases} -d\Gamma_t = l\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, q_t, P_{q_t}, u_t\right) dt \\ \qquad -\bar{Q}_t dW_t - Q_t d\widetilde{W}_t, \\ \Gamma_T = M(x_T, P_{x_T}). \end{cases} \tag{2.12}$$

Under conditions (**H1**) and (**H2**), it is easy to prove that equation (2.12) admits a unique strong solution. We define the Hamiltonian function

$$H : [0,T] \times \mathbb{R}^n \times Q_2\left(\mathbb{R}^n\right) \times \mathbb{R}^m \times Q_2\left(\mathbb{R}^m\right) \times \mathbb{R}^{n\times d} \times Q_2\left(\mathbb{R}^{n\times d}\right) \times \mathbb{R}^{n\times d} \times Q_2\left(\mathbb{R}^{n\times d}\right)$$
$$\times l^2\left(\mathbb{R}^n\right) \times l^2\left(\mathbb{R}^n\right) \times U \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times d} \times \mathbb{R}^{n\times d} \times l^2\left(\mathbb{R}^n\right) \times \mathbb{R}^{n\times d} \to \mathbb{R},$$

associated with the McKean–Vlasov stochastic control problem (2.1)-(2.7) by

$$\begin{aligned} & H(t, x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, q, P_q, v, \Psi, \Phi, k, \bar{k}, n, Q) \\ & = \Psi\left(b\left(t, x, P_x, v\right) - \widetilde{\sigma}\left(t, x, P_x, v\right)\xi\left(t, x, P_x\right)\right) - \Phi f\left(t, x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, q, P_q, v\right) \\ & \quad + k\sigma\left(t, x, P_x, v\right) + \bar{k}\widetilde{\sigma}\left(t, x, P_x, v\right) + \sum_{i=1}^{\infty} n_t^i g^i\left(t, x, P_x, v\right) + Q\xi\left(t, x, P_x\right) \\ & \quad + l\left(t, x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, q, P_q, v\right). \end{aligned} \tag{2.13}$$

2.3 Necessary conditions of optimality

In this section, we prove the necessary conditions of optimality for our system of McKean–Vlasov type, satisfied by a partially observed optimal control, assuming that the solution exists. The proof is based on convex perturbation and on some estimates of the state processes of system and observed process. Suppose that u is an optimal control with the optimal trajectory $(x, y, z, \bar{z}, q)$ of FBSDE (2.1). For any $0 \leq \theta \leq 1$ and $v + u \in \mathcal{U}_{ad}$, we define a perturbed control $u_t^\theta = u_t + \theta v_t$.

Lemma 2.1

*Under conditions (**H1**) and (**H2**), the following estimations holds*

$$\lim_{\theta\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{x}_t^\theta\right|^2\right] = 0, \tag{2.14}$$

$$\lim_{\theta\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{y}_t^\theta\right|^2 + \int_0^T\left(\left|\widetilde{z}_t^\theta\right|^2 + \left|\widetilde{\bar{z}}_t^\theta\right|^2 + \left\|\widetilde{q}_t^\theta\right\|_{l^2(\mathbb{R}^n)}^2\right) ds\right] = 0, \tag{2.15}$$

$$\mathbb{E}\int_0^T\left|\widetilde{\rho}_t^\theta\right|^2 dt = 0. \tag{2.16}$$

Proof. For notational ease, we introduce the following notations.

For $t \in [0,T]$, $\theta > 0$, we set

$$\widetilde{x}_t^\theta = \theta^{-1}\left(x_t^\theta - x_t\right) - x_t^1, \quad \widetilde{\bar{z}}_t^\theta = \theta^{-1}\left(\bar{z}_t^\theta - \bar{z}_t\right) - \bar{z}_t^1,$$
$$\widetilde{y}_t^\theta = \theta^{-1}\left(y_t^\theta - y_t\right) - y_t^1, \quad \widetilde{q}_t^\theta = \theta^{-1}\left(q_t^\theta - q_t\right) - q_t^1,$$
$$\widetilde{z}_t^\theta = \theta^{-1}\left(z_t^\theta - z_t\right) - z_t^1, \quad \widetilde{\rho}_t^\theta = \theta^{-1}\left(\rho_t^\theta - \rho_t\right) - \rho_t^1,$$

and we denote by

$$\widetilde{x}_t^{\lambda,\theta} = x_t + \lambda\theta\left(\widetilde{x}_t^\theta + x_t^1\right), \quad \widetilde{z}_t^{\lambda,\theta} = z_t + \lambda\theta\left(\widetilde{z}_t^\theta + z_t^1\right),$$
$$\widetilde{y}_t^{\lambda,\theta} = y_t + \lambda\theta\left(\widetilde{y}_t^\theta + y_t^1\right), \quad \widetilde{\bar{z}}_t^{\lambda,\theta} = \bar{z}_t + \lambda\theta\left(\widetilde{\bar{z}}_t^\theta + \bar{z}_t^1\right),$$
$$\gamma_t^{\lambda,\theta} = \left(\widetilde{x}_t^{\lambda,\theta}, P_{\widetilde{x}_t^{\lambda,\theta}}, u_t^\theta\right), \quad \widetilde{q}_t^{\lambda,\theta} = q_t + \lambda\theta\left(\widetilde{q}_t^\theta + q_t^1\right).$$

First, we have

$$\begin{cases} d\widetilde{x}_t^\theta = \left([b_t^x - \widetilde{\sigma}_t^x\xi_t - \widetilde{\sigma}_t\xi_t^x]\,\widetilde{x}_t^\theta + [b_t^{\mu,x} - \widetilde{\sigma}_t\xi_t^{\mu,x} - \xi_t\widetilde{\sigma}_t^{\mu,x}] + \beta_1^\theta\right)dt \\ \qquad + \left(\sigma_t^x\widetilde{x}_t^\theta + \sigma_t^{\mu,x} + \beta_2^\theta\right)dW_t + \left(\widetilde{\sigma}_t^x\widetilde{x}_t^\theta dt + \widetilde{\sigma}_t^{\mu,x} + \beta_3^\theta\right)dY_t \\ \qquad + \sum_{i=1}^{\infty}\left(g_t^{i,x}\widetilde{x}_t^\theta + g_t^{i,\mu,x} + \beta_4^{i,\theta}\right)dH_t^i, \\ \widetilde{x}_0^\theta = 0, \end{cases} \tag{2.17}$$

where

$$b_t^x = \int_0^1 b_x\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, \quad b_t^{\mu,x} = \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}_t^\theta}\right]d\lambda,$$
$$\widetilde{\sigma}_t^x = \int_0^1 \widetilde{\sigma}_x\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, \quad \widetilde{\sigma}_t^{\mu,x} = \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\widetilde{\sigma}\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}_t^\theta}\right]d\lambda,$$
$$\xi_t^x = \int_0^1 \xi_x\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, \quad \xi_t^{\mu,x} = \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}_t^\theta}\right]d\lambda,$$
$$\sigma_t^x = \int_0^1 \sigma_x\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, \quad \sigma_t^{\mu,x} = \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}_t^\theta}\right]d\lambda,$$
$$g_t^{i,x} = \int_0^1 g_x^i\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, \quad g^{i,\mu,x} = \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} g\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}_t^\theta}\right]d\lambda,$$

and

$$\begin{aligned} \beta_1^\theta &= \int_0^1 \left[b_x\left(t,\gamma_t^{\lambda,\theta}\right) - b_x(t)\right]d\lambda x_t^1 \\ &\quad - \xi_t\int_0^1\left[\widetilde{\sigma}_x\left(t,\gamma_t^{\lambda,\theta}\right) - \widetilde{\sigma}_x(t)\right]d\lambda x_t^1 - \widetilde{\sigma}_t\int_0^1\left[\xi_x\left(t,\gamma_t^{\lambda,\theta}\right) - \xi_x(t)\right]d\lambda x_t^1 \\ &\quad + \int_0^1\left[b_v\left(t,\gamma_t^{\lambda,\theta}\right) - b_v(t)\right]d\lambda v_t - \xi_t\int_0^1\left[\widetilde{\sigma}_v\left(t,\gamma_t^{\lambda,\theta}\right) - \widetilde{\sigma}_v(t)\right]d\lambda v_t \\ &\quad + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} b\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} b\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda \\ &\quad - \xi_t\int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x}\widetilde{\sigma}\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x}\widetilde{\sigma}\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda \\ &\quad - \widetilde{\sigma}_t\int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x}\xi\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x}\xi\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda, \end{aligned}$$

$$
\begin{aligned}
\beta_2^\theta =& \int_0^1 \left[\sigma_x\left(t,\gamma_t^{\lambda,\theta}\right)-\sigma_x(t)\right]d\lambda x_t^1+\int_0^1\left[\sigma_v\left(t,\gamma_t^{\lambda,\theta}\right)-\sigma_v(t)\right]d\lambda v_t \\
&+\int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x}\sigma\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)-\partial_\mu^{P_x}\sigma\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda,
\end{aligned}
$$

$$
\begin{aligned}
\beta_3^\theta =& \int_0^1 \left[\widetilde{\sigma}_x\left(t,\gamma_t^{\lambda,\theta}\right)-\widetilde{\sigma}_x(t)\right]d\lambda x_t^1+\int_0^1\left[\widetilde{\sigma}_v\left(t,\gamma_t^{\lambda,\theta}\right)-\widetilde{\sigma}_v(t)\right]d\lambda v_t \\
&+\int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x}\widetilde{\sigma}\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)-\partial_\mu^{P_x}\widetilde{\sigma}\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda,
\end{aligned}
$$

$$
\begin{aligned}
\beta_4^{i,\theta} =& \int_0^1 \left[g_x^i\left(t,\gamma_t^{\lambda,\theta}\right)-g_x^i(t)\right]d\lambda x_t^1+\int_0^1\left[g_v^i\left(t,\gamma_t^{\lambda,\theta}\right)-g_v^i(t)\right]d\lambda v_t \\
&+\int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x}g^i\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)-\partial_\mu^{P_x}g^i\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda.
\end{aligned}
$$

Under conditions (**H1**) and (**H2**), it is not difficult to see that

$$
\lim_{\theta\to 0}\mathbb{E}\left[\left|\beta_1^\theta\right|^2+\left|\beta_2^\theta\right|^2+\left|\beta_3^\theta\right|^2+\left|\beta_4^{i,\theta}\right|^2\right]=0.
$$

Applying Itô's formula to $\left|\widetilde{x}_t^\theta\right|^2$, we have

$$
\begin{aligned}
\mathbb{E}\left|\widetilde{x}_t^\theta\right|^2 =& 2\mathbb{E}\int_0^T \widetilde{x}_t^\theta\left(\left[b_t^x-\widetilde{\sigma}_t^x\xi_t-\widetilde{\sigma}_t\xi_t^x\right]\widetilde{x}_t^\theta+\left[b_t^{\mu,x}-\widetilde{\sigma}_t\xi_t^{\mu,x}-\xi_t\widetilde{\sigma}_t^{\mu,x}\right]+\beta_1^\theta\right)dt \\
&+\mathbb{E}\int_0^T\left|\sigma_t^x\widetilde{x}_t^\theta+\sigma_t^{\mu,x}+\beta_2^\theta\right|^2dt+\mathbb{E}\int_0^T\left|\widetilde{\sigma}_t^x\widetilde{x}_t^\theta+\widetilde{\sigma}_t^{\mu,x}+\beta_3^\theta\right|^2dt \\
&+\sum_{i=1}^\infty\mathbb{E}\int_0^T\left|g_t^{i,x}\widetilde{x}_t^\theta+g_t^{i,\mu,x}+\beta_4^{i,\theta}\right|^2dt \\
\leq& C\mathbb{E}\int_0^T\left|\widetilde{x}_t^\theta\right|^2dt+\int_0^T\mathbb{E}\left[\left|\beta_1^\theta\right|^2+\left|\beta_2^\theta\right|^2+\left|\beta_3^\theta\right|^2+\left|\beta_4^{i,\theta}\right|^2\right]dt.
\end{aligned}
$$

Finally, estimate (2.14) now follows easily from the Gronwall inequality.

Let $\left(\widetilde{y}_t^\theta,\widetilde{z}_t^\theta,\widetilde{\bar{z}}_t^\theta,\widetilde{q}_t^\theta\right)$ be the solution of the following BSDE

$$
\begin{cases}
d\widetilde{y}_t^\theta=\left[f_t^x\widetilde{x}_t^\theta+f_t^{\mu,x}+f_t^y\widetilde{y}_t^\theta+f_t^{\mu,y}+f_t^z\widetilde{z}_t^\theta+f_t^{\mu,z}+f_t^{\bar{z}}\widetilde{\bar{z}}_t^\theta+f_t^{\mu,\bar{z}}+f_t^q\widetilde{q}_t^\theta+f_t^{\mu,q}+\Upsilon_t^\theta\right]dt \\
\qquad\qquad +\widetilde{z}_t^\theta dW_t+\widetilde{\bar{z}}_t^\theta dY_t+\sum_{i=1}^\infty\widetilde{q}_t^\theta dH_t^i, \\
\widetilde{y}_T^\theta=\theta^{-1}\left[\varphi\left(x_T^\theta,P_{x_T^\theta}\right)-\varphi\left(x_T,P_{x_T}\right)\right]-\varphi_x\left(x_T,P_{x_T}\right)x_T^1-\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(x_T,P_{x_T},\widehat{x}_T\right)\widehat{x}_T^1\right],
\end{cases}
$$

where $\tilde{x}_t^\theta$ satisfies equation (2.17), and

$$
\begin{aligned}
f_t^\alpha &= -\int_0^1 f_\alpha\left(t,\chi_t^{\lambda,\theta}\right)d\lambda,\ \text{ for } \alpha=x,y,z,\bar{z},q, \\
f_t^{\mu,\alpha} &= -\int_0^1\widehat{\mathbb{E}}\left[\partial_\mu^{P_\alpha}f\left(t,\chi_t^{\lambda,\theta},\widehat{\widetilde{\alpha}_t^{\lambda,\theta}}\right)\widehat{\widetilde{\alpha}}_t^\theta\right]d\lambda,\ \text{ for } \alpha=x,y,z,\bar{z},q,
\end{aligned}
$$

where

$$\chi_t^{\lambda,\theta} = \left(\widetilde{x}_t^{\lambda,\theta}, P_{\widetilde{x}_t^{\lambda,\theta}}, \widetilde{y}_t^{\lambda,\theta}, P_{\widetilde{y}_t^{\lambda,\theta}}, \widetilde{z}_t^{\lambda,\theta}, P_{\widetilde{z}_t^{\lambda,\theta}}, \widetilde{\bar{z}}_t^{\lambda,\theta}, P_{\widetilde{\bar{z}}_t^{\lambda,\theta}}, \widetilde{q}_t^{\lambda,\theta}, P_{\widetilde{q}_t^{\lambda,\theta}}, u_t^{\lambda,\theta}\right),$$

and Υ_t^θ is given by

$$\begin{aligned}
\Upsilon_t^\theta &= \int_0^1 \left[f_x\left(t, \chi_t^{\lambda,\theta}\right) - f_x(t)\right] d\lambda x_t^1 + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} f\left(t, \chi_t^{\lambda,\theta}, \widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} f\left(t, \chi_t, \widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right] d\lambda \\
&+ \int_0^1 \left[f_y\left(t, \chi_t^{\lambda,\theta}\right) - f_y(t)\right] d\lambda y_t^1 + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_y} f\left(t, \chi_t^{\lambda,\theta}, \widehat{\widetilde{y}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_y} f\left(t, \chi_t, \widehat{\widetilde{y}}_t\right)\right)\widehat{y}_t^1\right] d\lambda \\
&+ \int_0^1 \left[f_z\left(t, \chi_t^{\lambda,\theta}\right) - f_z(t)\right] d\lambda z_t^1 + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_z} f\left(t, \chi_t^{\lambda,\theta}, \widehat{\widetilde{z}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_z} f\left(t, \chi_t, \widehat{\widetilde{z}}_t\right)\right)\widehat{z}_t^1\right] d\lambda \\
&+ \int_0^1 \left[f_{\bar{z}}\left(t, \chi_t^{\lambda,\theta}\right) - f_{\bar{z}}(t)\right] d\lambda \bar{z}_t^1 + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_{\bar{z}}} f\left(t, \chi_t^{\lambda,\theta}, \widehat{\widetilde{\bar{z}}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} f\left(t, \chi_t, \widehat{\widetilde{\bar{z}}}_t\right)\right)\widehat{\bar{z}}_t^1\right] d\lambda \\
&+ \int_0^1 \left[f_q\left(t, \chi_t^{\lambda,\theta}\right) - f_q(t)\right] d\lambda q_t^1 + \int_0^1 \left[f_v\left(t, \chi_t^{\lambda,\theta}\right) - f_v(t)\right] d\lambda v_t \\
&+ \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_q} f\left(t, \chi_t^{\lambda,\theta}, \widehat{\widetilde{q}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_q} f\left(t, \chi_t, \widehat{\widetilde{q}}_t\right)\right)\widehat{q}_t^1\right] d\lambda.
\end{aligned}$$

Due the fact that $f_t^x, f_t^{\mu,x}, f_t^y, f_t^{\mu,y}, f_t^z, f_t^{\mu,z}, f_t^{\bar{z}}, f_t^{\mu,\bar{z}}, f_t^q$ and $f_t^{\mu,q}$ are continuous, we have

$$\lim_{\theta\to 0} \mathbb{E}\left|\Upsilon_t^\theta\right|^2 = 0. \tag{2.18}$$

Appying Itô's formula to $\left|\widetilde{y}_t^\theta\right|^2$, we have

$$\begin{aligned}
&\mathbb{E}\left|\widetilde{y}_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|\widetilde{z}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left|\widetilde{\bar{z}}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left\|\widetilde{q}_s^\theta\right\|_{l^2(\mathbb{R}^n)}^2 ds \\
&= \mathbb{E}\left|\widetilde{y}_T^\theta\right|^2 + 2\mathbb{E}\int_t^T \widetilde{y}_s^\theta \left(f_s^x \widetilde{x}_s^\theta + f_s^{\mu,x} + f_s^y \widetilde{y}_s^\theta + f_s^{\mu,y} + f_s^z \widetilde{z}_s^\theta + f_s^{\mu,z} + f_s^{\bar{z}} \widetilde{\bar{z}}_s^\theta + f_s^{\mu,\bar{z}} + f_s^q \widetilde{q}_s^\theta + f_s^{\mu,q} + \Upsilon_s^\theta\right) ds.
\end{aligned}$$

By Young's inequality, for each $\varepsilon > 0$, we get

$$\begin{aligned}
&\mathbb{E}\left|\widetilde{y}_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|\widetilde{z}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left|\widetilde{\bar{z}}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left\|\widetilde{q}_s^\theta\right\|_{l^2(\mathbb{R}^n)}^2 ds \\
&\leq \mathbb{E}\left|\widetilde{y}_T^\theta\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|\widetilde{y}_s^\theta\right|^2 ds \\
&+ \varepsilon\mathbb{E}\int_t^T \left|\left(f_s^x \widetilde{x}_s^\theta + f_s^{\mu,x} + f_s^y \widetilde{y}_s^\theta + f_s^{\mu,y} + f_s^z \widetilde{z}_s^\theta + f_s^{\mu,z} + f_s^{\bar{z}} \widetilde{\bar{z}}_s^\theta + f_s^{\mu,\bar{z}} + f_s^q \widetilde{q}_s^\theta + f_s^{\mu,q} + \Upsilon_s^\theta\right)\right|^2 ds
\end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E}\left|\widetilde{y}_T^\theta\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|\widetilde{y}_s^\theta\right|^2 ds + C_\varepsilon\mathbb{E}\int_t^T \left|f_s^x \widetilde{x}_s^\theta\right|^2 ds + C_\varepsilon\mathbb{E}\int_t^T \left|f_s^{\mu,x}\right|^2 ds + C_\varepsilon\mathbb{E}\int_t^T \left|f_s^y \widetilde{y}_s^\theta\right|^2 ds \\
&+ C_\varepsilon\mathbb{E}\int_t^T \left|f_s^{\mu,y}\right|^2 ds + C_\varepsilon\mathbb{E}\int_t^T \left|f_s^z \widetilde{z}_s^\theta\right|^2 ds + C_\varepsilon\mathbb{E}\int_t^T \left|f_s^{\mu,z}\right|^2 ds + C_\varepsilon\mathbb{E}\int_t^T \left|f_s^{\bar{z}} \widetilde{\bar{z}}_s^\theta\right|^2 ds \\
&+ C_\varepsilon\mathbb{E}\int_t^T \left|f_s^{\mu,\bar{z}}\right|^2 ds + C_\varepsilon\mathbb{E}\int_t^T \left|f_s^q \widetilde{q}_s^\theta\right|^2 ds + C_\varepsilon\mathbb{E}\int_t^T \left|f_s^{\mu,q}\right|^2 ds.
\end{aligned}$$

By the boundedness of $f_t^x, f_t^{\mu,x}, f_t^y, f_t^{\mu,y}, f_t^z, f_t^{\mu,z}, f_t^{\bar{z}}, f_t^{\mu,\bar{z}}, f_t^q$ and $f_t^{\mu,q}$, we obtain

$$\begin{aligned}
&\mathbb{E}\left|\widetilde{y}_t^\theta\right|^2+\mathbb{E}\int_t^T\left|\widetilde{z}_s^\theta\right|^2 ds+\mathbb{E}\int_t^T\left|\widetilde{\bar{z}}_s^\theta\right|^2 ds+\mathbb{E}\int_t^T\left\|\widetilde{q}_s^\theta\right\|_{l^2(\mathbb{R}^n)}^2 ds\\
&\leq\left(\frac{1}{\varepsilon}+C_\varepsilon\right)\mathbb{E}\int_t^T\left|\widetilde{y}_s^\theta\right|^2 ds+C_\varepsilon\mathbb{E}\int_t^T\left|\widetilde{z}_s^\theta\right|^2 ds+C_\varepsilon\mathbb{E}\int_t^T\left|\widetilde{\bar{z}}_s^\theta\right|^2 ds+C_\varepsilon\mathbb{E}\int_t^T\left\|\widetilde{q}_s^\theta\right\|_{l^2(\mathbb{R}^n)}^2 ds\\
&+\mathbb{E}\left|\widetilde{y}_T^\theta\right|^2+C_\varepsilon\mathbb{E}\int_t^T\left|f_s^x\widetilde{x}_s^\theta\right|^2 ds+C_\varepsilon\mathbb{E}\int_t^T\left|\Upsilon_s^\theta\right|^2 ds.
\end{aligned}$$

Hence, in view of equations (2.14), (2.18), the fact that $f_t^x, f_t^{\mu,x}$ are continuous and bounded, by Gronwall's inequality, we obtain (2.15).

Now, we proceed to prove (2.16). It is plain to check that $\widetilde{\rho}_t^\theta$ satisfies the following equality

$$d\widetilde{\rho}_t^\theta=\left[\widetilde{\rho}_t^\theta\xi\left(t,x_t^\theta,P_{x_t^\theta}\right)+\bar{\Upsilon}_t^\theta\right]dY_t+\rho_t\left[\xi_t^x\widetilde{x}_t^\theta+\xi_t^{\mu,x}\right]dY_t,$$

where

$$\begin{aligned}
\xi_t^x&=\int_0^1\xi_x\left(t,\widetilde{x}_t^{\lambda,\theta},P_{\widetilde{x}_t^{\lambda,\theta}}\right)d\lambda,\\
\xi_t^{\mu,x}&=\int_0^1\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t,\widetilde{x}_t^{\lambda,\theta},P_{\widetilde{x}_t^{\lambda,\theta}},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}_t^\theta}\right]d\lambda,
\end{aligned}$$

and $\bar{\Upsilon}_t^\theta$ is given by

$$\begin{aligned}
\bar{\Upsilon}_t^\theta&=\rho_t\int_0^1\left[\xi_x\left(t,\widetilde{x}_t^{\lambda,\theta},P_{\widetilde{x}_t^{\lambda,\theta}}\right)-\xi_x\left(t\right)\right]d\lambda x_t^1\\
&+\rho_t\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x}\xi\left(t,\widetilde{x}_t^{\lambda,\theta},P_{\widetilde{x}_t^{\lambda,\theta}},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)-\partial_\mu^{P_x}\xi\left(t,\widetilde{x}_t,P_{\widetilde{x}_t},\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda\\
&+\rho_t^1\left[\xi\left(t,x_t^\theta,P_{x_t^\theta}\right)-\xi\left(t\right)\right].
\end{aligned}$$

Taking into account the fact that ξ_t^x and $\xi_t^{\mu,x}$ are continuous, we deduce

$$\lim_{\theta\to 0}\mathbb{E}\left|\bar{\Upsilon}_t^\theta\right|^2=0. \tag{2.19}$$

Then, applying Itô's formula to $\left|\widetilde{\rho}_t^\theta\right|^2$ and taking expectation, we have

$$\mathbb{E}\left|\widetilde{\rho}_t^\theta\right|^2\leq C\mathbb{E}\int_0^T\left|\widetilde{\rho}_t^\theta\right|^2 dt+C\mathbb{E}\int_0^T\left|\widetilde{x}_t^\theta\right|^2 dt+C\mathbb{E}\int_0^T|\xi_t^{\mu,x}|^2 dt+C\mathbb{E}\int_0^T\left|\bar{\Upsilon}_t^\theta\right|^2 dt.$$

Finally, by Gronwall's inequality, estimates (2.14) and recall to the Wasserstein metric, the above convergence result (2.16) holds. □

Since u is an optimal control, then, we have the following lemma.

Lemma 2.2

*Let conditions (**H1**) and (**H2**) hold. Then, we have*

$$
\begin{aligned}
0 \leq\ & \mathbb{E}\left[\rho_T M_x\left(x_T, P_{x_T}\right) x_T^1+\rho_T \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}, \widehat{x}_T\right) \widehat{x}_T^1\right]\right] \\
& +\mathbb{E}\left[\rho_T^1 M\left(x_T, P_{x_T}\right)\right]+\mathbb{E}\left[h_y\left(y_0, P_{y_0}\right) y_0^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(y_0, P_{y_0}, \widehat{y}_0) \widehat{y}_0^1\right]\right] \\
& +\mathbb{E} \int_0^T \rho_t^1 l(t) dt+\mathbb{E} \int_0^T \rho_t l_v(t) v_t dt+\mathbb{E} \int_0^T \rho_t\left[l_x(t) x_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t, \widehat{x}_t\right) \widehat{x}_t^1\right]\right] dt \qquad (2.20) \\
& +\mathbb{E} \int_0^T \rho_t\left[l_y(t) y_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t, \widehat{y}_t\right) \widehat{y}_t^1\right]\right] dt+\mathbb{E} \int_0^T \rho_t\left[l_z(t) z_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{z}_t\right) \widehat{z}_t^1\right]\right] dt \\
& +\mathbb{E} \int_0^T \rho_t\left[l_{\bar{z}}(t) \bar{z}_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t, \widehat{\bar{z}}_t\right) \widehat{\bar{z}}_t^1\right]\right] dt+\mathbb{E} \int_0^T \rho_t\left[l_q(t) q_t^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_q} l\left(t, \widehat{q}_t\right) \widehat{q}_t^1\right]\right] dt.
\end{aligned}
$$

Proof. Using Lemmas 2.1 and Taylor expansion, we get

$$
\begin{aligned}
0 & \leq \theta^{-1}\left[J\left(u_t^\theta\right)-J\left(u_t\right)\right] \\
& =\theta^{-1} \mathbb{E}\left[\rho_T^\theta M\left(x_T^\theta, P_{x_T^\theta}\right)-\rho_T M\left(x_T, P_{x_T}\right)\right] \\
& +\theta^{-1} \mathbb{E}\left[h\left(y_0^\theta\right)-h\left(y_0\right)\right] \\
& +\theta^{-1} \mathbb{E} \int_0^T\left[\rho_t^\theta l^\theta(t)-\rho_t l(t)\right] dt \\
& =J_1+J_2+J_3,
\end{aligned}
$$

where $l^\theta(t)=l\left(t, x_t^\theta, P_{x_t^\theta}, y_t^\theta, P_{y_t^\theta}, z_t^\theta, P_{z_t^\theta}, \bar{z}_t^\theta, P_{\bar{z}_t^\theta}, q_t^\theta, P_{q_t^\theta}, u_t^\theta\right)$.

Then, from the results of (2.14), (2.15) and (2.16), we derive

$$
\begin{aligned}
J_1 & =\theta^{-1} \mathbb{E}\left[\rho_T^\theta M\left(x_T^\theta, P_{x_T^\theta}\right)-\rho_T M\left(x_T, P_{x_T}\right)\right] \\
& =\theta^{-1} \mathbb{E}\left[\left(\rho_T^\theta-\rho_T\right) M\left(x_T^\theta, P_{x_T^\theta}\right)\right] \\
& +\theta^{-1} \mathbb{E}\left[\rho_T \int_0^1 M_x\left(x_T+\lambda\left(x_T^\theta-x_T\right), P_{x_T+\lambda\left(\widehat{x}_T^\theta-\widehat{x}_T\right)}\right)\left(x_T^\theta-x_T\right) d\lambda\right] \\
& +\theta^{-1} \mathbb{E}\left[\rho_T \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T+\lambda\left(\widehat{x}_T^\theta-\widehat{x}_T\right), P_{x_T+\lambda\left(\widehat{x}_T^\theta-\widehat{x}_T\right)}, \widehat{x}_T\right)\left(\widehat{x}_T^\theta-\widehat{x}_T\right)\right] d\lambda\right] \\
& \longrightarrow \mathbb{E}^u\left[\vartheta_T M\left(x_T, P_{x_T}\right)\right]+\mathbb{E}^u\left[\left(M_x\left(x_T, P_{x_T}\right)\right) x_T^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}, \widehat{x}_T\right) \widehat{x}_T^1\right]\right].
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
J_2 &= \theta^{-1}\mathbb{E}\left[h\left(y_0^\theta, P_{y_0^\theta}\right) - h\left(y_0, P_{y_0}\right)\right] \\
&= \theta^{-1}\mathbb{E}\left[\int_0^1 h_y\left(y_0 + \lambda\left(y_0^\theta - y_0\right), P_{y_0+\lambda\left(\widehat{y}_0^\theta - \widehat{y}_0\right)}\right)\left(y_0^\theta - y_0\right) d\lambda\right] \\
&+ \theta^{-1}\mathbb{E}\left[\int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h\left(y_0 + \lambda\left(\widehat{y}_0^\theta - \widehat{y}_0\right), P_{y_0+\lambda\left(\widehat{y}_0^\theta - \widehat{y}_0\right)}, \widehat{y}_0\right)\left(\widehat{y}_0^\theta - \widehat{y}_0\right)\right] d\lambda\right] \\
&\longrightarrow \mathbb{E}^u\left[\left(h_y\left(y_0, P_{y_0}\right)\right) y_0^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(y_0, P_{y_0}, \widehat{y}_0)\widehat{y}_0^1\right]\right],
\end{aligned}
$$

and

$$
\begin{aligned}
J_3 &= \theta^{-1}\mathbb{E}\left[\int_0^T \left(\rho_t^\theta l^\theta(t) - \rho_t l(t)\right) dt\right] \\
&\longrightarrow \mathbb{E}^u \int_0^T \Big[\vartheta_t l(t) + l_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right] + l_y(t)y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t, \widehat{y}_t\right)\widehat{y}_t^1\right] \\
&+ l_z(t)z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{z}_t\right)\widehat{z}_t^1\right] + l_{\bar{z}}(t)\bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right] \\
&+ l_q(t)q_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_q} l\left(t, \widehat{q}_t\right)\widehat{q}_t^1\right] + l_v(t)v_t\Big] dt.
\end{aligned}
$$

Then, the variational inequality (2.20) can be rewritten as

$$
\begin{aligned}
0 &\le \mathbb{E}^u\left[M_x\left(x_T, P_{x_T}\right)x^1(T) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}, \widehat{x}_T\right)\widehat{x}_T^1\right]\right] \\
&+ \mathbb{E}^u\left[\vartheta_T M\left(x_T, P_{x_T}\right)\right] + \mathbb{E}^u\left[h_y\left(y_0, P_{y_0}\right)y^1(0) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(y_0, P_{y_0}, \widehat{y}_0)\widehat{y}_0^1\right]\right] \\
&+ \mathbb{E}^u\int_0^T \vartheta_t l(t)dt + \mathbb{E}^u\int_0^T l_v(t)v_t dt + \mathbb{E}^u\int_0^T \left[l_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right]\right] dt \qquad (2.21) \\
&+ \mathbb{E}^u\int_0^T \left[l_y(t)y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t, \widehat{y}_t\right)\widehat{y}_t^1\right]\right] dt + \mathbb{E}^u\int_0^T \left[l_z(t)z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{z}_t\right)\widehat{z}_t^1\right]\right] dt \\
&+ \mathbb{E}^u\int_0^T \widehat{\mathbb{E}}\left[l_{\bar{z}}(t)\bar{z}_t^1 + \partial_\mu^{P_{\bar{z}}} l\left(t, \widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right] dt + \mathbb{E}^u\int_0^T \left[l_q(t)q_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_q} l\left(t, \widehat{q}_t\right)\widehat{q}_t^1\right]\right] dt.
\end{aligned}
$$

□

Theorem 2.1

*(Partial necessary conditions of optimality) Let conditions (**H1**) and (**H2**) hold and let $(x, y, z, \bar{z}, q, u)$ be an optimal solution of our partially observed optimal control problem of McKean–Vlasov type. Then, there are $(\Psi, \Phi, k, \bar{k}, n)$ and $(\Gamma, \bar{Q}, Q)$ of $\mathbb{F}$-adapted processes that satisfy equations (2.11), (2.12) respectively, and that for all $v \in \mathcal{U}_{ad}$, we have*

$$
\mathbb{E}^u\left[H_v(t)\left(v_t - u_t\right)/\mathcal{F}_t^Y\right] \ge 0, a.\mathbb{E}, a.s, \qquad (2.22)
$$

where the Hamiltonian function

$$H(t) = H\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, q_t, P_{q_t}, u_t, \Psi_t, \Phi_t, k_t, \bar{k}_t, n_t, Q_t\right),$$

is defined by (2.13).

Proof. Applying Itô's formula to $\Psi_t x_t^1$ and $\Phi_t y_t^1$ such that,

$$\begin{aligned}
\Phi_0 &= -h_y(y_0, P_{y_0}) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right], \\
\Psi_T &= M_x\left(x_T, P_{x_T}\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(\widehat{x}_T, P_{x_T}, x_T\right)\right] \\
&\quad -\varphi_x\left(x_T, P_{x_T}\right)\Phi_T - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{\Phi}_T\right],
\end{aligned}$$

and using Fubini's theorem, we get

$$\begin{aligned}
\mathbb{E}^u\left[\Psi_T x_T^1\right] &= \mathbb{E}^u \int_0^T \left[\Psi_t\left(b_v(t) - \widetilde{\sigma}_v(t)\xi(t)\right)v_t + \bar{k}_t\widetilde{\sigma}_v(t)v_t + k_t g_v(t)v_t + \sum_{i=1}^{\infty} n_t^i g_v^i(t)v_t\right]dt \\
&\quad + \mathbb{E}^u \int_0^T x_t^1\left[f_x(t)\Phi_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f(t)\widehat{\Phi}_t\right] - l_x(t) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l(t)\right]\right]dt \\
&\quad - \mathbb{E}^u \int_0^T x_t^1\left[\xi_x(t)Q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t)\widehat{Q}_t\right]\right]dt,
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
&\mathbb{E}^u\left[\Phi_T y_T^1\right] + \mathbb{E}^u\left[h_y(y_0, P_{y_0}) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right]\right] \\
&= -\mathbb{E}^u \int_0^T \Phi_t\left[f_v(t)v_t + f_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f(t, \widehat{x}_t)\widehat{x}_t^1\right]\right]dt \\
&- \mathbb{E}^u \int_0^T y_t^1\left[l_y(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l(t)\right]\right]dt - \mathbb{E}^u \int_0^T z_t^1\left[l_z(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l(t)\right]\right]dt \\
&- \mathbb{E}^u \int_0^T \bar{z}_t^1\left[l_{\bar{z}}(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l(t)\right]\right]dt - \mathbb{E}^u \int_0^T q_t^1\left[l_q(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_q} l(t)\right]\right]dt.
\end{aligned} \tag{2.24}$$

Now, applying Itô's formula to $\vartheta_t\Gamma_t$ and using also Fubini's theorem, we have

$$\begin{aligned}
\mathbb{E}^u\left[\vartheta_T M(x_T)\right] &= -\mathbb{E}^u \int_0^T \vartheta_t l(t)\,dt \\
&\quad + \mathbb{E}^u \int_0^T Q_t\left[\xi_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t, \widehat{x}_t)\widehat{x}_t^1\right]\right]dt.
\end{aligned} \tag{2.25}$$

From equations (2.23), (2.24), and (2.25), we obtain

$$
\begin{aligned}
&\mathbb{E}^u\left[M_x\left(x_T, P_{x_T}\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}\right)\right]\right] \\
&+ \mathbb{E}^u\left[h_y(y_0, P_{y_0}) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right] + \vartheta_T M(x_T)\right] \\
&= \mathbb{E}^u \int_0^T \left[\Psi_t \left[b_v\left(t\right) - \widetilde{\sigma}_v \xi(t)\right] v_t + \bar{k}_t \widetilde{\sigma}_v\left(t\right) v_t + k_t g_v\left(t\right) v_t + \sum_{i=1}^{\infty} n_t^i g_v^i\left(t\right) v_t - \Phi_t f_v\left(t\right) v_t\right] dt \\
&- \mathbb{E}^u \int_0^T \vartheta_t l\left(t\right) dt - \mathbb{E}^u \int_0^T x_t^1 \left[l_x(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t\right)\right]\right] dt \qquad (2.26) \\
&- \mathbb{E}^u \int_0^T y_t^1 \left[l_y\left(t\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t\right)\right]\right] dt - \mathbb{E}^u \int_0^T z_t^1 \left[l_z\left(t\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t\right)\right]\right] dt \\
&- \mathbb{E}^u \int_0^T \bar{z}_t^1 \left[l_{\bar{z}}\left(t\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t\right)\right]\right] dt - \mathbb{E}^u \int_0^T q_t^1 \left[l_q\left(t\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_q} l\left(t\right)\right]\right] dt,
\end{aligned}
$$

thus

$$
\begin{aligned}
&\mathbb{E}^u\left[M_x\left(x_T, P_{x_T}\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}\right)\right]\right] \\
&+ \mathbb{E}^u\left[h_y(y_0, P_{y_0}) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right] + \vartheta_T M(x_T)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}^u \int_0^T H_v\left(t\right) v_t - \mathbb{E}^u \int_0^T l_v(t) v_t dt - \mathbb{E}^u \int_0^T \vartheta_t l\left(t\right) dt - \mathbb{E}^u \int_0^T x_t^1 \left[l_x(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t\right)\right]\right] dt \\
&- \mathbb{E}^u \int_0^T y_t^1 \left[l_y\left(t\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t\right)\right]\right] dt - \mathbb{E}^u \int_0^T z_t^1 \left[l_z\left(t\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t\right)\right]\right] dt \\
&- \mathbb{E}^u \int_0^T \bar{z}_t^1 \left[l_{\bar{z}}\left(t\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t\right)\right]\right] dt - \mathbb{E}^u \int_0^T q_t^1 \left[l_q\left(t\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_q} l\left(t\right)\right]\right] dt.
\end{aligned}
$$

This together with the variational inequality (2.21) imply (2.22), the proof is then completed. □

2.4 Partially observed linear-quadratic control problem of McKean-Vlasov FBSDEs

In this section, we are going to consider a partially observed linear-quadratic control problem of McKean–Vlasov type. We find an explicit expression of the corresponding optimal control by applying our partial necessary conditions established in the previous section.

We consider the following forward-backward system:

$$\begin{cases} dx_t = \left(b_t^1 x_t + b_t^2 \mathbb{E}\left[x_t\right] + b_t^3 v_t - \sigma_t^2 \gamma_t\right) dt + \sigma_t^1 dW_t + \sigma_t^2 dY_t + \sum_{i=1}^{\infty} g_t^i dH_t^i, \\ -dy_t = \left(f_t^1 x_t + f_t^2 \mathbb{E}\left[x_t\right] + f_t^3 y_t + f_t^4 \mathbb{E}\left[y_t\right] + f_t^5 z_t + f_t^6 \mathbb{E}\left[z_t\right] + f_t^7 \overline{z}_t + f_t^8 \mathbb{E}\left[\overline{z}_t\right] \right. \\ \qquad \left. + f_t^9 q_t + f_t^{10} \mathbb{E}\left[q_t\right] + f_t^{11} v_t\right) dt - z_t dW_t - \overline{z} dY_t - \sum_{i=1}^{\infty} q_t^i dH_t^i, \\ x\left(0\right) = x_0, \; y_T = \phi_1 x_T + \phi_2 \mathbb{E}\left[x_T\right], \end{cases} \tag{2.27}$$

and the SDE

$$\begin{cases} dY_t = \gamma_t dt + d\widetilde{W}_t \\ Y_0 = 0, \end{cases} \tag{2.28}$$

where

$$\begin{aligned} b_t^1 x_t + b_t^2 \mathbb{E}\left[x_t\right] + b_t^3 v_t &= b\left(t, x_t^v, P_{x_t^v}, v_t\right), \\ \sigma_t^1 &= \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right), \\ \sigma_t^2 &= \widetilde{\sigma}\left(t, x_t^v, P_{x_t^v}, v_t\right), \\ g_t^i &= g^i\left(t, x_{t_-}^v, P_{x_{t_-}^v}, v_t\right), \\ \gamma_t &= \xi\left(t, x_t^v, P_{x_t^v}\right), \end{aligned}$$

and

$$\begin{aligned} f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \overline{z}_t^v, P_{\overline{z}_t^v}, q_t^v, P_{q_t^v}, v_t\right) &= f_t^1 x_t + f_t^2 \mathbb{E}\left[x_t\right] + f_t^3 y_t + f_t^4 \mathbb{E}\left[y_t\right] \\ &+ f_t^5 z_t + f_t^6 \mathbb{E}\left[z_t\right] + f_t^7 \overline{z}_t + f_t^8 \mathbb{E}\left[\overline{z}_t\right] \\ &+ f_t^9 q_t + f_t^{10} \mathbb{E}\left[q_t\right] + f_t^{11} v_t. \end{aligned}$$

The quadratic cost function to be minimized

$$\begin{aligned} J\left(v\left(\cdot\right)\right) = \mathbb{E}^u \int_0^T &\left[L_t^1 x_t^2 + L_t^2 \left(\mathbb{E}\left[x_t\right]\right)^2 + L_t^3 y_t^2 + L_t^4 \left(\mathbb{E}\left[y_t\right]\right)^2 + L_t^5 v_t^2\right] dt \\ &+ \mathbb{E}^u \left[M_1 x_T^2 + M_2 \left(\mathbb{E}\left[x_T\right]\right)^2 + h_t y_0^2\right]. \end{aligned} \tag{2.29}$$

Here, all the coefficients $b^1(\cdot), b^2(\cdot), b^3(\cdot), \sigma^1(\cdot), \sigma^2(\cdot), g(\cdot), \gamma(\cdot), f^{j_1}(\cdot)$ are bounded and deterministic functions for $j_1 = 1, ..., 11$, $L^{j_2}(\cdot)$ is positive function and bounded for $j_2 = 1, 2, 3, 4, 5, 6$, and $M_1(\cdot), M_2(\cdot), h(\cdot)$ are positive constants. Then for any $v \in \mathcal{U}_{ad}$, equations (2.27) and (2.28) have unique solutions, respectively. Now, we introduce

$$\rho_t = \exp\left\{\int_0^t \gamma_s dY_s - \frac{1}{2}\int_0^t \left|\gamma_s\right|^2 ds\right\},$$

which is the unique $\mathcal{F}_t^Y$-adapted solution of the SDE:

$$\begin{cases} d\rho_t = \rho_t \gamma_t dY_t, \\ \rho_0 = 1, \end{cases}$$

and we define the probability measure P^v by $dP^v = \rho_t^v dP$.

In this setting, the Hamiltonian function is defined as

$$\begin{aligned} & H(t, x, y, z, \bar{z}, q, v, \Psi, \Phi, k, \bar{k}, n, Q) \\ & = \Psi\left(b_t^1 x_t + b_t^2 \mathbb{E}\left[x_t\right] + b_t^3 v_t - \sigma_t^2 \gamma_t\right) - \Phi\left(f_t^1 x_t + f_t^2 \mathbb{E}\left[x_t\right] + f_t^3 y_t + f_t^4 \mathbb{E}\left[y_t\right] + f_t^5 z_t\right. \\ & + \left. f_t^6 \mathbb{E}\left[z_t\right] + f_t^7 \bar{z}_t + f_t^8 \mathbb{E}\left[\bar{z}_t\right] + f_t^9 q_t + f_t^{10} \mathbb{E}\left[q_t\right] + f_t^{11} v_t\right) + k\sigma_t^1 + \bar{k}\sigma_t^2 \\ & \quad + \sum_{i=1}^{\infty} n_t^i g_t^i + Q\gamma_t + L_t^1 x_t^2 + L_t^2 \left(\mathbb{E}\left[x_t\right]\right)^2 + L_t^3 y_t^2 + L_t^4 \left(\mathbb{E}\left[y_t\right]\right)^2 + L_t^5 v_t^2. \end{aligned} \tag{2.30}$$

Further due to equations (2.11) and (2.12), the corresponding adjoint equations will be given by

$$\begin{cases} -d\Gamma_t = \left(L_t^1 x_t^2 + L_t^2 \left(\mathbb{E}\left[x_t\right]\right)^2 + L_t^3 y_t^2 + L_t^4 \left(\mathbb{E}\left[y_t\right]\right)^2 + L_t^5 v_t^2\right) dt \\ \qquad\quad -\bar{Q}_t dW_t - Q_t d\widetilde{W}_t, \\ \Gamma_T = M(x_T, P_{x_T}), \end{cases} \tag{2.31}$$

and

$$\begin{cases} -d\Psi_t = \left[b_t^1 \Psi_t + b_t^2 \mathbb{E}\left[\Psi_t\right] - f_t^1 \Phi_t - f_t^2 \mathbb{E}\left[\Phi_t\right] + 2L_t^1 x_t + 2L_t^2 \mathbb{E}\left[x_t\right]\right] dt \\ \qquad\qquad -k_t dW_t - \bar{k}_t d\widetilde{W}_t - \sum_{i=1}^{\infty} n_t^i dH_t^i, \\ d\Phi_t = \left(f_t^3 \Phi_t + f_t^4 \mathbb{E}\left[\Phi_t\right] - 2L_t^3 y_t - 2L_t^4 \mathbb{E}\left[y_t\right]\right) dt + \left(f_t^5 \Phi_t + f_t^6 \mathbb{E}\left[\Phi_t\right]\right) dW_t \\ \qquad\qquad + \left[f_t^7 q_t + f_t^8 \mathbb{E}\left[\Phi_t\right]\right] d\widetilde{W}_t + \sum_{i=1}^{\infty} \left(f_t^{i,9} \Phi_t + f_t^{i,10} \mathbb{E}\left[\Phi_t\right]\right) dH_t^i \\ \Psi_T = 2M_1 x_T + 2M_2 \mathbb{E}\left[x_T\right] - \phi_1 x_T - \phi_2 \mathbb{E}\left[x_T\right], \\ \Phi_0 = -2h_t y_0. \end{cases} \tag{2.32}$$

According to Theorem 2.1, the necessary condition for optimality (2.22) will be

$$\mathbb{E}^u \left[\Psi_t b_t^3 - \Phi_t f_t^{11} + 2L_t^5 u_t / \mathcal{F}_t^Y\right] = 0. \ \ a.s.a.e.$$

If $u(\cdot)$ is partial observed optimal control, then

$$u_t = -\frac{1}{2L_t^5} \left(b_t^3 \mathbb{E}^u \left[\Psi_t / \mathcal{F}_t^Y\right] - f_t^{11} \mathbb{E}^u \left[\Phi_t / \mathcal{F}_t^Y\right]\right). \tag{2.33}$$

Chapter 3

Maximum principle for partially observed Risk-sensitive optimal control problem of McKean–Vlasov FBSDEs involving impulse controls

3.1 Introduction

This chapter examines the second topic of this thesis, which is the maximum principle pertaining to risk-sensitive optimal control problems under partial observation, modeled by FBSDEs of the general McKean–Vlasov equations:

$$\begin{cases} dx_t^{v,\eta} = b(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, v_t)dt + \sigma(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, v_t)dW_t + \mathcal{C}_t d\eta_t, \\ -dy_t^{v,\eta} = f(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, y_t^{v,\eta}, P_{y_t^{v,\eta}}, z_t^{v,\eta}, P_{z_t^{v,\eta}}, v_t)dt - z_t^{v,\eta}dW_t + \mathcal{D}_t d\eta_t, \\ x_0^{v,\eta} = a, \quad y_T^{v,\eta} = \varphi(x_T^{v,\eta}, P_{x_T^{v,\eta}}), \end{cases} \tag{3.1}$$

where W_t is a one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and $\eta(\cdot) = \sum_{i\geq 1} \eta_i \mathbf{1}_{[\tau_i, T]}$ such that each $\eta_i \in \mathbb{R}^n$. $P_{x_t^{v,\eta}}, P_{y_t^{v,\eta}}$ and $P_{z_t^{v,\eta}}$ denotes the law of the random variable $x_t^{v,\eta}, y_t^{v,\eta}$ and $z_t^{v,\eta}$ respectively. The coefficients $b : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times U \to \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times U \to \mathbb{R}^{n\times d}, \varphi : \mathbb{R}^n \times Q_2(\mathbb{R}^n) \to \mathbb{R}^m$ are given deterministic functions, and $\mathcal{C} : [0,T] \to \mathbb{R}^{n\times n}, \mathcal{D} : [0,T] \to \mathbb{R}^{n\times d}$ are continuous functions. $Q_2\left(\mathbb{R}^d\right)$ is the space of all probability measures μ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$, endowed with the following 2-Wasserstein metric.

It's important to highlight that the aforementioned forward-backward stochastic differential equation (3.1) of the McKean-Vlasov type is quite general, as the coefficients can exhibit nonlinear dependence on the $P_{x_t^{v,\eta}}, P_{y_t^{v,\eta}}$ and $P_{z_t^{v,\eta}}$ which are considered as elements within the space of probability measures.

Consider state processes $(x_t^{v,\eta}, y_t^{v,\eta}, z_t^{v,\eta})$ are not fully observable. Rather, they are only partially observed through a noisy process Y, which is described by the following equation:

$$\begin{cases} dY_t = \xi(t, x_t^{v,\eta}, P_{x_t^{v,\eta}})dt + d\widetilde{W}_t^v, \\ Y_0 = 0, \end{cases} \tag{3.2}$$

here $\xi : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \to \mathbb{R}^n$ and $\widetilde{W}_t^v$ represents stochastic processes that rely on the control variable v.

The corresponding cost functional to be maximized is of the McKean-Vlasov type and is defined as follows

$$\begin{aligned} J^\theta(v,\eta) = \mathbb{E}^v \Bigg[\exp\theta \Bigg(& \int_0^T l(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, y_t^{v,\eta}, P_{y_t^{v,\eta}}, z_t^{v,\eta}, P_{z_t^{v,\eta}}, v_t)dt \\ & + M(x_T^{v,\eta}, P_{x_T^{v,\eta}}) + h(y_0^{v,\eta}, P_{y_0^{v,\eta}}) + \sum_{i\geq 1} c(\tau_i, \eta_i) \Bigg) \Bigg], \end{aligned} \tag{3.3}$$

here $\mathbb{E}^v$ represents expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P^v)$ and y_0^v is deterministic. θ represents risk-sensitive index for $\theta \in (0,1]$. The coefficients $M : \mathbb{R}^n \times Q_2(\mathbb{R}^n) \to \mathbb{R}$, $h : \mathbb{R}^m \times Q_2(\mathbb{R}^m) \to \mathbb{R}$, $c : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $l : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times \mathbb{R}^m \times Q_2(\mathbb{R}^m) \times \mathbb{R}^{m\times d} \times Q_2\left(\mathbb{R}^{m\times d}\right) \times U \to \mathbb{R}$ are deterministic functions.

The objective of our partially observed risk-sensitive optimal control problem is to maximize the cost functional (3.3) over $(v_\cdot, \eta_\cdot) \in \mathcal{A}$, subject to (3.1) and (3.2). A control $(u_\cdot, \zeta_\cdot) \in \mathcal{A}$ that satisfies

$$J^\theta(u,\zeta) = \max_{(v_\cdot,\eta_\cdot)\in\mathcal{A}} J^\theta(v,\eta),$$

is called a risk-sensitive optimal control.

The chapter mainly concentrates on formulating the risk-sensitive maximum principle for McKean-Vlasov forward-backward stochastic differential equations with impulse control. In this framework, impulse control is modeled as a piecewise process that does not necessarily need to be monotonic. Additionally, this work introduces further concavity conditions under which the partial necessary risk-sensitive conditions of optimality are sufficient. As an example, a linear quadratic (LQ) risk-sensitive optimal control problem of the McKean-Vlasov type is studied. It is noteworthy that the results offered in this study build upon the research conducted by Ma and Wang [27].

3.2 Statement of the problem

Let $T > 0$ be a real constant, and let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ represent a complete filtered probability space equipped with two independent standard one-dimensional Brownian motions W and Y. Define $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ and $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^Y \vee \mathcal{N}$, where $\mathcal{N}$ represents the collection of P-null set, $\mathcal{F}_t^W$ and $\mathcal{F}_t^Y$ represents the P-completed natural filtration generated by W and Y respectively. Let $\mathcal{U}_{ad}$ be the set of the admissible control variables. Let τ_i be a given sequence of increasing $\mathbb{F}$-stopping times such that $\tau_i \uparrow +\infty$ as $i \to +\infty$. The assumption $\tau_i \uparrow +\infty$ implies that only a finite number of impulses can occur within the interval $[0, T]$. Define $\mathcal{I}$ as the set of processes $\eta(\cdot) = \sum_{i\geq 1} \eta_i \mathbf{1}_{[\tau_i, T]}$ where each $\eta_i \in \mathbb{R}^n$ is $\mathcal{F}_{\tau_i}$-measurable random variable. Let $\mathcal{K}$ represents the class of impulse processes $\eta(\cdot) \in \mathcal{I}$ with $\mathbb{E}(\sum_{i\geq 1} |\eta_i|)^2 < \infty$.
The admissible control set is denoted by $\mathcal{A} = \mathcal{U}_{ad} \times \mathcal{K}$. The notation $\mathbb{R}^n$ represents the n-dimensional Euclidean space, and by $(\cdot, \cdot)$ (resp. $|\cdot|$) the inner product (resp. norm).

We will introduce the following spaces:

- The set $\mathcal{G}^2(\Omega, \mathcal{F}, P, \mathbb{R}^n)$ consists of all $\mathbb{R}^n$-valued random variables that are square-integrable and $\mathcal{F}_T$-measurable.
- $\mathcal{G}^2_{\mathcal{F}}(0, T, \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued square integrable $\mathbb{F}$-adapted processes.
- $\mathbb{S}^2(0, T, \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued $\mathbb{F}$-adapted and continuous processes,

$$\mathbb{E}(\sup_{0\leq t\leq T} |f_t|^2) < \infty.$$

- The space $\mathcal{G}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ is a Hilbert space with the inner product defined by $(x, y)_2 = \mathbb{E}[x.y]$ for $x, y \in \mathcal{G}^2\left(\mathcal{F}; \mathbb{R}^d\right)$. Its norm is given by $\|x\|_2 = \sqrt{(x, x)_2}$.
- $Q_2\left(\mathbb{R}^d\right)$ the space of all probability measures μ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$ with finite second moment, i.e, $\int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty$, endowed with the following 2-Wasserstein metric: for $\mu, \nu \in Q_2\left(\mathbb{R}^d\right)$,

$$\mathbb{D}_2(\mu_1, \mu_2) = \inf\left\{\left[\int_{\mathbb{R}^d} |x-y|^2 \kappa(dx, dy)\right]^{\frac{1}{2}} : \kappa \in Q_2\left(\mathbb{R}^{2d}\right), \kappa\left(\cdot, \mathbb{R}^d\right) = \mu_1, \kappa\left(\mathbb{R}^d, \cdot\right) = \mu_2\right\}.$$

Let $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{P}\right)$ represents a copy of the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. For any random variable $(\vartheta_1, \vartheta_2) \in \mathcal{G}^2\left(\mathcal{F}; \mathbb{R}^d\right) \times \mathcal{G}^2\left(\mathcal{F}; \mathbb{R}^d\right)$, let $\left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right)$ represents independent copy of the random variable $(\vartheta_1, \vartheta_2)$ defined on $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{P}\right)$. Let $(\widehat{u}_t, \widehat{x}_t, \widehat{y}_t, \widehat{z}_t)$ represents independent copy of (u_t, x_t, y_t, z_t) with $P_{x_t}=\widehat{P}_{\widehat{x}_t}$, $P_{y_t}=\widehat{P}_{\widehat{y}_t}$ and $P_{z_t}=\widehat{P}_{\widehat{z}_t}$. We use $\widehat{\mathbb{E}}[\cdot]$ to represent the expectation under probability measure $\widehat{P}$, and define $P_X = P \circ X^{-1}$ as the distribution of the random variable X. The core concept of differentiability concerning probability measures involves associating a distribution $\mu \in Q_2\left(\mathbb{R}^d\right)$ with a random variables $\vartheta_1 \in \mathcal{G}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ such that $\mu = P_{\vartheta_1}$. Specifically, let the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is sufficiently rich such that for every $\mu \in Q_2\left(\mathbb{R}^d\right)$, there exists a random variable $\vartheta_1 \in \mathcal{G}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ satisfying $\mu = P_{\vartheta_1}$. It is well established that the probability space $([0,1], \mathcal{B}[0,1], dx)$, where dx is the Borel measure, possesses this characteristic. For further details on differentiability with respect to probability measures, refer to, Abba and Lakhdari [1] (Definition 1.1 and Definition 2.2).

Definition 3.1 Let U be a nonempty convex subset of $\mathbb{R}^k$. The control $v : \Omega \times [0,T] \to U$ will be admissible if it is $\mathcal{F}_t^Y$-adapted and holds

$$\sup_{0 \le t \le T} \mathbb{E}\left|v_t\right|^2 < \infty.$$

Utilizing Girsanov's theorem, we get that:

$$\rho_t^v = \exp\left\{\int_0^t \xi(s, x_s^{v,\eta}, P_{x_s^{v,\eta}})dY_s - \frac{1}{2}\int_0^t \left|\xi(s, x_s^{v,\eta}, P_{x_s^{v,\eta}})\right|^2 ds\right\},$$

where ρ^v represents unique $\mathcal{F}_t^Y$-adapted solution of the SDE of McKean–Vlasov type

$$\begin{cases} d\rho_t^v = \rho_t^v \xi\left(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}\right) dY_t, \\ \rho_0^v = 1, \end{cases} \tag{3.4}$$

and if $dP^v = \rho_t^v dP$, then P^v forms a new probability measure and $\left(W_t, \widetilde{W}_t^v\right)$ is a standard $\mathbb{R}^2$-valued Brownian motion under this probability. By Bayes formula, cost functional (3.3) can be stated as

$$J^\theta(v,\eta) = \mathbb{E}\left[\rho_T^v \exp\theta\left(\int_0^T l(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, y_t^{v,\eta}, P_{y_t^{v,\eta}}, z_t^{v,\eta}, P_{z_t^{v,\eta}}, v_t)dt \right.\right.$$
$$\left.\left. + \; M(x_T^{v,\eta}, P_{x_T^{v,\eta}}) + h(y_0^{v,\eta}, P_{y_0^{v,\eta}}) + \sum_{i\ge1} c(\tau_i, \eta_i)\right)\right]. \tag{3.5}$$

Next, we define

$$\begin{cases} dg_t^v = l\left(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, y_t^{v,\eta}, P_{y_t^{v,\eta}}, z_t^{v,\eta}, P_{z_t^{v,\eta}}, v_t\right) dt, \\ g_0^v = 0. \end{cases} \tag{3.6}$$

Therefore the original problem (3.3) is equivalent to maximize (3.5) over $(v_\cdot, \eta_\cdot) \in \mathcal{A}$ subject to (3.1), (3.2) and (3.6).

We denote for $\phi = b, \sigma$ and ξ

$$\begin{aligned} &\xi(t) = \xi(t, x_t^{u,\eta}, P_{x_t^{u,\eta}}), \\ &\xi_x(t) = \xi_x(t, x_t^{u,\eta}, P_{x_t^{u,\eta}}), \\ &\phi(t) = \phi(t, x_t^{u,\eta}, P_{x_t^{u,\eta}}, u_t), \\ &\phi_\varsigma(t) = \phi_\varsigma(t, x_t^{u,\eta}, P_{x_t^{u,\eta}}, u_t), \text{ for } \varsigma = x, v, \end{aligned}$$

and their derivative processes

$$\begin{aligned} &\partial_\mu^{P_x} \xi\left(t\right) = \partial_\mu^{P_x} \xi\left(t, \widehat{x}_t, P_{x_t}; x_t\right), \\ &\partial_\mu^{P_x} \xi\left(t, \widehat{x}_t\right) = \partial_\mu^{P_x} \xi\left(t, x_t, P_{x_t}; \widehat{x}_t\right), \\ &\partial_\mu^{P_x} \phi\left(t\right) = \partial_\mu^{P_x} \phi\left(t, \widehat{x}_t, P_{x_t}, \widehat{u}_t; x_t\right), \\ &\partial_\mu^{P_x} \phi\left(t, \widehat{x}_t\right) = \partial_\mu^{P_x} \phi\left(t, x_t, P_{x_t}, u_t; \widehat{x}_t\right). \end{aligned}$$

Moreover, we represents $\Lambda = f, l$ and $\varsigma = x, y, z, v$

$$\begin{aligned} \Lambda(t) &= \Lambda(t, x_t^{u,\eta}, P_{x_t^{u,\eta}}, y_t^{u,\eta}, P_{y_t^{u,\eta}}, z_t^{u,\eta}, P_{z_t^{u,\eta}}, u_t), \\ \Lambda_\varsigma(t) &= \Lambda_\varsigma(t, x_t^{u,\eta}, P_{x_t^{u,\eta}}, y_t^{u,\eta}, P_{y_t^{u,\eta}}, z_t^{u,\eta}, P_{z_t^{u,\eta}}, u_t). \end{aligned}$$

Thus, we have $\varsigma = x, y, z$

$$\begin{aligned} \partial_\mu^{P_\varsigma} \Lambda\left(t\right) &= \partial_\mu^{P_\varsigma} \Lambda\left(t, \widehat{x}_t, P_{x_t}, \widehat{y}_t, P_{y_t}, \widehat{z}_t, P_{z_t}, \widehat{u}_t; \varsigma_t\right), \\ \partial_\mu^{P_\varsigma} \Lambda\left(t, \widehat{\varsigma}_t\right) &= \partial_\mu^{P_\varsigma} \Lambda\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, u_t; \widehat{\varsigma}_t\right). \end{aligned}$$

We will utilize the following conditions.

Condition (H1)

1. $\forall t \in [0, T]$, the function $b(\cdot, 0, \delta_0, 0) \in \mathcal{G}_{\mathcal{F}}^2(0, T, \mathbb{R})$, $\sigma(\cdot, 0, \delta_0, 0) \in \mathcal{G}_{\mathcal{F}}^2(0, T, \mathbb{R})$, $\xi(\cdot, 0, \delta_0) \in \mathcal{G}_{\mathcal{F}}^2(0, T, \mathbb{R})$, $f(\cdot, 0, \delta_0, 0, \delta_0, 0, \delta_0, 0) \in \mathcal{G}_{\mathcal{F}}^2(0, T, \mathbb{R})$ and $\varphi(0, \delta_0) \in \mathbb{L}^2(\Omega, \mathcal{F}, P, \mathbb{R})$, where δ_0 is the Dirac measure at 0. The function $\mathcal{C} : [0, T] \to \mathbb{R}$ and $\mathcal{D} : [0, T] \to \mathbb{R}$ are continuous.

2. Functions b and σ are continuously differentiable in (x, v) and also bounded by $C(1 + |x| + |v|)$.

3. Functions f and l are continuously differentiable in (x, y, z, v), and also bounded by $C(1 + |x| + |y| + |z| + |v|)$ and $C\left(1 + |x|^2 + |y|^2 + |z|^2 + |v|^2\right)$, respectively. Derivatives of f and l with respect to (x, y, z, v) are uniformly bounded.

4. Function c is continuous and continuously differentiable in η and their derivative is bounded by $C(1 + |\eta|)$.

5. Functions φ, M and ξ are continuously differentiable in x, and the function h is continuously differentiable in y. Derivatives of M, h are bounded by $C(1 + |x|)$ and $C(1 + |y|)$ respectively.

6. Derivatives $b_x, b_v, \sigma_x, \sigma_v, \xi_x$ are uniformly bounded and continuous.

Condition (H2)

1. Functions $b, \sigma, f, l, \xi, M, h, \varphi \in \mathbb{C}_b^{1,1}\left(Q_2\left(\mathbb{R}\right)\right)$.
2. Derivatives $\partial_\mu^{P_x} b, \partial_\mu^{P_x} \sigma, \partial_\mu^{P_x} \xi, \left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}\right)(f, l)$ are Lipchitz continuous and bounded, such that, for some $C > 0$, it satisfies

(i) For $\Pi = b, \sigma, \xi$ and $\forall \mu, \mu' \in Q_2\left(\mathbb{R}\right), \forall x, x' \in \mathbb{R}$,

$$\begin{aligned}
&\left|\partial_\mu^{P_x} \Pi\left(t, x, \mu\right)\right| \leq C, \\
&\left|\partial_\mu^{P_x} \Pi\left(t, x, \mu\right) - \partial_\mu^{P_x} \Pi\left(t, x', \mu'\right)\right| \leq C\left(\mathbb{D}_2\left(\mu, \mu'\right) + |x - x'|\right),
\end{aligned}$$

(ii) For $\Pi = M, \varphi$, and $\forall \mu, \mu' \in Q_2\left(\mathbb{R}\right), \forall x, x' \in \mathbb{R}$,

$$\begin{aligned}
&\left|\partial_\mu^{P_x} \Pi\left(x, \mu\right)\right| \leq C, \\
&\left|\partial_\mu^{P_x} \Pi\left(x, \mu\right) - \partial_\mu^{P_x} \Pi\left(x', \mu'\right)\right| \leq C\left(\mathbb{D}_2\left(\mu, \mu'\right) + |x - x'|\right).
\end{aligned}$$

(iii) For $\Pi = f, l$, and $\forall \mu_1, \mu_1', \mu_2, \mu_2', \mu_3, \mu_3' \in Q_2\left(\mathbb{R}\right)$ and $\forall x, x', y, y', z, z' \in \mathbb{R}$,

$$\begin{aligned}
&\left|\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}\right) \Pi\left(t, x, \mu_1, y, \mu_2, z, \mu_3\right)\right| \leq C, \\
&\left|\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}\right) \Pi\left(t, x, \mu_1, y, \mu_2, z, \mu_3\right) - \left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}\right) \Pi\left(t, x', \mu_1', y', \mu_2', z', \mu_3'\right)\right| \\
&\leq C\left(|x - x'| + |y - y'| + |z - z'| + \mathbb{D}_2\left(\mu_1, \mu_1'\right) + \mathbb{D}_2\left(\mu_2, \mu_2'\right) + \mathbb{D}_2\left(\mu_3, \mu_3'\right)\right).
\end{aligned}$$

By using (**H1**) and (**H2**), Theorem 3.1 in [9], Theorem 5.1 in [11], and Propositions 2.1 and 2.2 in Wu and Zhang [43], we get that for each $(v_\cdot, \eta_\cdot) \in \mathcal{A}$, there is a unique solution $(x^{v,\eta}, y^{v,\eta}, z^{v,\eta}) \in \mathcal{G}^2_{\mathcal{F}}(0,T,\mathbb{R}) \times \mathcal{G}^2_{\mathcal{F}}(0,T,\mathbb{R}) \times \mathcal{G}^2_{\mathcal{F}}(0,,\mathbb{R})$ which solves equation (3.1).

Now, introducing the first-order variational equations as:

$$\begin{cases} dx_t^1 = \left[b_x(t)\, x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b(t,\widehat{x}_t)\,\widehat{x}_t^1\right] + b_v(t)\, v_t\right] dt \\ \qquad + \left[\sigma_x(t)\, x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma(t,\widehat{x}_t)\,\widehat{x}_t^1\right] + \sigma_v(t)\, v_t\right] dW_t + \mathcal{C}_t d\eta_t, \\ -dy_t^1 = \left[f_x(t)\, x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f(t,\widehat{x}_t)\,\widehat{x}_t^1\right] + f_y(t)\, y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} f(t,\widehat{y}_t)\,\widehat{y}_t^1\right] \right. \\ \qquad \left. + f_z(t)\, z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} f(t,\widehat{z}_t)\,\widehat{z}_t^1\right] + f_v(t)\, v_t\right] dt - z_t^1 dW_t + \mathcal{D}_t d\eta_t, \\ x_0^1 = 0, \quad y_T^1 = \varphi_x(x_T, P_{x_T})\, x_T^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi(x_T, P_{x_T}, \widehat{x}_t)\,\widehat{x}_T^1\right], \end{cases} \tag{3.7}$$

$$\begin{cases} dg_t^1 = \left[l_x(t)\, g_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l(t,\widehat{x}_t)\,\widehat{g}_t^1\right] + l_y(t)\, g_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l(t,\widehat{y}_t)\,\widehat{g}_t^1\right] \right. \\ \qquad \left. + l_z(t)\, g_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l(t,\widehat{z}_t)\,\widehat{g}_t^1\right] + l_v(t)\, v_t\right] dt, \\ g_0^1 = 0, \end{cases} \tag{3.8}$$

and

$$\begin{cases} d\rho_t^1 = \left[\rho_t^1 \xi(t) + \rho_t \xi_x(t)\, x_t^1 + \rho_t \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t,\widehat{x}_t)\,\widehat{x}_t^1\right]\right] dY_t, \\ \rho_0^1 = 0. \end{cases} \tag{3.9}$$

Set $\vartheta = \rho^{-1}\rho^1$, utilizing Itô's formula, we get

$$\begin{cases} d\vartheta_t = \left[\xi_x(t)\, x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t,\widehat{x}_t)\,\widehat{x}_t^1\right]\right] d\widetilde{W}_t, \\ \vartheta_0 = 0. \end{cases} \tag{3.10}$$

3.3 Necessary conditions of optimality

For any $0 \leq \varepsilon \leq 1$ and $v_\cdot + u_\cdot \in \mathcal{U}_{ad}, \zeta_\cdot + \eta_\cdot \in \mathcal{K}$, we define a perturbed control $u_t^\varepsilon = u_t + \varepsilon v_t$ and $\zeta_t^\varepsilon = \zeta_t + \varepsilon\eta_t$. Let $\left(x^\varepsilon = x^{u^\varepsilon,\zeta^\varepsilon}, y^\varepsilon = y^{u^\varepsilon,\zeta^\varepsilon}, z^\varepsilon = z^{u^\varepsilon,\zeta^\varepsilon}\right)$ represents the solution of equation (3.1) corresponding to admissible control $(u^\varepsilon, \zeta^\varepsilon)$ and consider (u, ζ) as an optimal control having optimal trajectory (x, y, z), then we get.

Lemma 3.1

*By (**H1**) and (**H2**), we get that the following estimations satisfies:*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{x}_t^\varepsilon\right|^2\right] = 0, \tag{3.11}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{y}_t^\varepsilon\right|^2 + \int_0^T \left|\widetilde{z}_t^\varepsilon\right|^2 ds\right] = 0, \tag{3.12}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{\rho}_t^\varepsilon\right|\right] = 0, \tag{3.13}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{g}_t^\varepsilon\right|^2\right] = 0, \tag{3.14}$$

where

$$\widetilde{x}_t^{\varepsilon} = \frac{x_t^{\varepsilon} - x_t}{\varepsilon} - x_t^1, \text{ for } t \in [0,T], \varepsilon > 0,$$
$$\widetilde{y}_t^{\varepsilon} = \frac{y_t^{\varepsilon} - y_t}{\varepsilon} - y_t^1, \quad \widetilde{z}_t^{\varepsilon} = \frac{z_t^{\varepsilon} - z_t}{\varepsilon} - z_t^1,$$
$$\widetilde{g}_t^{\varepsilon} = \frac{g_t^{\varepsilon} - g_t}{\varepsilon} - g_t^1, \quad \widetilde{\rho}_t^{\varepsilon} = \frac{\rho_t^{\varepsilon} - \rho_t}{\varepsilon} - \rho_t^1.$$

Proof. We denote

$$\widetilde{x}_t^{\lambda,\varepsilon} = x_t + \lambda\varepsilon\left(\widetilde{x}_t^{\varepsilon} + x_t^1\right), \quad \widetilde{z}_t^{\lambda,\varepsilon} = z_t + \lambda\varepsilon\left(\widetilde{z}_t^{\varepsilon} + z_t^1\right),$$
$$\widetilde{y}_t^{\lambda,\varepsilon} = y_t + \lambda\varepsilon\left(\widetilde{y}_t^{\varepsilon} + y_t^1\right), \quad \gamma_t^{\lambda,\varepsilon} = \left(\tilde{x}_t^{\lambda,\varepsilon}, P_{\tilde{x}_t^{\lambda,\varepsilon}}, u_t^{\lambda,\varepsilon}\right).$$

First, we get

$$\begin{cases} d\widetilde{x}_t^{\varepsilon} = \left(b_t^x \widetilde{x}_t^{\varepsilon} + b_t^{\mu,x} + \beta_{1,t}^{\varepsilon}\right) dt + \left(\sigma_t^x \widetilde{x}_t^{\varepsilon} + \sigma_t^{\mu,x} + \beta_{2,t}^{\varepsilon}\right) dW_t, \\ \tilde{x}_0^{\varepsilon} = 0, \end{cases} \tag{3.15}$$

where

$$b_t^x = \int_0^1 b_x\left(t, \gamma_t^{\lambda,\varepsilon}\right) d\lambda, \quad b_t^{\mu,x} = \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b\left(t, \gamma_t^{\lambda,\varepsilon}, \widehat{\widetilde{x}_t^{\lambda,\varepsilon}}\right) \widehat{\widetilde{x}_t^{\varepsilon}}\right] d\lambda,$$
$$\sigma_t^x = \int_0^1 \sigma_x\left(t, \gamma_t^{\lambda,\varepsilon}\right) d\lambda, \quad \sigma_t^{\mu,x} = \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \sigma\left(t, \gamma_t^{\lambda,\varepsilon}, \widehat{\widetilde{x}_t^{\lambda,\varepsilon}}\right) \widehat{\widetilde{x}_t^{\varepsilon}}\right] d\lambda,$$

and

$$\begin{aligned}\beta_{1,t}^{\varepsilon} = &\int_0^1 \left[b_x\left(t, \gamma_t^{\lambda,\varepsilon}\right) - b_x(t)\right] d\lambda x_t^1 + \int_0^1 \left[b_v\left(t, \gamma_t^{\lambda,\varepsilon}\right) - b_v(t)\right] d\lambda v_t \\ &+ \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} b\left(t, \gamma_t^{\lambda,\varepsilon}, \widehat{\widetilde{x}_t^{\lambda,\varepsilon}}\right) - \partial_\mu^{P_x} b\left(t, \widehat{\widetilde{x}}_t\right)\right) \widehat{x}_t^1\right] d\lambda,\end{aligned}$$

$$\begin{aligned}\beta_{2,t}^{\varepsilon} = &\int_0^1 \left[\sigma_x\left(t, \gamma_t^{\lambda,\varepsilon}\right) - \sigma_x(t)\right] d\lambda x_t^1 + \int_0^1 \left[\sigma_v\left(t, \gamma_t^{\lambda,\varepsilon}\right) - \sigma_v(t)\right] d\lambda v_t \\ &+ \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} \sigma\left(t, \gamma_t^{\lambda,\varepsilon}, \widehat{\widetilde{x}_t^{\lambda,\varepsilon}}\right) - \partial_\mu^{P_x} \sigma\left(t, \widehat{\widetilde{x}}_t\right)\right) \widehat{x}_t^1\right] d\lambda.\end{aligned}$$

Under conditions (**H1**) and (**H2**), we also get that

$$\lim_{\varepsilon \to 0} \mathbb{E}\left(\left|\beta_{1,t}^{\varepsilon}\right|^2 + \left|\beta_{2,t}^{\varepsilon}\right|^2\right) = 0.$$

Utilizing Itô's formula to $|\widetilde{x}_t^{\varepsilon}|^2$, we obtain

$$\begin{aligned}\mathbb{E}\left|\widetilde{x}_t^{\varepsilon}\right|^2 &= 2\mathbb{E}\int_0^T \widetilde{x}_t^{\varepsilon}\left(b_t^x \widetilde{x}_t^{\varepsilon} + b_t^{\mu,x} + \beta_{1,t}^{\varepsilon}\right) dt + \mathbb{E}\int_0^T \left|\sigma_t^x \widetilde{x}_t^{\varepsilon} + \sigma_t^{\mu,x} + \beta_{2,t}^{\varepsilon}\right|^2 dt \\ &\leq C\mathbb{E}\int_0^T \left|\widetilde{x}_t^{\varepsilon}\right|^2 dt + \int_0^T \mathbb{E}\left(\left|\beta_{1,t}^{\varepsilon}\right|^2 + \left|\beta_{2,t}^{\varepsilon}\right|^2\right) dt.\end{aligned}$$

Finally, estimate (3.11) now follows easily from the Burkholder–Davis–Gundy inequality and Gronwall's inequality.

Let $(\widetilde{y}_t^\varepsilon, \widetilde{z}_t^\varepsilon)$ be the solution of the below stated BSDE

$$\begin{cases} d\widetilde{y}_t^\varepsilon = \left[f_t^x \widetilde{x}_t^\varepsilon + f_t^{\mu,x} + f_t^y \widetilde{y}_t^\varepsilon + f_t^{\mu,y} + f_t^z \widetilde{z}_t^\varepsilon + f_t^{\mu,z} + \beta_{3,t}^\varepsilon \right] dt + \widetilde{z}_t^\varepsilon dW_t, \\ \widetilde{y}_T^\varepsilon = \varepsilon^{-1}(\varphi\left(x_T^\varepsilon, P_{x_T^\varepsilon}\right) - \varphi\left(x_T, P_{x_T}\right)) \\ \quad -\varphi_x\left(x_T, P_{x_T}\right) x_T^1 - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \varphi\left(x_T, P_{x_T}, \widehat{x}_T\right) \widehat{x}_T^1\right], \end{cases}$$

where $\tilde{x}_t^\varepsilon$ satisfies SDE (3.15), and

$$\begin{aligned} f_t^\alpha &= -\int_0^1 f_\alpha\left(t, \chi_t^{\lambda,\varepsilon}\right) d\lambda, \text{ for } \alpha = x, y, z, \\ f_t^{\mu,\alpha} &= -\int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_\alpha} f\left(t, \chi_t^{\lambda,\varepsilon}, \widehat{\alpha_t^{\lambda,\varepsilon}}\right) \widehat{\alpha}_t^\varepsilon\right] d\lambda, \text{ for } \alpha = x, y, z, \end{aligned}$$

where

$$\chi_t^{\lambda,\varepsilon} = \left(\widetilde{x}_t^{\lambda,\varepsilon}, P_{\widetilde{x}_t^{\lambda,\varepsilon}}, \widetilde{y}_t^{\lambda,\varepsilon}, P_{\widetilde{y}_t^{\lambda,\varepsilon}}, \widetilde{z}_t^{\lambda,\varepsilon}, P_{\widetilde{z}_t^{\lambda,\varepsilon}}, u_t^{\lambda,\varepsilon}\right),$$

and $\beta_{3,t}^\varepsilon$ is stated as

$$\begin{aligned} \beta_{3,t}^\varepsilon = &\int_0^1 \left[f_x\left(t, \chi_t^{\lambda,\varepsilon}\right) - f_x(t)\right] d\lambda x_t^1 + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} f\left(t, \chi_t^{\lambda,\varepsilon}, \widehat{\widetilde{x}_t^{\lambda,\varepsilon}}\right) - \partial_\mu^{P_x} f\left(t, \chi_t, \widehat{\widetilde{x}}_t\right)\right) \widehat{x}_t^1\right] d\lambda \\ &+ \int_0^1 \left[f_y\left(t, \chi_t^{\lambda,\varepsilon}\right) - f_y(t)\right] d\lambda y_t^1 + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_y} f\left(t, \chi_t^{\lambda,\varepsilon}, \widehat{\widetilde{y}_t^{\lambda,\varepsilon}}\right) - \partial_\mu^{P_y} f\left(t, \chi_t, \widehat{\widetilde{y}}_t\right)\right) \widehat{y}_t^1\right] d\lambda \\ &+ \int_0^1 \left[f_z\left(t, \chi_t^{\lambda,\varepsilon}\right) - f_z(t)\right] d\lambda z_t^1 + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_z} f\left(t, \chi_t^{\lambda,\varepsilon}, \widehat{\widetilde{z}_t^{\lambda,\varepsilon}}\right) - \partial_\mu^{P_z} f\left(t, \chi_t, \widehat{\widetilde{z}}_t\right)\right) \widehat{z}_t^1\right] d\lambda \\ &+ \int_0^1 \left[f_v\left(t, \chi_t^{\lambda,\varepsilon}\right) - f_v(t)\right] d\lambda v_t. \end{aligned}$$

In view of the fact that $f_t^x, f_t^{\mu,x}, f_t^y, f_t^{\mu,y}, f_t^z$ and $f_t^{\mu,z}$ are continuous, we have

$$\lim_{\varepsilon \to 0} \mathbb{E}\left|\beta_{3,t}^\varepsilon\right|^2 = 0. \tag{3.16}$$

Utilizing Itô's formula to $|\widetilde{y}_t^\varepsilon|^2$, we obtain

$$\begin{aligned} &\mathbb{E}\left|\widetilde{y}_t^\varepsilon\right|^2 + \mathbb{E}\int_t^T \left|\widetilde{z}_s^\varepsilon\right|^2 ds \\ &= \mathbb{E}\left|\widetilde{y}_T^\varepsilon\right|^2 + 2\mathbb{E}\int_t^T \widetilde{y}_s^\varepsilon \left(f_s^x \widetilde{x}_s^\varepsilon + f_s^{\mu,x} + f_s^y \widetilde{y}_s^\varepsilon + f_s^{\mu,y} + f_s^z \widetilde{z}_s^\varepsilon + f_s^{\mu,z} + \beta_{3,t}^\varepsilon\right) ds. \end{aligned}$$

By Young's inequality, and conditions (**H1**) and (**H2**), we get

$$\begin{aligned} &\mathbb{E}\left|\widetilde{y}_t^\varepsilon\right|^2 + \mathbb{E}\int_t^T \left|\widetilde{z}_s^\varepsilon\right|^2 ds \\ &\leq C_1 \mathbb{E}\int_t^T \left|\widetilde{y}_s^\varepsilon\right|^2 ds + C\mathbb{E}\int_t^T \left|\widetilde{z}_s^\varepsilon\right|^2 ds + \mathbb{E}\left|\widetilde{y}_T^\varepsilon\right|^2 \\ &+ C\mathbb{E}\int_t^T \left|f_s^x \widetilde{x}_s^\varepsilon\right|^2 ds + C\mathbb{E}\int_t^T \left|\beta_{3,t}^\varepsilon\right|^2 ds. \end{aligned}$$

Applying Gronwall's inequality and in view of (3.11), (3.16), we obtain (3.12).

Finally, in a similar way, the estimate (3.13) and (3.14) can also be obtained. Which completes the proof of Lemma 1. □

Now, introducing adjoint equations of McKean-Vlasov type which rely on the risk-sensitive parameter θ,

$$\begin{cases} d\Gamma_t = \bar{Q}_t dW_t + Q_t d\widetilde{W}_t, \\ \Gamma_T = e^{\theta\Theta_T}, \end{cases} \tag{3.17}$$

$$\begin{cases} -d\alpha_t = \Big[l_x(t)\alpha_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l(t)\widehat{\alpha}_t\right] + l_y(t)\alpha_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l(t)\widehat{\alpha}_t\right] \\ \qquad + l_z(t)\alpha_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l(t)\widehat{\alpha}_t\right]\Big] dt - \beta_t dW_t, \\ \alpha_T = \theta e^{\theta\Theta_T}, \end{cases} \tag{3.18}$$

and

$$\begin{cases} -d\Psi_t = \Big[b_x(t)\Psi_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b\,(t)\,\widehat{\Psi}_t\right] + \sigma_x(t)k_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma\,(t)\,\widehat{k}_t\right] \\ \qquad + \xi_x\,(t)\,Q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\,(t)\,\widehat{Q}_t\right] - f_x(t)\Phi_t - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f\,(t)\,\widehat{\Phi}_t\right]\Big] dt - k_t dW_t, \\ d\Phi_t = \Big[f_y(t)\Phi_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} f\,(t)\,\widehat{\Phi}_t\right]\Big] dt + \Big[f_z\,(t)\,\Phi_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} f\,(t)\,\widehat{\Phi}_t\right]\Big] dW_t \\ \Psi_T = \theta e^{\theta\Theta_T}\Big[M_x\,(x_T, P_{x_T}) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\,(\widehat{x}_T, P_{x_T}, x_T)\right]\Big] \\ \qquad -\varphi_x\,(x_T, P_{x_T})\,\Phi_T - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\,(\widehat{x}_T, P_{x_T}, x_T)\,\widehat{\Phi}_T\right], \\ \Phi_0 = \theta e^{\theta\Theta_T}\Big[-h_y(y_0, P_{y_0}) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right]\Big], \end{cases} \tag{3.19}$$

where

$$\Theta_T = \int_0^T l\,(t)\,dt + M\,(x_T, P_{x_T}) + h\,(y_0, P_{y_0}) + \sum_{i\geq 1} c\,(\tau_i, \eta_i)\,.$$

Clearly, under (**H1**) and (**H2**), there exists a unique solutions $\left(\Gamma, \bar{Q}, Q\right) \in \mathcal{G}_{\mathcal{F}}^2\,(0, T, \mathbb{R}) \times \mathcal{G}_{\mathcal{F}}^2\,(0, T, \mathbb{R}) \times \mathcal{G}_{\mathcal{F}}^2\,(0, T, \mathbb{R})\,; (\alpha, \beta) \in \mathcal{G}_{\mathcal{F}}^2\,(0, T, \mathbb{R}) \times \mathcal{G}_{\mathcal{F}}^2\,(0, T, \mathbb{R})\,; (\Psi, k, \Phi) \in \mathcal{G}_{\mathcal{F}}^2\,(0, T, \mathbb{R}) \times \mathcal{G}_{\mathcal{F}}^2\,(0, T, \mathbb{R}) \times \mathcal{G}_{\mathcal{F}}^2\,(0, T, \mathbb{R})$ satisfying the McKean-Vlasov equations (3.17), (3.18) and (3.19) respectively.

Remark 3.1

In contrast to partially observed risk-neutral studies, a notable distinction in our findings is that the adjoint equations and variational inequalities of the McKean-Vlasov type are significantly influenced by the risk-sensitive parameter θ and the impulse control.

Theorem 3.1

*Suppose (**H1**) and (**H2**) satisfies, then we have*

$$\mathbb{E}^u\left[e^{\theta\Theta_T}\rho_T^{-1}\rho_T^1\right]+\mathbb{E}^u\left[\theta e^{\theta\Theta_T}g_T^1\right]+\mathbb{E}^u\left[\theta e^{\theta\Theta_T}\sum_{i\geq 1}c_\zeta\left(\tau_i,\zeta_i\right)\eta_i\right]$$
$$+\mathbb{E}^u\left[\theta e^{\theta\Theta_T}\left(M_x\left(x_T,P_{x_T}\right)x_T^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}M\left(x_T,P_{x_T},\widehat{x}_T\right)\widehat{x}_T^1\right]\right)\right]$$
$$+\mathbb{E}^u\left[\theta e^{\theta\Theta_T}\left(h_y\left(y_0,P_{y_0}\right)y_0^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}h(y_0,P_{y_0},\widehat{y}_0)\widehat{y}_0^1\right]\right)\right]\leq 0.$$

Proof. Using Lemma 3.1, Taylor expansion, and the fact that $\varepsilon^{-1}\left[J\left(u_t^\varepsilon,\zeta_t^\varepsilon\right)-J\left(u_t,\zeta_t\right)\right]\leq 0$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-1}\left[J\left(u_t^\varepsilon,\zeta_t^\varepsilon\right)-J\left(u_t,\zeta_t\right)\right]\\
&=\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{E}\left[\left(\rho_T^\varepsilon-\rho_T\right)e^{\theta\Theta_T}\right]+\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{E}\left[\rho_T^\varepsilon\left(e^{\theta\Theta_T^\varepsilon}-e^{\theta\Theta_T}\right)\right]\\
&=\mathbb{E}\left[\rho_T^1e^{\theta\Theta_T}\right]+\mathbb{E}\left[\rho_T\theta e^{\theta\Theta_T}g_T^1\right]+\mathbb{E}\left[\rho_T\theta e^{\theta\Theta_T}\sum_{i\geq 1}c_\zeta\left(\tau_i,\zeta_i\right)\eta_i\right]\\
&\quad+\mathbb{E}\left[\rho_T\theta e^{\theta\Theta_T}\left(M_x\left(x_T,P_{x_T}\right)x_T^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}M\left(x_T,P_{x_T},\widehat{x}_T\right)\widehat{x}_T^1\right]\right)\right]\\
&\quad+\mathbb{E}\left[\rho_T\theta e^{\theta\Theta_T}\left(h_y\left(y_0,P_{y_0}\right)y_0^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}h(y_0,P_{y_0},\widehat{y}_0)\widehat{y}_0^1\right]\right)\right]
\end{aligned}$$

$$\begin{aligned}
&=\mathbb{E}^u\left[e^{\theta\Theta_T}\rho_T^{-1}\rho_T^1\right]+\mathbb{E}^u\left[\theta e^{\theta\Theta_T}g_T^1\right]+\mathbb{E}^u\left[\theta e^{\theta\Theta_T}\sum_{i\geq 1}c_\zeta\left(\tau_i,\zeta_i\right)\eta_i\right]\\
&\quad+\mathbb{E}^u\left[\theta e^{\theta\Theta_T}\left(M_x\left(x_T,P_{x_T}\right)x_T^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}M\left(x_T,P_{x_T},\widehat{x}_T\right)\widehat{x}_T^1\right]\right)\right]\\
&\quad+\mathbb{E}^u\left[\theta e^{\theta\Theta_T}\left(h_y\left(y_0,P_{y_0}\right)y_0^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}h(y_0,P_{y_0},\widehat{y}_0)\widehat{y}_0^1\right]\right)\right]\leq 0.
\end{aligned}\tag{3.20}$$

This completes the proof. □

Let the Hamiltonian function related to the stochastic control problem defined by equations (3.1)-(3.5) be given by

$$\begin{aligned}
&H(t,x,P_x,y,P_y,z,P_z,v,\Psi,\Phi,k,\alpha,Q)\\
&=\Psi b\left(t,x,P_x,v\right)-\Phi f\left(t,x,P_x,y,P_y,z,P_z,v\right)\\
&\quad+k\sigma\left(t,x,P_x,v\right)+Q\xi\left(t,x,P_x\right)+\alpha l\left(t,x,P_x,y,P_y,z,P_z,v\right).
\end{aligned}\tag{3.21}$$

The following theorem presents the first main result of this paper.

Theorem 3.2

*(Partial necessary risk-sensitive conditions of optimality) Assuming conditions (**H1**) and (**H2**) are met, let (x, y, z, u) denote an optimal solution to the partially observed risk-sensitive optimal control problem of McKean–Vlasov type. Then, there exist unique solution $\left(\Gamma, \bar{Q}, Q\right) \in \mathcal{G}^2_{\mathcal{F}}(0, T, \mathbb{R}) \times \mathcal{G}^2_{\mathcal{F}}(0, T, \mathbb{R}) \times \mathcal{G}^2_{\mathcal{F}}(0, T, \mathbb{R}); (\alpha, \beta) \in \mathcal{G}^2_{\mathcal{F}}(0, T, \mathbb{R}) \times \mathcal{G}^2_{\mathcal{F}}(0, T, \mathbb{R}); (\Psi, k, \Phi) \in \mathcal{G}^2_{\mathcal{F}}(0, T, \mathbb{R}) \times \mathcal{G}^2_{\mathcal{F}}(0, T, \mathbb{R}) \times \mathcal{G}^2_{\mathcal{F}}(0, T, \mathbb{R})$ to (3.17), (3.18) and (3.19) respectively, and that for all $(v., \eta.) \in \mathcal{A}$, we have*

$$\mathbb{E}^u \left[H_v(t)(v_t - u_t) / \mathcal{F}^Y_t\right] \leq 0, \quad a.e, a.s, \tag{3.22}$$

$$\mathbb{E}^u \left[\sum_{i \geq 1} \left(\Psi_{\tau_i} \mathcal{C}_{\tau_i} - \Phi_{\tau_i} \mathcal{D}_{\tau_i} + \alpha_{\tau_i} c_\zeta(\tau_i, \zeta_i)\right)(\eta_i - \zeta_i) / \mathcal{F}^Y_t\right] \leq 0, \quad \forall \eta. \in \mathcal{K}, \tag{3.23}$$

where $H(t)$ represents Hamiltonian function defined by (3.21).

Proof. Utilizing Itô's formula to $\Gamma_t \vartheta_t$, we have

$$\mathbb{E}^u \left[\Gamma_T \vartheta_T\right] = \mathbb{E}^u \int_0^T Q_t \left[\xi_x(t) x^1_t + \widehat{\mathbb{E}}\left[\partial^{P_x}_\mu \xi(t, \widehat{x}_t) \widehat{x}^1_t\right]\right] dt. \tag{3.24}$$

Employing Itô's formula to $\Psi_t x^1_t$ and $\Phi_t y^1_t$, and utilizing Fubini's theorem, we get

$$\begin{aligned}
\mathbb{E}^u \left[\Psi_T x^1_T\right] &= \mathbb{E}^u \int_0^T \Psi_t b_v(t) v_t dt + \mathbb{E}^u \int_0^T k_t \sigma_v(t) v_t dt \\
&- \mathbb{E}^u \int_0^T x^1_t \left[\xi_x(t) Q_t + \widehat{\mathbb{E}}\left[\partial^{P_x}_\mu \xi(t) \widehat{Q}_t\right] - f_x(t)\Phi_t - \widehat{\mathbb{E}}\left[\partial^{P_x}_\mu f(t) \widehat{\Phi}_t\right] dt \\
&+ \mathbb{E}^u \int_0^T \Psi_t \mathcal{C}_t d\eta_t,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}^u \left[\Phi_T y^1_T\right] - \mathbb{E}^u \left[\Phi_0 y^1_0\right] &= -\mathbb{E}^u \int_0^T \Phi_t \left[f_x(t) x^1_t + \widehat{\mathbb{E}}\left[\partial^{P_x}_\mu f(t, \widehat{x}_t) \widehat{x}^1_t\right] + f_v(t) v_t\right] dt \\
&- \mathbb{E}^u \int_0^T \Phi_t \mathcal{D}_t d\eta_t.
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbb{E}^u \left[\Psi_T x^1_T\right] + \mathbb{E}^u \left[\Phi_T y^1_T\right] - \mathbb{E}^u \left[\Phi_0 y^1_0\right] + \mathbb{E}^u \left[\Gamma_T \vartheta_T\right] \\
&= \mathbb{E}^u \int_0^T \Psi_t b_v(t) v_t dt + \mathbb{E}^u \int_0^T k_t \sigma_v(t) v_t dt - \mathbb{E}^u \int_0^T \Phi_t f_v(t) v_t dt \\
&+ \mathbb{E}^u \int_0^T \Psi_t \mathcal{C}_t d\eta_t - \mathbb{E}^u \int_0^T \Phi_t \mathcal{D}_t d\eta_t.
\end{aligned} \tag{3.25}$$

We notice that

$$
\begin{aligned}
\Gamma_T &= e^{\theta\Theta_T}, \\
y_T^1 &= \varphi_x\left(x_T, P_{x_T}\right) x_T^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{x}_T^1\right], \\
\Phi_0 &= -\theta e^{\Theta_T}\left(h_y(y_0, P_{y_0}) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right]\right), \\
\Psi_T &= \theta e^{\theta\Theta_T}\left(M_x\left(x_T, P_{x_T}\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(\widehat{x}_T, P_{x_T}, x_T\right)\right]\right) \\
&\quad -\varphi_x\left(x_T, P_{x_T}\right)\Phi_T - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{\Phi}_T\right].
\end{aligned}
\tag{3.26}
$$

Substituting (3.26) into (3.25), we obtain

$$
\begin{aligned}
&\mathbb{E}^u\left[x_T^1\theta e^{\theta\Theta_T}\left(M_x\left(x_T, P_{x_T}\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(\widehat{x}_T, P_{x_T}, x_T\right)\right]\right)\right] \\
&- \mathbb{E}^u\left[x_T^1\left(\varphi_x\left(x_T, P_{x_T}\right)\Phi_T + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{\Phi}_T\right]\right)\right] \\
&+ \mathbb{E}^u\left[\Phi_T\left(\varphi_x\left(x_T, P_{x_T}\right)x_T^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{x}_T^1\right]\right)\right] \\
&+ \mathbb{E}^u\left[y_0^1\theta e^{\theta\Theta_T}\left(h_y(y_0, P_{y_0}) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right]\right)\right] + \mathbb{E}^u\left[\vartheta_T e^{\theta\Theta_T}\right] \\
&= \mathbb{E}^u\left[\theta e^{\theta\Theta_T}\left(M_x\left(x_T, P_{x_T}\right)x_T^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{x}_T^1\right]\right)\right] \\
&+ \mathbb{E}^u\left[\theta e^{\theta\Theta_T}\left(h_y\left(y_0, P_{y_0}\right)y_0^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\widehat{y}_0^1\right]\right)\right] + \mathbb{E}^u\left[e^{\theta\Theta_T}\vartheta_T\right] \\
&= \mathbb{E}^u\int_0^T \Psi_t b_v(t)\, v_t dt + \mathbb{E}^u\int_0^T k_t\sigma_v(t)\, v_t dt - \mathbb{E}^u\int_0^T \Phi_t f_v(t)\, v_t dt \\
&+ \mathbb{E}^u\int_0^T \Psi_t\mathcal{C}_t d\eta_t - \mathbb{E}^u\int_0^T \Phi_t\mathcal{D}_t d\eta_t.
\end{aligned}
\tag{3.27}
$$

From (3.20) and $\vartheta = \rho^{-1}\rho^1$, we get

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[J\left(u_t^\varepsilon, \zeta_t^\varepsilon\right) - J\left(u_t, \zeta_t\right)\right] &= \mathbb{E}^u\left[e^{\theta\Theta_T}\vartheta_T\right] + \mathbb{E}^u\left[\theta e^{\theta\Theta_T} g_T^1\right] + \mathbb{E}^u\left[\theta e^{\theta\Theta_T}\sum_{i\geq 1} c_\zeta\left(\tau_i, \zeta_i\right)\eta_i\right] \\
&+\mathbb{E}^u\left[\theta e^{\theta\Theta_T}\left(M_x\left(x_T, P_{x_T}\right)x_T^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{x}_T^1\right]\right)\right] \\
&+\mathbb{E}^u\left[\theta e^{\theta\Theta_T}\left(h_y\left(y_0, P_{y_0}\right)y_0^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\widehat{y}_0^1\right]\right)\right] \leq 0.
\end{aligned}
\tag{3.28}
$$

Substituting (3.27) into (3.28), we obtain

$$
\begin{aligned}
&\mathbb{E}^u\left[\theta e^{\theta\Theta_T} g_T^1\right] + \mathbb{E}^u\int_0^T \Psi_t b_v(t)\, v_t dt + \mathbb{E}^u\int_0^T k_t\sigma_v(t)\, v_t dt - \mathbb{E}^u\int_0^T \Phi_t f_v(t)\, v_t dt \\
&+\mathbb{E}^u\int_0^T \Psi_t\mathcal{C}_t d\eta_t - \mathbb{E}^u\int_0^T \Phi_t\mathcal{D}_t d\eta_t + \mathbb{E}^u\left[\sum_{i\geq 1}\alpha_{\tau_i} c_\zeta\left(\tau_i, \zeta_i\right)\eta_i\right] \leq 0.
\end{aligned}
\tag{3.29}
$$

Applying once more Itô's formula to $g_t^1\alpha_t$ and utilizing Fubini's theorem, we get

$$
\mathbb{E}^u\left[g_T^1\alpha_T\right] = \mathbb{E}^u\int_0^T \alpha_t l_v(t) v_t dt. \tag{3.30}
$$

Substituting (3.30) into (3.29), we obtain

$$\mathbb{E}^u \int_0^T \Psi_t b_v(t) v_t dt + \mathbb{E}^u \int_0^T k_t \sigma_v(t) v_t dt - \mathbb{E}^u \int_0^T \Phi_t f_v(t) v_t dt + \mathbb{E}^u \int_0^T \alpha_t l_v(t) v_t dt$$
$$+ \mathbb{E}^u \left[\sum_{i \geq 1} \Psi_{\tau_i} \mathcal{C}_{\tau_i} \eta_i \right] - \mathbb{E}^u \left[\sum_{i \geq 1} \Phi_{\tau_i} \mathcal{D}_{\tau_i} \eta_i \right] + \mathbb{E}^u \left[\sum_{i \geq 1} \alpha_{\tau_i} c_\zeta(\tau_i, \zeta_i) \eta_i \right] \leq 0,$$

where

$$\mathbb{E}^u \int_0^T (\Psi_t \mathcal{C}_t - \Phi_t \mathcal{D}_t) d\eta_t = \mathbb{E}^u \left[\sum_{i \geq 1} (\Psi_{\tau_i} \mathcal{C}_{\tau_i} - \Phi_{\tau_i} \mathcal{D}_{\tau_i}) \eta_i \right].$$

Then, we can write

$$\mathbb{E}^u \left[H_v(t)(v_t - u_t) / \mathcal{F}_t^Y \right] + \mathbb{E}^u \left[\sum_{i \geq 1} (\Psi_{\tau_i} \mathcal{C}_{\tau_i} - \Phi_{\tau_i} \mathcal{D}_{\tau_i} + \alpha_{\tau_i} c_\zeta(\tau_i, \zeta_i))(\eta_i - \zeta_i) / \mathcal{F}_t^Y \right] \leq 0, \tag{3.31}$$

for any $v_\cdot + u_\cdot \in \mathcal{U}_{ad}$ and $\zeta_\cdot + \eta_\cdot \in \mathcal{K}$.

By choosing $v \equiv 0$ in (3.31), we get (3.23). If we select $\eta \equiv 0$, then we get (3.22). This completes the proof. □

3.4 Sufficient conditions of optimality:

In this section, we show that under specific additional concavity conditions (**H3**), the necessary conditions for partially observed risk-sensitive optimal control, as described in Theorem 3.2, are also sufficient.

The function $\Xi : \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}$ is concave if, for every $(x^u, P_x^u), (x^v, P_x^v) \in \mathbb{R} \times Q_2(\mathbb{R})$,

$$\Xi(x^v, P_x^v) - \Xi(x^u, P_x^u) \leq \Xi_x(x^u, P_x^u)(x^v - x^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \Xi(x^u, P_x^u)(x^v - x^u) \right].$$

Here, we required an extra condition (**H3**), given below:

Condition (H3)

1. Function l does not depend on (x, P_x, y, P_y, z, P_z), and $\varphi(x, P_x) = \phi x$, where ϕ is a constant.
2. Functions M, h are concave in (x, P_x) and (y, P_y) respectively.
3. Function $\eta \to c(t, \eta)$ is concave.

4. $H(t,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\Psi^u,\Phi^u,k^u,\alpha^u,Q^u)$ is concave in $(x^u,P_x^u,y^u,P_{y^u},z^u,P_{z^u},u)$.

$$\begin{aligned} H^v(t)-H^u(t) \leq\ & H_x^u(t)(x^v-x^u)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}H^u(t)(\widehat{x}^v-\widehat{x}^u)\right] \\ & +H_y^u(t)(y^v-y^u)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}H^u(t)(\widehat{y}^v-\widehat{y}^u)\right] \\ & +H_z^u(t)(z^v-z^u)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z}H^u(t)(\widehat{z}^v-\widehat{z}^u)\right] \\ & +H_v^u(t)(v-u), \end{aligned}$$

where $H^\varsigma(t)=H(t,x^v,P_x^v,y^v,P_{y^v},z^v,P_{z^v},\zeta,\Psi^u,\Phi^u,k^u,\alpha^u,Q^u)$, for $\varsigma=v,u$.

We now introduce the adjoint equations for controlled system (3.1) as outlined below:

$$\begin{cases} d\alpha_t=\beta_t dW_t, \\ \alpha_T=\theta e^{\theta\Theta_T}, \\ d\Gamma_t=\bar{Q}_t dW_t+Q_t d\widetilde{W}_t, \\ \Gamma_T=e^{\theta\Theta_T}, \end{cases} \tag{3.32}$$

and

$$\begin{cases} -d\Psi_t=\Big[b_x(t)\Psi_t+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}b(t)\widehat{\Psi}_t\right]+\sigma_x(t)k_t+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma(t)\widehat{k}_t\right] \\ \qquad +\xi_x(t)Q_t+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t)\widehat{Q}_t\right]-f_x(t)\Phi_t-\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}f(t)\widehat{\Phi}_t\right]\Big]dt-k_t dW_t, \\ -d\Phi_t=\left[f_y(t)\Phi_t+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}f(t)\widehat{\Phi}_t\right]\right]dt+\left[f_z(t)\Phi_t+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z}f(t)\widehat{\Phi}_t\right]\right]dW_t \\ \Psi_T=\theta e^{\theta\Theta_T}\left[M_x(x_T,P_{x_T})+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}M(\widehat{x}_T,P_{x_T},x_T)\right]\right]-\phi\Phi_T, \\ \Phi_0=\theta e^{\theta\Theta_T}\left[-h_y(y_0,P_{y_0})-\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}h(\widehat{y}_0,P_{y_0},y_0)\right]\right]. \end{cases} \tag{3.33}$$

In this cases, the Hamiltonian function defined as

$$\begin{aligned} & H(t,x,P_x,y,P_y,z,P_z,v,\Psi,\Phi,k,\alpha,Q) \\ & =\Psi b(t,x,P_x,v)-\Phi f(t,x,P_x,y,P_y,z,P_z,v) \\ & \quad +k\sigma(t,x,P_x,v)+Q\xi(t,x,P_x)+\alpha l(t,v). \end{aligned} \tag{3.34}$$

Theorem 3.3

*Let (**H1**), (**H2**) and (**H3**) satisfies. Suppose ρ^v be $\mathcal{F}_t^Y$-adapted, $(u.,\zeta.)\in\mathcal{U}_{ad}\times\mathcal{K}$ be an admissible control, and (x,y,z) represents the corresponding trajectories. Assume that (α,β,Γ) and (Ψ,k,Φ) holds (3.32) and (3.33), respectively. Additionally, the Hamiltonian function H is concave in (x,P_x,y,P_y,z,P_z,v), and*

$$\mathbb{E}^u\left[H_v(t)(v_t-u_t)/\mathcal{F}_t^Y\right]\le 0, \quad a.e, a.s, \tag{3.35}$$

$$\mathbb{E}^u\left[\sum_{i\ge1}\left(\Psi_{\tau_i}\mathcal{C}_{\tau_i}-\Phi_{\tau_i}\mathcal{D}_{\tau_i}+\alpha_{\tau_i}c_\zeta(\tau_i,\zeta_i)\right)(\eta_i-\zeta_i)/\mathcal{F}_t^Y\right]\le 0, \quad \forall \eta. \in \mathcal{K}. \tag{3.36}$$

Then (u,ζ) will be an optimal control.

Proof. For any control $(v.,\eta.)\in\mathcal{U}_{ad}\times\mathcal{K}$, we denote by

$$\Theta_T^v=\int_0^T l^v(t)\,dt+M\left(x_T^v,P_{x_T^v}\right)+h\left(y_0^v,P_{y_0^v}\right)+\sum_{i\ge1}c(\tau_i,\eta_i),$$
$$\Theta_T^u=\int_0^T l^u(t)\,dt+M\left(x_T^u,P_{x_T^u}\right)+h\left(y_0^u,P_{y_0^u}\right)+\sum_{i\ge1}c(\tau_i,\zeta_i),$$

then, we have

$$\begin{aligned} J^\theta(v_t,\eta_t)-J^\theta(u_t,\zeta_t)&=\mathbb{E}^v\left[e^{\theta\Theta_T^v}\right]-\mathbb{E}^u\left[e^{\theta\Theta_T^u}\right]\\ &=\mathbb{E}^v\left[e^{\theta\Theta_T^v}\right]-\mathbb{E}\left[\rho_T^u e^{\theta\Theta_T^u}\right]\\ &\quad+\mathbb{E}\left[\rho_T^v e^{\theta\Theta_T^u}\right]-\mathbb{E}\left[\rho_T^v e^{\theta\Theta_T^u}\right]\\ &=\mathbb{E}\left[(\rho_T^v-\rho_T^u)e^{\theta\Theta_T^u}\right]\\ &\quad+\mathbb{E}^v\left[e^{\theta\Theta_T^v}\right]-\mathbb{E}^v\left[e^{\theta\Theta_T^u}\right]. \end{aligned} \tag{3.37}$$

By concavity property of the function M,h and c, (3.37) can be written as

$$\begin{aligned} J^\theta(v_t,\eta_t)-J^\theta(u_t,\zeta_t)&\le\mathbb{E}\left[(\rho_T^v-\rho_T^u)e^{\theta\Theta_T^u}\right]\\ &\quad+\mathbb{E}^v\left[\theta e^{\theta\Theta_T^u}(g_T^v-g_T^u)\right]\\ &\quad+\mathbb{E}^v\left[\theta e^{\theta\Theta_T^u}\sum_{i\ge1}c_\zeta(\tau_i,\zeta_i)(\eta_i-\zeta_i)\right]\\ &\quad+\mathbb{E}^v\left[\theta e^{\theta\Theta_T^u}\left(M_x\left(x_T^u,P_{x_T^u}\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}M\left(x_T^u,P_{x_T^u}\right)\right]\right)(x_T^v-x_T^u)\right]\\ &\quad+\mathbb{E}^v\left[\theta e^{\theta\Theta_T^u}\left(h_y\left(y_0^u,P_{y_0^u}\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}h(y_0^u,P_{y_0^u})\right]\right)(y_0^v-y_0^u)\right]\\ &=I_1+I_2+I_3+I_4+I_5. \end{aligned}$$

Employing Itô's formula to $(\rho_t^v-\rho_t^u)\Gamma_t^u$ and $(g_t^v-g_t^u)\alpha_t^u$, we have

$$I_1=\mathbb{E}^v\left[\int_0^T Q_t^u(\xi^v(t)-\xi^u(t))\,dt\right], \tag{3.38}$$

and

$$I_2 = \mathbb{E}^v \left[\int_0^T \alpha_t^u \left(l^v \left(t \right) - l^u \left(t \right) \right) dt \right]. \tag{3.39}$$

Note that

$$I_3 = \mathbb{E}^v \left[\sum_{i \geq 1} \alpha_{\tau_i} c_\zeta \left(\tau_i, \zeta_i \right) \left(\eta_i - \zeta_i \right) \right]. \tag{3.40}$$

Utilizing Itô's formula to $\left(x_t^v - x_t^u \right) \Psi_t^u$ and $\rho_t^u \Phi_t^u \left(y_t^v - y_t^u \right)$, we have

$$\begin{aligned} I_4 &= \mathbb{E}^v \left[\left(x_T^v - x_T^u \right) \phi \Phi_T^u \right] \\ &- \mathbb{E}^v \left[\int_0^T \left(x_t^v - x_t^u \right) \left[b_x(t) \Psi_t + \widehat{\mathbb{E}} \left[\partial_\mu^{P_x} b \left(t \right) \widehat{\Psi}_t \right] + \sigma_x(t) k_t + \widehat{\mathbb{E}} \left[\partial_\mu^{P_x} \sigma \left(t \right) \widehat{k}_t \right] \right. \right. \\ &+ \left. \left. \xi_x \left(t \right) Q_t + \widehat{\mathbb{E}} \left[\partial_\mu^{P_x} \xi \left(t \right) \widehat{Q}_t \right] - f_x(t) \Phi_t - \widehat{\mathbb{E}} \left[\partial_\mu^{P_x} f \left(t \right) \widehat{\Phi}_t \right] \right] dt \right] \\ &+ \mathbb{E}^v \left[\int_0^T \Psi_t^u \left(b^v(t) - b^u(t) \right) dt \right] + \mathbb{E}^v \left[\int_0^T k_t^u \left(\sigma^v(t) - \sigma^u(t) \right) dt \right] + \mathbb{E}^v \left[\int_0^T \Psi_t^u \mathcal{C}_t d \left(\eta_t - \zeta_t \right) \right], \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} \mathbb{E} \left[\rho_T^v \Phi_T^u \left(y_T^v - y_T^u \right) \right] - \mathbb{E} \left[\rho_0^v \Phi_0^u \left(y_0^v - y_0^u \right) \right] &= \mathbb{E}^v \left[\int_0^T \left(y_t^v - y_t^u \right) \left(f_y(t) \Phi_t + \widehat{\mathbb{E}} \left[\partial_\mu^{P_y} f \left(t \right) \widehat{\Phi}_t \right] \right) dt \right] \\ &+ \mathbb{E}^v \left[\int_0^T \left(z_t^v - z_t^u \right) \left(f_z \left(t \right) \Phi_t + \widehat{\mathbb{E}} \left[\partial_\mu^{P_z} f \left(t \right) \widehat{\Phi}_t \right] \right) dt \right] \\ &+ \mathbb{E}^v \left[\int_0^T \Phi_t^u \left(f^u \left(t \right) - f^v \left(t \right) \right) dt \right] - \mathbb{E}^v \left[\int_0^T \Phi_t^u \mathcal{D}_t d \left(\eta_t - \zeta_t \right) \right]. \end{aligned}$$

Then

$$\begin{aligned} I_5 = &- \mathbb{E} \left[\rho_T^v \Phi_T^u \left(y_T^v - y_T^u \right) \right] + \mathbb{E}^v \left[\int_0^T \left(y_t^v - y_t^u \right) \left(f_y(t) \Phi_t + \widehat{\mathbb{E}} \left[\partial_\mu^{P_y} f \left(t \right) \widehat{\Phi}_t \right] \right) dt \right] \\ &+ \mathbb{E}^v \left[\int_0^T \left(z_t^v - z_t^u \right) \left(f_z \left(t \right) \Phi_t + \widehat{\mathbb{E}} \left[\partial_\mu^{P_z} f \left(t \right) \widehat{\Phi}_t \right] \right) dt \right] \\ &+ \mathbb{E}^v \left[\int_0^T \Phi_t^u \left(f^u \left(t \right) - f^v \left(t \right) \right) dt \right] - \mathbb{E}^v \left[\int_0^T \Phi_t^u \mathcal{D}_t d \left(\eta_t - \zeta_t \right) \right]. \end{aligned} \tag{3.42}$$

Noting that

$$\begin{aligned} \mathbb{E}^v \left[\int_0^T \Psi_t^u \mathcal{C}_t d \left(\eta_t - \zeta_t \right) \right] &= \mathbb{E}^v \left[\sum_{i \geq 1} \Psi_{\tau_i} \mathcal{C}_{\tau_i} \left(\eta_i - \zeta_i \right) \right], \\ \mathbb{E}^v \left[\int_0^T \Phi_t^u \mathcal{D}_t d \left(\eta_t - \zeta_t \right) \right] &= \mathbb{E}^v \left[\sum_{i \geq 1} \Phi_{\tau_i} \mathcal{D}_{\tau_i} \left(\eta_i - \zeta_i \right) \right]. \end{aligned}$$

From (3.38),(3.39), (3.40), (3.41) and (3.42), we get

$$
\begin{aligned}
J^{\theta}\left(v_t,\eta_t\right)-J^{\theta}\left(u_t,\zeta_t\right) \leq\ & \mathbb{E}^v \int_0^T \left(H^v\left(t\right)-H^u\left(t\right)\right) dt \\
& -\mathbb{E}^v\left[\int_0^T \left(H_x^u\left(t\right)\left(x_t^v-x_t^u\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} H^u\left(t\right)\left(\widehat{x}_t^v-\widehat{x}_t^u\right)\right]\right) dt\right] \\
& -\mathbb{E}^v\left[\int_0^T \left(H_y^u\left(t\right)\left(y_t^v-y_t^u\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} H^u\left(t\right)\left(\widehat{y}_t^v-\widehat{y}_t^u\right)\right]\right) dt\right] \\
& -\mathbb{E}^v\left[\int_0^T \left(H_z^u\left(t\right)\left(z_t^v-z_t^u\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z} H^u\left(t\right)\left(\widehat{z}_t^v-\widehat{z}_t^u\right)\right]\right) dt\right] \\
& +\mathbb{E}^v\left[\sum_{i\geq 1}\left(\Psi_{\tau_i}\mathcal{C}_{\tau_i}-\Phi_{\tau_i}\mathcal{D}_{\tau_i}+\alpha_{\tau_i}c_\zeta\left(\tau_i,\zeta_i\right)\right)\left(\eta_i-\zeta_i\right)\right], \qquad (3.43)
\end{aligned}
$$

where

$$
\begin{aligned}
& H_x^u\left(t\right)\left(x_t^v-x_t^u\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} H^u\left(t\right)\left(\widehat{x}_t^v-\widehat{x}_t^u\right)\right] \\
& =\left(x_t^v-x_t^u\right)\left(b_x(t)\Psi_t+\sigma_x(t)k_t+\xi_x\left(t\right)Q_t-f_x(t)\Phi_t\right) \\
& \quad +\left(x_t^v-x_t^u\right)\left(\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b\left(t\right)\widehat{\Psi}_t\right]+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma\left(t\right)\widehat{k}_t\right]+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t\right)\widehat{Q}_t\right]-\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f\left(t\right)\widehat{\Phi}_t\right]\right),
\end{aligned}
$$

and

$$
\begin{aligned}
H_y^u\left(t\right)\left(y^v-y^u\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} H^u\left(t\right)\left(\widehat{y}_t^v-\widehat{y}_t^u\right)\right] &= \left(y_t^v-y_t^u\right)\left(f_y(t)\Phi_t+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} f\left(t\right)\widehat{\Phi}_t\right]\right), \\
H_z^u\left(t\right)\left(z^v-z^u\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z} H^u\left(t\right)\left(\widehat{z}_t^v-\widehat{z}_t^u\right)\right] &= \left(z_t^v-z_t^u\right)\left(f_z\left(t\right)\Phi_t+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z} f\left(t\right)\widehat{\Phi}_t\right]\right).
\end{aligned}
$$

By the concavity of the functional H in (t,x,P_x,y,P_y,z,P_z,v), and from (3.43) we have

$$
\begin{aligned}
J^{\theta}\left(v_t,\eta_t\right)-J^{\theta}\left(u_t,\zeta_t\right) \leq\ & \mathbb{E}^v \int_0^T H_v\left(t\right)\left(v_t-u_t\right) dt \\
& +\mathbb{E}^v\left[\sum_{i\geq 1}\left(\Psi_{\tau_i}\mathcal{C}_{\tau_i}-\Phi_{\tau_i}\mathcal{D}_{\tau_i}+\alpha_{\tau_i}c_\zeta\left(\tau_i,\zeta_i\right)\right)\left(\eta_i-\zeta_i\right)\right] \\
& \leq \mathbb{E}\int_0^T \rho_t^v H_v\left(t\right)\left(v_t-u_t\right) dt \\
& +\mathbb{E}\left[\sum_{i\geq 1}\rho_{\tau_i}^v\left(\Psi_{\tau_i}\mathcal{C}_{\tau_i}-\Phi_{\tau_i}\mathcal{D}_{\tau_i}+\alpha_{\tau_i}c_\zeta\left(\tau_i,\zeta_i\right)\right)\left(\eta_i-\zeta_i\right)\right] \\
& \leq \mathbb{E}\int_0^T \mathbb{E}\left[\rho_t^v H_v\left(t\right)\left(v_t-u_t\right)/\mathcal{F}_t^Y\right] dt \\
& +\mathbb{E}\left[\sum_{i\geq 1}\rho_{\tau_i}^v\left(\Psi_{\tau_i}\mathcal{C}_{\tau_i}-\Phi_{\tau_i}\mathcal{D}_{\tau_i}+\alpha_{\tau_i}c_\zeta\left(\tau_i,\zeta_i\right)\right)\left(\eta_i-\zeta_i\right)/\mathcal{F}_t^Y\right].
\end{aligned}
$$

Since $\rho_t^v>0$, and utilizing condition (3.35), (3.36) we obtain

$$
J^{\theta}\left(v_t,\eta_t\right)-J^{\theta}\left(u_t,\zeta_t\right)\leq 0,
$$

that is (u, ζ) is an optimal control. $\square$

3.5 Application to LQ risk-sensitive control problem

3.5.1 Partially observed LQ risk-sensitive optimal control problem

In this subsection, we apply the partial necessary risk-sensitive optimality conditions from Theorem 3.2 to examine a mean-field type partially observed linear-quadratic (LQ) risk-sensitive optimal control problem. Let $U = \mathbb{R}$. The dynamic state is described by the following mean-field type forward-backward stochastic differential equation

$$\begin{cases} dx_t = \left(A_t^1 x_t + A_t^2 \mathbb{E}\left[x_t\right] + A_t^3 v_t\right) dt + \left(B_t^1 x_t + B_t^2 \mathbb{E}\left[x_t\right] + B_t^3 v_t\right) dW_t + \mathcal{C}_t^1 d\eta_t, \\ -dy_t = \left(D_t^1 x_t + D_t^2 \mathbb{E}\left[x_t\right] + D_t^3 y_t + D_t^4 \mathbb{E}\left[y_t\right] + D_t^5 z_t + D_t^6 \mathbb{E}\left[z_t\right] + D_t^7 v_t\right) dt - z_t dW_t + \mathcal{C}_t^2 d\eta_t, \\ x\left(0\right) = x_0, \quad y_T = 0, \end{cases}$$

and the observation stochastic equation satisfies

$$\begin{cases} dY_t = G_t dt + d\widetilde{W}_t, \\ Y_0 = 0, \end{cases}$$

where

$$\begin{aligned} &b(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, v_t) = A_t^1 x_t + A_t^2 \mathbb{E}\left[x_t\right] + A_t^3 v_t, \\ &\sigma(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, v_t) = B_t^1 x_t + B_t^2 \mathbb{E}\left[x_t\right] + B_t^3 v_t, \\ &G_t = \xi\left(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}\right), \\ &\mathcal{C}_t = \mathcal{C}_t^1, \\ &\mathcal{D}_t = \mathcal{C}_t^2, \end{aligned}$$

and

$$\begin{aligned} f\left(t, x_t^{v,\eta}, P_{x_t^{v,\eta}}, y_t^{v,\eta}, P_{y_t^{v,\eta}}, z_t^{v,\eta}, P_{z_t^{v,\eta}}, v_t\right) = &\, D_t^1 x_t + D_t^2 \mathbb{E}\left[x_t\right] + D_t^3 y_t + D_t^4 \mathbb{E}\left[y_t\right] \\ &+ D_t^5 z_t + D_t^6 \mathbb{E}\left[z_t\right] + D_t^7 v_t. \end{aligned}$$

Here, $\left(\mathcal{C}^1\left(\cdot\right), \mathcal{C}^2\left(\cdot\right)\right) \geq 0$, and the coefficients $A^i\left(\cdot\right), B^i\left(\cdot\right), D^j\left(\cdot\right), G\left(\cdot\right)$ are bounded and deterministic functions for $i = 1, \ldots, 3$ and $j = 1, \ldots, 7$.

We specify the cost functional in the form of an expected exponential function as follows

$$J(v,\eta) = \mathbb{E}^{v}\left[\exp\frac{\theta}{2}\left(x_T^2 + y_0^2 + \int_0^T R_t v_t^2 dt + \sum_{i\geq 1}\eta_i^2\right)\right].$$

Assume that $(u_\cdot, \zeta_\cdot) \in \mathcal{A}$ represents optimal control and (x, y, z) represents the corresponding trajectory. Denote by

$$\Theta_T = \frac{1}{2}\left(x_T^2 + y_0^2 + \int_0^T R_t u_t^2 dt + \sum_{i\geq 1}\zeta_i^2\right).$$

The corresponding adjoint equations of mean-field type which depend on the risk-sensitive parameter is given by

$$\begin{cases} d\Gamma_t = \bar{Q}_t dW_t + Q_t d\widetilde{W}_t, \\ \Gamma_T = e^{\theta\Theta_T}, \end{cases} \tag{3.44}$$

$$\begin{cases} -d\Psi_t = \left(A_t^1\Psi_t + A_t^2\mathbb{E}[\Psi_t] + B_t^1 k_t + B_t^2\mathbb{E}[k_t] - D_t^1\Phi_t - D_t^2\mathbb{E}[\Phi_t]\right)dt - k_t dW_t, \\ -d\Phi_t = \left(D_t^3\Phi_t + D_t^4\mathbb{E}[\Phi_t]\right)dt + \left(D_t^5\Phi_t + D_t^6\mathbb{E}[\Phi_t]\right)dW_t \\ \Psi_T = \theta e^{\theta\Theta_T} x_T, \\ \Phi_0 = \theta e^{\theta\Theta_T}(-y_0), \end{cases} \tag{3.45}$$

where Q_t and (Ψ_t, Φ_t, k_t) are adjoint processes satisfy (3.44) and (3.45), respectively. In this case, the Hamiltonian function is defined as follows

$$\begin{aligned} &H(t, x, \mathbb{E}[x_t], y_t, \mathbb{E}[y_t], z_t, \mathbb{E}[z_t], v_t, \Psi_t, \Phi_t, k_t, \alpha_t, Q_t) \\ &= \Psi_t\left(A_t^1 x_t + A_t^2\mathbb{E}[x_t] + A_t^3 v_t\right) + k\left(B_t^1 x_t + B_t^2\mathbb{E}[x_t] + B_t^3 v_t\right) + G_t Q_t \\ &\quad -\Phi_t\left(D_t^1 x_t + D_t^2\mathbb{E}[x_t] + D_t^3 y_t + D_t^4\mathbb{E}[y_t] + D_t^5 z_t + D_t^6\mathbb{E}[z_t] + D_t^7 v_t\right) + \frac{1}{2}\alpha_t R_t v_t^2. \end{aligned} \tag{3.46}$$

Theorem 3.4

*Suppose (**H1**) and (**H2**) satisfies, and Q_t and (Ψ_t, Φ_t, k_t) are $\mathbb{F}$-adapted solutions of (3.44) and (3.45), respectively. Consider (u, ζ) as optimal, then the maximum principle*

$$\mathbb{E}\left[\left(A_t^3\Psi_t + B_t^3 k_t - D_t^7\Phi_t + \alpha_t R_t u_t\right)(v_t - u_t)/\mathcal{F}_t^Y\right] \leq 0, \;\; a.e.\; t \in [0,T], P - a.s,$$

$$\mathbb{E}\left[\sum_{i\geq 1}\left(\Psi_{\tau_i}\mathcal{C}_{\tau_i}^1 - \Phi_{\tau_i}\mathcal{C}_{\tau_i}^2 + \alpha_{\tau_i}\zeta_i\right)(\eta_i - \zeta_i)\right] \leq 0,$$

holds, for all $(v_\cdot, \eta_\cdot) \in \mathcal{A}$.

Remark 3.2

By Theorem 3.4, (u, ζ) *can be stated as*

$$u_t = -\frac{1}{\alpha_t R_t}\left(A_t^3 \Psi_t + B_t^3 k_t - D_t^7 \Phi_t\right), \quad a.e., a.s.$$
$$\zeta_i = \frac{1}{\alpha_{\tau_i}}\left(\Phi_{\tau_i} \mathcal{C}_{\tau_i}^2 - \Psi_{\tau_i} \mathcal{C}_{\tau_i}^1\right), \forall i \geq 1, \quad a.s.$$

3.5.2 Fully observed LQ risk-sensitive optimal control problem

In this subsection, we investigate the maximum principle for a fully observed risk-sensitive optimal control problem and, using the findings from Theorem 3.4, derive an explicit formula for the optimal control.

We examine the following mean-field type forward-backward system

$$\begin{cases} dx_t = A_t^3 u_t dt + B_t^3 u_t dW_t, \\ -dy_t = \left(D_t^3 y_t + D_t^4 \mathbb{E}\left[y_t\right] + D_t^5 z_t + D_t^6 \mathbb{E}\left[z_t\right] + D_t^7 v_t\right) dt - z_t dW_t, \\ x\left(0\right) = x_0, \quad y_T = 0. \end{cases}$$

We will find the optimal control u such that

$$J\left(u\right) = \max_{v_\cdot \in \mathcal{U}_{ad}} \mathbb{E}\left[\exp \theta\left(x_T + y_0\right)\right].$$

Here, Hamiltonian function (3.46) is defined by

$$\begin{aligned} &H(t, x, \mathbb{E}\left[x_t\right], y_t, \mathbb{E}\left[y_t\right], z_t, \mathbb{E}\left[z_t\right], v_t, \Psi_t, \Phi_t, k_t) \\ &= A_t^3 \Psi_t v_t + B_t^3 k v_t - \Phi_t\left(D_t^3 y_t + D_t^4 \mathbb{E}\left[y_t\right] + D_t^5 z_t + D_t^6 \mathbb{E}\left[z_t\right] + D_t^7 v_t\right), \end{aligned} \tag{3.47}$$

where the adjoint equation (3.44) disappears.

The related adjoint equations, which depend on the risk-sensitive parameter, are expressed as follows

$$\begin{cases} d\Psi_t = k_t dW_t, \\ d\Phi_t = \left(D_t^3 \Phi_t + D_t^4 \mathbb{E}\left[\Phi_t\right]\right) dt + \left(D_t^5 \Phi_t + D_t^6 \mathbb{E}\left[\Phi_t\right]\right) dW_t, \\ \Psi_T = \theta e^{\theta(x_T + y_0)}, \\ \Phi_0 = \theta e^{\theta(x_T + y_0)}. \end{cases}$$

According to Theorem 4, we deduce that

$$A_t^3\Psi_t + B_t^3 k_t - D_t^7\Phi_t = 0. \tag{3.48}$$

We suppose

$$\Psi_t = \pi_t e^{\theta(x_t+y_t)}, \tag{3.49}$$

$$\Phi_t = \mu_t e^{\theta(x_t+y_t)}, \tag{3.50}$$

where π_t and μ_t are deterministic functions which are to be determined.

Now, applying Itô's formula to (3.49), we obtain

$$\begin{aligned} d\Psi_t = &\Big(\pi_t' + \pi_t\theta\left(A_t^3 u_t - D_t^3 y_t - D_t^4\mathbb{E}\left[y_t\right] - D_t^5 z_t - D_t^6\mathbb{E}\left[z_t\right]\right. \\ &\left. - D_t^7 u_t + \frac{1}{2}\theta z B_t^3 u_t\right)\Big) e^{\theta(x_t+y_t)}dt + \pi_t\theta\left(B_t^3 u_t + z_t\right) e^{\theta(x_t+y_t)}dW_t. \end{aligned} \tag{3.51}$$

Comparing the coefficients of (3.49) and (3.51), we have

$$\begin{cases} \pi_t' + \pi_t\theta\Big(A_t^3 u_t - D_t^3 y_t - D_t^4\mathbb{E}\left[y_t\right] - D_t^5 z_t \\ \quad - D_t^6\mathbb{E}\left[z_t\right] - D_t^7 u_t + \theta z B_t^3 u_t\Big) = 0, \\ \pi_T = \theta, \end{cases} \tag{3.52}$$

and

$$k_t = \pi_t\theta\left(B_t^3 u_t + z_t\right) e^{\theta(x_t+y_t)}. \tag{3.53}$$

Similarly, utilizing Itô's formula to (3.50), we get

$$\begin{cases} \mu_t' + \mu_t\theta\left(A_t^3 u_t - D_t^3 y_t - D_t^4\mathbb{E}\left[y_t\right] - D_t^5 z_t - D_t^6\mathbb{E}\left[z_t\right] - D_t^7 u_t + \theta z B_t^3 u_t\right) e^{\theta(x_t+y_t)} \\ \qquad + D_t^3\Phi_t + D_t^4\mathbb{E}\left[\Phi_t\right] = 0 \\ \mu_0 = 0. \end{cases} \tag{3.54}$$

By substituting (3.49), (3.53) and (3.50) into (3.48), the optimal control, which is fully observed, can be represented in the state feedback form

$$u_t = -\frac{1}{B_t^3 B_t^3\pi_t\theta}\left(A_t^3\pi_t - D_t^7\mu_t + B_t^3\pi_t z_t\right), \tag{3.55}$$

where π_t and μ_t holds (3.52) and (3.54), respectively.

Conclusion

In this thesis, we have investigated stochastic optimal control problems in two different topics. In the first one, we have proved the stochastic maximum principle for a partially observed optimal control problem of forward-backward stochastic differential equations governed by both a family of Teugels martingales and an independent Brownian motion in which the control domain is convex. As an application of the general theory, a partially observed linear-quadratic control problem of McKean–Vlasov type is studied in terms of stochastic filtering. The second one, we have investigated the maximum principle pertaining to risk-sensitive optimal control problems under partial observation, modeled by FBSDEs of the general McKean–Vlasov equations. The control variable consists of two components: a continuous control and an impulse control. The cost functional is an exponential of integral type based on the regularity McKean–Vlasov framework. Moreover, the sufficient conditions of optimality are obtained under certain concavity assumptions. As an application, the main outcomes are used to solve a linear-quadratic risk-sensitive optimal control problem of the regularity McKean–Vlasov type, both under partial and full observation conditions.

Following this study, several perspectives are considered:

- Study the risk-sensitive optimal control problems for systems governed by both a family of Teugels martingales and an independent Brownian motion.
- Explore the risk-sensitive optimal control problems under partial observation, modeled by fully coupled FBSDEs of the general McKean–Vlasov equations, which could have valuable applications in finance.

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