People ′s Democratic Republic

of Algeria Ministry of Higher

Education and Scientific

Research

THIRD CYCLE LMD FORMATION

A Thesis submitted in partial execution of the requirements for the degree of

DOCTOR IN MATHEMATICS

Suggested by

Mohamed Kheider University Biskra

Presented by

Dounia BAHLALI

Titled

Consistent control of stochastic systems with delay

Supervisor: **Pr. Farid CHIGHOUB**

Examination Committee:

Mr. Mouloud CHERFAOUI	**Professor**	**University of Biskra**	**President**
Mr. Farid CHIGHOUB	**Professor**	**University of Biskra**	**Supervisor**
Mr. Salah-Eddine REBIAI	**Professor**	**University of Batna2**	**Examiner**
Mr. Nabil KHELFALLAH	**Professor**	**University of Biskra**	**Examiner**
Mr. Lazhar TAMER	**MCA**	**University of Biskra**	**Examiner**

College year 2024 / 2025

Dedication

For my mother and my father

For my brother Raid

For my sisters

For all my family

For all my friends

For all who supported me

Thank you from my heart

Dounia Bahlali

Acknowledgments

First and foremost, I thank Allah Almighty who enabled me to do this work.

I would like to express my deepest gratitude to my supervisor Pr. Farid Chighoub,

I would like to thank my committee members

Pr. Mouloud Cherfaoui,

Pr. Salah Eddine Rebiai,

Pr, Nabil Khellfallah

Dr. Tamer Lazhar

Thank you for serving as my committee members.

Dounia Bahlali

ملخص

تركز هذه الأطروحة على حل موضوعين بحثيين في سياقات متميزة باستخدام طرق التحكم العشوائي.

يطور الموضوع الأول نظرية تتناول فئة واسعة من مشاكل التحكم العشوائي غير المتسقة مع الزمن والتي تتميز بالمعادلات المتأخرة التفاضلية العشوائية، مما يشير إلى عدم وجود مبدأ بيلمان الأمثل. يتضمن النهج تأطير هذه المشكلات ضمن إطار نظري للعبة والبحث عن استراتيجيات توازن ناش المثالية للعبة الفرعية. بالنسبة لعملية عامة خاضعة للرقابة مع تأخير ووظيفة موضوعية واسعة بشكل معقول، فإننا نوسع معادلة بيلمان القياسية إلى نظام من المعادلات غير الخطية.

يسهل هذا التمديد تحديد كل من استراتيجية التوازن ودالة قيمة التوازن. الأهم من ذلك، لتجسيد قابلية تطبيق النظرية، نتعمق في مثال محدد مثل محفظة التباين المتوسط مع مشكلة النفور من المخاطر المعتمدة مع التأخير.

من خلال توسيع الأسس النظرية، يوفر هذا التحليل رؤى ثاقبة لمعالجة وحل التناقضات الزمنية في مثال عملي.

يدرس الموضوع الثاني استراتيجية توازن الاستثمار وإعادة التأمين/الأعمال التجارية والاستثمار الجديدة لشركات التأمين ذات التباين المتوسط مع النفور من المخاطر المعتمدة على الدولة، ويُسمح لشركات التأمين بشراء إعادة تأمين متناسبة، والحصول على أعمال جديدة والاستثمار في سوق مالية.

الكلمات الرئيسية : عدم تناسق الزمن، متوسط التباين، معادلات بلمان الموسعة ، اعادة التأمين و الاستثمار، شركة التأمين، إستراتيجية التوازن ، معادلات تفاضلية عشوائية مع تأخير .

Résumé

Cette thèse se concentre sur la résolution de deux sujets de recherche dans des contextes distincts en utilisant des méthodes de contrôle stochastique. Le premier sujet développe une théorie traitant d'une large classe de problèmes de contrôle stochastique inconsistant dans le temps, caractérisés par des équations différentielles stochastiques avec retard (EDSD), indiquant l'absence d'un principe d'optimalité de Bellman. L'approche consiste à situer ces problèmes dans un cadre de théorie des jeux et à rechercher des stratégies d'équilibre de Nash parfaites en sous-jeu. Pour un processus général contrôlé avec délai et une fonction objective raisonnablement large, nous étendons l'équation de Bellman standard dans un système d'équations non linéaires. Cette extension facilite la détermination de la stratégie d'équilibre et de la fonction de valeur d'équilibre. Pour illustrer l'applicabilité de la théorie, nous approfondissons un exemple spécifique d'un tel portefeuille à variance moyenne avec un problème d'aversion au risque dépendant de l'état avec un délai. En élargissant les fondements théoriques , cette analyse fournit des informations sur la façon de traiter et de résoudre les inconsistances temporelles dans un exemple pratique. Dans le deuxième thème, qui étudie une stratégie d'équilibre investissement-réassurance/nouvelle entreprise et de placement pour les assureurs à variance moyenne ayant une aversion au risque dépendant de l'État, les assureurs sont autorisés à acheter une réassurance proportionnelle, à acquérir de nouvelles entreprises et à investir sur un marché financier, où l'excédent et le processus de tarification des stocks à risque des assureurs sont supposés suivre un processus de prélèvement géométrique. Sous L'influence de l'entrée/sortie de capitaux liées à la performance sur le processus de richesse de l'investisseur est modélisée par une équation de retard différentiel stochastique (SDDE).

Mots clés: Inconsistance temporelle, variance- moyenne, equations HJB étendues, réassurance et investissement, assureur, stratégie d'équilibre, équation différentielle stochastique avec retard.

Abstract

This thesis focuses on solving two research topics in distinct contexts using stochastic control methods. The first topic develops a theory addressing a broad class of time-inconsistent stochastic control problems characterized by stochastic differential delayed equations (SDDEs), indicating the absence of a Bellman optimality principle. The approach involves framing these problems within a game theoretic framework and seeking subgame perfect Nash equilibrium strategies. For a general controlled process with delay and a reasonably broad objective functional, we extend the standard Bellman equation into a system of nonlinear equations. This extension facilitates the determination of both the equilibrium strategy and the equilibrium value function. Importantly, to exemplify the theory's applicability, we delve into specific example such mean-variance portfolio with state dependent risk aversion problem with delay. By extending the theoretical foundations, this analysis provides insights into addressing and resolving time inconsistencies in a practical example. In the second topic studies an equilibrium investment-reinsurance /new business and investment strategy for mean-variance insurers with state dependent risk aversion, the insurers are allowed to purchase proportional reinsurance, acquire new business and invest in a financial market, where both the surplus and the price process of risky stocks of the insurers are assumed to follow geometric Levy process. Under the influence of performance-related capital inflow/outflow, the wealth process of the investor is modeled by a stochastic differential delay equation (SDDE).

Keys words : Time inconsistency, Mean-Variance, Extended HJB equations, Reinsurance and Investment, Insurer, Equilibrium Strategy, Stochastic Differential Equation With Delay.

Contents

Symbols

In this thesis we use the following symbols:

- S^n : the set of $n \times n$ symmetric real matrices.
- $C^\top$: the transpose of the vector (or matrix) C.
- $\langle \cdot, \cdot \rangle$: the inner product in some Euclidean space.

For any Euclidean space $H = \mathbb{R}^n$, or S^n with Frobenius norm $|\cdot|$, and $p, l, d \in \mathbb{N}$ we let for any $t \in [0, T]$

- $\mathbb{L}^p(\Omega, \mathcal{F}_t, \mathbb{P}; H) = \{\xi : \Omega \to H \mid \xi \text{ is } \mathcal{F}_t - \text{measurable, s.t. } \mathbb{E}[|\xi|^p] < \infty\}$, for any $p \geq 1$.
- $\mathbb{L}^2\left(\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*), \theta; H^l\right) = \Big\{ r(\cdot) : \mathbb{R}^* \to H^l | r(\cdot) = (r_k(\cdot))_{k=1,2,\ldots,l}$ is
$$\mathcal{B}(\mathbb{R}^*) - \text{measurable with } \sum_{k=1}^{l} \int_{\mathbb{R}^*} |r_k(z)|^2 \theta_\alpha^k(dz)\, ds < \infty \Big\}.$$
- $\mathcal{S}_\mathcal{F}^2(t, T; H) = \Big\{ \mathcal{Y}(\cdot) : [t, T] \times \Omega \to H \mid \mathcal{Y}(\cdot) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} - \text{adapted},$
$$s \mapsto \mathcal{Y}(s) \text{ is càdlàg, with } \mathbb{E}\left[\sup_{s \in [t,T]} |\mathcal{Y}(s)|^2 ds\right] < \infty \Big\}.$$
- $\mathcal{C}_\mathcal{F}^2(t, T; H) = \Big\{ \mathcal{Y}(\cdot) : [t, T] \times \Omega \to H \mid \mathcal{Y}(\cdot) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} - \text{adapted},$
$$s \mapsto \mathcal{Y}(s) \text{ is continuous, with } \mathbb{E}\left[\sup_{s \in [t,T]} |\mathcal{Y}(s)|^2 ds\right] < \infty \Big\}.$$
- $L_\mathcal{F}^p(t, T; H) = \Big\{ \mathcal{Y}(\cdot) : [t, T] \times \Omega \to H \mid \mathcal{Y}(\cdot) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} - \text{adapted},$
$$s \mapsto \mathcal{Y}(s), \text{ with } \mathbb{E}\left[\sup_{s \in [t,T]} |\mathcal{Y}(s)|^p ds\right] < \infty \Big\}, \text{for any } p \geq 1.$$
- $\mathcal{L}_\mathcal{F}^2(t, T; H^p) = \Big\{ \mathcal{Y}(\cdot) : [t, T] \times \Omega \to H^p | \mathcal{Y}(\cdot) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} - \text{adapted},$
$$\text{with } \mathbb{E}\left[\int_t^T |\mathcal{Y}(s)|^2 ds\right] < \infty \Big\}.$$
- $\mathcal{L}_{\mathcal{F},p}^2(t, T; H) = \Big\{ \mathcal{Y}(\cdot) : [t, T] \times \Omega \to H | \mathcal{Y}(\cdot) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} - \text{predictable},$
$$\text{with } \mathbb{E}\left[\int_t^T |\mathcal{Y}(s)|^2 ds\right] < \infty \Big\}.$$

Acronyms

- $a.e$: almost everywhere.
- $a.s$: almost surely.
- $e.g$: for example.
- $i.e$: that is.
- SDE: Stochastic differential equations.
- $SDDE$: Stochastic differential delayed equations..
- $BSDE$: Backward stochastic differential equation.
- PDE: Partial differential equation.
- HJB: Hamilton-Bellman-Jacobi
- ODE: Ordinary differential equation.
- $\frac{\partial f}{\partial x}, f_x$: The derivatives with respect to x.
- f_{xx} : The second derivative with respect to x.
- $\mathbb{P}\otimes dt$: The product measure of P with the Lebesgue measure dt on $[0, T]$.
- $W(\cdot)$: Brownian motion.

Introduction

Over the past two decades, numerous scholars have delved into stochastic control problems involving delays and explored their applications in diverse fields such as life science, engineering and financial mathematics. Notable contributors include researchers like [24], [28], [30], [33], [45], [56], [63], [66] and [68]. Stochastic models incorporating delays capture phenomena characterized by past-dependence, where their behavior at the present time t relies not only on the situation at t but also on a finite portion of their past history. These models are often denoted as stochastic differential delayed equations (SDDEs). In the work by Chang et al. [23], a portfolio management problem akin to Merton's type was considered, incorporating a risky asset return linked to the return history. Using the dynamic programming approach, they derived an explicit solution for the CRRA utility case. Shi [66] extended this work to a recursive utility framework. However, there is limited literature addressing the mean-variance portfolio problem with delays. David [28] pioneered the investigation of the optimal investment problem for a single-objective mean-variance scenario under a jump-diffusion delayed system. Employing a sufficient maximum principle on the quadratic loss minimization problem associated with the single-objective mean-variance problem, he obtained an optimal investment strategy in closed-loop form. Shen et al. [68] established two versions of sufficient maximum principles for a stochastic optimal control problem with delay and mean-field characteristics. By applying the second version to the mean-variance portfolio problem with delay, they derived efficient portfolios and efficient frontiers based on solutions to two systems of linear ordinary differential equations. In another work [69], Shen and Zeng explored an optimal investment and reinsurance problem with delay for an insurer under the mean-variance criterion. The game-theoretic approach to addressing time inconsistency, utilizing Nash equilibrium points, has a longstanding history. Notably, when the discount function deviates from exponential discounted utility models lose their time-consistency, meaning they no longer adhere to Bellman's optimality principle. Consequently, the classical dynamic programming approach cannot be directly applied to solve these problems. Given this limitation, there are two primary methods for addressing time inconsistency in non-exponential discounted utility models. The first approach involves considering naive agents, who make decisions without accounting for potential changes in their preferences in the near future. At any given time $t \in [0, T]$, the agent solves the problem as a standard optimal control problem with the initial condition $X(t) = x$. If we assume that the naive agent at time 0 solves the problem, their solution corresponds to the so-called pre-commitment solution. This pre-committed strategy remains optimal as long as the agent can commit to their future behavior at $t = 0$. The second approach involves formulating a time-inconsistent decision problem as a non-cooperative game among different instances of the decision maker at various points in time. Nash equilibrium strategies are then considered to define a new concept of solution for the original problem. Strotz, as referenced in [70], was the first to propose a game-theoretic formulation to address dynamic time-inconsistent optimal decision problems, specifically focusing on the deterministic Ramsey problem, as mentioned in [60]. By introducing the concept of non-commitment and allowing for an infinitesimally small commitment period, Strotz provided a primitive notion of Nash equilibrium strategy. Subsequent research along this line, in both continuous and discrete time, has been conducted by Pollak [58], Phelps and Pollak [57]. Continuing with the game-theoretic approach, Ekland & Lazrak [31] and Marin-Solano & Navas [71] addressed the optimal consumption problem in a deterministic framework where the utility function incorporates a non-exponential discount function. They characterized equilibrium strategies using a value function that must satisfy an "extended HJB equation," a nonlinear differential equation with a non-local term dependent on the global behavior of the solution. In this scenario, each decision at time t is made by a t-agent, representing the controller's incarnation at that time, referred to as a "sophisticated t-agent" in

[71]. Bjork & Murgoci, as referenced in [15], extended this idea to the stochastic setting, where the controlled dynamics are driven by a general class of Markov processes and a general objective function. Yong, in [74], studied a class of time-inconsistent deterministic linear quadratic models by discretizing time and deriving equilibrium controls via a class of Riccati-Volterra equations. In [77], Yong investigated a general time-inconsistent stochastic optimal control problem with discounting, also by discretizing time, and characterized a feedback time-consistent Nash equilibrium control using the "equilibrium HJB equation".

Regarding equilibrium strategies for an optimal consumption-investment problem with a general discount function, Ekeland & Pirvu [32] were the first to investigate Nash equilibrium strategies, where the price process of the risky asset is driven by geometric Brownian motion. They characterize the equilibrium strategies through the solutions of a flow of backward stochastic differential equations (BSDEs) and demonstrate, for a special form of the discount function, that the BSDEs reduce to a system of two ordinary differential equations (ODEs) with a solution. In [77], Yong discussed the case of a time-inconsistent consumption-investment problem under a power utility function.

The aim of this thesis is to investigate a category of stochastic control problems in continuous time characterized by time inconsistency, meaning they lack adherence to a Bellman optimality principle. This property introduces challenges to defining optimality, as a strategy deemed optimal at a specific time and state may not retain optimality when observed from a later date and different state. The approach to addressing this class of time-inconsistent problems employs a game-theoretic framework. Instead of seeking optimal strategies, we focus on identifying subgame perfect Nash equilibrium strategies. This thesis extends the continuous time theory developed in [15] to a general controlled system with delay, building on the foundational concepts established in the earlier work.

In contrast to Bjork et al.[16], we expand the scope of the state-dependent risk aversion mean-variance optimization problem to include delays, where the state system is governed by a stochastic delay differential equation. Our primary focus is on the combination $X(T) + \Theta Y(T)$, referred to as terminal state and everage of terminal wealth where the state-dependent risk aversion is including. By employing stochastic control theory with delays, we derive extended Hamilton-Jacobi-Bellman equations. However, the solution to the mean-variance optimization problem with state-dependent risk aversion is not fully explicit, requiring the construction of an exponential martingale process related to the wealth evolution process. Since the wealth process in this thesis is described by a stochastic delay differential equation, constructing the exponential martingale using the approach in Bjork et al.[16] is not feasible. Consequently, we transform the wealth process $X(t)$ into the combinatorial wealth dynamic $X(t) + \Theta Y(t)$ described by a stochastic differential equation. Subsequently, we construct the exponential martingale over $X(t) + \Theta Y(t)$ and seek the optimal strategy based on historical performance over a the time. Formulating our problem within a game-theoretic framework, we establish general sufficient conditions for Nash equilibrium strategies, in the extended HJB equation sense. This thesis is organized as follows:

- **Chapter 1:** In this introductory chapter, we give a short introduction to stochastic control problem and the principal concepts used in this thesis.

- **Chapter 2:** In this chapter, we develop a theory addressing a broad class of time-inconsistent stochastic control problems characterized by stochastic differential delayed equations (SDDEs), indicating the absence of a Bellman optimality principle. The approach involves framing these problems within a game theoretic framework and seeking subgame perfect Nash equilibrium strategies. For a general controlled process with delay and a reasonably broad objective functional, we extend the standard Bellman equation into a system of nonlinear equations. This extension facilitates the determination of both the equilibrium strategy and the equilibrium value function. Importantly, to exemplify the theory's applicability, we delve into specific example such mean-variance portfolio with state dependent risk aversion problem with delay. This analysis not only extends the theoretical foundations but also provides insights into addressing and resolving time inconsistency in a practical example. It is worth mentioning that the content of this Chapter is the subject of our paper Bahlali et al [5]

- **Chapter 3:** In this chapter, we study the equilibrium investment-reinsurance /new business and investment strategy for mean-variance insurers with state dependent risk aversion, the insurers are allowed to purchase proportional reinsurance, acquire new business and invest in a financial market, where both the surplus and the price process of risky stocks of the insurers are assumed to follow geometric Levy process. Under the consideration of the performance related capital inflow/outflow, the wealth process of the investor is modeled by a stochastic differential delay equation (SDDE). The insurers aim to optimize the mean-variance utility of the combination of terminal wealth and average performance wealth. We formulate the optimal investment and reinsurance mean-variance problem within a game theoretic framework, seeking subgame perfect Nash equilibrium then applying the stochastic control theory with delay. Next by solving the extended HJB equation and constructing the exponential martingale process the explicit- form solutions of the optimal investment-reinsurance strategy and the corresponding equilibrium value function are derived. It is worth mentioning that the content of this third Chapter is the subject of our working paper .

.

Scientific Contributions

Publications based on this thesis

- Bahlali, D., & Chighoub, F. (2024). A general time-inconsistent stochastic optimal control problem with delay. Studies in Engineering and Exact Sciences, 5(2), e6922-e6922. DOI: 10.54021/seesv5n2-115

Conference papers and awards based on this thesis

- Bahlali, D., & Chighoub, F, Extended HJB system for mean variance insurers under Delay, the first National Conference on Pure Applied Mathematics NC-PAM 2021, Laghouat, December 11-12, 2021.

- Bahlali, D., & Chighoub, F, Stochastic Optimal Time Inconsistent with Mean Variance Insurer, La conference Nationale Nouvelle Tendance en Mathematiques Theoriques et Computationnelles qui a lieu à l'univers de Tamanghasset du 8-9 novembre 2022.

- Bahlali, D., & Chighoub, F, Optimal Time Reinsurance-Investment Strategies for mean variance creterion, BAYMAT Conference: Bringing Young Mathematicians Together,Universitat de València and Universitat Politècnica de València on the 9th-11th November 2022,

- Bahlali, D., & Chighoub, F,Time Inconsistency Problem and Hamilton Jacobi Bellman, The First Workshop on Applied Mathematics, 6-8 December 2022 Constantine, Algeria.

- Bahlali, D., & Chighoub, F, General Time Inconsistent Problems In Delayed Stochastic Differential Equations, during the first International Conference on Mathematics Sciences and Applications May 2-3, 2023, Guelma University, Algeria.

- Bahlali, D., & Chighoub, F, Extended HJB Equations For Solving Time Inconsistent Problems in SDDEs, the first National Applied Mathematics Seminar 14-15 May 2023, Biskra, Algeria.

- Bahlali, D., & Chighoub, F, Time-Inconsistent Investment And Consumption Strategies Under A General Discount Function in 7th International Conference on Mathematics, 11-13 July 2023, Fatih Sultan Mehmet Vakif University, Istanbul, Turky.

- Bahlali, D., & Chighoub, F, Hamilton Bellman Jacobi systems in financial market problem, 1st International Conference Mohand Moussaoui on Applied Mathematics and Modeling, Laboratory of Applied Mathematics and Modeling (LMAM) University 8 Mai 1945 Guelma, 19-20th Nov, 2024

- Bahlali, D., & Chighoub, F, SDEs with past dependence, «The First National Symposium on Innovative Mathematics: Retrospective and Perspectives 2024, organized by the Faculty of Exact Sciences and Computer Science at UMAB, held on December 11-12, 2024, in Mostaganem, Algeria »

Chapter 1

Preliminaries in Classical Stochastic Control Problems

1.1 Introduction

In this chapter, in the first section we give a preliminaries in the classical stochastic control problems and the principle methods of solving optimization problems, then, in the second section, we give the major approaches to handle time inconsistency problems.

1.2 Classical Stochastic Control Problems

The mathematical optimization discipline known as optimal control theory concentrates on identifying a control for a dynamical system over time that maximizes an objective function. It possesses multiple applications in operations research, engineering, and science. One of two approaches can be used to determine the optimal control: Pontryagin's Maximum Principle or Bellman's Dynamic Programming

1.2.1 Formulation of the control problem

This subsection presents two mathematical formulations (strong and weak) of stochastic optimum control problems in the subsequent two subsections.

Strong formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, satisfying the usual condition, on which an d-dimensional standard Brownian motion $W(\cdot)$ is defined, denote by U the separable metric space and $T \in (0, +\infty)$ being fixed. Consider the following controlled stochastic differential equation

$$\begin{cases} dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dW(t), \\ X(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$.

The process $u(\cdot)$ is called the control expressing the action of the decision-makers (controller). At any time instant the controller has some information (as specified by the information field $\mathcal{F}_t$) of what has occurred up to that moment, but not able to predict what is going to happen afterwards due to the uncertainty of the system (as a consequence, for any t the controller cannot exercise his/her decision $u(t)$ before the time t really comes). This non anticipative restriction in mathematical terms can be expressed as "$u(\cdot)$ is $\{\mathcal{F}_t\}_{t\geq 0}$ −adapted".

The control $u(\cdot)$ is an element of the set

$$\mathcal{U}[0, T] = \left\{u : [0, T] \times \Omega \to U \text{ such that } u(\cdot) \text{ is } \{\mathcal{F}_t\}_{t\geq 0} - \text{adapted}\right\}.$$

We define the cost functional

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T f(t, X(t), u(t))dt + h(X(T))\right], \tag{1.2}$$

where $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$.

Definition 1.1

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, satisfying the usual conditions and let $W(\cdot)$ be a given d-dimensional standard $\{\mathcal{F}_t\}_{t\in[0.T]}$-Brownian motion. A control $u(\cdot)$ is called an s-admissible control, and $(x(\cdot), u(\cdot))$ an s-admissible pair, if

1. $u(\cdot) \in \mathcal{U}[0, T]$;

2. $x(\cdot)$ *is the unique solution of equation (1.1)*

3. $f(\cdot, X(\cdot), u(\cdot)) \in \mathcal{L}^1_{\mathcal{F}}(0, T; \mathbb{R})$ *and* $h(X(T)) \in \mathbb{L}^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

We denote by $\mathcal{U}^s_{ad}[0, T]$ the set of all admissible controls. Our stochastic optimal control problem under strong formulation can be stated as follows:

Problem 1.1

Minimize (1.2) over $\mathcal{U}^s_{ad}[0, T]$. The goal is to find $\hat{u}(\cdot) \in \mathcal{U}^s_{ad}[0, T]$ (if it ever exists), such that

$$J(\hat{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}^s_{ad}[0,T]} J(u(\cdot)). \tag{1.3}$$

Any $\hat{u}(\cdot) \in \mathcal{U}_{ad}^{s}[0,T]$ satisfying (1.3) is called an s-optimal control. The corresponding state process $\hat{X}(\cdot)$ and the state-control pair $\left(\hat{X}(\cdot), \hat{u}(\cdot)\right)$ are called an s-optimal state process and an s-optimal pair, respectively.

Weak formulation

We note that in the strong formulation the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ along with the Brownian motion $W(\cdot)$ are all fixed, however it is not the case in the weak formulation, where we consider them as a parts of the control.

Definition 1.2

A 6-tuple $\pi = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W(\cdot), u(\cdot)\right)$ is called a w-admissible control, and $(X(\cdot), u(\cdot))$ a w-admissible pair, if

1. $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *is a filtered probability space satisfying the usual conditions*;
2. $W(\cdot)$ *is a d-dimensional standard Brownian motion defined on* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$;
3. $u(\cdot)$ *is an* $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted process on* $(\Omega, \mathcal{F}, \mathbb{P})$ *taking values in* U;
4. $X(\cdot)$ *is the unique solution of equation (1.1) on* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *under* $u(\cdot)$ *and some prescribed state constraints are satisfied*;
5. $f(\cdot, X(\cdot), u(\cdot)) \in \mathcal{L}_{\mathcal{F}}^{1}(0,T;\mathbb{R})$ *and* $h(X(T)) \in \mathbb{L}^{1}(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.
6. *The spaces* $\mathcal{L}_{\mathcal{F}}^{1}(0,T;\mathbb{R})$ *and* $\mathbb{L}^{1}(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ *are defined on the given filtered probability space* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *associated with the 6-tuple* π.
7. *The set of all w-admissible controls is denoted by* $\mathcal{U}_{ad}^{w}[0,T]$.
8. *Sometimes, might write* $u(\cdot) \in \mathcal{U}_{ad}^{w}[0,T]$ *instead of* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W(\cdot), u(\cdot)\right) \in \mathcal{U}_{ad}^{w}[0,T]$.

The following is a statement of our stochastic optimum control issue under weak formulation:

Problem 1.2

The objective is to minimize the cost functional given by equation (1.2) over the set *of admissible controls* $\mathcal{U}_{ad}^{w}[0,T]$. *Namely, one seeks* $\hat{\pi}(\cdot) \in \mathcal{U}_{ad}^{w}[0,T]$ *such that*

$$J(\hat{\pi}(\cdot)) = \inf_{\pi(\cdot)\in\mathcal{U}_{ad}^{w}[0,T]} J(\pi(\cdot)).$$

1.2.2 Methods to solving optimal control problem

Pontryagin's maximal principle and Bellman's dynamic programming method are two major approaches for studying an optimal control.

Dynamic Programming Method

In this subsection, we study an approach to solving optimal control problems, namely, the method of dynamic programming. Dynamic programming, originated by R. Bellman [11] in the early 1950's, is a mathematical method for making a sequence of interrelated decisions. It is applicable to a wide range of optimization issues, including optimal control issues. When applied to optimal controls, the fundamental idea is to take a family of optimal control problems with various initial times and states and use the so-called Hamilton-Jacobi-Bellman equation (abbreviated HJB), a nonlinear first-order (in the deterministic case) or second-order (in the stochastic case) partial differential equation, to establish relationships between these problems. By maximizing or minimizing the Hamiltonian or generalized Hamiltonian involved in the HJB equation, one can obtain an optimal feedback control if the equation can be solved analytically or numerically. The so-called verification technique is this. pointing out that a full family of problems (with various initial times and states) are truly solved by this method.

The Bellman principle Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, satisfying the usual conditions, $T > 0$ a finite time, and W a d-dimensional Brownian motion defined on the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$.

We consider the state stochastic differential equation

$$dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), s \in [0,T]. \tag{1.4}$$

The control $u = u(s)_{0\leq s\leq T}$ is a progressively measurable process valued in the control subset $U \subset \mathbb{R}^k$, satisfies a square integrability condition. We denote by A the set of control processes u. The Borelian functions $b : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}^n$ and $\sigma : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}^{n\times d}$ satisfying, for some constant $C > 0$ the following conditions:

$$|b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| \leq C|x-y|, \tag{1.5}$$

$$|b(t,x,u)| + |\sigma(t,x,u)| \leq C\left[1+|x|\right]. \tag{1.6}$$

Under (1.5) and (1.6) the SDE (1.4) has a unique solution x.

The cost functional associated with (1.4) is the following:

$$J(t,x,u) = \mathbb{E}_{t,x}\left[\int_t^T f(s,X(s),u(s))ds + h(X(T))\right], \tag{1.7}$$

where $\mathbb{E}_{t,x}$ is the expectation operator conditional on $X(t) = x$, $f : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$, be given functions, we assume that

$$|f(t,x,u)| + |h(x)| \leq C\left[1+|x|^2\right], \tag{1.8}$$

for some constant C. The quadratic growth condition (1.8), ensure that J is well defined.

The objective is to minimize the cost functional

$$V(t,x) = \inf_{u\in U} J(t,x,u), \text{ for } (t,x) \in [0,T]\times\mathbb{R}^n, \tag{1.9}$$

which is called the value function of the problem (1.1) and (1.2). In the theory of stochastic control, dynamic programming is a key idea. We offer a variant of the stochastic Bellman's principle of optimality.

Theorem 1.1

Let $(t,x) \in [0,T]\times \mathbb{R}^n$ be given. Then we have

$$V(t,x) = \inf_{u\in U}\mathbb{E}_{t,x}\left[\int_t^{t+h} f(s,X(s),u(s))ds + V(t+h,X(t+h))\right], \text{ for } t \leq t+h \leq T. \tag{1.10}$$

Proof: The dynamic programming principle is demonstrated by Yong and Zhou [76]. ∎

The Hamilton-Jacobi-Bellman equation The HJB equation is the infinitesimal version of the dynamic programming principle. It is derived under smoothness assumptions on the value function. We define the generalized Hamiltonian $\forall(t,x,u,p,P) \in [0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n\times\mathbb{R}^{n\times d}$

$$G(t,x,u,p,P) = \frac{1}{2}tr\left(\sigma(t,x,u)\sigma(t,x,u)^\top P + b(t,x,u)^\top p + f(t,x,u),\right) \tag{1.11}$$

We also need to introduce the second-order infinitesimal generator $\mathcal{L}^u$ associated to the diffusion x with control u

$$\mathcal{L}^u\varphi(t,x) = b(t,x,u).D_x\varphi(t,x) + \frac{1}{2}tr\left(\sigma(t,x,u)\sigma(t,x,u)^\top D_x^2\varphi(t,x)\right). \tag{1.12}$$

The classical HJB equation associated to the stochastic control problem (1.9) is

$$-V_t(t,x) - \inf_{u\in A}\left[\mathcal{L}^u V(t,x) + f(t,x,u)\right] = 0, \text{on } [0,T]\times\mathbb{R}^n. \tag{1.13}$$

We give sufficient conditions which enable to conclude that the smooth solution of the HJB equation coincides with the value function this is the so-called verification result.

Theorem 1.2

Let W be a $C^{1,2}([0,T],\mathbb{R}^n)\cap C([0,T],\mathbb{R}^n)$ function. Assume that f and h are quadratic growth, i.e. there is a constant C such that

$$|f(t,x,u)| + |h(x)| \le C\left[1+|x|^2\right], \text{ for all } (t,x,u)\in[0,T)\times\mathbb{R}^n\times U.$$

1. *Suppose that $W(T,.)\le h$, and*

$$W_t(t,x) + G(t,x,W(t,x),D_xW(t,x),D_x^2W(t,x)) \ge 0, \tag{1.14}$$

on $[0,T)\times\mathbb{R}^n$, then $W\le V$ on $[0,T)\times\mathbb{R}^n$.

2. *Assume further that $W(T,\cdot)=h$, and there exists a minimizer $\hat{u}(t,x)$ of $\mathcal{L}^uV(t,x)+f(t,x,u)$, such that*

$$\begin{aligned} 0 &= W_t(t,x) + G(t,x,W(t,x),D_xW(t,x),D_x^2W(t,x)) \\ &= W_t(t,x) + \mathcal{L}^{\hat{u}(t,x)}W(t,x) + f(t,x,u), \end{aligned} \tag{1.15}$$

the stochastic differential equation

$$dX(s) = b(s,X(s),\hat{u}(s,x))ds + \sigma(s,X(s),\hat{u}(s,x))dW(s),$$

defines a unique solution $X(t)$ for each given initial data $X(t) = x$, and the process $\hat{u}(s,x)$ is a well-defined control process in U. Then $W = V$, and $\hat{u}$ is an optimal Markov control process.

Proof: The proof of this verification theorem is found in the book by Yong and Zhou [76]. ■

1.3 Time-inconsistent problem

In a typical dynamic programming issue situation, a controller just needs to choose his current course of action if he wants to maximize an objective function by selecting the optimal plan. This is due to the dynamic programming principle, also referred to as Belman's optimality principle, which makes the assumption that the controller's subsequent iterations will resolve the lingering issues from today and behave optimally in the future. In many cases, however, the DPP does not hold, therefore an optimal control selected at any initial pair (of time and state) may not continue to be optimal over time. In these situations, the controller's subsequent incarnations might behave as opponents of their current self by having altered tastes or preferences or choosing to base their actions on other goal functions. Dynamic inconsistency is the term for the aforementioned conundrum, which economists have long noted and researched, particularly in relation to non-exponential (or hyperbolic) type discount functions. In [70] Strotz demonstrated that when a discount function was added to consumption plans, a person can initially choose one plan but then switch to another. With the exception of exponential discount functions, this would be true for a wide variety of discount functions. The majority of literature, however, uses exponential discounting by default because none of the other types could provide clear solutions. A hyperbolic discount function would be more realistic, according to experimental research findings that contradict this assumption. For instance, Loewenstein and Prelec [48] suggest that the discount rates for the near future are significantly lower than the discount rates for the time farther in the future. Another significant example of time inconsistent problems is the mean-variance optimization problems, which were introduced by Markowitz [55], in addition to the non-exponential discounted utility maximization.

The mean-variance criterion's concept is that it quantifies risk using variance, enabling decision makers to determine their acceptable risk level and then maximize return. However, the iterated expectation property is absent from the mean-variance criterion since the objective functional contains a non-linear function of the expectation. Consequently, multi-period and continuous-time mean-variance problems are inconsistent with time.

1.3.1 Approaches to handle time inconsistency

Given the inapplicability of standard DPP on these problems, there are three approaches of handling various forms of time inconsistency in optimal control problems.

Pre-committed optimal strategies

One possibility is to investigate the pre-committed problem: we fix one initial point, such as $(0, x_0)$, and then try to find the control process $\bar{u}(.)$ that optimizes $J(0, x_0, .)$. We then simply ignore the fact that at a later points in time like as $(s, X(s; 0, x_0, \bar{u}(.)))$ the control $\bar{u}(.)$ will not be optimal for the functional $J(s, X(s; \bar{u}(.)), .)$. Kydland and Prescott [35] argue that a pre-committed strategy may be economically meaningful in certain cases. In the context of MV optimization problem, pre-committed optimal solution have been widely investigated in different situations. [62] is probably the first paper that studies a pre-committed MV model in a continuous-time setting (although he only considers one single stock with a constant risk-free rate), followed by [6]. In a discrete-time setting, [52] developed an embedding approach to change the originally time-inconsistent MV problem into a stochastic LQ control problem. This approach was extended in [82], together with an indefinite stochastic linear– quadratic control approach, to the continuous-time case. Further extensions and modifications are carried out in, among many others, [51] and [14]. Markowitz's problem with transaction cost has recently solved in [26]. For general mean field control problems, Andersson & Djehiche [4] and Li [47] proposed a mean field type stochastic maximum principle to characterize "pre-commited" optimal control when both the state dynamics and the cost functional are of a mean-field type. The linear-quadratic optimal control problem for mean-field SDEs has been investigated by Yong [74]. The maximum principle for a jump-diffusion mean-field model have been studied in Shen and Siu [67].

Game theoretic approach

We use the game theoretic approach to handle the time inconsistency in the identical viewpoint as Ekeland et al. [31] and Bjork and Murgoci [15]. Let us briefly explain the game perspective that we will consider as follows:

- We consider a game with one player at every point t in the interval $[0, T)$. This player corresponds to the incarnation of the controller on instant t and referred to "player t".
- The $t-th$ player can control the scheme just at time t by taking his/her policy $u(t, \cdot) : \Omega \to \mathbb{R}^m$.
- A control process $u(\cdot)$ is then viewed as a complete explanation of the selected strategies of all players in the game.
- The reward to the player t is specified by the functional $J(t, \xi; u(\cdot))$.

The dynamic optimality approach

The dynamically optimum strategy is an alternate method for the mean-variance portfolio selection problem that was put forth by Pedersen and Peskir [49]. Although Karnam et al. [22] have analogous work, this is an innovative technique to treating time inconsistencies. Although it differs from the subgame perfect equilibrium method, the strategy proposed by Pedersen & Peskir [49] is time-consistent in that it is independent of the initial time and initial state variable. The behavior of an optimizer who continuously reassesses his position and solves an unlimited number of issues in an instantaneously optimal manner is also described by their policy, which is intuitive and formalizes a very simple approach to time inconsistency. The continuous variant of the naive individual proposed by [58] is comparable to the dynamically optimum individual. The dynamically optimal investor is the reincarnation of the precommitted investor at each time t. This is because at time t, he adopts the same strategy as the time-t precommitted investor, disregards his past and future, and immediately departs from it by dressing like the time t+ precommitted investor. Powell [59] notes that the dynamically optimal approach shares similarities with the well-known methods of repeated optimization along a rolling horizon for engineering optimization problems with an infinite time horizon, such as the receding horizon procedure or the model predictive control known as rolling horizon procedures.

CHAPTER 2

A General Time-Inconsistent Stochastic Optimal Control Problem with Delay

2.1 Introduction

In this chapter, we develop a theory addressing a broad class of time-inconsistent stochastic control problems characterized by stochastic differential delayed equations (SDDEs). Importantly, to exemplify the theory's applicability, we delve into specific example such mean-variance portfolio with state dependent risk aversion problem with delay.

2.2 Model and problem formulations

Throughout this chapter $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is a filtered complete probability space on which a one dimensional standard Brownian motion W is defined, whose natural filtration is $(\mathcal{F}_t)_{t\in[0,T]}$ such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets and $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed finite time horizon $T > 0$. Given a closed subset $U \subset \mathbb{R}$, let $b : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$, $\widetilde{b} : [0,T] \times \mathbb{R}^2 \times U \to \mathbb{R}$ and $\sigma : [0,T] \times \mathbb{R}^2 \times U \to \mathbb{R}$, be a deterministic functions. We consider on the time interval $[0,T]$ the following controlled SDDE

$$\begin{cases} dX^{\xi,\pi}(s) = & \left\{ b\left(s, X^{\xi,\pi}(s), Y^{\xi,\pi}(s), Z^{\xi,\pi}(s)\right) + \widetilde{b}\left(s, X^{\xi,\pi}(s), Y^{\xi,\pi}(s), \pi(s)\right) \right\} ds \\ & +\sigma\left(s, X^{\xi,\pi}(s), Y^{\xi,\pi}(s), \pi(s)\right) dW(s), \ s \in [0,T], \\ X^{\xi,\pi}(s) = & \xi(s), \ s \in [-\delta, 0], \end{cases} \tag{2.1}$$

where $\pi : [0,T] \times \Omega \to U$ represents the control process, ξ is the initial path, $Y(s) = \int_{-\delta}^{0} e^{\lambda\tau} X(s+\tau)\, d\tau$ and $Z(s) = X(s-\delta)$ are some functionals of the path segments $\{X(s+\tau)\}_{\tau\in[-\delta,0]}$, $\lambda \in \mathbb{R}$ is a given averaging parameter and $\delta > 0$ is a fixed delay. As time evolves, we need to consider the following controlled stochastic differential equation with delay starting from the situation $(t, \xi) \in [0,T] \times \mathcal{C}([-\delta, 0]; \mathbb{R})$

$$\begin{cases} dX^{\xi,\pi}(s) = & \left\{b\left(s, X^{\xi,\pi}(s), Y^{\xi,\pi}(s), Z^{\xi,\pi}(s)\right) + \widetilde{b}\left(s, X^{\xi,\pi}(s), Y^{\xi,\pi}(s), \pi(s)\right)\right\} ds \\ & +\sigma\left(t, X^{\xi,\pi}(s), Y^{\xi,\pi}(s), \pi(s)\right) dW(s), \ s \in [t,T] \\ X^{\xi,\pi}(s) = & \xi(t-s), \ s \in [t-\delta, t]. \end{cases} \tag{2.2}$$

For any initial state $(t,\xi) \to [0,T] \times \mathcal{C}([-\delta,0];\mathbb{R})$, in order to measure the performance of a control process π we introduce the following payoff functional by

$$\bar{J}(t,\xi,\pi) = \mathbb{E}_{t,\xi}\left[F\left(\xi, X^{\xi,\pi}(T) + \Theta Y^{\xi,\pi}(T)\right)\right] + G\left(\xi, \mathbb{E}_{t,\xi}\left[X^{\xi,\pi}(T) + \Theta Y^{\xi,\pi}(T)\right]\right), \tag{2.3}$$

where, $\mathbb{E}_{t,\xi}[\cdot]$ is the conditional expectation, given that the initial path of X is ξ, $F : \mathbb{R}^3 \to \mathbb{R}$, $G : \mathbb{R}^3 \to \mathbb{R}$ are a deterministic functions, $\Theta \in \mathbb{R}$ is the weight between $X(T)$ and $Y(T)$. The terms in the payoff functional $\bar{J}$ are unconventional. Specifically, $G(\xi, \mathbb{E}_{t,\xi}[X(T) + \Theta Y(T)])$ which can be motivated by the variance term in a mean-variance portfolio problem with delay model [68].

Definition 2.1 (*Admissible control*)

An admissible control π over $[t,T]$ is a U-valued measurable $(\mathcal{F}_s)_{s\in[t,T]}$ adapted process such that

1. *For each initial state $(t,\xi(t))$, the SDDE (2.2) admits unique strong solution denoted by X such that $X \in \mathcal{S}^2_{\mathcal{F}}(t,T,\mathbb{R})$.*
2. $\mathbb{E}_{t,\xi}\left[F\left(\xi, X^{\xi,\pi}(T) + \Theta Y^{\xi,\pi}(T)\right)\right] + G\left(\xi, \mathbb{E}_{t,\xi}\left[X^{\xi,\pi}(T) + \Theta Y^{\xi,\pi}(T)\right]\right) < \infty.$

Remark 2.1

In the rest of this chapter we denote by $\mathcal{U}^F[t,T]$ the set of all admissible control over $[t,T]$.

The stochastic optimal control problem can be stated as follows:

Problem 2.1 (*N*)

For any given initial pair $(t,\xi) \in [0,T] \times \mathcal{C}([-\delta,0];\mathbb{R})$, find a $\hat{\pi} \in \mathcal{U}^F[t,T]$ such that

$$\bar{J}(t,\xi,\hat{\pi}) = \min_{\pi\in\mathcal{U}[t,T]} \bar{J}(t,\xi,\pi). \tag{2.4}$$

Remark 2.2

For given initial state $(t, \xi(t))$, any admissible strategy $\hat{\pi}$ satisfying (2.4) is called a pre-commitment optimal solution to Problem 2.1 at $(t, \xi(t))$.

In general, the above control problem is infinite-dimensional since the objective function may depend on the initial path in a complicated way. Inspired by [45] to make the problem finite-dimensional, it is required that the objective function depends only on the initial path ξ through the following two functionals

$$x = \xi(0), \;\; y = \int_{-\delta}^{0} e^{\lambda\tau} \xi(\tau)\, d\tau. \tag{2.5}$$

Consequently, in the sequel, we will work with a new objective function which is, by hypothesis, only depending on x and y instead of the whole initial path. More precisely, we assume that

$$\begin{aligned} \bar{J}(t, \xi, \pi) \;&= \bar{J}(t, x, y, \pi) \\ &= \mathbb{E}_{t,x,y}\left[F\left(x, y, X^{x,y,\pi}(T) + \Theta Y^{x,y,\pi}(T)\right)\right] \\ &+ G\left(x, y, \mathbb{E}_{t,x,y}\left[X^{x,y,\pi}(T) + \Theta Y^{x,y,\pi}(T)\right]\right), \end{aligned} \tag{2.6}$$

where $\mathbb{E}_{t,x,y}[\cdot] = \mathbb{E}[\cdot \mid X(t) = x, Y(t) = y]$.

2.3 Optimal time-consistent solution

The dynamic optimization problem (2.6) exhibits time inconsistency due to the non-linear term in the objective functional J, which depends on the combination of terminal wealth and average performance wealth. Recognizing the significance of time consistency for rational decision-making, our objective throughout this study is to characterize the optimal time-consistent solution (also referred to as equilibrium) to Problem 2.1. To achieve this goal, we will leverage the extended Hamilton-Jacobi-Bellman (HJB) equations, following a methodology similar to Bjork et al. [15]. It's worth noting that in [15], the authors assumed that the state variable is controlled through general stochastic differential equations without delay. Therefore, we must adapt their definition of equilibriums and their extended HJB equations to align with our specific problem, which involves past dependence. To articulate the definition of feedback equilibriums, we first need to introduce the class of admissible feedback controls (also known as control laws in [15]).

Definition 2.2

An admissible feedback strategy is a map $\pi : [0,T] \times \mathbb{R}^2 \to U$ such that, for any $(t,\xi) \in [0,T] \times \mathcal{C}([-\delta,0];\mathbb{R})$, the SDDE

$$\begin{cases} dX^{\xi,\pi}(s) = & \left\{ b\left(s, X^{\xi,\pi}(s), Y^{\xi,\pi}(s), Z^{\xi,\pi}(s)\right) + \widetilde{b}(s, X^{\xi,\pi}(s), Y^{\xi,\pi}(s), \pi(s) \right\} ds \\ & +\sigma\left(s, X^{\xi,\pi}(s), Y^{\xi,\pi}(s), \pi(s)\right) dW(s), \; s \in [0,T], \\ X^{\xi,\pi}(s) = & \xi(s) = x, \; y = \int_{-\delta}^{0} e^{\lambda\tau}\xi(\tau)\, d\tau, \end{cases} \tag{2.7}$$

has a unique strong solution denoted by X such that $X \in \mathcal{S}^2_{\mathcal{F}}(t,T,\mathbb{R})$.

We denote by $\mathcal{U}^F[t,T]$ the set of all admissible feedback control. In addition, we will sometimes use the notation $\pi(t)$ instead of $\pi(t,x,y)$.

Remark 2.3

It is crucial to note that our assumption entails that the feedback controls are independent of z. This assumption can be broadly understood through the representation (2.6), which stipulates that the objective function J is solely dependent on x and y.

We refer the readers to [15] and [32] for the the intuition behind this definition.

Definition 2.3 (*Feedback Equilibrium control*)

An admissible feedback control $\hat{\pi} \in \mathcal{U}^F[0,T]$ is a feedback equilibrium control if the following condition holds

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ J(t,x,y;\hat{\pi}) - J(t,x,y;\pi^{\varepsilon}) \right\} \geq 0, \tag{2.8}$$

where for any $\varepsilon \in [0, T-t]$,

$$\pi^{\varepsilon}(s,x,y) = \begin{cases} \pi(s,x,y) & \text{for } (s,x,y) \in [t,t+\varepsilon] \times \mathbb{R}^2, \\ \hat{\pi}(s,x,y) & \text{for } (s,x,y) \in [t+\varepsilon,T] \times \mathbb{R}^2. \end{cases} \tag{2.9}$$

The deterministic function $V : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$,

$$V(t,x,y) = J(t,x,y,\hat{\pi}), \tag{2.10}$$

*is called the equilibrium value function of the Problem (**N**).*

Remark 2.4

We remark also that the equilibrium value function V do not depend on z, which is due to (2.6).

Before giving the extended HJB equations and the associated verification theorem for equilibriums, we define the infinitesimal generator associated to our model [33]. For any feedback control $\pi \in \mathcal{U}^F[t,T]$ the operator $\mathcal{A}^\pi$ is defined for any $\phi \in C^{1,2,1}\left([0,T]\times\mathbb{R}^2\right)$ as follows

$$\begin{aligned}
&\mathcal{A}^\pi \phi(t,x,y) \\
&= \frac{\partial \phi}{\partial t}(t,x,y) + \frac{\partial \phi}{\partial x}(t,x,y)\left\{b(t,x,y,z) + \widetilde{b}(t,x,y,\pi)\right\} \\
&+ \frac{\partial \phi}{\partial y}(t,x,y)\left\{x - e^{-\delta\lambda} z - \lambda y\right\} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x,y)\,\sigma^2(t,x,y,z,\pi).
\end{aligned} \tag{2.11}$$

Inspired from [15], we formulate the extended HJB equations as follows,
$\forall (t,x,y) \in [0,T]\times\mathbb{R}^2$, we have

$$\begin{cases}
\sup_\pi \{\mathcal{A}^\pi V(t,x,y) - \mathcal{A}^\pi f(t,x,y,x,y) + \mathcal{A}^\pi f^{x,y}(t,x,y) \\
\qquad -\mathcal{A}^\pi (G \diamond g)(t,x,y) + \mathcal{H}^\pi g(t,x,y)\} = 0, \\
\mathcal{A}^{\hat{\pi}} f^{x_1,y_1}(t,x,y) = 0, \\
\mathcal{A}^{\hat{\pi}} g(t,x,y) = 0, \\
V(T,x,y) = F(x,y,x+\Theta y) + G(x,y,x+\Theta y), \\
f^{x_1,y_1}(T,x,y) = F(x_1,y_1,x+\Theta y), \\
g(T,x,y) = x + \Theta y.
\end{cases} \tag{2.12}$$

Where V, g, $f^{x_1,y_1} \in C^{1,2,1}\left([0,T]\times\mathbb{R}^2\right)$ and $f \in C^{1,2,1,2,1}\left([0,T]\times\mathbb{R}^4\right)$ are deterministic functions, with $\hat{\pi}$ is the feedback control which realizes the supermum in the V-equation i.e. $\forall (t,x,y) \in [0,T]\times\mathbb{R}^2$

$$\begin{aligned}
\hat{\pi}(t) = \arg\sup_\pi \{&\mathcal{A}^\pi V(t,x,y) - \mathcal{A}^\pi f(t,x,y,x,y) + \mathcal{A}^\pi f^{x,y}(t,x,y) \\
&-\mathcal{A}^\pi (G \diamond g)(t,x,y) + \mathcal{H}^\pi g(t,x,y)\},
\end{aligned} \tag{2.13}$$

1. The function $f^{x_1,y_1}$is defined by

$$f^{x_1,y_1}(t,x,y) = f(t,x,y,x_1,y_1),\ \forall (t,x,y,x_1,y_1) \in [0,T]\times\mathbb{R}^4.$$

2. The function $G \diamond g$ is defined by

$$(G \diamond g)(t,x,y) = G(x,y,g(t,x,y)),\ \forall (t,x,y) \in [0,T]\times\mathbb{R}^2.$$

3. For any $\pi \in U$

$$\mathcal{H}^{\pi} g\left(t,x,y\right) = \frac{\partial G}{\partial g}\left(x,y,g\left(t,x,y\right)\right)\mathcal{A}^{\pi} g\left(t,x,y\right), \ \forall \left(t,x,y\right) \in \left[0,T\right] \times \mathbb{R}^2.$$

Remark 2.5

In this context, the operator $\mathcal{A}^{\pi}$ exclusively operates on variables within parentheses. Consequently, the expression $\mathcal{A}^{\pi} f\left(t,x,y,x,y\right)$ is interpreted as $\mathcal{A}^{\pi} h\left(t,x,y\right)$, where h is defined by $h(t,x,y) = f\left(t,x,y,x,y\right)$. For the expression $\mathcal{A}^{\pi} f^{x_1,y_1}\left(t,x,y\right)$, the operator does not act on the uppercase index (x_1,y_1), considering it as a fixed parameter. Similarly, in the expression $\mathcal{A}^{\pi} f^{x,y}\left(t,x,y\right)$, the operator solely affects the variables (t,x,y) within the parentheses and does not operate on the uppercase index (x,y). In cases where F is independent of (x_1,y_1) and there is no G term, the problem simplifies to a standard time-consistent problem. The terms $f^{x,y}\left(t,x,y\right) + f\left(t,x,y,x,y\right)$ in the $V-$equation cancel out, leading the system to reduce to the standard Bellman equation

$$\begin{cases} \mathcal{A}^{\pi} V\left(t,x,y\right) & = 0, \\ V\left(T,x,y\right) & = F\left(x + \Theta y\right). \end{cases} \tag{2.14}$$

2.4 Extended HJB equations and verification theorem

Theorem 2.6

We assume that there exist four functions V, f^{x_1,y_1}, f and g which have the following properties conditions

1. *V, f^{x_1,y_1}, f and g do not depend on z.*

2. *V, f^{x_1,y_1} and g solve the extended HJB system (2.12).*

3. *V, f^{x_1,y_1}, $g \in C_p^{1,2,1}\left(\left[0,T\right] \times \mathbb{R}^2\right)$ and $f \in C_p^{1,2,1,2,1}\left(\left[0,T\right] \times \mathbb{R}^4\right)$.*

4. *$\hat{\pi} \in \mathcal{U}^F\left[t,T\right]$ realizes the supermum in the V-equation.*

Then, $\hat{\pi}$ is an equilibrium control law and V is the corresponding equilibrium value *function i.e*

$$V(t,x,y) = \mathbb{E}_{t,x,y}\left[F\left(x,y,X^{\hat{\pi}}(T)+\Theta Y^{\hat{\pi}}(T)\right)\right]+G\left(x,y,\mathbb{E}_{t,x,y}\left[X^{\hat{\pi}}(T)+\Theta Y^{\hat{\pi}}(T)\right]\right). \tag{2.15}$$

Furthermore, g and f have the following Probabilistic representations

$$f(t,x,y,x_1,y_1) = \mathbb{E}_{t,x,y}\left[F\left(x_1,y_1,X^{\hat{\pi}}(T)+\Theta Y^{\hat{\pi}}(T)\right)\right], \tag{2.16}$$

and

$$g(t,x,y) = \mathbb{E}_{t,x,y}\left[X^{\hat{\pi}}(T)+\Theta Y^{\hat{\pi}}(T)\right]. \tag{2.17}$$

From this, it follows that

$$V(t,x,y) = f(t,x,y,x,y)+G(x,y,g(t,x,y)). \tag{2.18}$$

Proof: We suppose that V, g, f^{x_1,y_1} and $\hat{\pi}$ satisfy the conditions in this theorem, we start by showing that g , f^{x_1,y_1} have the Feynman-Kac representation and that V is the equilibrium value function corresponding to $\hat{\pi}$, (i.e. that $V(t,x,y) = J(t,x,y;\hat{\pi})$). Then, we will prove that $\hat{\pi}$ is indeed a feedback equilibrium control.

Step I) To show that g has the interpretation (2.17), we apply the Itô formula See [33] to the process $\tau \to g\left(\tau, X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau)\right)$, we obtain

$$\begin{aligned} dg\left(\tau,X^{\hat{\pi}}(\tau),Y^{\hat{\pi}}(\tau)\right) &= \mathcal{A}^{\hat{\pi}}g\left(\tau,X^{\hat{\pi}}(\tau),Y^{\hat{\pi}}(\tau)\right)d\tau \\ &\quad + \frac{\partial g}{\partial x}\left(\tau,X^{\hat{\pi}}(\tau),Y^{\hat{\pi}}(\tau)\right)\sigma(\tau,X(\tau),Y(\tau),\hat{\pi}(\tau))\,dW(\tau). \end{aligned} \tag{2.19}$$

From the third equation in (2.12) and from the boundary condition for g, it follows that the process $g\left(\tau,X^{\hat{\pi}}(\tau),Y^{\hat{\pi}}(\tau)\right)$ is a martingale, so we obtain our desired representation of g as

$$g(t,x,y) = \mathbb{E}_{t,x,y}\left[X^{\hat{\pi}}(T)+\Theta Y^{\hat{\pi}}(T)\right].$$

Now applying Itô formula to $\tau \to f^{x_1,y_1}\left(\tau,X^{\hat{\pi}}(\tau),Y^{\hat{\pi}}(\tau)\right)$, we obtain that

$$\begin{aligned} df^{x_1,y_1}\left(\tau,X^{\hat{\pi}}(\tau),Y^{\hat{\pi}}(\tau)\right) &= \mathcal{A}f^{x_1,y_1}\left(\tau,X^{\hat{\pi}}(\tau),Y^{\hat{\pi}}(\tau)\right)d\tau \\ &\quad + \frac{\partial f^{x_1,y_1}}{\partial x}\left(\tau,X^{\hat{\pi}}(\tau),Y^{\hat{\pi}}(\tau)\right)\sigma(\tau,X(\tau),Y(\tau),\hat{\pi}(\tau))\,dW(\tau). \end{aligned} \tag{2.20}$$

From the second equation in (2.12) and from the boundary condition for f^{x_1,y_1}, it follows that the process $f^{x_1,y_1}\left(\tau, X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau)\right)$ is a martingale, so we obtain our desired representation of f^{x_1,y_1} as

$$f^{x_1,y_1}(t,x,y) = \mathbb{E}_{t,x,y}\left[F\left(x_1, y_1, X^{\hat{\pi}}(T), Y^{\hat{\pi}}(T)\right)\right].$$

To show that $V(t,x,y) = J(t,x,y;\hat{\pi})$, we use the first equation in (2.12) to obtain

$$\begin{aligned}&\mathcal{A}^{\hat{\pi}}V(t,x,y) - \mathcal{A}^{\hat{\pi}}f(t,x,y,x,y) + \mathcal{A}^{\hat{\pi}}f^{x,y}(t,x,y)\\&-\mathcal{A}^{\hat{\pi}}(G\diamond g)(t,x,y) + \mathcal{H}^{\hat{\pi}}g(t,x,y)\\&= 0.\end{aligned} \tag{2.21}$$

From the second and third equations in (2.12), then (2.21) takes the form

$$\mathcal{A}^{\hat{\pi}}V(t,x,y) - \mathcal{A}^{\hat{\pi}}f(t,x,y,x,y) - \mathcal{A}^{\hat{\pi}}(G\diamond g)(t,x,y) = 0. \tag{2.22}$$

We now apply the Itô formula to the process $V\left(\tau, X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau)\right)$. Integrating and taking expectations we get

$$\begin{aligned}\mathbb{E}_{t,x,y}\left[V\left(T, X^{\hat{\pi}}(T), Y^{\hat{\pi}}(T)\right)\right] &= V(t,x,y)\\&+\mathbb{E}_{t,x,y}\left[\int_t^T \mathcal{A}^{\hat{\pi}}V\left(\tau, X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau)\right)d\tau\right].\end{aligned} \tag{2.23}$$

Using (2.22) thus, we obtain

$$\begin{aligned}&\mathbb{E}_{t,x,y}\left[V\left(T, X^{\hat{\pi}}(T), Y^{\hat{\pi}}(T)\right)\right] - V(t,x,y)\\&= \mathbb{E}_{t,x,y}\left[\left\{\mathcal{A}^{\hat{\pi}}f\left(\tau, X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau), X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau)\right)\right.\right.\\&\left.\left.+\mathcal{A}^{\hat{\pi}}(G\diamond g)\left(\tau, X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau)\right)\right\}d\tau\right].\end{aligned} \tag{2.24}$$

In the same way we obtain

$$\begin{aligned}&\mathbb{E}_{t,x,y}\left[f\left(T, X^{\hat{\pi}}(T), Y^{\hat{\pi}}(T), X^{\hat{\pi}}(T), Y^{\hat{\pi}}(T)\right)\right] - f(t,x,y,x,y)\\&= \mathbb{E}_{t,x,y}\left[\int_t^T \mathcal{A}^{\hat{\pi}}f\left(\tau, X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau), X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau)\right)d\tau\right],\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}_{t,x,y}\left[(G\diamond g)\left(T, X^{\hat{\pi}}(T), Y^{\hat{\pi}}(T)\right)\right] - (G\diamond g)(t,x,y)\\&= \mathbb{E}_{t,x,y}\left[\int_t^T \mathcal{A}^{\hat{\pi}}(G\diamond g)\left(\tau, X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau), X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau)\right)d\tau\right],\end{aligned}$$

using the two later inequalities and the boundary conditions for V, f and g we get

$$V(t,x,y) = \mathbb{E}_{t,x,y}\left[F\left(x,y,X^{\hat{\pi}}(T),Y^{\hat{\pi}}(T)\right)\right] + G\left(x,y,\mathbb{E}_{t,x,y}\left[X^{\hat{\pi}}(T)+\Theta Y^{\hat{\pi}}(T)\right]\right). \tag{2.25}$$

using the two later inequalities and the boundary conditions for V, f and g we get

$$V(t,x,y) = \mathbb{E}_{t,x,y}\left[F\left(x,y,X^{\hat{\pi}}(T),Y^{\hat{\pi}}(T)\right)\right] + G\left(x,y,\mathbb{E}_{t,x,y}\left[X^{\hat{\pi}}(T)+\Theta Y^{\hat{\pi}}(T)\right]\right). \tag{2.26}$$

Step II) The purpose of the second part of the proof is to emphasize that $\hat{\pi}$ is an equilibrium strategy. For any admissible strategy π, we define f^{π} and g^{π} by

$$\begin{aligned} f^{\pi}(t,x,y,x_1,y_1) &= \mathbb{E}_{t,x,y}[x_1,y_1,X^{\pi}(T)+\Theta Y^{\pi}(T)], \\ g^{\pi}(t,x,y) &= \mathbb{E}_{t,x,y}[X^{\pi}(T)+\Theta Y^{\pi}(T)], \end{aligned} \tag{2.27}$$

Noting that $f = f^{\hat{\pi}}$ and $g = g^{\hat{\pi}}$ for $\pi = \hat{\pi}$. For any $\varepsilon > 0$ and for any admissible strategy, we move to construct an admissible strategy as in definition **2.2**.

By Lemma **3.3** in [15] applied to the points t and $t+\varepsilon$, we get

$$\begin{aligned} &J(t,x,y,\pi^{\varepsilon}) \\ &= \mathbb{E}_{t,x,y}\left[J\left(t+\varepsilon, X^{\pi^{\varepsilon}}(t+\varepsilon), Y^{\pi^{\varepsilon}}(t+\varepsilon), \pi^{\varepsilon}\right)\right] \\ &- \left(\mathbb{E}_{t,x,y}\left[f^{\pi^{\varepsilon}}\left(t+\varepsilon, X^{\pi^{\varepsilon}}(t+\varepsilon), Y^{\pi^{\varepsilon}}(t+\varepsilon), t+\varepsilon, X^{\pi^{\varepsilon}}(t+\varepsilon), Y^{\pi^{\varepsilon}}(t+\varepsilon)\right)\right]\right. \\ &\left.-\mathbb{E}_{t,x,y}\left[f^{\pi^{\varepsilon}(.)}(t+\varepsilon, X^{\pi^{\varepsilon}}(t+\varepsilon), Y^{\pi^{\varepsilon}}(t+\varepsilon), t,x,y)\right]\right) \\ &- \left(\mathbb{E}_{t,x,y}\left[G(t+\varepsilon, X^{\pi^{\varepsilon}}(t+\varepsilon), Y^{\pi^{\varepsilon}}(t+\varepsilon), g^{\pi^{\varepsilon}}(t+\varepsilon, X^{\pi^{\varepsilon}}(t+\varepsilon), Y^{\pi^{\varepsilon}}(t+\varepsilon))\right]\right. \\ &\left.-G(t+\varepsilon, x, y, \mathbb{E}_{t,x,y}\left[g^{\pi^{\varepsilon}}(t+\varepsilon, X^{\pi^{\varepsilon}}(t+\varepsilon), Y^{\pi^{\varepsilon}}(t+\varepsilon)\right]\right). \end{aligned} \tag{2.28}$$

It is easy to remark that for any $\varepsilon \in [0, T-t]$

$$\pi^{\varepsilon}(s,x,y) = \begin{cases} \pi(s,x,y) & \text{for } (s,x,y) \in [t,t+\varepsilon] \times \mathbb{R}^2, \\ \hat{\pi}(s,x,y) & \text{for } (s,x,y) \in [t+\varepsilon,T] \times \mathbb{R}^2, \end{cases} \tag{2.29}$$

and by continuity, we have $X^{\pi^{\varepsilon}}(t+\varepsilon) = X^{\pi}(t+\varepsilon)$ and $Y^{\pi^{\varepsilon}}(t+\varepsilon) = Y^{\pi}(t+\varepsilon)$. Then we get that

$$J\left(t+\varepsilon, X^{\pi^{\varepsilon}}(t+\varepsilon), Y^{\pi^{\varepsilon}}(t+\varepsilon), \pi^{\varepsilon}\right) = V\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon)\right), \tag{2.30}$$

and

$$\begin{aligned} &f^{\pi^{\varepsilon}}\left(t+\varepsilon, X^{\pi^{\varepsilon}}(t+\varepsilon), Y^{\pi^{\varepsilon}}(t+\varepsilon), X^{\pi^{\varepsilon}}(t+\varepsilon), Y^{\pi^{\varepsilon}}(t+\varepsilon)\right) \\ &= f\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon), X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon)\right), \\ &f^{\pi^{\varepsilon}}\left(t+\varepsilon, X^{\pi^{\varepsilon}}(t+\varepsilon), Y^{\pi^{\varepsilon}}(t+\varepsilon), x, y\right) = f\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon), x, y\right), \end{aligned} \tag{2.31}$$

and

$$g^{\pi^{\varepsilon}}(t+\varepsilon, X^{\pi^{\varepsilon}}(t+\varepsilon), Y^{\pi^{\varepsilon}}(t+\varepsilon)) = g(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon)).$$

Consequently

$$\begin{aligned}
&J(t,x,y,\pi^{\varepsilon}) \\
&= \mathbb{E}_{t,x,y}\left[V\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon)\right)\right] \\
&-\left(\mathbb{E}_{t,x,y}\left[f^{\pi}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon), t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}\left(t+\varepsilon\right)\right)\right]\right. \\
&\left.-\mathbb{E}_{t,x,y}\left[f^{\pi}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}\left(t+\varepsilon\right), t, x, y\right)\right]\right) \\
&-\left(\mathbb{E}_{t,x,y}\left[G(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}\left(t+\varepsilon\right), g^{\pi}(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}\left(t+\varepsilon\right))\right]\right. \\
&\left.-G\left(t,x,y,\mathbb{E}_{t,x,y}\left[g^{\pi}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}\left(t+\varepsilon\right)\right)\right]\right)\right),
\end{aligned} \tag{2.32}$$

furthermore, including to the extended HJB equation we have that

$$\begin{aligned}
&\mathcal{A}^{\pi}V(t,x,y) - \mathcal{A}^{\pi}f(t,x,y,x,y) + \mathcal{A}^{\pi}f^{x,y}(t,x,y) \\
&-\mathcal{A}^{\pi}(G\circ g)(t,x,y) + \mathcal{H}^{\pi}g(t,x,y) \\
&\leq 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathbb{E}_{t,x,y}\left[V\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon)\right)\right] - V(t,x,y) \\
&-\left(\mathbb{E}_{t,x,y}\left[f^{\pi}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon), X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon)\right)\right]\right. \\
&\left.-f(t,x,y,t,x,y)\right) + \mathbb{E}_{t,x,y}\left[f^{\pi}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon), x, y\right)\right] \\
&-f(t,x,y,x,y) \\
&-\left(\mathbb{E}_{t,x,y}\left[G\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon), g^{\pi}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon)\right)\right)\right]\right. \\
&\left.-G\left(t,x,y,g(t,x,y)\right)\right) + G\left(t,x,y,\mathbb{E}_{t,x,y}\left[g^{\pi}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon)\right)\right]\right) \\
&-G(t,x,y,g(t,x,y)) \\
&\leq o(\varepsilon).
\end{aligned} \tag{2.33}$$

After numerous simplification, we get

$$\begin{aligned}
&V(t,x,y) \\
&\geq \mathbb{E}_{t,x,y}\left[V\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon)\right)\right] \\
&-\mathbb{E}_{t,x,y}\left[f^{\pi}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon), X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon)\right)\right] \\
&+\mathbb{E}_{t,x,y}\left[f^{\pi}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon), x, y\right)\right] \\
&-\mathbb{E}_{t,x,y}\left[G\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon), g^{\pi}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon)\right)\right)\right] \\
&+G\left(t,x,y,\mathbb{E}_{t,x,y}\left[g^{\pi}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), Y^{\pi}(t+\varepsilon)\right)\right]\right) + o(\varepsilon), \\
&= J(t,x,y,\pi^{\varepsilon}) + o(\varepsilon).
\end{aligned} \tag{2.34}$$

We have already proved in the first part that $V(t,x,y) = J(t,x,y,\hat{\pi})$. So, $J(t,x,y,\hat{\pi}) - J(t,x,y,\pi^{\varepsilon}) \geq o(\varepsilon)$, hence

$$\liminf_{\varepsilon \downarrow 0} \left\{ \frac{J(t,x,y,\hat{\pi}) - J(t,x,y,\pi^{\varepsilon})}{\varepsilon} \right\} \geq 0. \tag{2.35}$$

As a result, $\hat{\pi}$ is an equilibrium strategy. ■

Remark 2.7

Given that the infinitesimal generators $\mathcal{A}^{\pi}$ incorporates coefficients from the SDDE (2.7) that depend on z, it follows that the coefficients of the extended Hamilton-Jacobi-Bellman (HJB) system (2.12) are also contingent on z. Consequently, we cannot a priori anticipate solutions to the extended HJB equations to be independent of z in the general case. However, the following theorem outlines necessary conditions on the functions b, $\widetilde{b}$, σ, F, G for verifying condition (1) in the precedent theorem .

Theorem 2.8

If the extended HJB system (2.12) has a solution V, f and g which are independent of z. Then the following conditions have to be verified

$$b\left(t,x,y,z\right) = \alpha\left(t,x,y\right) + z\beta\left(t,x,y\right), \tag{2.36}$$

and that

$$\frac{\partial \hat{\alpha}}{\partial y}\left(t,x,y\right) = e^{\delta\lambda}\beta\frac{\partial \hat{\alpha}}{\partial x}, \tag{2.37}$$

where $\hat{\alpha}\left(t,x,y\right) = \alpha\left(t,x,y\right) - \lambda\beta y$

and

$$\begin{cases} \dfrac{\partial b}{\partial y}\left(t,x,y,z\right) = e^{\delta\lambda}\beta\dfrac{\partial b}{\partial x}\left(t,x,y,z\right), \\ \dfrac{\partial \widetilde{b}}{\partial y}\left(t,x,y,\hat{\pi}\left(t\right)\right) = e^{\delta\lambda}\beta\dfrac{\partial \widetilde{b}}{\partial x}\left(t,x,y,\hat{\pi}\left(t\right)\right), \\ \dfrac{\partial \sigma}{\partial y}\left(t,x,y,\hat{\pi}\left(t\right)\right) = e^{\delta\lambda}\beta\dfrac{\partial \sigma}{\partial x}\left(t,x,y,\hat{\pi}\left(t\right)\right), \\ \Theta\left(\dfrac{\partial F}{\partial y}\left(x,y,x+\Theta y\right) + \dfrac{\partial G}{\partial y}\left(x,y,x+\Theta y\right)\right) \\ = e^{\delta\lambda}\beta\left(\dfrac{\partial F}{\partial x}\left(x,y,x+\Theta y\right) + \dfrac{\partial G}{\partial x}\left(x,y,x+\Theta y\right)\right), \\ \Theta = e^{\delta\lambda}. \end{cases} \tag{2.38}$$

Proof: According to the infinitesimal generator given in (2.11), we get

$$\begin{aligned}
&\mathcal{A}^{\pi} V(t,x,y) \\
&= \frac{\partial V}{\partial t}(t,x,y) + \frac{\partial V}{\partial x}(t,x,y)\left(b(t,x,y,z) + \widetilde{b}(t,x,y,\pi(t))\right) \\
&+ \frac{\partial V}{\partial y}(t,x,y)\left\{x - e^{-\delta\lambda} z - \lambda y\right\} + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,x,y)\,\sigma^2(t,x,y,\pi(t)),
\end{aligned}$$

for f

$$\begin{aligned}
&\mathcal{A}^{\pi} f(t,x,y,x,y) \\
&= \frac{\partial f}{\partial t}(t,x,y,x,y) + \left(\frac{\partial f}{\partial y}(t,x,y,x,y) + \frac{\partial f}{\partial y_1}(t,x,y,x,y)\right)\left\{x - e^{-\delta\lambda} z - \lambda y\right\} \\
&+ \left(\frac{\partial f}{\partial x}(t,x,y,x,y) + \frac{\partial f}{\partial x_1}(t,x,y,x,y)\right)\left(b(t,x,y,z) + \widetilde{b}(t,x,y,\pi(t))\right) \\
&+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(t,x,y,x,y) + \frac{\partial^2 f}{\partial x_1^2}(t,x,y,x,y) + 2\frac{\partial^2 f}{\partial x \partial x_1}(t,x,y,x,y)\right)\sigma^2(t,x,y,\pi(t)),
\end{aligned}$$

for f^{x_1,y_1}

$$\begin{aligned}
&\mathcal{A}^{\pi} f^{x_1,y_1}(t,x,y) \\
&= \frac{\partial f^{x_1,y_1}}{\partial t}(t,x,y) + \frac{\partial f^{x_1,y_1}}{\partial y}(t,x,y)\left\{x - e^{-\delta\lambda} z - \lambda y\right\} \\
&+ \frac{\partial f^{x_1,y_1}}{\partial x}(t,x,y)\left(b(t,x,y,z) + \widetilde{b}(t,x,y,\pi(t))\right) + \frac{1}{2}\frac{\partial^2 f^{x_1,y_1}}{\partial x^2}(t,x,y)\,\sigma^2(t,x,y,\pi(t)),
\end{aligned}$$

for $G \diamond g$,

$$\begin{aligned}
&\mathcal{A}^{\pi}(G \diamond g)(t,x,y) \\
&= \mathcal{A}^{\pi} G(x,y,g(t,x,y)) \\
&= \frac{\partial G}{\partial g}(x,y,g(t,x,y))\frac{\partial g}{\partial t}(t,x,y) \\
&+ \left(\frac{\partial G}{\partial y}(x,y,g(t,x,y)) + \frac{\partial G}{\partial g}(x,y,g(t,x,y))\frac{\partial g}{\partial y}(t,x,y)\right)\left\{x - e^{-\delta\lambda} z - \lambda y\right\} \\
&+ \left(\frac{\partial G}{\partial x}(x,y,g(t,x,y)) + \frac{\partial G}{\partial g}(x,y,g(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\right)\left(b(t,x,y,z) + \widetilde{b}(t,x,y,\pi(t))\right) \\
&+ \frac{1}{2}\left(\frac{\partial^2 G}{\partial x^2}(x,y,g(t,x,y)) + \frac{\partial^2 G}{\partial g^2}(x,y,g(t,x,y))\frac{\partial^2 g}{\partial x^2}(t,x,y)\right. \\
&\left. + \frac{\partial G}{\partial g}(x,y,g(t,x,y))\frac{\partial^2 g}{\partial x^2}(t,x,y) + 2\frac{\partial^2 G}{\partial x \partial g}(x,y,g(t,x,y))\frac{\partial g}{\partial x}(t,x,y)\right)\sigma^2(t,x,y,\pi(t)),
\end{aligned}$$

and

$$
\begin{aligned}
&\mathcal{H}^{\pi} g(t,x,y)\\
&= \frac{\partial G}{\partial g}(x,y,g(t,x,y))\mathcal{A}^{\pi} g(t,x,y)\\
&= \frac{\partial G}{\partial g}(x,y,g(t,x,y))\left(\frac{\partial g}{\partial t}(t,x,y) + \frac{\partial g}{\partial x}(t,x,y)\left(b(t,x,y,z) + \widetilde{b}(t,x,y,\pi(t))\right)\right.\\
&\left. + \frac{\partial g}{\partial y}(t,x,y)\left\{x - e^{-\delta\lambda}z - \lambda y\right\} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x,y)\sigma^2(t,x,y,\pi(t))\right).
\end{aligned}
$$

Next, substituting the above results in the extended HJB equations (2.12), the extended HJB system becomes

$$
\begin{cases}
\frac{\partial V}{\partial t}(t,x,y) + \left(\frac{\partial V}{\partial y}(t,x,y) - \frac{\partial f}{\partial y_1}(t,x,y,x,y) - \frac{\partial G}{\partial y}(t,x,y)\frac{\partial g}{\partial y}(t,x,y)\right)\\
\left\{x - e^{-\delta\lambda}z - \lambda y\right\} + b(t,x,y,z)\\
\left(\frac{\partial V}{\partial x}(t,x,y) - \frac{\partial f}{\partial x_1}(t,x,y,x,y) - \frac{\partial G}{\partial x}(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\right)\\
+\sup_{\pi\in U}\left\{\left(\frac{\partial V}{\partial x}(t,x,y) - \frac{\partial f}{\partial x_1}(t,x,y,x,y) - \frac{\partial G}{\partial x}(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\right)\widetilde{b}(t,x,y,\pi)\right.\\
+\frac{1}{2}\left(\frac{\partial^2 V}{\partial x^2}(t,x,y) - \frac{\partial^2 f}{\partial x_1^2}(t,x,y,x,y) - 2\frac{\partial^2 f}{\partial x\partial x_1}(t,x,y,x,y) - \frac{\partial^2 G}{\partial x^2}(t,x,y)\right.\\
\left.\left.-\frac{\partial^2 G}{\partial g}(t,x,y)\frac{\partial^2 g}{\partial x_1^2}(t,x,y) - 2\frac{\partial^2 G}{\partial x\partial g}(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\right)\sigma^2(t,x,y,\pi(t))\right\} = 0,\\
\\
\frac{\partial f^{x_1,y_1}}{\partial t}(t,x,y) + \frac{\partial f^{x_1,y_1}}{\partial x}(t,x,y)\left(b(t,x,y,z) + \widetilde{b}(t,x,y,\hat{\pi}(t))\right)\\
+\frac{\partial f^{x_1,y_1}}{\partial y}(t,x,y)\left\{x - e^{-\delta\lambda}y - \lambda z\right\} + \frac{1}{2}\frac{\partial^2 f^{x_1,y_1}}{\partial x^2}(t,x,y)\sigma^2(t,x,y,\hat{\pi}(t)) = 0,\\
\\
\frac{\partial g}{\partial t}(t,x,y) + \frac{\partial g}{\partial x}(t,x,y)\left(b(t,x,y,z) + \widetilde{b}(t,x,y,\hat{\pi}(t))\right) + \frac{\partial g}{\partial y}(t,x,y)\\
\left\{x - e^{-\delta\lambda}z - \lambda y\right\} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x,y)\sigma^2(t,x,y,\hat{\pi}(t)) = 0,\\
\\
V(T,x,y) = F(x,y,x+\Theta y) + G(x,y,x+\Theta y),\\
f^{x_1,y_1}(T,x,y) = F(x_1,y_1,x+\Theta y),\\
g(T,x,y) = x + \Theta y.
\end{cases}
\tag{2.39}
$$

We wish to obtain necessary conditions on b, $\widetilde{b}$, σ, F, G, for ensuring that the system of equations (2.39) has a solution independent of z. So, differentiating the equations of the system (2.39) with respect to z we obtain

$$\begin{aligned}
&\left(\frac{\partial V}{\partial y}(t,x,y)-\frac{\partial f}{\partial y_1}(t,x,y,x,y)-\frac{\partial G}{\partial y}(t,x,y)\frac{\partial g}{\partial y}(t,x,y)\right)\\
&=e^{\delta\lambda}\left(\frac{\partial V}{\partial x}(t,x,y)-\frac{\partial f}{\partial x_1}(t,x,y,x,y)-\frac{\partial G}{\partial x}(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\right)\frac{\partial b}{\partial z}(t,x,y,z),\\
&\frac{\partial f}{\partial y}(t,x,y,x,y)=e^{\delta\lambda}\frac{\partial f}{\partial x}(t,x,y,x,y)\frac{\partial b}{\partial z}(t,x,y,z),\\
&\frac{\partial g}{\partial y}(t,x,y)=e^{\delta\lambda}\frac{\partial g}{\partial x}(t,x,y)\frac{\partial b}{\partial z}(t,x,y,z),
\end{aligned}$$

Replacing $\left(\frac{\partial V}{\partial y}(t,x,y)-\frac{\partial f}{\partial y_1}(t,x,y,x,y)-\frac{\partial G}{\partial y}(t,x,y)\frac{\partial g}{\partial y}(t,x,y)\right)$, $\frac{\partial f}{\partial y}(t,x,y,x_1,y_1)$ and $\frac{\partial g}{\partial y}(t,x,y)$ in the equations of the system (2.39) we get

$$\begin{cases}
\frac{\partial V}{\partial t}(t,x,y)+\Big\{e^{\delta\lambda}\left(\frac{\partial V}{\partial x}(t,x,y)-\frac{\partial f}{\partial x_1}(t,x,y,x,y)-\frac{\partial G}{\partial x}(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\right)\\
\frac{\partial b}{\partial z}(t,x,y,z)\left\{x-e^{-\delta\lambda}z-\lambda y\right\}\\
+\left(\frac{\partial V}{\partial x}(t,x,y)-\frac{\partial f}{\partial x_1}(t,x,y)-\frac{\partial G}{\partial x}(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\right)b(t,x,y,z)\\
+\sup_{\pi\in U}\Big\{\left(\frac{\partial V}{\partial x}(t,x,y)-\frac{\partial f}{\partial x_1}(t,x,y)-\frac{\partial G}{\partial x}(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\right)\widetilde{b}(t,x,y,\pi)\\
+\frac{1}{2}\sigma^2(t,x,y,z,\pi(t))\Big(\frac{\partial^2 V}{\partial x^2}(t,x,y)-\frac{\partial^2 f}{\partial x_1^2}(t,x,y)-2\frac{\partial^2 f}{\partial x\partial x_1}(t,x,y)\\
-\frac{\partial^2 G}{\partial x^2}(t,x,y)-\frac{\partial^2 G}{\partial g}(t,x,y).\frac{\partial^2 g}{\partial x_1^2}(t,x,y)-2\frac{\partial^2 G}{\partial x\partial g}(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\Big)\Big\}=0,\\
\\
\frac{\partial f}{\partial t}(t,x,y,x,y)+e^{\delta\lambda}\frac{\partial f}{\partial x}(t,x,y,x,y)\frac{\partial b}{\partial z}(t,x,y,z)\left\{x-e^{-\delta\lambda}y-\lambda z\right\}\\
+\frac{\partial f}{\partial x}(t,x,y,x,y)\left\{b(t,x,y,z)+\widetilde{b}(t,x,y,\hat{\pi}(t))\right\}\\
+\frac{1}{2}\sigma^2(t,x,y,\hat{\pi}(t))\frac{\partial^2 f}{\partial x^2}(t,x,y,x,y)=0,\\
\\
\frac{\partial g}{\partial t}(t,x,y)+e^{\delta\lambda}\frac{\partial g}{\partial x}(t,x,y)\frac{\partial b}{\partial z}(t,x,y,z)\left\{x-e^{-\delta\lambda}z-\lambda y\right\}\\
+\frac{\partial g}{\partial x}(t,x,y)\left\{b(t,x,y,z)+\widetilde{b}(t,x,y,\hat{\pi}(t))\right\}+\frac{1}{2}\sigma^2(t,x,y,\hat{\pi}(t))\frac{\partial^2 g}{\partial x^2}(t,x,y)=0,
\end{cases}\tag{2.40}$$

We take

$$b(t,x,y,z)=\alpha(t,x,y)+z\beta(t,x,y).\tag{2.41}$$

We obtain

$$\begin{cases} \frac{\partial V}{\partial t}(t,x,y) + \left(\frac{\partial V}{\partial x}(t,x,y) - \frac{\partial f}{\partial x_1}(t,x,y,x,y) - \frac{\partial G}{\partial x}(t,x,y)\frac{\partial g}{\partial x}(t,x,y) \right) \\ e^{\delta\lambda}\beta(t,x,y)\{x-\lambda y\} + \alpha(t,x,y) \\ \left(\frac{\partial V}{\partial x}(t,x,y) - \frac{\partial f}{\partial x_1}(t,x,y,x,y) - \frac{\partial G}{\partial x}(t,x,y)\frac{\partial g}{\partial x}(t,x,y) \right) \\ +\sup_{\pi} \left\{ \left(\frac{\partial V}{\partial x}(t,x,y) - \frac{\partial f}{\partial x_1}(t,x,y,x,y) - \frac{\partial G}{\partial x}(t,x,y)\frac{\partial g}{\partial x}(t,x,y) \right) \widetilde{b}(t,x,y,\pi(t)) \right. \\ +\frac{1}{2}\sigma^2(t,x,y,\pi(t)) \left(\frac{\partial^2 V}{\partial x^2}(t,x,y) - \frac{\partial^2 f}{\partial x_1^2}(t,x,y,x,y) - 2\frac{\partial^2 f}{\partial x\partial x_1}(t,x,y,x,y) \right. \\ \left. \left. -\frac{\partial^2 G}{\partial x^2}(t,x,y) - \frac{\partial^2 G}{\partial g}(t,x,y)\frac{\partial^2 g}{\partial x_1^2}(t,x,y) - 2\frac{\partial^2 G}{\partial x\partial g}(t,x,y)\frac{\partial g}{\partial x}(t,x,y) \right) \right\} = 0, \\ \\ \frac{\partial f}{\partial t}(t,x,y,x_1,y_1) + +e^{\delta\lambda}\frac{\partial f}{\partial x}(t,x,y,x_1,y_1)\beta(t,x,y)\{x-\lambda y\} \\ +\left(\alpha(t,x,y) + \widetilde{b}(t,x,y,\hat{\pi}(t)) \right) \frac{\partial f}{\partial x}(t,x,y,x_1,y_1) \\ +\frac{1}{2}\sigma^2(t,x,y,\hat{\pi}(t))\frac{\partial^2 f}{\partial x^2}(t,x,y,x_1,y_1) = 0, \\ \\ \frac{\partial g}{\partial t}(t,x,y) + e^{\delta\lambda}\frac{\partial g}{\partial x}(t,x,y)\beta(t,x,y)\{x-\lambda y\} + \frac{\partial g}{\partial x}(t,x,y) \\ \left(\alpha(t,x,y) + \widetilde{b}(t,x,y,\hat{\pi}(t)) \right) + \frac{1}{2}\sigma^2(t,x,y,\hat{\pi}(t))\frac{\partial^2 g}{\partial x^2}(t,x,y) = 0. \end{cases} \tag{2.42}$$

Which does not contain any z. The last step is to ensure the equalities

$$\begin{array}{l} \left(\frac{\partial V}{\partial y}(t,x,y) - \frac{\partial f}{\partial y_1}(t,x,y,x,y) - \frac{\partial G}{\partial y}(t,x,y)\frac{\partial g}{\partial y}(t,x,y) \right) \\ = e^{\delta\lambda}\left(\frac{\partial V}{\partial x}(t,x,y) - \frac{\partial f}{\partial x}(t,x,y,x,y) - \frac{\partial G}{\partial x}(t,x,y)\frac{\partial g}{\partial x}(t,x,y) \right)\frac{\partial b}{\partial z}(t,x,y,z), \\ \frac{\partial f}{\partial y}(t,x,y,x_1,y_1) = e^{\delta\lambda}\frac{\partial f}{\partial x}(t,x,y,x_1,y_1)\frac{\partial b}{\partial z}(t,x,y,z), \\ \frac{\partial g}{\partial y}(t,x,y) = e^{\delta\lambda}\frac{\partial g}{\partial x}(t,x,y)\frac{\partial b}{\partial z}(t,x,y,z). \end{array} \tag{2.43}$$

If we introduce a new variable $\tilde{y}$ such that

$$\frac{\partial}{\partial \tilde{y}} = \frac{\partial}{\partial y} - e^{\delta\lambda}\beta(t,x,y)\frac{\partial}{\partial x}, \tag{2.44}$$

then (2.43) states that

$$\begin{array}{l} \frac{\partial V}{\partial \tilde{y}}(t,x,y) - \frac{\partial f}{\partial \tilde{y}}(t,x,y,x,y) - \frac{\partial G}{\partial g}(t,x,y)\frac{\partial g}{\partial \tilde{y}}(t,x,y) = 0, \\ \frac{\partial f}{\partial \tilde{y}}(t,x,y,x_1,y_1) = 0\ , \\ \frac{\partial g}{\partial \tilde{y}}(t,x,y) = 0\ . \end{array} \tag{2.45}$$

Hence, V , f and g have to be independents of $\tilde{y}$. Consequently, differentiating the equations in the system (2.39) as well as the terminal conditions of this system, we find that our conditions which are supposed in the beginning must be verified. ■

In the next section we will apply the above theory in a mean variance portfolio problem

2.5 Application in mean-variance portfolio with state dependent risk aversion with delay

In this section, we assume that an investor can invest in a financial market, in which two securities are treated continuously, one of them is a bond with price $P_0(s)$ as time $s \in [0,T]$ governed by

$$\frac{dP_0(s)}{P_0(s)} = r_0(s)ds, \;\; P_0(0) = p_0 > 0, \tag{2.46}$$

where $r_0 : [0,T] \to (0,+\infty)$ represents a deterministic function denoting the risk-free rate. The additional asset, termed as risky stocks, is characterized by its price process P_1 which follows the following stochastic differential equation

$$\frac{dP_1(s)}{P_1(s^-)} = r_1(s)ds + \sigma(s)dW(s), \;\; P_1(0) = p_1 > 0, \tag{2.47}$$

where $r_1 : [0,T] \to (0,+\infty)$ and $\sigma : [0,T] \longrightarrow \mathbb{R}$ represent the appreciation rate and the volatility of the risky stock, respectively. W is a one-dimensional standard Brownian motion.

2.5.1 Wealth process

Starting from an initial $x_0 > 0$ at time 0, a trading strategy is one-dimensional stochastic process denoted by π, which represents the amount invested in the risky stock at time $s \in [0,T]$. The dollar amount invested in the bond at time s is given by $X^{x_0,\pi}(s) - \pi(s)$, where $X^{x_0,\pi}$ is the wealth process associated with the strategy π and the initial capital x_0. So, the evolution of $X^{x_0,\pi}$ can be described as

$$\begin{cases} dX^{x_0,\pi}(s) = & \{X^{x_0,\pi}(s) - \pi(s)\} \dfrac{dP_0(s)}{P_0(s)} + \pi(s)\dfrac{dP_1(s)}{P_1(s^-)}, \;\; s \in [0,T], \\ X^{x_0,\pi}(0) = & x_0. \end{cases} \tag{2.48}$$

Accordingly, the wealth process solves the following SDE

$$\begin{cases} dX^{x_0,\pi}(s) = & \{r_0(s)X^{x_0,\pi}(s) + \rho(s)\pi(s)\}\, ds + \pi(s)\sigma(s)dW(s), \ s \in [0,T], \\ X^{x_0,\pi}(0) = & x_0, \end{cases} \tag{2.49}$$

where $\rho(s) = r_1(s) - r_0(s)$.

Noting that the wealth process of the investor is traditionally formulated as in (2.49) is the stochastic differential equation without delay. We refer readers to [16] for the optimal time-consistent solutions of the above model.

In this formulation, we define a wealth process with delay, influenced by instantaneous capital inflow or outflow from the investor's current wealth, as discussed in [68] and [69]. Still denoting the investor's wealth process by X^π, we introduce the processes $Y^\pi(s) = \int_{-\delta}^{0} e^{\lambda\tau} X^\pi(s+\tau)d\tau$ and $Z^\pi(s) = X^\pi(s-\delta)$, where $\lambda > 0$ is an average parameter and $\delta \in \mathbb{R}$ represents the delay period. The process $Y^\pi(s)$ denotes the average of the wealth over the past period $[s-\delta, s]$, while $Z^\pi(s)$ provides point wise delayed information about the wealth at time $s \in [0,T]$.

We consider a function $h(s, X^\pi(s) - Y^\pi(s), X^\pi(s) - Z^\pi(s))$ representing the amount of capital inflow/outflow. Here, $X^\pi(s) - Y^\pi(s)$ signifies the average performance of the wealth in the delay period $[s-\delta, s]$, and $X^\pi(s) - Z^\pi(s)$ represents the absolute performance of the wealth throughout the delay period $[s-\delta, s]$. Such capital inflow/outflow, linked to the past performance of the wealth, can manifest in various scenarios. For instance, favorable past performance may lead to increased gains, enabling the investor to distribute dividends to stakeholders. Conversely, poor past performance may necessitate seeking additional capital injection to cover losses, ensuring the achievement of the final performance objective. As in [69], taking into account a capital inflow/outflow function h, we suppose that the insurer's wealth process is governed by the following SDDE

$$\begin{cases} dX^\pi(s) = & \{r_0(s)X^\pi(s) + \pi(s)\rho(s) - h(s, X^\pi(s) - Y^\pi(s), X^\pi(s) - Z^\pi(s)))\}\, ds \\ & +\sigma(s)\pi(s)dW(s) \quad \text{, for } s \in [0,T] \\ X^\pi(s) = & \xi(s), \ s \in [-\delta, 0], \ \xi(s) \in \mathcal{C}([-\delta, 0]; \mathbb{R}). \end{cases} \tag{2.50}$$

To make the problem affordable, we assume that h has a linear structure as follows

$$h(s, X^\pi(s) - Y^\pi(s), X^\pi(s) - Z^\pi(s)) = \alpha(s)(X^\pi(s) - Y^\pi(s)) + \beta\left(X^\pi(s) - Z^\pi(s)\right), \tag{2.51}$$

where $\alpha : [0,T] \longrightarrow \mathbb{R}_+$ is a deterministic uniformly bounded function, $\beta \geq 0$ is a constant such that $r_0(s) - \alpha(s) - \beta > 0$. Invoking (2.51) in equation (2.50) we obtain the wealth process should satisfies the following SDDE

$$\begin{cases} dX^\pi(s) = & \{\mu(s)X^\pi(s) + \rho(s)\pi(s) + \alpha Y^\pi(s) + \beta Z^\pi(s)\} \, ds + \sigma(s)\pi(s)dW(s), \; s \in [t,T] \\ X^\pi(s) = & \xi(s-t), \text{ for } s \in [t-\delta, t], \end{cases} \tag{2.52}$$

where $\xi \in \mathcal{C}([-\delta, 0]; \mathbb{R})$ and $\mu(s) = r_0(s) - \alpha(s) - \beta$. According to Lemma 2.1 in [68], for any admissible strategy π, the state equation (2.52) has a unique solution X^π .

2.5.2 Equilibrium investment strategy solution

For any fixed initial state $(t, \xi) \in [0,T] \times \mathcal{C}([-\delta, 0]; \mathbb{R})$, the purpose is to choose an investment strategy π by maximization of the conditional expectation of terminal wealth and average wealth over the period $[t - \delta, T]$, while trying at the same time minimize financial risk. Interpreting risk as the conditional variance. So the optimization problem is therefore to maximize the following utility

$$\begin{aligned} \bar{J}(t, \xi, \pi) &= J(t, x, y, \pi) \\ &= \mathbb{E}_{t,x,y}[X^\pi(T) + \Theta Y^\pi(T)] - \frac{\gamma(x,y)}{2} Var_{t,x,y}[X^\pi(T) + \Theta Y^\pi(T)], \end{aligned} \tag{2.53}$$

where $\mathbb{E}_{t,x,y}[\cdot] = \mathbb{E}[\cdot \mid X(t) = x, \, Y(t) = y]$ and $Var_{t,x,y}[\cdot] = Var[\cdot \mid X(t) = x, \, Y(t) = y]$, subject to $\mathcal{U}^F[0,T]$, where X^π satisfies (2.52), $\Theta \in \mathbb{R}$ is the weight between $X(T)$ and $Y(T)$. As in Björk et al. [16], let the deterministic function

$$\gamma(x,y) = \frac{\gamma}{x + \Theta y}, \tag{2.54}$$

as a state dependent risk aversion where $\gamma > 0$. Consequently, in this subsection, we interest by an objective function which is only depending on x and y instead of the whole initial path. More precisely, we suppose that The mean-variance optimization problem becomes

$$V(t,x,y) = \sup_\pi \left\{ \mathbb{E}_{t,x,y}[X^\pi(T) + \Theta Y^\pi(T)] - \frac{\gamma(x,y)}{2} Var_{t,x,y}[X^\pi(T) + \Theta Y^\pi(T)] \right\}. \tag{2.55}$$

Before formulating the extended HJB equations and the associated verification theorem for the equilibrium, we give firstly the infinitesimal generator corresponding to the above model.

For any feedback strategy π the operator $\mathcal{A}^{\pi}$ is defined for any function $\Phi \in \mathcal{C}^{1,2,1}(\ [0,T] \times \mathbb{R}^2)$ by

$$\begin{aligned}\mathcal{A}^{\pi}\Phi(t,x,y) = & \frac{\partial \Phi}{\partial t}(t,x,y) + \frac{\partial \Phi}{\partial y}(t,x,y)\left\{x - e^{-\delta\lambda}z - \lambda y\right\} \\ & + \frac{\partial \Phi}{\partial x}(t,x,y)\left\{\mu(t)x + \pi(t)\rho(t) + \alpha(t)y + \beta z\right\} \\ & + \frac{1}{2}\frac{\partial^2 \Phi}{\partial x^2}\left(\pi(t)\sigma(t)\right)^2 .\end{aligned} \tag{2.56}$$

By (2.54) , we get the following derivatives $\dfrac{\partial \gamma}{\partial x}(x,y) = -\dfrac{\gamma}{(x+\Theta y)^2}$, $\dfrac{\partial \gamma}{\partial y}(x,y) = -\dfrac{\Theta\gamma}{(x+\Theta y)^2}$ and $\dfrac{\partial^2 \gamma}{\partial x^2}(x,y) = \dfrac{2\gamma}{(x+\Theta y)^3}$, where the value function V which is given by

$$V(t,x,y) = f(t,x,y,x,y) + \frac{\gamma}{2(x+\Theta y)}g^2(t,x,y). \tag{2.57}$$

its derivatives are the following

$$\begin{aligned}\frac{\partial V}{\partial t}(t,x,y) = & \frac{\partial f}{\partial t}(t,x,y,x,y) + \frac{\gamma}{x+\Theta y}g(t,x,y)\frac{\partial g}{\partial t}(t,x,y), \\ \frac{\partial V}{\partial x}(t,x,y) = & \frac{\partial f}{\partial x}(t,x,y,x,y) + \frac{\partial f}{\partial x_1}(t,x,y,x,y) - \frac{\gamma}{2(x+\Theta y)^2}g^2(t,x,y) \\ & + \frac{\gamma}{x+\Theta y}g(t,x,y)\frac{\partial g}{\partial x}(t,x,y), \\ \frac{\partial^2 V}{\partial x^2}(t,x,y) = & \frac{\partial^2 f}{\partial x^2}(t,x,y,x,y) + \frac{\partial^2 f}{\partial x_1^2}(t,x,y,x,y) + 2\frac{\partial^2 f}{\partial x \partial x_1}(t,x,y,x,y) \\ & + \frac{\gamma}{(x+\Theta y)^3}g^2(t,x,y) - 2\frac{\gamma}{(x+\Theta y)^2}g(t,x,y)\frac{\partial g}{\partial x}(t,x,y) \\ & + \frac{\gamma}{x+\Theta y}\frac{\partial^2 g}{\partial x^2}(t,x,y) + \frac{\gamma}{x+\Theta y}g(t,x,y)\frac{\partial^2 g}{\partial x^2}(t,x,y), \\ \frac{\partial V}{\partial y}(t,x,y) = & \frac{\partial f}{\partial y}(t,x,y,x,y) + \frac{\partial f}{\partial y_1}(t,x,y,x,y) + \frac{\gamma}{x+\Theta y}g(t,x,y)\frac{\partial g}{\partial y}(t,x,y) \\ & - \frac{\gamma\Theta}{2(x+\Theta y)^2}g^2(t,x,y),\end{aligned} \tag{2.58}$$

where f and it's derivatives are evaluated at (t,x,y,x,y) while g and it's derivatives are evaluated at (t,x,y). Substituting the above expressions into (2.39) , we get

$$\left\{\begin{array}{l}\frac{\partial f}{\partial t}(t,x,y,x,y)+\frac{\gamma}{x+\Theta y}g(t,x,y)\frac{\partial g}{\partial t}(t,x,y)\\+\sup_{\pi}\left\{\left(\frac{\partial f}{\partial y_1}(t,x,y,x,y)+\frac{\gamma}{x+\Theta y}g(t,x,y)\frac{\partial g}{\partial y}(t,x,y)\right)\left\{x-e^{-\delta\lambda}z-\lambda y\right\}\right.\\+\left(\frac{\partial f}{\partial x}(t,x,y,x,y)+\frac{\gamma}{x+\Theta y}g(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\right)\{\mu(t)x+\pi(t)\rho(t)+\alpha(t)y+\beta z\}\\\left.+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(t,x,y,x,y)+\frac{\gamma}{x+\Theta y}g(t,x,y)\frac{\partial^2 g}{\partial x^2}(t,x,y)\right)(\pi(t)\sigma(t))^2\right\}=0,\\ \\ \frac{\partial f}{\partial t}(t,x,y,x,y)+\frac{\partial f}{\partial y}(t,x,y,x,y)\left\{x-e^{-\delta\lambda}z-\lambda y\right\}\\+\frac{\partial f}{\partial x}(t,x,y,x,y)\{\mu(t)x+\hat{\pi}(t)\rho(t)+\alpha(t)y+\beta z\}\\+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x,y,x,y)(\hat{\pi}(t)\sigma(t))^2=0,\\ \\ \frac{\partial g}{\partial t}(t,x,y)+\frac{\partial g}{\partial y}(t,x,y)\left\{x-e^{-\delta\lambda}z-\lambda y\right\}+\frac{\partial g}{\partial x}(t,x,y)\{\mu(t)x+\hat{\pi}(t)\rho(t)+\alpha(t)y+\beta z\}\\+\frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x,y)(\hat{\pi}(t)\sigma(t))^2=0.\end{array}\right. \tag{2.59}$$

Remark 2.9

For finding explicitly the equilibrium investment strategy we consider the following theorem .

Theorem 2.10

For the mean-variance problem (2.55), we assume that the equilibrium investment strategy is given by

$$\hat{\pi}(t,x,y)=c(t)(x+\Theta y)+k(t), \tag{2.60}$$

$$\left\{\begin{array}{ll}c(t)= & \frac{\rho(t)}{\sigma^2(t)\gamma}\left[e^{\int_t^T -\eta(t)du}+\gamma e^{\int_t^T (A(u)-\eta(u))du}-\gamma\right],\\ k(t)= & \frac{\rho(t)}{\sigma^2(t)}\left[e^{-\int_t^T \eta(t)du}\int_t^T e^{\int_s^T A(u)du}\left(B(s)+C(s)\chi(s)\right)ds-\int_t^T e^{-\int_t^s \eta(u)du}ds\right].\end{array}\right. \tag{2.61}$$

Where $A(t)=\mu(t)+\Theta+\rho(t)c(t), \mu(t)=r_0(s)-\alpha(t)-\beta,\ B(t)=\rho(s)k(t),$ $C(t)=\sigma(t)c(t)\ ,\ \chi(t)=\sigma(t)k(t)$ *and* $\eta(t)=A(t)+C^2(t)$ *.*

Proof: By assuming that $\hat{\pi}(t,x,y) = c(t)(x+\Theta y)+k(t)$, where c and k are a deterministic functions, by substituting the value of $\hat{\pi}(t,x,y)$ into (2.52) , we derive the following wealth process

$$\begin{aligned} dX^{\hat{\pi}}(s) = & \Big\{\mu(s)X^{\hat{\pi}}(s) + \rho(s)c(s)(X^{\hat{\pi}}(s)+\Theta Y^{\hat{\pi}}(s)) + \rho(s)k(s) + \alpha(s)Y^{\hat{\pi}}(s) \\ & +\beta Z^{\hat{\pi}}(s)\Big\}\, ds + \Big\{\sigma(s)c(s)(X^{\hat{\pi}}(s)+\Theta Y^{\hat{\pi}}(s)(s)) + \sigma(s)k(s)\Big\}\, dW(s)\ . \end{aligned} \tag{2.62}$$

Noting that from Theorem 2.8 the following conditions hold

$$\Theta = \beta e^{\delta\lambda}, \qquad \alpha(t) - \lambda\Theta = (r_0(s) - \alpha(t) - \beta + \Theta)\Theta, \tag{2.63}$$

where $\qquad \alpha(t) = \beta e^{\delta\lambda}(\mu(t) + \beta e^{\delta\lambda} + \lambda),\ \mu(t) = r_0(s) - \alpha(t) - \beta$

and $\qquad dY^{\pi}(t) = \Big(X^{\pi}(t) - \lambda Y^{\pi}(t) - e^{-\delta\lambda}Z^{\pi}(t)\Big)\, dt,$

Hence, we get

$$\begin{aligned} d\left(X^{\hat{\pi}}(s)+\Theta Y^{\hat{\pi}}(s)\right) = & \Big\{A(s)\left(X^{\hat{\pi}}(s)+\Theta Y^{\hat{\pi}}(s)\right) + B(t)\Big\}\, ds \\ & + \Big\{C(s)\left(X^{\hat{\pi}}(s)+\Theta Y^{\hat{\pi}}(s)\right) + \chi(t)\Big\}\, dW(s), \end{aligned} \tag{2.64}$$

where $A(t) = \mu(t) + \Theta + \rho(t)c(t)$, $B(t) = \rho(s)k(t)$, $C(t) = \sigma(t)c(t)$ and $\chi(t) = \sigma(t)k(t)$.

Next, we calculate $\mathbb{E}\left[X^{\hat{\pi}}(T)+\Theta Y^{\hat{\pi}}(T)\right]$ and $\mathbb{E}\left[\left(X^{\hat{\pi}}(T)+\Theta Y^{\hat{\pi}}(T)\right)^2\right]$here $X^{\hat{\pi}}(s) = x$ and $Y^{\hat{\pi}}(s) = y$. To calculate those we construct the following exponential martingale

$$d\mathcal{M}(t) = \mathcal{M}(t)\left(\Big\{-A(t) + C^2(t)\Big\}\, dt - C(t)dW(t)\right),$$

this implies that

$$\mathcal{M}(t) = \mathcal{M}(0)\exp\left\{\int_0^t \left(\Big\{-A(s) + \frac{1}{2}C^2(s)\Big\}\, ds + C(s)dW(s)\right)\right\},$$

then

$$\frac{\mathcal{M}(t)}{\mathcal{M}(T)} = \exp\left\{\int_t^T \left(\Big\{A(s) - \frac{1}{2}C^2(s)\Big\}\, ds + C(s)dW(s)\right)\right\}. \tag{2.65}$$

Moving now to apply Itô's formula to $\left(X^{\hat{\pi}}(s)+\Theta Y^{\hat{\pi}}(s)\right)\mathcal{M}(t)$ we get

$$d\left(\left(X^{\hat{\pi}}(t)+\Theta Y^{\hat{\pi}}(t)\right)\mathcal{M}(t)\right) = \mathcal{M}(t)\left\{(B(t) + C(t)\chi(t))\, dt + \chi(t)dW(t)\right\}.$$

Next we take expectations, integrating from t to T on the above equation then rearranging it, we get

$$\begin{aligned} X^{\hat{\pi}}(T)+\Theta Y^{\hat{\pi}}(T) = & \ (x+\Theta y)\frac{\mathcal{M}(t)}{\mathcal{M}(T)} \\ & + \int_t^T \left\{\left(\frac{\mathcal{M}(s)}{\mathcal{M}(T)}\right)\left\{(B\left(s\right) + C\left(s\right)\chi(s))\, ds + \chi(s)dW(s)\right\}\right\}. \end{aligned} \tag{2.66}$$

With $X^{\hat{\pi}}(s) = x$ and $Y^{\hat{\pi}}(s) = y$.

Consequently,

$$\begin{aligned}\mathbb{E}\left[X^{\hat{\pi}}(T) + \Theta Y^{\hat{\pi}}(T)\right] &= P_1(t)(x+\Theta y) + Q_1(t),\\ \mathbb{E}\left[\left(X^{\hat{\pi}}(T) + \Theta Y^{\hat{\pi}}(T)\right)^2\right] &= S(t)\,(x+\Theta y)^2 + P_2(t)(x+\Theta y) + Q_2\,(t)\,,\end{aligned}$$

and noting $\mathbb{E}\left[\dfrac{\mathcal{M}(t)}{\mathcal{M}(T)}\right] = e^{\int_t^T A(u)du}$. So, we obtain

$$P_1(t) = e^{\int_t^T A(u)du}, \tag{2.67}$$

and

$$Q_1(t) = \int_t^T e^{\int_s^T A(u)du}\left(B\,(s) + C\,(s)\,\chi(s)\right)ds. \tag{2.68}$$

By (2.66) we can derive

$$\begin{aligned}&\left(X^{\hat{\pi}}(T) + \Theta Y^{\hat{\pi}}(T)\right)^2\\ &= (x+\Theta y)^2\left(\frac{\mathcal{M}(t)}{\mathcal{M}(T)}\right)^2 + \left(\int_t^T \frac{\mathcal{M}(s)}{\mathcal{M}(T)}\left\{\left(B\,(s) + C\,(s)\,\chi(s)\right)ds + \int_t^T \chi(s)dW(s)\right\}\right)^2\\ &+ 2(x+\Theta y)\frac{\mathcal{M}(t)}{\mathcal{M}(T)}\left(\int_t^T\left\{\frac{\mathcal{M}(s)}{\mathcal{M}(T)}\left\{\left(B\,(s) + C\,(s)\,\chi(s)\right)ds + \chi(s)dW(s)\right\}\right\}\right),\end{aligned}$$

Hence, we get

$$S(t) = e^{\int_t^T \left(A(u) + \left(A(u) + C^2(u)\right)\right)du} = e^{\int_t^T \left(A(u)+\eta(u)\right)du}, \tag{2.69}$$

where $\eta(t) = A(t) + C^2\,(t)\,.$ And

$$\begin{aligned}P_2(t) &= 2\mathbb{E}\left[\frac{\mathcal{M}(t)}{\mathcal{M}(T)}\left(\int_t^T \frac{\mathcal{M}(s)}{\mathcal{M}(T)}\left\{\left(B\,(s) + C\,(s)\,\chi(s)\right)ds + \int_t^T \chi(s)dW(s)\right\}\right)\right]\\ &= 2\int_t^T e^{\int_t^s A(u)du}\, e^{\int_s^T (A(u)+\eta(u))du}\left(B\,(s) + C\,(s)\,\chi(s)\right)ds.\end{aligned} \tag{2.70}$$

And

$$Q_2\,(t) = \mathbb{E}\left[\int_t^T \frac{\mathcal{M}(s)}{\mathcal{M}(T)}\left(B\,(s) + C\,(s)\,\chi(s)\right)ds + \chi(s)dW(s)\right]^2. \tag{2.71}$$

We have already that

$$\begin{aligned}&f(t,x,y,x_1,y_1) \\ &= \mathbb{E}_{t,x,y}\left[\left(X^{\hat{\pi}}(T) + \Theta Y^{\hat{\pi}}(T))\right)\right] - \frac{\gamma}{2(x_1+\Theta y_1)}\mathbb{E}_{t,x,y}\left[\left(X^{\hat{\pi}}(T) + \Theta Y^{\hat{\pi}}(T)\right)^2\right]\\ &= P_1(t)(x+\Theta y) + Q_1(t) - \frac{\gamma}{2(x_1+\Theta y_1)}\left[S(t)\,(x+\Theta y)^2 + P_2(t)(x+\Theta y) + Q_2(t)\right],\end{aligned} \tag{2.72}$$

with

$$g(t,x,y) = \mathbb{E}_{t,x,y}[X^{\hat{\pi}}(T) + \Theta Y^{\hat{\pi}}(T)] = P_1(t)(x+\Theta y) + Q_1(t). \tag{2.73}$$

As $\hat{\pi}$ is the feedback control which realizes the supermum in the V- equation . Let we define

$$\begin{aligned}\Psi(\pi) &= \left(\frac{\partial f}{\partial y_1}(t,x,y,x,y) + \frac{\gamma}{x+\Theta y} g(t,x,y)\frac{\partial g}{\partial y}(t,x,y)\right)\left\{x - e^{-\delta\lambda}z - \lambda y\right\} \\ &+ \left(\frac{\partial f}{\partial x}(t,x,y,x,y) + \frac{\gamma}{x+\Theta y} g(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\right)\{\mu(t)x + \pi(t)\rho(t) + \alpha(t)y + \beta z\} \\ &+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(t,x,y,x,y) + \frac{\gamma}{x+\Theta y} g(t,x,y)\frac{\partial^2 g}{\partial x^2}(t,x,y)\right)(\pi(t,x,y)\sigma(t))^2.\end{aligned} \tag{2.74}$$

Let differentiating the function Ψ with respect π we obtain

$$\begin{aligned}\frac{\partial \Psi}{\partial \pi}(\pi) &= \left(\frac{\partial f}{\partial x}(t,x,y,x,y) + \frac{\gamma}{x+\Theta y} g(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\right)\rho(t) \\ &\quad + \left(\frac{\partial^2 f}{\partial x^2}(t,x,y,x,y) + \frac{\gamma}{x+\Theta y} g(t,x,y)\frac{\partial^2 g}{\partial x^2}(t,x,y)\right)\pi(t)\sigma^2(t).\end{aligned} \tag{2.75}$$

By the first order condition of optimum, we get

$$\begin{aligned}&\left(\frac{\partial f}{\partial x}(t,x,y,x,y) + \frac{\gamma}{x+\Theta y} g(t,x,y)\frac{\partial g}{\partial x}(t,x,y)\right)\rho(t) \\ &+ \left(\frac{\partial^2 f}{\partial x^2}(t,x,y,x,y) + \frac{\gamma}{x+\Theta y} g(t,x,y)\frac{\partial^2 g}{\partial x^2}(t,x,y)\right)\pi(t)\sigma^2(t) \\ &= 0.\end{aligned} \tag{2.76}$$

Which implies that

$$\hat{\pi}(t) = -\frac{\rho(t)}{\sigma^2(t)} \frac{\frac{\partial f}{\partial x}(t,x,y,x,y) + \frac{\gamma}{x+\Theta y} g(t,x,y)\frac{\partial g}{\partial x}(t,x,y)}{\frac{\partial^2 f}{\partial x^2}(t,x,y,x,y) + \frac{\gamma}{x+\Theta y} g(t,x,y)\frac{\partial^2 g}{\partial x^2}(t,x,y)}. \tag{2.77}$$

We move now to calculate the following derivatives

$$\begin{aligned}&\frac{\partial f}{\partial x}(t,x,y,x_1,y_1) = -\frac{\gamma}{(x_1+\Theta y_1)} S(t)(x+\Theta y) + P_1(t) - \frac{\gamma}{2(x_1+\Theta y_1)} P_2(t), \\ &\frac{\partial^2 f}{\partial x^2}(t,x,y,x_1,y_1) = -\frac{\gamma}{(x_1+\Theta y_1)} S(t), \\ &\frac{\partial g}{\partial x}(t,x,y) = P_1(t), \\ &\frac{\partial^2 g}{\partial x^2}(t,x,y) = 0.\end{aligned} \tag{2.78}$$

Consequently, by substituting the derivatives in (2.78) , so, the equilibrium investment strategy is given by

$$\hat{\pi}(t) = -\frac{\rho(t)}{\sigma^2(t)}\left[\frac{\left(P_1(t) - \gamma S(t) + \gamma P_1^2(t)\right)(x+\Theta y)}{-\gamma S(t)} + \frac{Q_1(t)P_1(t) - \frac{1}{2}P_2(t)}{-S(t)}\right]. \tag{2.79}$$

Hence, by comparing with our assumption we find finally the values of the functions c and k as following

$$\begin{cases} c(t) = & -\dfrac{\rho(t)}{\sigma^2(t)}\left[\dfrac{P_1(t) - \gamma S(t) + \gamma P_1^2(t)}{-\gamma S(t)}\right], \\ k(t) = & -\dfrac{\rho(t)}{\sigma^2(t)}\left[\dfrac{Q_1(t)P_1(t) - \dfrac{1}{2}P_2(t)}{-S(t)}\right]. \end{cases} \tag{2.80}$$

Where

$$\begin{aligned} \frac{P_1(t) - \gamma S(t) + \gamma P_1^2(t)}{-\gamma S(t)} &= -\frac{P_1(t)}{\gamma S(t)} - \frac{P_1^2(t)}{S(t)} + 1 \\ &= \frac{-1}{\gamma} e^{\int_t^T -\eta(u)du} - e^{\int_t^T (A(u)-\eta(u))du} + 1 \\ &= \frac{-1}{\gamma}\left[e^{\int_t^T -\eta(u)du} + \gamma e^{\int_t^T (A(u)-\eta(u))du} - \gamma\right], \end{aligned}$$

and

$$\begin{aligned} \frac{Q_1(t)P_1(t) - \frac{1}{2}P_2(t)}{-S(t)} = -\Bigg[& e^{-\int_t^T \eta(u)du} \int_t^T e^{\int_s^T A(u)du} \left(B\left(s\right) + C\left(s\right)\chi(s)\right) ds \\ & - \int_t^T e^{-\int_t^s \eta(u)du} ds. \Bigg] \end{aligned}$$

Thus, the proof is completed. ■

Remark 2.11

The corresponding equilibrium value function is given by

$$\begin{aligned} & V(t,x,y) = f(t,x,y,x,y) + \frac{\gamma(x,y)}{2} g^2(t,x,y) \\ & = P_1(t)\ (x + \Theta y) + Q_1(t) - \frac{\gamma}{2\left(x + \Theta y\right)}\left[S(t)\left(x + \Theta y\right)^2 + P_2(t)(x + \Theta y) + Q_2(t)\right] \\ & + \frac{\gamma}{2\left(x + \Theta y\right)}\left(P_1(t)(x + \Theta y) + Q_1(t)\right)^2. \end{aligned}$$

Remark 2.12

- *From the precedent theorem, it is easy to see that the investment equilibrium strategy does not have completely explicit expression because* $c(t)$ *and* $k(t)$ *satisfy* *integral equations (2.61).*

- *The existence and uniqueness of solutions for non linear integral equations (2.61) will be discussed and proved in the next section .*
- *The equilibrium strategy is obtained by constructing exponential martingale. Theoretically, it is difficult to find the martingale process for the wealth evolution equation with delay . In the case of two conditions (2.63) we obtain the stochastic differential equation corresponding to the terminal wealth* $X^{\pi}(t) + \Theta\, Y^{\pi}(t)$ *,on the basic , we construct the exponential martingale corresponding to the terminal wealth process* $X^{\pi}(t) + \Theta\, Y^{\pi}(t)$ *, where the two conditions (2.63) play a key role in seeking the equilibrium solution .*

2.6 Existence and Uniqueness of solutions for integral equations

In this section, in the following theorem we prove that the system of integral equations (2.61) has a unique global solution.

Theorem 2.13

The system of integral equations (2.61) admits a unique solutions $c(t), k(t) \in \mathcal{C}([0,T])$ *where* $\mathcal{C}([0,T])$ *is the space of continuous functions defined on* $[0,T]$.

Proof: Firstly, we consider the integral equation for c and constructing the following sequence $c_i(t) \in \mathbb{N}, i \in \mathbb{N}$

$$\begin{cases} c_0(t) = & 1, \\ c_i(t) = & \dfrac{\rho(t)}{\sigma^2(t)\gamma}\left[e^{-\int_t^T \eta_{i-1}(u)du} + \gamma e^{\int_t^T (A_{i-1}(u)-\eta_{i-1}(u))du} - \gamma\right], \end{cases} \tag{2.81}$$

where

$$\begin{aligned} A_{i-1}(t) &= \mu(t) + \Theta + \rho(t)c_{i-1}(t), \\ \eta_{i-1}(t) &= \mu(t) + \Theta + \rho(t)c_{i-1}(t) + \sigma^2(t)c_{i-1}^2(t). \end{aligned} \tag{2.82}$$

Next, by the following (i) , (ii) and (iii) three steps, we show that the sequence $\{c_i(t)\}_{i\in\mathbb{N}}$ converge to $c(t)$ in $\mathcal{C}([0,T])$.

(i) Our purpose in this step is proving that $c_i(t)$ is uniformly bounded in $\mathcal{C}([0,T])$. So, noting that

$$\eta_{i-1}(s) - \quad A_{i-1}(s) = \sigma^2(s)c_{i-1}^2(s),$$

and remarking that

$$\eta_{i-1}(s) - \quad A_{i-1}(s) \geq 0,$$

for all $i \in \mathbb{N}$ and for any $s \in [0,T]$, by applying Cauchy-Schwartz inequality hence,

$$\begin{aligned} -\frac{\rho(t)}{\sigma^2(t)\gamma}\gamma &\leq \left[e^{-\displaystyle\int_t^T (A_{i-1}(s)+\eta_{i-1}(s)-A_{i-1}(s))ds} + \gamma e^{\displaystyle\int_t^T (A_{i-1}(s)-\eta_{i-1}(s))ds} - \gamma \right] \\ &\leq \frac{\rho(t)}{\sigma^2(t)\gamma} e^{-\displaystyle\int_t^T A_{i-1}(s)ds}, \end{aligned} \tag{2.83}$$

where $\dfrac{\rho(t)}{\sigma^2(t)\gamma}$ is a deterministic positive and uniformly bounded function while $\rho : [0,T] \to (0,+\infty)$. Thus to prove that $\{c_i(t)\}_{i\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}([0,T])$, we only need to show that $e^{-\int_t^T A_{i-1}(s)ds}$ has upper bond for all $i \in \mathbb{N}$ and for any $t \in [0,T]$. Let $\Delta_i(t) = \rho(t)c_i(t)$, then we arrive at

$$\Delta_i(t) = \kappa(t)\left\{ e^{-\displaystyle\int_t^T (\mu(s)+\Theta+\Delta_{i-1}(s)+\eta_{i-1}(s)-A_{i-1}(s))ds} + \gamma e^{\displaystyle\int_t^T (A_{i-1}(s)-\eta_{i-1}(s))ds} - \gamma \right\}, \tag{2.84}$$

where $\kappa(t) = \rho(t)\dfrac{\rho(t)}{\sigma^2(t)\gamma} \geq 0$. So, we have the inequality

$$-\kappa(t)\gamma \leq \Delta_i(t) \leq \kappa(t) e^{-\displaystyle\int_t^T [\Delta_{i-1}(s)]ds} \leq \kappa(t)e^{\kappa(T-t)\gamma(T-t)} \leq \kappa(t)e^{\kappa(T)\gamma T}, \tag{2.85}$$

for all $i \in \mathbb{N}$ and for $t \in [0,T]$. That is

$$e^{-\displaystyle\int_t^T A_{i-1}(s)ds} = e^{-\left(\displaystyle\int_t^T \mu(s)+\Theta+\rho(s)c_{i-1}(s)\right)ds} \leq Me^{\kappa(T)\gamma T}, \tag{2.86}$$

holds for any $t \in [0,T]$

(ii) We show that $\{c_i(t)\}_{i\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}([0,T])$.Then according to the definition of recursion, it is not difficult to see that $c_i(t)$ is continuously differentiable for all $i \in \mathbb{N}$ and we derive, we get

$$c_i^{'}(t) = \frac{\rho(t)}{\sigma^2(t)\gamma}\left[\eta_{i-1}(t)e^{-\int_t^T \eta_{i-1}(u)du} + \gamma\left(\eta_{i-1}(t) - A_{i-1}(t)\right)e^{\int_t^T (A_{i-1}(u)-\eta_{i-1}(u))du}\right] \tag{2.87}$$
$$+ \frac{\rho'(t)}{\sigma^2(t)\gamma^3}\left[e^{-\int_t^T \eta_{i-1}(u)du} + \gamma e^{\int_t^T (A_{i-1}(u)-\eta_{i-1}(u))du} - \gamma\right]$$
$$- \frac{2\sigma'(t)\rho(t)}{\sigma^3(t)\gamma^3}\left[e^{-\int_t^T \eta_{i-1}(u)du} + \gamma e^{\int_t^T (A_{i-1}(u)-\eta_{i-1}(u))du} - \gamma\right].$$

Thus, because $\{c_i(t)\}_{i\in\mathbb{N}}$ is shown uniformly bounded on $[0,T]$ in the precedent step. Consequently, we conclude that$\left\{c_i^{'}(t)\right\}_{i\in\mathbb{N}}$is uniformly bounded on $[0,T]$.

(iii) In this step, we prove the existence and uniqueness for $c(t)$, for any $s,t \in [0,T]$ and applying the result from step (ii), we obtain

$$|c_i(t) - c_i(s)| = \left|\int_0^1 \frac{d}{du}c_i\left(s + u\left(t-s\right)\right)du\right| = \left|(t-s)\int_0^1 c_i^{'}(s+u\left(t-s\right))du\right| \leq M_0\left(t-s\right), \tag{2.88}$$

Where M_0 is constant independent of $i \in \mathbb{N}$. Hence the sequence $\{c_i(t)\}_{i\in\mathbb{N}}$ is equicontinuous, since we have already proved uniform boundedness in step (i), The Arzela-Ascoli Theorem implies that there exists $c \in [0,T]$, and exists a subsequence$\{c_{i_k}\}$ such that $c_{i_k} \to c$. Taking limit in (2.81) shows c is a solution for the first integral equation in (2.61). So, to prove uniqueness, assume that c, c^* are two solutions to the first integral equation (2.61). Noting that c, c^* are both bounded and that the function $\vartheta(x) = e^x$ is globally Lipchitz on any given bounded set, we can show easily that

$$|c(t) - c^*(t)| \leq M_0 \int_t^T |c(t) - c^*(t)|\, dt, \tag{2.89}$$

The Gronwall inequality now implies that

$$c(t) \equiv c^*(t), \tag{2.90}$$

Now turn to the second integral equation of (2.61) specifically, we interest now by $k(t)$ which is a linear integral (or differential) equation with respect to $k(t)$. According to Standard linear integral equation theory See [34] , it is easy to see that the second integral equation of (2.61) has a unique global solution by fixed point Theorem or Picard iteration method. Finally the proof is completed. ■

Remark 2.14

From the proof above, for any subsequence $\{c_i(t)\}$ there is further subsequence that converges to the same function c which is the solution to the first integral equation of (2.61). In fact, it provides a numerical algorithm for the determination of $c(t)$. Similarly, we have the following iteration schema to compute $k(t)$ numerically

$$\begin{cases} k_0(t) = 1, \\ k_i(t) = \dfrac{\rho(t)}{\sigma^2(t)} \left[e^{-\int_t^T \eta_{i-1}(s)ds} \displaystyle\int_t^T e^{\int_s^T A_{i-1}(u)du} \left(B_{i-1}(s) + C_{i-1}(s)\,\chi_{i-1}(s)\right) ds \right. \\ \qquad \left. - \displaystyle\int_t^T e^{-\int_t^s \eta_{i-1}(u)du} ds \right], \end{cases} \tag{2.91}$$

where $A_{i-1}(s) = r_0(s) - \alpha(s) - \beta + \Theta + \rho(s)c_{i-1}(s)$, $B_{i-1}(s) = \rho(s)k_{i-1}(s)$, $C_{i-1}(s) = \sigma(s)c_{i-1}(s)$, $\chi_{i-1}(s) = \sigma(s)k_{i-1}(s)$ *and* $\eta(t) = A(s) + C_{i-1}^2(s)$.

Equilibrium Reinsurance-Investment Strategies for Mean-Variance Insurers with Delay

3.1 Introduction

In this Chapter, we study an optimal time inconsistent reinsurance-investment problem under mean-variance insurers with state dependent risk aversion for SDDE with jumps-diffusion.

Throughout this chapter $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is a complete filtred probability space such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed finite time horizon $T > 0$. $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions, i.e. the filtration contains all $\mathbb{P}$-null sets and is right continuous. $\mathcal{F}_t$ stands for the information available up to time t and any decision made at time t is based on this information. All stochastic processes in this chapter are assumed to be well defined and adapted processes in this probability space.

3.2 Surplus process and financial market

We consider an insurer whose surplus process (without reinsurance and investment) is described by the following jump-diffusion model:

$$dR(s) = cds + \sigma_0 dW_0(s) - d\left\{\sum_{i=1}^{N(s)} Y_i\right\}, \tag{3.1}$$

which describes an insurer whose surplus process (without reinsurance and investment) where $c > 0$ is the premium rate, σ_0 is a positive constant, $W_0(.)$ is a one dimensional standard Brownien motion, $N(s)$ is a poisson process with intensity $\lambda_N > 0$, representing

the number of claims occurring up time s, Y_i is the size of the $i-th$ claims and $\{Y_i\}_{i\ \in\mathbb{N}-\{0\}}$ are assumed to be independents and identically distributed positive random variables with common distribution $\mathbb{P}_Y$ having finite first and finite second moments $\mu_Y = \int\limits_0^{+\infty} y\mathbb{P}_Y(dy)$ and $\sigma_Y = \int\limits_0^{+\infty} y^2\mathbb{P}_Y(dy)$ respectively. The term $\sigma_0 dW_0(s)$ can be regarded as the uncertainty from the premium income of the insurer. We assume that the premium income c is assumed to be calculated via the expected value principle, where $c = (1+\varkappa)\lambda_N\mu_Y$ with safety loading $\varkappa > 0$. We refer the readers to [81] and [29] their references for more information about the above model . Suppose that the insurer can purchase proportional reinsurance or acquire new business (for example acting as a reinsurer of other insurers, readers can see Bäuerle, N. (2005)[10]) , at each moment in order to contrôl the insurance business risk. Let $\pi_R(s) \geq 0$ the retention level of reinsurance or new business acquired at time $s \in [0,T]$. when $\pi_R(s) \in [0,1]$, it corresponds to a proportional reinsurance cover and shoes that the cedent should divert part of the premium to the reinsurer at the rate of $(1-\pi_R(s))(\theta_0+1)\lambda_N\mu_Y$, where θ_0 is the relative safety loading of the reinsurer satisfying $\theta_0 \geq \varkappa$. Meanwhile, for each claim occurring at time s , the reinsurer pays $100(1-\pi_R(s))\%$ of the claim, while the insurer pays the rest. The case where $\pi_R(s) \in]1,\infty[$ corresponds to acquiring new business. The process $\pi_R(s)$ is called a reinsurance strategy. Incorporation purchasing proportional reinsurance and acquiring new business into the surplus process, then the expression becomes as follows

$$dR^{\pi_R(s)}(s) = \left\{(\varkappa - \theta_0 + (1+\theta_0)\pi_R(s))\,\lambda_N\mu_Y\right\} ds + \sigma_0\pi_R(s)dW_0(s) - \pi_R(s)d\left\{\sum_{i=1}^{\tilde{N}(s)} Y_i\right\}. \tag{3.2}$$

Readers can see [81] for more information about the above model.

Beside purchasing proportional reinsurance or acquiring new business, we assume also that the insurers can invest in a financial market, in which two securities are trated continuously, one of them is a bond with price $P_0(s)$ as time $s \in [0,T]$ governed by

$$\frac{dP_0(s)}{P_0(s)} = r_0(s)ds,\ \ P_0(0) = p_0 > 0, \tag{3.3}$$

where $r_0(.) : [0,T] \to (0,+\infty)$ is a deterministic function which represents the risk-free rate. The other asset is called risky stocks, which price processe $P_1(.)$ satisfy the following stochastic differential equation

$$\frac{dP_1(s)}{P_1(s^-)} = r_1(s)ds + \sigma(s)dW_1(s) + d\left\{\sum_{i=1}^{\tilde{N}(s)} Z_i\right\},\quad P_1(0) = p_1 > 0, \tag{3.4}$$

where $r_1(.) : [0,T] \to (0,+\infty)$ and $\sigma(.) : [0,T] \longrightarrow \mathbb{R}^d$ represent the appreciation rate and the volatility of the risky stock, respectively. $W_1(.)$ is a one-dimensional standard Brownian motion, $\tilde{N}(s)$ representing the number of the jumps of the risky asset's price occurring up time s is a Poisson process with intensity $\lambda_{\tilde{N}} > 0$, Z_i is the size of the $i-th$ jumps amplitude of the risky asset's price and $\{Z_i\}_{i\in\mathbb{N}-\{0\}}$ are assumed to be $i.i.d$ random variables taking values in $[-1,+\infty[$ with common distribution $\mathbb{P}_Z$ having finite first and finite second moments $\mu_Z = \int_{-1}^{\infty} z\mathbb{P}z(dz)$ and $\sigma_Z = \int_{-1}^{\infty} z^2\mathbb{P}z(dz)$, respectively.

3.2.1 Wealth process

Starting from an initial state $x_0 > 0$ at time 0, the insurer is allowed to dynamically purchase proportional reinsurance, acquire new business and invest in the financial market. A trading strategy is a two-dimensional stochastic process $\pi(.) = (\pi_R(.), \pi_I(.))$, where $\pi_R(.) \geq 0$ represents the retention level of reinsurance or new business acquired at time $s \in [0,T]$ and $\pi_I(.)$ represents the amount invested in the risky stock at time $s \in [0,T]$. The dollar amount invested in the bond at s is $X^{x_0,\pi(.)}(s) - \pi_I(s)$, where $X^{x_0,\pi(.)}(.)$ is the wealth process associated with the strategy $\pi(.)$ and the initial capital x_0. Then the evolution of $X^{x_0,\pi(.)}(.)$ can be described as

$$\begin{cases} dX^{x_0,\pi(.)}(s) = & \left\{X^{x_0,\pi(.)}(s) - \pi_I(s)\right\} \dfrac{dP_0(s)}{P_0(s)} + \pi_I(s)\dfrac{dP_1(s)}{P_1(s^-)} + dR^{\pi_R(s)}(s), \text{ for } s \in [0,T], \\ X^{x_0,\pi(.)}(0) = & x_0. \end{cases} \tag{3.5}$$

Accordingly, the wealth process solves the following SDE with jumps, for $s \in [0,T]$,

$$\begin{cases} dX^{x_0,\pi(.)}(s) = & \left\{r_0(s)X^{x_0,\pi(.)}(s) + (\eta - \theta_0 + (1+\theta_0)\pi_R(s))\lambda_N\mu_Y \right. \\ & \left. +\pi_I(s)(r_1(s) - r_0(s))\right\} ds \\ & +\sigma_0\pi_R(s)dW_0(s) + \pi_I(s)\sigma(s)dW_1(s) \\ & -\pi_R(s^-)d\left\{\sum_{i=1}^{N(s)} Y_i\right\} + \pi_I(s^-)d\left\{\sum_{i=1}^{\tilde{N}(s)} Z_i\right\}, \\ X^{x_0,\pi(.)}(0) = & x_0. \end{cases} \tag{3.6}$$

Note that, following [64] the compound Poisson processes $\sum_{i=1}^{N(s)} Y_i$ and $\sum_{i=1}^{\tilde{N}(s)} Z_i$ can also be defined as follows

$$\sum_{i=1}^{N(s)} Y_i = \int_0^s \int_{\mathbb{R}^*} y N_0(dr, dz) \quad \text{and} \quad \sum_{i=1}^{\tilde{N}(s)} Z_i = \int_0^s \int_{\mathbb{R}^*} z N_1(dr, dz), \tag{3.7}$$

where $N_0(.,.)$ and $N_1(.,.)$ are finite poisson are finite poisson random measures on the space $[0,T] \times \mathbb{R}^*$ endwed with its Borel $\sigma-$field $B([0,T]) \otimes B(\mathbb{R}^*))$, with a compensators having the form

$$\nu_0(dy)ds = \lambda_N \mathbb{P}_Y(dy)ds \quad \text{and} \quad \nu_1(dz)ds = \lambda_{\tilde{N}} \mathbb{P}_Z(dz)ds, \qquad \text{respectively.}$$

We use the notations $\tilde{N}_0(dr, dz) = N_0(dr, dz) - \nu_0(dz)dr$ and $\tilde{N}_1(dr, dz) = N_1(dr, dz) - \nu_1(dz)dr$ for the compensated jump martingale random measures of $N_0(dr, dz)$ and $N_1(dr, dz)$, respectively. Obviously, we have

$$\int_{\mathbb{R}^*} y \nu_0(dy)ds = \lambda_N \int_{\mathbb{R}^*} y \mathbb{P}_Y(dy)ds = \lambda_N \mu_Y ds,$$
$$\int_{\mathbb{R}^*} z \nu_1(dz)ds = \lambda_{\tilde{N}} \int_{\mathbb{R}^*} z \mathbb{P}_Z(dz)ds = \lambda_{\tilde{N}} \mu_Z ds.$$

Hence, the dynamics for the wealth process above can be rewrited as

$$\begin{cases} dX^{x_0,\pi(.)}(s) = & \left\{ r_0(s) X^{x_0,\pi(.)}(s) + (\delta + \theta_0 \pi_R(s)) \lambda_N \mu_Y + \rho(s) \pi_I(s) \right\} ds \\ & + \sigma_0 \pi_R(s) dW_0(s) + \pi_I(s) \sigma(s) dW_1(s) \\ & - \pi_R(s^-) \int_0^\infty z \tilde{N}_0(ds, dz) + \pi_I(s^-) \int_{-1}^\infty z \tilde{N}_1(ds, dz), \\ X^{x_0,\pi(.)}(0) = & x_0, \end{cases} \tag{3.8}$$

where $\rho(s) = r_1(s) - r_0(s) + \lambda_{\tilde{N}} \mu_Z$ and $\delta = \eta - \theta_0$.

Remark 3.1

Noting that the wealth process of the insurers is traditionally formulated as in (3.6) *or evantually* (3.8) *that is, the stochastic differential equation without delay. We refer readers to [80] for the optimal time-consistent solutions of the above model.*

Now, we formulate a wealth process with delay, which may be caused by the instantaneous capital inflow into or outflow from the insurers's current wealth, see for example

[68, 69]. Still denoting the wealth process of the insurers by $X^{x_0,\pi(.)}(.)$, we introduce the following processes

$$X_1^{x_0,\pi(.)}(s) = \int_{-\delta}^{0} e^{\lambda\tau} X^{x_0,\pi(.)}(s+\tau)d\tau \quad \text{and} \quad X_2^{x_0,\pi(.)}(s) = X^{\varsigma_0,\pi(.)}(s-\delta). \tag{3.9}$$

Where $\lambda > 0$, is an average parameter and $\delta > 0$ represents the delay periode. $\bar{X}_1^{x_0,\pi(.)}(s)$ represents the average of the wealth process in past period $[s-\delta, s]$ where and $X_2^{x_0,\pi(.)}(s)$ is pointwise delay information of the wealth at time $s \in [0, T]$.

We consider a function $f(s, X^{x_0,\pi(.)}(s) - X_1^{x_0,\pi(.)}(s), X^{x_0,\pi(.)}(s) - X_2^{x_0,\pi(.)})$ which represents the capital inflow/outflow amount, where $X^{x_0,\pi(.)}(s) - X_1^{x_0,\pi(.)}(s)$ implies the average performance of the wealth in the delay period $[s-\delta, s]$ and $X^{x_0,\pi(.)}(s) - X_2^{x_0,\pi(.)}$ gives for the absolute performance of the wealth between throught the delay period $[s-\delta, s]$. Such capital inflow/outflow, which is related to the past performance of the wealth, may be encontoured in various situations. For example, a good past performance may bring the insurer more gain and further the insurer can pay a part of the gain as dividend to stakeholders. Contrarily, a poor past performance forces the insurer to seek further capital injection to cover the loss so that the final performance objective is still achievable.

Arguing as [69], taking into a account a capital inflow/outflow function f, we suppose that the insurer's wealth process is governed by the following SDDE with jump-diffusion

$$\begin{cases} dX^{\varsigma,\pi(.)}(s) & = \Big\{ r_0(s)X^{\varsigma,\pi(.)}(s) + (\delta + \theta_0\pi_R(s))\lambda_N\mu_Y + \rho(s)\pi_I(s) \\ & \quad -f(s, X^{\varsigma,\pi(.)}(s) - X_1^{\varsigma,\pi(.)}(s), X^{\varsigma,\pi(.)}(s) - X_2^{\varsigma,\pi(.)}(s)) \Big\} ds \\ & \quad +\sigma_0\pi_R(s)dW_0(s) + \sigma(s)\pi_I(s)dW_1(s) \\ & \quad -\pi_R(s^-)\int_0^{+\infty} z\tilde{N}_0(ds,dz) + \pi_I(s^-)\int_{-1}^{+\infty} z\tilde{N}_1(ds,dz), \text{ for } s \in [0,T], \\ X^{\varsigma,\pi(.)}(s) & = \varsigma(s), \; s \in [-\delta, 0], \; \varsigma(s) \in \mathcal{C}([-\delta,0];\mathbb{R}). \end{cases} \tag{3.10}$$

Here $X^{\varsigma,\pi(.)}(s) = \varsigma(s)$, for $s \in [-\delta, 0]$, represents the initial path of wealth process on time interval $[-\delta, 0]$. To make the problem affordable, we assume that $f(s, X^{\varsigma,\pi(.)}(s) - X_1^{\varsigma,\pi(.)}(s), X^{\varsigma,\pi(.)}(s) - X_2^{\varsigma,\pi(.)}(s))$ has a linear structure as follows

$$\begin{aligned} & f(s, X^{\varsigma,\pi(.)}(s) - X_1^{\varsigma,\pi(.)}(s), X^{\varsigma,\pi(.)}(s) - X_2^{\varsigma_0,\pi(.)}(s)) \\ & = \alpha(s)(X^{\varsigma,\pi(.)}(s) - X_1^{\varsigma,\pi(.)}(s)) + \beta\left(X^{\varsigma,\pi(.)}(s) - X_2^{\varsigma_0,\pi(.)}(s)\right) \end{aligned} \tag{3.11}$$

where $\alpha(.) : [0,T] \longrightarrow \mathbb{R}_+$ is a deterministic uniformly bounded function, $\beta \geq 0$ is a constant such that $r_0(s) - \alpha(s) - \beta > 0$. Invoking (3.11) in equation (3.10) , we obtain the insurer's wealth process should satisfies the following SDDE with jumps

$$\begin{cases} dX^{\varsigma_0,\pi(.)}(s) = & \Big\{\mu(s)X^{\pi(.)}(s) + (\delta + \theta_0\pi_R(s))\lambda_N\mu_Y + \rho(s)\pi_I(s) + \alpha(s)X_1^{\varsigma,\pi(.)}(s) \\ & +\beta X_2^{\varsigma,\pi(.)}(s)\Big\}\,ds + \sigma_0\pi_R(s)dW_0(s) + \sigma(s)\pi_I(s)dW_1(s) \\ & -\pi_R(s^-)\int\limits_0^{+\infty} z\tilde{N}_0(ds,dz) + \pi_I(s^-)\int\limits_{-1}^{+\infty} z\tilde{N}_1(ds,dz), \\ X^{\varsigma_0,\pi(.)}(s) = & \varsigma(s),\ \varsigma(s) \in \mathcal{C}([-\delta,0];\mathbb{R}), \quad \text{for} \quad s \in [-\delta,0], \end{cases} \tag{3.12}$$

where $\mu(s) = r_0(s) - \alpha(s) - \beta$. As time envolves, we need to consider the controlled jump-diffusion stochastic delay differential equation parameterized by
$(t,\varsigma_0) \in [0,T] \times \mathcal{C}([-\delta,0];\mathbb{R})$,

$$\begin{cases} dX^{\varsigma,\pi(.)}(s) = & \Big\{(\mu(s)X^{\varsigma,\pi(.)}(s) + (\delta + \theta_0\pi_R(s))\lambda_N\mu_Y + \rho(s)\pi_I(s) + \alpha\left(s\right)X_1^{\varsigma,\pi(.)}(s) \\ & +\beta X_2^{\varsigma,\pi(.)}(s)\Big\}\,ds + \sigma_0\pi_R(s)dW_0(s) + \sigma(s)\pi_I(s)dW_1(s) \\ & -\pi_R(s^-)\int\limits_0^{+\infty} z\tilde{N}_0(ds,dz) + \pi_I(s^-)\int\limits_{-1}^{+\infty} z\tilde{N}_1(ds,dz), \\ X^{\varsigma,\pi(.)}(s) = & \varsigma(s-t), \qquad \text{for } s \in [t-\delta,t]. \end{cases} \tag{3.1}$$

According to Lemma (2.1) in [68], since all the parameters are constants, for any $\pi(.) = (\pi_R(.),\pi_I(.)) \in \mathcal{L}^2_{\mathcal{F},p}(t,T,\mathbb{R}_+) \times \mathcal{L}^2_{\mathcal{F},p}(t,T,\mathbb{R})$, the state equation (3.13) has a unique solution $X\left(.\right) \in \mathcal{S}^2_{\mathcal{F}}(t,T,\mathbb{R})$.

Definition 3.1 (*Admissible strategy*)

An admissible strategy $\pi(.) = (\pi_R(.),\pi_I(.))$ *over* $[t,T]$ *is a* $\mathbb{R}^2-$*valued measurable ,* $(\mathcal{F}_s)_{s\in[t,T]}-$*predictible process, such that* $(\pi_R(.),\pi_I(.)) \in \mathcal{L}^2_{\mathcal{F},p}(t,T,\mathbb{R}_+) \times \mathcal{L}^2_{\mathcal{F},p}(t,T,\mathbb{R})$ and $\pi_R(.) \geq 0, \forall s \in [t,T]$.

Remark 3.2

In the rest of this chapter we denote by $\mathcal{R}^L[t,T] \times \Pi^L[t,T]$ *the set of the admissible reinsurance-investment strategies .*

3.3 Mean variance Criterion with state dependent risk aversion

For any fixed initial state $(t, \varsigma(.)) \in [0,T] \times \mathcal{C}([-\delta,0];\mathbb{R})$, the purpose of an insurer is to choose a reinsurance-investment strategy $\pi(.)$ by maximization of the conditional expectation of terminal wealth and average wealth over the period $[t-\delta,T]$, while trying at the same time minimize financial risk. Interpreting risk as the conditional variance. So the optimization problem is therefore to maximize the following utility

$$\bar{J}(t,\varsigma(.),\pi(.)) = \mathbb{E}_{t,\varsigma}[X^{\varsigma,\pi(.)}(T) + \Theta X_1^{\varsigma,\pi(.)}(T)] - \frac{\eta(x,x_1)}{2} Var_{t,\varsigma}[X^{\varsigma,\pi(.)}(T) + \Theta X_1^{\varsigma,\pi(.)}(T)], \tag{3.14}$$

subject to $\mathcal{R}^L[t,T] \times \Pi^L[t,T]$, where $X(.) = X^{\varsigma,\pi(.)}(.)$ satisfies (3.13). Here $\mathbb{E}_{t,\varsigma}[.]$ and $Var_{t,\varsigma}[.]$ are the conditional expectation and conditional variance given that the initial path of $X(.)$ is $\varsigma(.)$, $\Theta \in \mathbb{R}$ is the weight between $X(T)$ and $X_1(T)$. Here, let the deterministic function

$$\eta(x,x_1) = \frac{\eta}{x + \Theta x_1}, \tag{3.15}$$

as a state dependent insurer's risk aversion , $\eta > 0$, and it is more reasonable in finance market than a constant risk aversion. It is known that $\frac{\eta}{x}$ is a suitable choice of the state dependent risk aversion function that is proposed by Bjork et al. [16]. Note that this problem can be viewed as a dynamic optimization problem, since the objective of the insurer updates as state $\varsigma(.)$ changes.

In general, the above control problem is infinite-dimensional since the objective function may depend in the initial path in a complicated way. Inspired by [28, 45] to make the problem finite-dimensional; it is required that the objective function depends only on the initial path $\varsigma(.)$ through the following two functionals

$$x = \varsigma(0), \; x_1 = \int_{-\delta}^{0} e^{\lambda\tau} \varsigma(\tau)\, d\tau. \tag{3.16}$$

Thus, we will work with a new objective function which is by hypothesis, only depending on x and x_1 instead of the whole initial path. More precisely, we assume that

$$\begin{aligned}\bar{J}(t,\varsigma,\pi(.)) &= J(t,x,x_1,\pi(.))\\ &= \mathbb{E}_{t,x,x_1}[X^{\varsigma,\pi(.)}(T)+\Theta_1^{\varsigma,\pi(.)}(T)]-\frac{\eta(x,x_1)}{2}Var_{t,x,x_1}[X^{\varsigma,\pi(.)}(T)+\Theta_1^{\varsigma,\pi(.)}(T)],\end{aligned} \tag{3.17}$$

where

$$\mathbb{E}_{t,x,x_1}[.] = \mathbb{E}[.\mid X(t)=x,\ X_1(t)=x_1] \ \text{ and } \ Var_{t,x,x_1}[.] = Var[.\mid X(t)=x,\ X_1(t)=x_1]. \tag{3.18}$$

3.4 Optimal time-consistent solution

The objective functional $\bar{J}$ involves a non-linear term of expectation on combination of terminal wealth and average performance wealth. The optimization problem (3.17) is clearly time inconsistent and we have a lack of time consistency, and Bellman's stochastic principle of optimality, which says that if a contrôl law is optimal on the full time intervall $[0,T]$ then it is also optimal for any subinterval $[t,T]$ fails in this case, so we can not guarantee the optimality of $\bar{J}$, since lack of time inconsistency, our objective through this study, is to derive feedback equilibriums and the equilibrium value function to problem (3.17) via an extended Hamilton-Jacobi-Bellman (HJB) in a similar way as Bjork et al. [16]. In the aim to state the definition of subgame Nash equilibriums,we should introduce the class of admissible control.

Definition 3.2

An admissible control law is a map $\pi(.)=(\pi_R(.),\pi_I(.)):[0,T]\times\mathbb{R}^2\to\mathbb{R}_+\times\mathbb{R}$, such that the following SDDE with jump-diffusion

$$\begin{cases} dX^{\pi(.)}(s) = & \Big\{\mu(s)X^{\pi(.)}(s)+(\delta+\theta_0\pi_R(s)\lambda_N\mu_Y+\rho(s)\pi_I(s)+\alpha(s)X_1^{\pi(.)}(s)\\ & +\beta X_2^{\pi(.)}(s)\Big\}ds+\sigma_0\pi_R(s)dW_0(s)+\sigma(s)\pi_I(s)dW_1(s)\\ & -\pi_R(s^-)\int_0^{+\infty} z\tilde{N}_0(ds,dz)+\pi_I(s^-)\int_{-1}^{+\infty} z\tilde{N}_1(ds,dz),\\ X^{\pi(.)}(s) = & \varsigma(s)=x, \quad and \quad x_1=\int_{-\delta}^{0}\exp(\lambda\tau)\varsigma(\tau)d\tau \quad , \quad for\ s\in[t-\delta,t]. \end{cases} \tag{3.19}$$

has a unique strong solution denoted by $X(.)=X^{\pi(.)}(.)\in\mathcal{S}^2_{\mathcal{F}}(t,T,\mathbb{R})$

Definition 3.3 (***Equilibrium control***)

An admissible control law $\hat{\pi} = (\hat{\pi}_R, \hat{\pi}_I)$ *is an equilibrium control if the following condition holds, for any* $(\pi_R, \pi_I) \in \mathbb{R}_+ \times \mathbb{R}$

$$\liminf_{\varepsilon \downarrow 0} \left\{ \frac{J(t,x,x_1,\hat{\pi}) - J(t,x,x_1,\pi^\varepsilon)}{\varepsilon} \right\} \geq 0,$$

where for any $\varepsilon \in [0, T-t]$

$$\begin{aligned} \pi^\varepsilon(s,x,x_1) &= (\pi_R^\varepsilon, \pi_I^\varepsilon)(s,x,x_1) \\ &= \begin{cases} (\pi_R, \pi_I)\ (s,x,x_1) & \text{for } (s,x,x_1) \in [t, t+\varepsilon] \times \mathbb{R}^2 \\ (\hat{\pi}_R, \hat{\pi}_I)(s,x,x_1) & \text{for } (s,x,x_1) \in [t+\varepsilon, T] \times \mathbb{R}^2. \end{cases} \end{aligned}$$

The deterministic function $V : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$*, defined by*

$$V(t,x,x_1) = J(t,x,x_1,\hat{\pi}) = \sup_{\pi(.)} J(t,x,x_1,\pi(.))$$

is called the equilibrium value function of the optimization problem (3.17) *.*

In order to solve the optimization problem, we will apply the game theoretic framework as in [15, 16].

3.5 Extended HJB equations and verification theorem

Before formulating the extended HJB equations and the associated verification theorem for equilibriums, we give firstly the infinitesimal generator corresponding to the above model. For any feedback strategy $\pi = (\pi_R, \pi_I) \in \mathcal{R}^L[t,T] \times \Pi^L[t,T]$ the operator is defined for any function $\Phi \in \mathcal{C}^{1,2,1}(\ [0,T] \times \mathbb{R}^2)$ where $\Phi_t, \Phi_x, \Phi_{xx}$ and Φ_{x_1} its derivatives, so, the generator $\mathcal{A}^{(\pi_R,\pi_I)}$ is defined as following:

$$\begin{aligned}\mathcal{A}^{(\pi_R,\pi_I)}\Phi(t,x,x_1) &= \Phi_t(t,x,x_1)+\Phi_{x_1}(t,x,x_1)\left\{x-e^{-\delta\lambda}x_2-\lambda x_1\right\}\\ &+\Phi_x(t,x,x_1)\left\{\mu(t)x+\left[\delta+(1+\theta_0)\,\pi_R(t,x,x_1)\right]\lambda_N\mu_Y\right.\\ &\left.+\pi_I(t,x,x_1)\rho(t)+\alpha(t)x_1+\beta x_2-\pi_I(t)\lambda_{\tilde{N}}\mu_Z\right\}\\ &+\frac{1}{2}\Phi_{xx}(t,x,x_1)\left\{(\pi_R(t,x,x_1)\sigma_0)^2+(\pi_I(t,x,x_1)\sigma(t))^2\right\}\\ &+\textstyle\int_0^\infty\left\{\Phi(t,x-\pi_R(t,x,x_1)z,x_1)-\Phi(t,x,x_1)\right\}\nu_0(dz)\\ &+\textstyle\int_{-1}^\infty\left\{\Phi(t,x+\pi_I(t,x,x_1)z,x_1)-\Phi(t,x,x_1)\right\}\nu_1(dz).\end{aligned} \tag{3.20}$$

In this section and for the aim to establish explicitly the reinsurance-investment equilibriums for our optimization problem (3.17) according to its definition, we will formulate the extended HJB equations and we derive the system with their corresponding theorem verification. So, let $V,g:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ and $h:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ where $(y,y_1)\in\mathbb{R}^2$, satisfying the following system of HJB equations

$$\begin{cases}\sup\limits_{\pi(.)}\left\{\mathcal{A}^{(\pi_R,\pi_I)}V(t,x,x_1)-\mathcal{A}^{(\pi_R,\pi_I)}h(t,x,x_1,x,x_1)+\mathcal{A}^{(\pi_R,\pi_I)}h^{x,x_1}(t,x,x_1)\right.\\ \left.-\mathcal{A}^{(\pi_R,\pi_I)}(G\diamond g)(t,x,x_1)+\mathcal{D}^{(\pi_R,\pi_I)}g(t,x,x_1)\right\}=0, \quad \text{for } t\in[0,T],\\ \mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}h^{y,y_1}(t,x,x_1)=0, \quad \text{for } t\in[0,T],\\ \mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}g(t,x,x_1)=0, \qquad \text{for } t\in[0,T],\\ V(T,x,x_1)=F(x,x_1,x+\Theta x_1)+G(x,x_1,x+\Theta x_1),\\ h(T,x,x_1,y,y_1)=F(y,y_1,x+\Theta x_1),\\ g(T,x,x_1)=x+\Theta x_1.\end{cases} \tag{3.21}$$

We notice that

$$\hat{\pi}(.,.)\in\arg\sup\left\{\mathcal{A}^{(\pi_R,\pi_I)}V(t,x,x_1)-\mathcal{A}^{(\pi_R,\pi_I)}h(t,x,x_1,x,x_1)+\mathcal{A}^{(\pi_R,\pi_I)}h^{x,x_1}(t,x,x_1)\right. \tag{3.22}$$

$$\left.-\mathcal{A}^{(\pi_R,\pi_I)}\left(G\diamond g\right)(t,x,x_1)+\mathcal{D}^{(\pi_R,\pi_I)}g(t,x,x_1)\right\},$$

and that

$$\begin{aligned}\left(G\diamond g\right)(t,x,x_1) &= G(x,x_1,g(t,x,x_1))=\frac{\eta(x,x_1)}{2}g^2(t,x,x_1),\\ \mathcal{D}^{(\pi_R,\pi_I)}g(t,x,x_1) &= G_g(x,x_1,g(t,x,x_1)).\mathcal{A}^{(\pi_R,\pi_I)}g(t,x,x_1)\\ &= \eta(x,x_1).g(t,x,x_1).\mathcal{A}^{(\pi_R,\pi_I)}g(t,x,x_1).\end{aligned} \tag{3.23}$$

Theorem 3.3

We assume that there exist four functions V, h^{y,y_1}, h and g which have the following properties

1. *V, h^{y,y_1}, h and g do not depend on x_2.*

2. *V, h^{y,y_1} and g solve the extended HJB system (3.21).*

3. *V, h^{y,y_1} , $g \in C_p^{1,2,1}\left([0,T]\times\mathbb{R}^2\right)$ and $h \in C_p^{1,2,1,2,1}\left([0,T]\times\mathbb{R}^4\right)$.*

4. *$\hat{\pi} = (\hat{\pi}_R, \hat{\pi}_I) \in \mathcal{R}^L[t,T]\times\Pi^L[t,T]$ is an equilibrium control law and V is the corresponding value function which realizes the supermum in the V-equation. Then, $\hat{\pi}$ is a feedback equilibrium control and V is the corresponding equilibrium value function i.e.*

$$V(t,x,x_1) = J(t,x,x_1,\hat{\pi}), \tag{3.24}$$

So,

$$\begin{aligned}&V(t,x,x_1)\\&=\mathbb{E}_{t,x,x_1}[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)]\\&-\frac{\eta(x,x_1)}{2}Var_{t,x,x_1}[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)],\end{aligned}\tag{3.25}$$

and h, g have the following probabilistic representations

$$\begin{aligned}&h(t,x,x_1,x,x_1)\\&=\mathbb{E}_{t,x,x_1}\Big[\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)\\&-\frac{\eta(x,x_1)}{2}\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)^2\Big]\end{aligned}\tag{3.26}$$

and

$$g(t,x,x_1) = \mathbb{E}_{t,x,x_1}\left[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T))\right]. \tag{3.27}$$

Thus

$$V(t,x,x_1) = h(t,x,x_1,x,x_1)+\frac{\eta(x,x_1)}{2}g^2(t,x,x_1). \tag{3.28}$$

Proof: Similar to [5], we start by showing that V, h, g, have the Feynman-Kac representation and that V is the equilibrium value function corresponding to $\hat{\pi} = (\hat{\pi}_R, \hat{\pi}_I)$, (i.e. that $V(t,x,x_1) = J(t,x,x_1;\hat{\pi})$). Next, we prove that $\hat{\pi} = (\hat{\pi}_R, \hat{\pi}_I)$ is indeed a feedback equilibrium control.

Step 01. To show that g has the interpretation (3.27) , we apply the Itô formula (See [33]) to the process $\tau \to g\left(\tau, X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau)\right)$ we obtain

$$
\begin{aligned}
& dg\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right) \\
& = \mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)} g\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right) d\tau \\
& +\sigma_0 g_x\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right) \hat{\pi}_R\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right) dW_0(s) \\
& +\sigma(\tau) g_x\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right) \hat{\pi}_I\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right) dW_1(s) \\
& +\int_0^{\infty}\left\{g(t, X^{(\hat{\pi}_R,\hat{\pi}_I)} - z\pi_R\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right), X_1^{(\hat{\pi}_R,\hat{\pi}_I)})\right. \\
& \left.-g\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right)\right\} \tilde{N}_0(d\tau, dz) \\
& +\int_{-1}^{\infty} g\left\{(t, X^{(\hat{\pi}_R,\hat{\pi}_I)} + z\pi_I\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right), X_1^{(\hat{\pi}_R,\hat{\pi}_I)})\right. \\
& \left.-g\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right)\right\} \tilde{N}_1(d\tau, dz).
\end{aligned} \tag{3.29}
$$

From the third equation in (3.21) and from the boundary condition for g, it follows that the process $g\left(\tau, X^{\hat{\pi}}(\tau), Y^{\hat{\pi}}(\tau)\right)$ is a martingale, so we obtain our desired representation of g as

$$
g(t, x, x_1) = \mathbb{E}_{t,x,x_1}\left[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right].
$$

Now applying Itô formula to $\tau \to h^{x,x_1}\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right)$, (See [33]), we obtain that

$$
\begin{aligned}
& dh^{x,x_1}\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right) \\
& = \mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)} h^{x,x_1}\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right) d\tau \\
& +\sigma_0 h_x\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right) \hat{\pi}_R\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right) dW_0(s) \\
& +\sigma(\tau) h_x\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right) \hat{\pi}_I\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right) dW_1(s) \\
& +\int_0^{\infty}\left\{h^{x,x_1}(t, X^{(\hat{\pi}_R,\hat{\pi}_I)} - z\pi_R\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right), X_1^{(\hat{\pi}_R,\hat{\pi}_I)})\right. \\
& \left.-h^{x,x_1}\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right)\right\} \tilde{N}_0(d\tau, dz) \\
& +\int_{-1}^{\infty}\left\{h^{x,x_1}(t, X^{(\hat{\pi}_R,\hat{\pi}_I)} + z\pi_I\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right), X_1^{(\hat{\pi}_R,\hat{\pi}_I)})\right. \\
& \left.-h^{x,x_1}\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right)\right\} \tilde{N}_1(d\tau, dz).
\end{aligned} \tag{3.30}
$$

From the second equation in (3.21) and from the boundary condition for h^{x,x_1}, it follows that the process $h^{x,x_1}\left(\tau, X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau), X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right)$ is a martingale, so we obtain our desired representation of h^{x,x_1} as

$$\begin{aligned} h^{x,x_1}(t,x,x_1) &= \mathbb{E}_{t,x,x_1}\left[F\left(x_1,y_1,X^{(\hat{\pi}_R,\hat{\pi}_I)}(T),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)\right] \\ &= \mathbb{E}_{t,x,x_1}\left[\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)\right. \\ &\left. -\frac{\eta(x,x_1)}{2}\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)^2\right]. \end{aligned}$$

To show that $V(t,x,x_1)=J(t,x,x_1;\hat{\pi})$, we use the first equation in (3.21) to obtain

$$\begin{aligned} &\mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}V(t,x,x_1)-\mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}h(t,x,x_1,x,x_1)+\mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}h^{x,x_1}(t,x,x_1) \\ &-\mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}\left(G\diamond g\right)(t,x,x_1)+\mathcal{D}^{(\hat{\pi}_R,\hat{\pi}_I)}g\left(t,x,x_1\right) \\ &=0. \end{aligned} \tag{3.31}$$

From the second and third equations in (3.21), then (3.31) takes the form

$$\mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}V(t,x,x_1)-\mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}h(t,x,x_1,x,x_1)-\mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}\left(G\diamond g\right)(t,x,x_1)=0. \tag{3.32}$$

We now apply the Itô formula to the process $V\left(\tau,X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right)$. Integrating and taking expectations we get

$$\begin{aligned} &\mathbb{E}_{t,x,x_1}\left[V\left(T,X^{(\hat{\pi}_R,\hat{\pi}_I)}(T),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)\right] \\ &=V(t,x,x_1)+\mathbb{E}_{t,x,x_1}\left[\int_t^T\mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}V\left(\tau,X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right)d\tau\right]. \end{aligned} \tag{3.33}$$

Using (3.32), we thus obtain

$$\begin{aligned} &\mathbb{E}_{t,x,x_1}\left[V\left(T,X^{(\hat{\pi}_R,\hat{\pi}_I)}(T),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)\right]-V(t,x,x_1) \\ &=\mathbb{E}_{t,x,x_1}\left[\int_t^T\left\{\mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}h\left(\tau,X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau),X^{\hat{\pi}}(\tau),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right)\right.\right. \\ &\left.\left.+\mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}\left(G\diamond g\right)\left(\tau,X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right)\right\}d\tau\right] \end{aligned} \tag{3.34}$$

In the same way we obtain

$$\begin{aligned} &\mathbb{E}_{t,x,x_1}\left[h\left(T,X^{(\hat{\pi}_R,\hat{\pi}_I)}(T),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T),X^{(\hat{\pi}_R,\hat{\pi}_I)}(T),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)\right]-h(t,x,x_1,x,x_1) \\ &=\mathbb{E}_{t,x,x_1}\left[\int_t^T\mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}h\left(\tau,X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau),X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right)d\tau\right]. \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} &\mathbb{E}_{t,x,x_1}\left[\left(G\diamond g\right)\left(T,X^{(\hat{\pi}_R,\hat{\pi}_I)}(T),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)\right]-\left(G\diamond g\right)(t,x,x_1) \\ &=\mathbb{E}_{t,x,x_1}\left[\int_t^T\mathcal{A}^{(\hat{\pi}_R,\hat{\pi}_I)}\left(G\diamond g\right)\left(\tau,X^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau),X^{\hat{\pi}}(\tau),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(\tau)\right)d\tau\right], \end{aligned} \tag{3.35}$$

using the two later inequalities and the boundary conditions for V, h and g we get

$$
\begin{aligned}
&V(t,x,x_1)\\
&=\mathbb{E}_{t,x,x_1}\left[F\left(x,x_1,X^{(\hat{\pi}_R,\hat{\pi}_I)}(T),X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)\right]\\
&+G\left(x,x_1,\mathbb{E}_{t,x,x_1}\left[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right]\right)\\
&=\mathbb{E}_{t,x,x_1}[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)]-\frac{\eta(x,x_1)}{2}Var_{t,x,x_1}[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)],
\end{aligned}
\tag{3.36}
$$

Step 02 . The aim of the second part of the proof is to emphasize that $\hat{\pi}=(\hat{\pi}_R,\hat{\pi}_I)$ is an equilibrium strategy.

For any admissible strategies (π_R,π_I), we define $h^{(\pi_R,\pi_I)}$ and $g^{(\pi_R,\pi_I)}$ by

$$
\begin{aligned}
&h^{(\pi_R,\pi_I)}(t,x,x_1,y,y_1)=\mathbb{E}_{t,x,x_1}[y,y_1,X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)],\\
&g^{(\pi_R,\pi_I)}(t,x,x_1)=\mathbb{E}_{t,x,x_1}[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)],
\end{aligned}
\tag{3.37}
$$

Noting that $h=h^{(\hat{\pi}_R,\hat{\pi}_I)}$ and $g=g^{(\hat{\pi}_R,\hat{\pi}_I)}$ for $(\hat{\pi}_R,\hat{\pi}_I)=(\pi_R,\pi_I)$. For any $\varepsilon>0$ and for any admissible strategy, we move to construct an admissible strategy . By Lemma 3.3 in [15] applied to the points t and $t+\varepsilon$, we get

$$
\begin{aligned}
&J(t,x,x_1,\pi^{\varepsilon})\\
&=\mathbb{E}_{t,x,y}\left[J\left(t+\varepsilon,X^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon),X_1^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon),\pi^{\varepsilon}\right)\right]\\
&-\left(\mathbb{E}_{t,x,x_1}\left[h^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}\left(t+\varepsilon,X^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon),X_1^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon),t+\varepsilon,X^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon),\right.\right.\right.\\
&\left.\left.X_1^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon)\right)\right]-\mathbb{E}_{t,x,x_1}\left[h^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon,X^{\pi^{\varepsilon}}(t+\varepsilon),X_1^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon),t,x,x_1)\right]\\
&-\left(\mathbb{E}_{t,x,x_1}\left[G(t+\varepsilon,X^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon),X_1^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon),g^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}t+\varepsilon,X^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon),\right.\right.\\
&\left.\left.X_1^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon)\right)\right]-G(t+\varepsilon,x,x_1,\mathbb{E}_{t,x,x_1}\left[g^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon,X^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon),X_1^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon)\right].
\end{aligned}
\tag{3.38}
$$

It is easy to remark that for any $\varepsilon\in[0,T-t]$

$$
(\pi_R^{\varepsilon},\pi_I^{\varepsilon})(s,x,x_1)=\begin{cases}(\pi_R,\pi_I)\ (s,x,x_1) & \text{for } (s,x,x_1)\in[t,t+\varepsilon]\times\mathbb{R}^2\\ (\hat{\pi}_R,\hat{\pi}_I)(s,x,x_1) & \text{for } (s,x,x_1)\in[t+\varepsilon,T]\times\mathbb{R}^2.\end{cases}
\tag{3.39}
$$

and by continuity, we have $X^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon)=X^{(\pi_R,\pi_I)}(t+\varepsilon)$ and $X_1^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon)=X_1^{(\pi_R,\pi_I)}(t+\varepsilon)$. Then we get that

$$
J\left(t+\varepsilon,X^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon),X_1^{(\pi_R^{\varepsilon},\pi_I^{\varepsilon})}(t+\varepsilon),\pi^{\varepsilon}\right)=V\left(t+\varepsilon,X^{(\pi_R,\pi_I)}(t+\varepsilon),X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\right),
\tag{3.40}
$$

and

$$\begin{aligned}
&h^{(\pi_R^\varepsilon,\pi_I^\varepsilon)}\left(t+\varepsilon, X^{(\pi_R^\varepsilon,\pi_I^\varepsilon)}(t+\varepsilon), X_1^{(\pi_R^\varepsilon,\pi_I^\varepsilon)}(t+\varepsilon), X^{(\pi_R^\varepsilon,\pi_I^\varepsilon)}(t+\varepsilon), X_1^{(\pi_R^\varepsilon,\pi_I^\varepsilon)}(t+\varepsilon)\right)\\
&= h\left(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon), X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\right),\\
&h^{(\pi_R^\varepsilon,\pi_I^\varepsilon)}\left(t+\varepsilon, X^{(\pi_R^\varepsilon,\pi_I^\varepsilon)}(t+\varepsilon), X_1^{(\pi_R^\varepsilon,\pi_I^\varepsilon)}(t+\varepsilon), x, x_1\right)\\
&= h\left(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon), x, x_1\right),
\end{aligned} \tag{3.41}$$

and

$$g^{(\pi_R^\varepsilon,\pi_I^\varepsilon)}(t+\varepsilon, X^{(\pi_R^\varepsilon,\pi_I^\varepsilon)}(t+\varepsilon), Y^{(\pi_R^\varepsilon,\pi_I^\varepsilon)}(t+\varepsilon)) = g(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)). \tag{3.42}$$

Consequently

$$\begin{aligned}
&J(t,x,x_1,\pi^\varepsilon)\\
&= \mathbb{E}_{t,x,x_1}\left[V\left(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\right)\right]\\
&-\left(\mathbb{E}_{t,x,x_1}\left[h^{(\pi_R,\pi_I)}\left(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon),\right.\right.\right.\\
&\left.\left.X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\right)\right]\\
&\left.-\mathbb{E}_{t,x,x_1}\left[h^{(\pi_R,\pi_I)}\left(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon), x, x_1\right)\right]\right)\\
&-\left(\mathbb{E}_{t,x,x_1}\left[G(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\right.\right.\\
&\left.g^{(\pi_R,\pi_I)}(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon))\right]\\
&\left.-G\left(t,x,x_1,\mathbb{E}_{t,x,x_1}\left[g^{(\pi_R,\pi_I)}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\right)\right]\right)\right).
\end{aligned} \tag{3.43}$$

Furthermore, including to the extended HJB equation we have that

$$\begin{aligned}
&\mathcal{A}^{(\pi_R,\pi_I)}V(t,x,x_1) - \mathcal{A}^{(\pi_R,\pi_I)}h(t,x,x_1,x,x_1) + \mathcal{A}^{(\pi_R,\pi_I)}h^{x,x_1}(t,x,x_1)\\
&-\mathcal{A}^{(\pi_R,\pi_I)}\left(G\diamond g\right)(t,x,x_1)) + \mathcal{D}^{(\pi_R,\pi_I)}g(t,x,x_1)\\
&\leq 0,
\end{aligned} \tag{3.44}$$

which implies that

$$
\begin{aligned}
&\mathbb{E}_{t,x,x_1}\left[V\left(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\right)\right] - V(t,x,x_1)\\
&-\Big(\mathbb{E}_{t,x,x_1}\Big[h^{(\pi_R,\pi_I)}\ \Big(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon),\\
&X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\Big) - h(t,x,x_1,x,x_1)\Big]\\
&+\mathbb{E}_{t,x,x_1}\left[h^{(\pi_R,\pi_I)}\left(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon), x, x_1\right)\right]\\
&-h(t,x,x_1,x,x_1)\\
&-\Big(\mathbb{E}_{t,x,x_1}\Big[G\Big(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\ ,\\
&g^{(\pi_R,\pi_I)}\left(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\right)\Big)\Big]\\
&-G(t,x,x_1,g(t,x,x_1))\\
&+G(t,x,x_1,\mathbb{E}_{t,x,x_1}\left[g^{(\pi_R,\pi_I)}\left(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\right)\right])\\
&-G(t,x,x_1,g(t,x,x_1))\\
&\leq o(\varepsilon).
\end{aligned}
\tag{3.45}
$$

After numerous simplification, we get

$$
\begin{aligned}
&V(t,x,x_1)\\
&\geq \mathbb{E}_{t,x,x_1}\left[V\left(t+\varepsilon, X^{(\pi_R,\pi_I)}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\right)\right]\\
&-\mathbb{E}_{t,x,x_1}\Big[h^{(\pi_R,\pi_I)}\Big(t+\varepsilon, X^{\pi}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon),\\
&X^{\pi}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\Big)\Big]\\
&+\mathbb{E}_{t,x,x_1}\left[h^{(\pi_R,\pi_I)}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon), x, x_1\right)\right]\\
&-\mathbb{E}_{t,x,x_1}\Big[G\Big(t+\varepsilon, X^{\pi}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon),\\
&g^{(\pi_R,\pi_I)}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\right)\Big]\\
&+G\left(t,x,x_1,\mathbb{E}_{t,x,x_1}\left[g^{(\pi_R,\pi_I)}\left(t+\varepsilon, X^{\pi}(t+\varepsilon), X_1^{(\pi_R,\pi_I)}(t+\varepsilon)\right)\right]\right) + o(\varepsilon)\\
&= J(t,x,x_1,\pi^{\varepsilon}) + o(\varepsilon).
\end{aligned}
\tag{3.46}
$$

We have already proved in the first part that $V(t,x,x_1) = J(t,x,x_1,\hat{\pi})$. So, $J(t,x,x_1,\hat{\pi}) - J(t,x,x_1,\pi^{\varepsilon}) \geq o(\varepsilon)$, hence

$$
\liminf_{\varepsilon\downarrow 0}\left\{\frac{J(t,x,x_1,\hat{\pi}) - J(t,x,x_1,\pi^{\varepsilon})}{\varepsilon}\right\} \geq 0. \tag{3.47}
$$

As a result, $\hat{\pi} = (\hat{\pi}_R, \hat{\pi}_I)$ is an equilibrium strategy. ■

Remark 3.4

Since the infinitesimal generator $\mathcal{A}^{(\pi_R,\pi_I)}$ involves the drift of the SDDE (3.13) *which depends on x_2 the coefficients of the extended HJB system* (3.21) *depends on*

x_2 .Consequently, we can not apriori expect the extended HJB equations to have solutions independent on x_2 in the general case. However , the following theorem provides necessary conditions on the delay parameters $\mu(t), \alpha(t), \beta, \Theta, \lambda$ where the condition V, h and g do not depend x_2 .

Theorem 3.5

If the extended HJB system (3.21) has a solution V, h and g which are independent of x_2 and the equilibrium feedback strategies $\hat{\pi} = (\hat{\pi}_R, \hat{\pi}_I) : [0,T] \times \mathbb{R}^2 \to \mathbb{R}_+ \times \mathbb{R}$. Then the following conditions have to be verified

$$\begin{cases} \Theta = & \beta e^{\delta\lambda}, \\ \alpha(t) = & \beta e^{\delta\lambda}(\mu(t) + \beta e^{\delta\lambda} + \lambda). \end{cases} \tag{3.48}$$

Proof: Noting that for brevity in this proof, we suppress the subscript (t, x, x_1) the functions $V(t, x, x_1)$, $h(t, x, x_1, y, y_1)$, $g(t, x, x_1)$, $\eta(x, x_1)$ and all their derivatives with respect t, x and x_1. We know that if V , h and g only depend on t, x and x_1, then, V, h and g satisfy the following HJB equations

$$\begin{aligned}
&V_t + \sup_{(\pi_R, \pi_I)} \Bigg\{ \left\{ V_{x_1} - h_{x_1} - \frac{\eta_{x_1}}{2} g^2 \right\} \left\{ x - e^{-\delta\lambda} x_2 - \lambda x_1 \right\} + \left\{ V_x - h_y - \frac{\eta_x}{2} g^2 \right\} \\
&\{\mu(t)x + [\delta + (1+\theta_0)\,\pi_R]\lambda_N \mu_Y + \pi_I \rho(t) + \alpha(t)x_1 + \beta x_2 - \pi_I \lambda_{\tilde{N}} \mu_Z\} \\
&+\frac{1}{2} \left\{ V_{xx} - h_{yy} - 2h_{xy} - \frac{\eta_{xx}}{2} g^2 - \eta g_x^2 - 2\eta_x g g_x \right\} \left\{ (\pi_R \sigma_0)^2 + (\pi_I \sigma(t))^2 \right\} \\
&-\int_0^\infty \left\{ V + \frac{\eta}{2} g^2 \right\} \nu_0(dz) - \int_{-1}^\infty \left\{ V + \frac{\eta}{2} g^2 \right\} \nu_1(dz) \\
&+\int_0^\infty \left\{ V(t, x - \pi_R z, x_1) - \tfrac{\eta(t, x - \pi_R z, x_1)}{2} g^2(t, x - \pi_R z, x_1) \right\} \nu_0(dz) \\
&+\int_{-1}^\infty \left\{ V(t, x + \pi_I z, x_1) - \tfrac{\eta(x + \pi_I z, x_1)}{2} g^2(t, x + \pi_I z, x_1) \right\} \nu_1(dz) \\
&+\eta g \int_0^\infty \{g(t, x - \pi_R z, x_1) - g\}\, \nu_0(dz) + \eta g \int_{-1}^\infty \{g(t, x + \pi_I z, x_1) - g\}\, \nu_1(dz) \Bigg\} = 0,
\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}
&h_t + h_{x_1} \left\{ x - e^{-\delta\lambda} x_2 - \lambda x_1 \right\} + h_x \{\mu(t)x + [\delta + (1+\theta_0)\,\hat{\pi}_R]\lambda_N \mu_Y + \hat{\pi}_I \rho(t) \\
&+\alpha(t)x_1 + \beta x_2 - \hat{\pi}_I \lambda_{\tilde{N}} \mu_Z\} + \frac{1}{2} h_{xx} \left\{ (\hat{\pi}_R \sigma_0)^2 + (\hat{\pi}_I \sigma(t))^2 \right\} \\
&+\int_0^\infty \{h(t, x - \hat{\pi}_R z, x_1, y - \hat{\pi}_R z, y_1) - h\}\, \nu_0(dz) \\
&+\int_{-1}^\infty \{h(t, x + \hat{\pi}_I z, y + \hat{\pi}_I z, y_1) - h\}\, \nu_1(dz) = 0,
\end{aligned} \tag{3.50}$$

and

$$
\begin{aligned}
& g_t + g_{x_1}\left\{x - e^{-\delta\lambda}x_2 - \lambda x_1\right\} + g_x\left\{\mu(t)x + [\delta + (1+\theta_0)\,\hat{\pi}_R]\lambda_N\mu_Y\right. \\
& \left. +\hat{\pi}_I\rho(t) + \alpha(t)x_1 + \beta x_2 - \pi_I\lambda_{\tilde{N}}\mu_Z\right\} + \frac{1}{2}g_{xx}\left\{(\hat{\pi}_R\sigma_0)^2 + (\hat{\pi}_I\sigma(t))^2\right\} \\
& +\int_0^{\infty}\left\{g(t, x - \hat{\pi}_R z, x - g(t,x,x)\right\}\nu_0(dz) + \int_{-1}^{\infty}\left\{g(t, x + \hat{\pi}_I z, x) - g\right\}\nu_1(dz) = 0,
\end{aligned}
\tag{3.51}
$$

With the terminal conditions

$$
\begin{aligned}
& V(T,x,x_1) = F(x, x_1, x+\Theta x_1) + G(x, x_1, x + \Theta x_1), \\
& h(T,x,x_1,y,y_1) = F(y, y_1, x + \Theta x_1), \\
& g(T,x,x_1) = x + \Theta x_1.
\end{aligned}
\tag{3.52}
$$

We wish to obtain necessary conditions on $\mu(t)$, $\alpha(t)$, β, Θ, λ to guarantee that the above equations have a solutions independent to x_2. Differentiating the above equations with respect to x_2 we obtain

$$
\begin{aligned}
V_{x_1} - \frac{\eta_{x_1}}{2}g^2 &= \left(V_x - \frac{\eta_x}{2}g^2\right)\beta e^{\delta\lambda}, \\
h_{x_1} &= h_x\beta e^{\delta\lambda}, \\
g_{x_1} &= g_x\beta e^{\delta\lambda}
\end{aligned}
\tag{3.53}
$$

remplacing h_{x_1}, g_{x_1} and $V_{x_1} - \dfrac{\eta_{x_1}}{2}g^2$ in the equations (3.49), (3.50) and (3.51) we find

$$
\begin{aligned}
& V_t + \sup_{(\pi_R,\pi_I)}\left\{\left\{V_{x_1} - h_{x_1} - \frac{\eta_{x_1}}{2}g^2\right\}\{x - \lambda x_1\} + \left\{V_x - h_y - \frac{\eta_x}{2}g^2\right\}\right. \\
& \{\mu(t)x + [\delta + (1+\theta_0)\,\pi_R]\lambda_N\mu_Y + \pi_I\rho(t) + \alpha(t)x_1 - \pi_I\lambda_{\tilde{N}}\mu_Z\} \\
& +\frac{1}{2}\left\{V_{xx} - h_{yy} - 2h_{xy} - \frac{\eta_{xx}}{2}g^2 - \gamma g_x^2 - 2\gamma_x g g_x\right\}\left\{(\pi_R\sigma_0)^2 + (\pi_I\sigma(t))^2\right\} \\
& -\int_0^{\infty}\left\{V + \frac{\gamma}{2}g^2\right\}\nu_0(dz) - \int_{-1}^{\infty}\left\{V + \frac{\gamma}{2}g^2\right\}\nu_1(dz) \\
& +\int_0^{\infty}\left\{V(t, x - \pi_R z, x_1) - \frac{\eta(x - \pi_R z, x_1)}{2}g^2(t, x - \pi_R z, x_1)\right\}\nu_0(dz) \\
& +\int_{-1}^{\infty}\left\{V(t, x + \pi_I z, x_1) - \frac{\eta(x + \pi_I z, x_1)}{2}g^2(t, x + \pi_I z, x_1)\right\}\nu_1(dz) \\
& \left. +\eta g\int_0^{\infty}\{g(t, x - \pi_R z, x_1) - g\}\nu_0(dz) + \eta g\int_{-1}^{\infty}\{g(t, x + \pi_I z, x_1) - g\}\nu_1(dz)\right\} = 0.
\end{aligned}
\tag{3.54}
$$

and

$$
\begin{aligned}
& h_t + h_{x_1}\{x - \lambda x_1\} + h_x\{\mu(t)x + [\delta + (1+\theta_0)\,\hat{\pi}_R]\lambda_N\mu_Y + \hat{\pi}_I\rho(t) + \alpha(t)x_1 - \hat{\pi}_I\lambda_{\tilde{N}}\mu_Z\} \\
& +\frac{1}{2}h_{xx}\left\{(\hat{\pi}_R\sigma_0)^2 + (\hat{\pi}_I\sigma(t))^2\right\} + \int_0^{\infty}\{h(t, x - \hat{\pi}_R z, x_1, y - \hat{\pi}_R z, y_1) - h\}\nu_0(dz) \\
& +\int_{-1}^{\infty}\{h(t, x + \hat{\pi}_I z, y + \hat{\pi}_I z, y_1) - h\}\nu_1(dz) = 0,
\end{aligned}
\tag{3.55}
$$

and

$$\begin{aligned}
&g_t + g_{x_1}\left\{x-\lambda x_1\right\} + g_x\left\{\mu(t)x + [\delta + (1+\theta_0)\,\hat{\pi}_R]\lambda_N\mu_Y + \hat{\pi}_I\rho(t) + \alpha(t)x_1 - \hat{\pi}_I\lambda_{\tilde{N}}\mu_Z\right\} \\
&+\frac{1}{2}g_{xx}\left\{(\hat{\pi}_R\sigma_0)^2) + (\hat{\pi}_I\sigma(t))^2\right\} + \int_0^\infty \left\{g(t,x-\hat{\pi}_R z, x_1) - g\right\}\nu_0(dz) \\
&+\int_{-1}^\infty \left\{g(t,x+\hat{\pi}_I z, x_1) - g\right\}\nu_1(dz) = 0,
\end{aligned} \tag{3.56}$$

which does not contain any x_2, the last step ensures the following equalities:

$$V_{x_1} - \frac{\eta_{x_1}}{2}g^2 = V_x - \frac{\eta_x}{2}g^2\beta e^{\delta\lambda}, \tag{3.57}$$

$$h_{x_1} = h_x\beta e^{\delta\lambda},$$

$$g_{x_1} = g_x\beta e^{\delta\lambda}.$$

If we introduce a new variable $\widetilde{x}$ where

$$\frac{\partial}{\partial\widetilde{x}} = \frac{\partial}{\partial x_1} - \beta e^{\delta\lambda}\frac{\partial}{\partial x}, \tag{3.58}$$

So, the equations of the system (3.52), (3.53) and $(3,54)$ state that

$$V_{\widetilde{x}} = 0, \tag{3.59}$$

$$h_{\widetilde{x}} = 0,$$

$$g_{\widetilde{x}} = 0.$$

Hence V, h and g have to be independent of x_2. Consequently, differentiating the equations (3.49), (3.50) and (3.51) and the terminal conditions in $\widetilde{x}$, we find that the conditions (3.46) must be verified. ■

Lemma 3.6

Let $\hat{\pi} = (\hat{\pi}_R, \hat{\pi}_I)$ the feedback equilibrium control which realizes the supermum, we suppress the subscript (t,x,x_1) the functions $V(t,x,x_1)$, $h(t,x,x_1,y,y_1)$, $g(t,x,x_1)$, $\eta(x,x_1)$ and all their derivatives with respect t, x and x_1. We know that if V, h and

g only depend on t, x and x_1,the HJB equations is reduced to the following

$$\begin{cases} h_t + \frac{\eta}{x+\Theta x_1} g g_t + \sup\limits_{(\pi_R,\pi_I)} \left\{ \left\{ h_{x_1} + \dfrac{\eta}{x+\Theta x_1} g g_{x_1} \right\} \left\{ x - e^{-\delta\lambda} x_2 - \lambda x_1 \right\} + \left\{ h_x + \frac{\eta}{x+\Theta x_1} g g_x \right\} \right. \\ \{\mu(t)x + [\delta + (1+\theta_0)\pi_R] \lambda_N \mu_Y + \pi_I \rho(t) + \alpha(t) x_1 + \beta x_2 - \pi_I \lambda_{\tilde{N}} \mu_Z\} \\ + \dfrac{1}{2} \left\{ h_{xx} + \frac{\eta}{x+\Theta x_1} g g_{xx} \right\} \left\{ (\pi_R \sigma_0)^2 + (\pi_I \sigma(t))^2 \right\} - \displaystyle\int_0^\infty h \nu_0(dz) - \int_{-1}^\infty h \nu_1(dz) \\ + \displaystyle\int_0^\infty h(t, x - \pi_R z, x_1, y, y_1) \nu_0(dz) + \int_{-1}^\infty h(t, x + \pi_I z, x_1, y, y_1) \nu_1(dz) \\ \left. + \eta g \left(\displaystyle\int_0^\infty \{ g(t, x - \pi_R z, x_1) - g \} \nu_0(dz) \right) + \eta g \left(\int_0^\infty \{ g(t, x + \pi_I z, x_1) - g \} \nu_1(dz) \right) \right\} = 0, \\ \\ h_t + h_{x_1} \left\{ x - e^{-\delta\lambda} x_2 - \lambda x_1 \right\} + h_x \{\mu(t)x + [\delta + (1+\theta_0)\hat{\pi}_R] \lambda_N \mu_Y + \hat{\pi}_I \rho(t) \\ + \alpha(t) x_1 + \beta x_2 - \hat{\pi}_I \lambda_{\tilde{N}} \mu_Z\} \\ + \dfrac{1}{2} h_{xx} \left\{ (\hat{\pi}_R \sigma_0)^2 + (\hat{\pi}_I \sigma(t))^2 \right\} + \displaystyle\int_0^\infty \{ h(t, x - \hat{\pi}_R z, x_{1,} y - \hat{\pi}_R z, y_1) - h \} \nu_0(dz) \\ + \displaystyle\int_{-1}^\infty \{ h(t, x + \hat{\pi}_I z, y + \hat{\pi}_I z, y_1) - h \} \nu_1(dz) = 0, \\ \\ g_t + g_{x_1} \left\{ x - e^{-\delta\lambda} x_2 - \lambda x_1 \right\} + g_x \{\mu(t)x + [\delta + (1+\theta_0)\hat{\pi}_R] \lambda_N \mu_Y + \hat{\pi}_I \rho(t) \\ + \alpha(t) x_1 + \beta x_2 - \hat{\pi}_I \lambda_{\tilde{N}} \mu_Z\} + \dfrac{1}{2} g_{xx} \left\{ (\hat{\pi}_R \sigma_0)^2 + (\hat{\pi}_I \sigma(t))^2) \right\} \\ + \displaystyle\int_0^\infty \{ g(t, x - \hat{\pi}_R z, x - g \} \nu_0(dz) + \int_{-1}^\infty \{ g(t, x + \hat{\pi}_I z, x) - g \} \nu_1(dz) = 0, \end{cases} \tag{3.60}$$

Proof: By applying the generator $\mathcal{A}^{(\pi_R,\pi_I)}$ and using (3.28). ■

Remark 3.7

For finding explicitly the equilibrium reinsurance and investment strategies we consider the following theorem .

Theorem 3.8

For our mean-variance problem (3.17)*, we assume that the equilibrium reinsurance-investment strategies are given by*

$$\begin{cases} \hat{\pi}_R(t,x,x_1) = c_0(t)(x+\Theta x_1) + k_0(t), \\ \hat{\pi}_I(t,x,x_1) = c_1(t)(x+\Theta x_1) + k_1(t), \end{cases} \tag{3.61}$$

where

$$\begin{cases} c_0(t) = & \dfrac{\theta_0 \lambda_N \mu_Y}{\eta\left[\sigma_0^2 + \lambda_N \sigma_Z^2\right]} \left(e^{\int_t^T -\varrho(u)du} + \eta e^{\int_t^T (A(u)-\varrho(u))du} - \eta \right), \\ k_0(t) = & \dfrac{\theta_0 \lambda_N \mu_Y}{\eta\left[\sigma_0^2 + \lambda_N \sigma_Z^2\right]} \left(e^{\int_t^T -\varrho(u)du} + \eta e^{\int_t^T (A(u)-\varrho(u))du} - \eta \right), \\ c_1(t) = & \dfrac{\rho(t)}{\eta\left[\sigma^2(t) + \lambda_{\tilde{N}} \sigma_Z^2\right]} \left(e^{\int_t^T -\varrho(u)du} + \eta e^{\int_t^T (A(u)-\varrho(u))du} - \eta \right), \\ k_1(t) = & \dfrac{\rho(t)}{\eta\left[\sigma^2(t) + \lambda_{\tilde{N}} \sigma_Z^2\right]} \left(e^{\int_t^T -\varrho(u)du} + \eta e^{\int_t^T (A(u)-\varrho(u))du} - \eta \right). \end{cases} \tag{3.62}$$

where

$$\begin{cases} \mu(t) = r_0(t) - \alpha(t) - \beta \ , \varrho(u) = A(u) + C_0^2(u) + C_1^2(u), \\ A(t) = \mu(s) + \Theta + \rho(t)c_1(t) + \lambda_N \mu_Y \theta_0 c_0(t), C_0(t) = \sigma_0 c_0(t) \\ C_1(t) = \sigma(t)c_1(t), \chi_0(t) = \sigma_0 k_0(t), \quad \chi_1(t) = \sigma(t)k_1(t). \end{cases} \tag{3.63}$$

Proof: We assume that $\hat{\pi}_R(t,x,x_1) = c_0(t)(x+\Theta x_1) + k_0(t)$ and $\hat{\pi}_I(t,x,x_1) = c_1(t)(x+\Theta x_1) + k_1(t)$. where c_0, c_1, k_0 and k_1 are a deterministic functions. we conjecture into the wealth process (3.13) , we derive the wealth process, we get

$$
\begin{aligned}
& dX^{(\hat{\pi}_R,\hat{\pi}_I)}(s) \\
& = \Big\{ \mu(s) X^{(\hat{\pi}_R,\hat{\pi}_I)}(s) + \delta\lambda_N\mu_Y + \lambda_N\mu_Y\ \theta_0 c_0(s)(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s)) \\
& + k_0(s)\theta_0\lambda_N\mu_Y + \rho(s)c_1(s)(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s)) + \rho(s)k_1(s) \\
& + \alpha(s)X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s) + \beta X_2^{(\hat{\pi}_R,\hat{\pi}_I)}(s) \Big\} ds + \sigma_0 c_0(s)(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s))dW_0(s) \\
& + \sigma_0 k_0(s)dW_0(s) + \sigma(s)c_1(s)(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s))dW_1(s) \\
& + \sigma(s)k_1(s)dW_1(s) - \Big\{ c_0(s^-)(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s^-) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s^-)) + k_0(s^-) \Big\} \int_0^{+\infty} z\tilde{N}_0(ds,dz) \\
& + \Big\{ c_1(s^-)(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s^-) + \Theta X^{(\hat{\pi}_R,\hat{\pi}_I)}(s^-)) + k_1(s^-) \Big\} \int_{-1}^{+\infty} z\tilde{N}_1(ds,dz),
\end{aligned}
\tag{3.64}
$$

So, Noting that

$$
\begin{aligned}
& \Theta = \beta e^{\delta\lambda}, \quad \alpha(s) = \beta e^{\delta\lambda}(\mu(s) + \beta e^{\delta\lambda} + \lambda), \quad \mu(t) = r_0(t) - \alpha(t) - \beta \\
& \alpha(t) - \lambda\Theta = \mu(s)\Theta, \\
& dX_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s) = \Big(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s) - \lambda X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s) - e^{-\delta\lambda}X_2^{(\hat{\pi}_R,\hat{\pi}_I)}(s) \Big) ds, \\
& A(t) = \mu(s) + \Theta + \rho(t)c_1(t) + \lambda_N\mu_Y\theta_0 c_0(t),\ C_0(t) = \sigma_0 c_0(t) \\
& C_1(t) = \sigma(t)c_1(t), \chi_0(t) = \sigma_0 k_0(t) \text{ and } \chi_1(t) = \sigma(t)k_1(t).
\end{aligned}
\tag{3.65}
$$

So (3.64) is rewritten as follow

$$
\begin{cases}
d\Big(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s) \Big) \\
= \Big\{ A(s) \Big(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s) \Big) + B(t) \Big\} ds \\
+ \Big\{ C_0(s) \Big(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s) \Big) + \chi_0(t) \Big\} dW_0(s) \\
+ \Big\{ C_1(s) \Big(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s) \Big) + \chi_1(t) \Big\} dW_1(s) \\
- \Big\{ c_0(s^-) \Big(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s^-) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s^-) \Big) + k_0(s^-) \Big\} \int_0^{+\infty} z\tilde{N}_0(ds,dz) \\
+ \Big\{ c_1(s^-) \Big(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s^-) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s^-) \Big) + k_1(s^-) \Big\} \int_{-1}^{+\infty} z\tilde{N}_1(ds,dz). \\
dX_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s) = \Big(X^{(\hat{\pi}_R,\hat{\pi}_I)}(s) - \lambda X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(s) - e^{-\delta\lambda}X_2^{(\hat{\pi}_R,\hat{\pi}_I)}(s) \Big) ds.
\end{cases}
\tag{3.66}
$$

Next, we calculate $\mathbb{E}\left[X^{(\hat{\pi}_R,\hat{\pi}_I)}(t) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(t) \right]$ and $\mathbb{E}\left[\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(t) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(t) \right)^2 \right]$. We start by constructing the following exponential martingale process as follows

$$
\begin{aligned}
d\Gamma(t) \quad & = \Gamma(t) \Big\{ \Big(-A(t) + C_0^2(t) + C_1^2(t) \Big) dt - C_0((t))dW_0(t) - C_1((t))dW_1(t) \\
& + \int_0^{\infty} \{ \ln(1 + c_0((t^-))z) - c_0((t^-))z \} \nu_0(dz)dt - \int_0^{\infty} \ln(1 + c_0((t^-))z)\tilde{N}_0(ds,dz) \\
& + \int_{-1}^{\infty} \{ \ln(1 + c_1((t^-))z) - c_1((t^-))z \} \nu_1(dz)dt - \int_{-1}^{\infty} \ln(1 + c_1((t^-))z)\tilde{N}_1(dt,dz) \Big\},
\end{aligned}
\tag{3.67}
$$

implies that

$$\begin{aligned}\Gamma(t) &= \Gamma(0)\exp\left\{\int\limits_0^t \left[\left\{-A(s)+\frac{1}{2}C_0^2(s)+\frac{1}{2}C_1^2(s))+\int\limits_0^\infty\left(\ln(1+c_0(s^-)z)-c_0(s^-)z\right)\nu_0(dz)\right.\right.\right.\\ &\left.-\int\limits_{-1}^\infty\left(\ln(1+c_1(s^-)z)-c_1(s^-)z\right)\ \nu_1(dz)\right\}ds+C_0(s)dW_0(s)+C_1(s)dW_1(s)\\ &\left.\left.-\int\limits_0^\infty\ln(1+c_0(s^-)z)\tilde{N}_0(ds,dz)+\int\limits_{-1}^\infty\ln(1+c_1(s^-)z)\tilde{N}_1(ds,dz)\right]\right\}.\end{aligned} \tag{3.68}$$

Then

$$\begin{aligned}\frac{\Gamma(t)}{\Gamma(T)} &= \exp\left\{\int\limits_t^T\left[\left\{A(s)-\frac{1}{2}C_0^2(s)-\frac{1}{2}C_1^2(s))-\int\limits_0^\infty\left(\ln(1+c_0(s^-)z)-c_0(s^-)z\right)\ \nu_0(dz)\right.\right.\right.\\ &\left.+\int\limits_{-1}^\infty\left(\ln(1+c_1(s^-)z)-c_1(s^-)z\right)\ \nu_1(dz)\right\}ds+C_0(s)dW_0(s)+C_1(s)dW_1(s)\\ &\left.\left.+\int\limits_0^\infty\ln(1+c_0(s^-)z)\tilde{N}_0(ds,dz)-\int\limits_{-1}^\infty\ln(1+c_1(s^-)z)\tilde{N}_1(ds,dz)\right]\right\}.\end{aligned} \tag{3.69}$$

Moving now to apply Itô formula to $\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(t)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(t)\right)\Gamma(t)$ then, we derive, we get

$$\begin{aligned}\Gamma(t) &= \Gamma(0)\exp\left\{\int\limits_0^t\left[\left\{-A(s)+\frac{1}{2}C_0^2(s)+\frac{1}{2}C_1^2(s))\right.\right.\right.\\ &+\int\limits_0^\infty\left(\ln(1+c_0(s^-)z)-c_0(s^-)z\right)\ \nu_0(dz)-\int\limits_{-1}^\infty\left(\ln(1+c_1(s^-)z)-c_1(s^-)z\right)\ \nu_1(dz)\\ &\left.-\int\limits_{-1}^\infty\left(\ln(1+c_1(s^-)z)-c_1(s^-)z\right)\ \nu_1(dz)\right\}ds+C_0(s)dW_0(s)+C_1(s)dW_1(s)\\ &\left.\left.-\int\limits_0^\infty\ln(1+c_0(s^-)z)\tilde{N}_0(ds,dz)+\int\limits_{-1}^\infty\ln(1+c_1(s^-)z)\tilde{N}_1(ds,dz)\right]\right\}.\end{aligned} \tag{3.70}$$

Hence, we find that

$$\begin{aligned}
&d\left(\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(t)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(t)\right)\Gamma(t)\right)\\
&=\Gamma(t)\left\{B(t)+C_0(t)\chi_0(t)+C_1(t)\chi_1(t)\right\}dt+\Gamma(t)\left\{C_0(t)\chi_0(t)dW_0(t)\right.\\
&\left.+C_1(t)\chi_1(t)dW_1(t)\right\}+\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(t)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I))}(t)\right)\\
&\left\{\int_0^{+\infty}\left\{-\ln(1+c_0(t^-)z)-c_0(t^-)z-c_0(t^-)z\ln(1+c_0(t^-)z)\ \right\}\ \tilde{N}_0(dt,dz)\right.\\
&+\int_{-1}^{+\infty}\left\{\ln(1+c_1(t^-)z)-c_1(t^-)z-c_1(t^-)z\ln(1+c_1(t^-)z)\right\}\tilde{N}_1(ds,dz)\\
&+\int_0^{+\infty}\left\{\ln(1+c_0(t^-)z)-c_0(t^-)z\right\}\nu_0(dz)dt\\
&\left.-\int_{-1}^{+\infty}\left\{\ln(1+c_1(t^-)z)-c_1(t^-)z\right\}\nu_1(dz)dt\right\}\Gamma(t).
\end{aligned}\tag{3.71}$$

We move to find $\mathbb{E}\left[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right]$ and $\mathbb{E}\left[\left((X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)^2\right]$, next we take expectations, integrating from t to T in the above equation then rearranging it. We pick out that

$$\begin{aligned}
&X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\\
&=(x+\Theta x_1)\left(\frac{\Gamma(t)}{\Gamma(T)}+\int_t^T\left(\frac{\Gamma(s)}{\Gamma(T)}\gamma(s)\right)ds\right)\\
&+\int_t^T\left(\frac{\Gamma(s)}{\Gamma(T)}M(s)\right)ds+\int_t^T\frac{\Gamma(s)}{\Gamma(T)}[C_0(s)\chi_0(s)dW_0(s)+\chi_1(s)dW_1(s)].
\end{aligned}\tag{3.72}$$

Where $X^{(\hat{\pi}_R,\hat{\pi}_I)}(t)=x$, $X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(t)=x_1$, $\mathcal{M}(t)=B(t)+C_0(t)\chi_0(t)+C_1(t)\chi_1(t)$ and

$$\begin{aligned}
\gamma(t)\ &=\int_0^{+\infty}\left(-\ln(1+c_0(t^-)z)-c_0(t^-)z\ -c_0(t^-)z\ln(1+c_0(t^-)z)\right)\tilde{N}_0(dt,dz)\\
&+\int_{-1}^{+\infty}\left(\ln(1+c_1(t^-)z)-c_1(t^-)z-c_1(t^-)z\ln(1+c_1(t^-)z)\right)\tilde{N}_1(dt,dz)\\
&+\int_0^{+\infty}\left(\ln(1+c_0(t^-)z)-c_0(t^-)z\right)\nu_0(dz)-\int_{-1}^{+\infty}\left(\ln(1+c_1(t^-)z)-c_1(t^-)z\right)\nu_1(dz).
\end{aligned}\tag{3.73}$$

So $\mathbb{E}\left[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right]$ is given by

$$\begin{aligned}
&\mathbb{E}\left[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right]\\
&=(x+\Theta x_1)\left[\mathbb{E}\left[\frac{\Gamma(t)}{\Gamma(T)}\right]+\mathbb{E}\left[\int_t^T\frac{\Gamma(s)}{\Gamma(T)}\gamma(t)ds\right]\right]\\
&+\mathbb{E}\left[\int_t^T\frac{\Gamma(s)}{\Gamma(T)}\mathcal{M}(s)ds+\int_t^T\frac{\Gamma(s)}{\Gamma(T)}\left(C_0(s)\chi_0(s)dW_0(s)+\chi_1(s)dW_1(s)\right)\right].
\end{aligned}\tag{3.74}$$

We calculate the expectations we get

$$\begin{aligned}
&\mathbb{E}\left[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right]\\
&=(x+\Theta x_1)\left\{e^{\int_s^T(A(u)+\Im(u))du}+\int_t^T\left(e^{\int_s^T(A(u)+\Im(u))du}\left[\int_0^{+\infty}\left(\ln(1+c_0(s)z)-c_0(s)z\right)\nu_0(dz)\right.\right.\right.\\
&\left.\left.\left.-\int_{-1}^{+\infty}\left(\ln(1+c_1(t)z)-c_1(t)z\right)\nu_1(dz)\right]\right)ds\right\}+\int_t^T e^{\int_s^T(A(u)+\Im(u))du}\mathcal{M}(s)ds.
\end{aligned}\tag{3.75}$$

Where

$$\mathbb{E}[\frac{\Gamma(t)}{\Gamma(T)}]=e^{\int_t^T(A(s)+\Im(s))ds}.\tag{3.76}$$

And

$$\Im(s)=\int_0^{\infty}\left(-\ln(1+c_0(s)z)+c_0(s)z\right)\nu_0(dz)+\int_{-1}^{\infty}\left(\ln(1+c_1(s)z)-c_1(s)z\right)\nu_1(dz).$$

$$\mathbb{E}\left[\gamma(s)\right]=\left(\int_0^{+\infty}\left(\ln(1+c_0(t)z)-c_0(t)z\right)\nu_0(dz)\right)-\left(\int_{-1}^{+\infty}\left(\ln(1+c_1(t)z)-c_1(t)z\right)\nu_1(dz)\right).$$

Hence

$$\mathbb{E}\left[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right]=\psi_1(t)\ (x+\Theta x_1)+Q_1(t).\tag{3.77}$$

Where

$$\psi_1(t)=e^{\int_t^T(A(u)+\Im(u))du}-\int_t^T e^{\int_s^T(A(u)+\Im(u))du}\Im(s)ds.\tag{3.78}$$

And

$$Q_1(t)=\int_t^T e^{\int_s^T(A(u)+\Im(u))du}\mathcal{M}(s)ds.\tag{3.79}$$

Next, we calculate $\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)^2$

$$\begin{aligned}
&\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)^2\\
&=(x+\Theta x_1)^2\left[\frac{\Gamma(t)}{\Gamma(T)}+\int_t^T\frac{\Gamma(s)}{\Gamma(T)}\gamma(s)ds\right]^2\\
&+\left[\int_t^T\frac{\Gamma(s)}{\Gamma(T)}M(s)ds+\int_t^T\frac{\Gamma(s)}{\Gamma(T)}\left(C_0(s)\chi_0(s)dW_0(s)+C_1(s)\chi_1(s)dW_1(s)\right)\right]^2 .\\
&+2(x+\Theta x_1)\left[\frac{\Gamma(t)}{\Gamma(T)}+\int_t^T\frac{\Gamma(s)}{\Gamma(T)}\gamma(s)ds\right]\left[\int_t^T\frac{\Gamma(s)}{\Gamma(T)}\mathcal{M}(s)ds\right.\\
&\left.+\int_t^T\frac{\Gamma(s)}{\Gamma(T)}\left(C_0(s)\chi_0(s)dW_0(s)+C_1(s)\chi_1(s)dW_1(s)\right)\right].
\end{aligned}\tag{3.80}$$

Hence, we calculate $\mathbb{E}\left[\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(T)+\Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)^2\right]$. We obtain

$$\begin{aligned}
&\mathbb{E}\left[\frac{\Gamma(t)}{\Gamma(T)}+\int_t^T\frac{\Gamma(s)}{\Gamma(T)}\gamma(s)ds\right]^2\\
&=\mathbb{E}\left[\left(\frac{\Gamma(t)}{\Gamma(T)}\right)^2\right]+\int_t^T\mathbb{E}\left[\frac{\Gamma(s)}{\Gamma(T)}\right]\mathbb{E}\left[\gamma(s)\right]ds\\
&\int_t^T\mathbb{E}\left[\frac{\Gamma(s)}{\Gamma(T)}\right]\mathbb{E}\left[\gamma(s)\right]ds+2\mathbb{E}\left[\left(\frac{\Gamma(t)}{\Gamma(T)}\int_t^T\frac{\Gamma(s)}{\Gamma(T)}\gamma(s)ds\right)\right].
\end{aligned}\tag{3.81}$$

We have

$$\begin{aligned}
&\int_t^T\mathbb{E}\left[\frac{\Gamma(s)}{\Gamma(T)}\gamma(s)\right]ds\int_t^T\mathbb{E}\left[\frac{\Gamma(s)}{\Gamma(T)}\gamma(s)\right]ds\\
&=\left(\int_t^T e^{\int_t^T(A(s)+\Im(s))ds}\left[\left(\int_0^{+\infty}\left(\ln(1+c_0(t)z)-c_0(t)z\right)\nu_0(dz)\right)\right.\right.\\
&\left.\left.-\left(\int_{-1}^{+\infty}\left(\ln(1+c_1(t)z)-c_1(t)z\right)\nu_1(dz)\right)ds\right]\right)^2.
\end{aligned}\tag{3.82}$$

And

$$\mathbb{E}\left[\left(\frac{\Gamma(t)}{\Gamma(T)}\int_t^T \frac{\Gamma(s)}{\Gamma(T)}\gamma(s)ds\right)\right]$$
$$= 2\int_t^T \left\{ e^{\int_t^s A(u)du} e^{\int_s^T (A(u)+\varsigma(u))du} \right.$$
$$\left.\left(\int_0^{+\infty} (\ln(1+c_0(t)z) - c_0(t)z)\,\nu_0(dz)\right) - \left(\int_{-1}^{+\infty} (\ln(1+c_1(t)z) - c_1(t)z)\,\nu_1(dz)\right)\right\} ds. \tag{3.83}$$

We note $\mathbb{E}\left[\left(\frac{\Gamma(t)}{\Gamma(T)}\right)^2\right] = e^{\int_s^T (A(u)+\varsigma(u)+2\Im(u))du}$, where $\varsigma(u) = A(u) + C_0^2(u) + C_1^2(u)$. Consequently,

$$\mathbb{E}\left[\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(T) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)\right)^2\right] = S(t)\left(x+\Theta x_1\right)^2 + \psi_2(t)(x+\Theta x_1) + Q_2\left(t\right). \tag{3.84}$$

Where

$$S(t) = e^{\int_s^T (A(u)+\varrho(u)+2\Im(u))du} + \int_t^T e^{\int_s^T 2(A(u)+\Im(u))du}\,\Im^2(s)ds, \tag{3.85}$$

and

$$Q_2\left(t\right) = \mathbb{E}\left[\int_t^T \frac{\Gamma(s)}{\Gamma(T)}\mathcal{M}(s)ds + \int_t^T \frac{\Gamma(s)}{\Gamma(T)}\left(C_0(s)\chi_0(s)dW_0(s) + C_1(s)\chi_1(s)dW_1(s)\right)^2\right]. \tag{3.86}$$

And

$$\psi_2(t) = 2e^{\int_s^T (A(u)+\Im(u))du} - \int_t^T e^{\int_s^T (A(u)+\Im(u))du}\,\Im(s).\mathcal{M}(s)ds. \tag{3.87}$$

Rendering , we have already that

$$\begin{aligned} &h(t,x,x_1,y,y_1) \\ &= \mathbb{E}_{t,x,x_1}\left[\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(T) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T))\right)\right] \\ &-\frac{\eta}{2(y+\Theta y_1)}\mathbb{E}_{t,x,x_1}\left[\left(X^{(\hat{\pi}_R,\hat{\pi}_I)}(T) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T))\right)^2\right]. \end{aligned} \tag{3.88}$$

So

$$\begin{aligned} &h(t,x,x_1,y,y_1) = \psi_1(t)\ (x+\Theta x_1) + Q_1(t) \\ &-\frac{\eta}{2(y+\Theta y_1)}\left[S(t)\left(x+\Theta x_1\right)^2 + \psi_2(t)\ (x+\Theta x_1) + Q_2(t)\right]. \end{aligned}$$

and

$$g(t,x,x_1) = \mathbb{E}_{t,x,x_1}[X^{(\hat{\pi}_R,\hat{\pi}_I)}(T) + \Theta X_1^{(\hat{\pi}_R,\hat{\pi}_I)}(T)] = \psi_1(t)\ (x+\Theta x_1) + Q_1(t). \tag{3.90}$$

Consequently, as $\hat{\pi} = (\hat{\pi}_R, \hat{\pi}_I)$ is the feedback control which realises the supermum in V- equation in the first equation in (3.61) , Let 's define the function Λ as follow

$$\begin{aligned}
&\Lambda(\pi_R, \pi_I) \\
&= \left\{h_{x_1} + \tfrac{\eta}{x+\Theta x_1} g g_{x_1}\right\} \left\{x - e^{-\delta\lambda} x_2 - \lambda x_1\right\} + \left\{h_x + \tfrac{\eta}{x+\Theta x_1} g g_x\right\} \\
&\{\mu(t)x + [\delta + (1+\theta_0)\,\pi_R]\,\lambda_N \mu_Y + \pi_I \rho(t) + \alpha(t) x_1 + \beta x_2 - \pi_I \lambda_{\tilde{N}} \mu_Z\} \\
&+\frac{1}{2}\left\{h_{xx} + \tfrac{\eta}{x+\Theta x_1} g g_{xx}\right\} \left\{(\pi_R \sigma_0)^2 + (\pi_I \sigma(t))^2\right\} - \int_0^\infty h\nu_0(dz) - \int_{-1}^\infty h\nu_1(dz) \\
&+\int_0^\infty h(t, x - \pi_R z, x_1, y, y_1)\nu_0(dz) + \int_{-1}^\infty h(t, x + \pi_I z, x_1, y, y_1)\nu_1(dz) \\
&+\gamma g\left(\int_0^\infty \{g(t, x - \pi_R(t,x,x_1)z, x_1) - g\}\,\nu_0(dz)\right) \\
&+\gamma g\left(\int_0^\infty \{g(t, x + \pi_I z, x_1) - g\}\,\nu_1(dz)\right) .
\end{aligned} \tag{3.91}$$

We move now to calculate the following derivatives

$$\begin{aligned}
&h_x = -\frac{\eta}{(x_1 + \Theta y_1)} S(t)\,(x + \Theta y) + \psi_1(t) - \frac{\eta}{2(x_1 + \Theta y_1)} Q_1(t),\ h_{xx} = -\tfrac{\eta}{(x_1+\Theta y_1)} S(t), \\
&g_t = \psi_1'(t)\ (x + \Theta x_1) + Q_1'(t), \quad g_x = \psi_1(t), \quad g_{xx} = 0.
\end{aligned} \tag{3.92}$$

While h and it's derivatives are evaluated at (t, x, x, x_1) and g and it's derivatives are evaluated at (t, x, x_1).

Then, by the first order condition of optimality, let's differentiating the function Λ with respect π_R and π_I we obtain

$$\begin{cases}
\dfrac{\partial \Lambda(\hat{\pi}_R, \hat{\pi}_I)}{\partial \hat{\pi}_R} = \left\{-\eta S(t) + \psi_1(t) + \dfrac{\eta}{2(x_1 + \Theta y_1)}\psi_2(t) + \eta\psi_1^2(t) \ + \dfrac{\eta}{x + \Theta x_1}\psi_1(t) Q_1(t)\right\} \\
\qquad .\theta_0 \lambda_N \mu_Y - \dfrac{1}{2}\dfrac{\eta}{(x_1 + \Theta y_1)} S(t)\left[\sigma_0{}^2 + \lambda_N \sigma_Z^2\right]\hat{\pi}_R\ , \\
\dfrac{\partial \Lambda(\hat{\pi}_R, \hat{\pi}_I)}{\partial \hat{\pi}_I} = \left\{-\eta S(t) + \psi_1(t) + \dfrac{\eta}{2(x_1 + \Theta y_1)}\psi_2(t) + \eta\psi_1^2(t) \ + \dfrac{\eta}{x + \Theta x_1}\psi_1(t) Q_1(t)\right\} \\
\qquad .\rho(t) - \dfrac{1}{2}\dfrac{\eta}{(y + \Theta y_1)} S(t)\left[\sigma^2(t) + \lambda_{\tilde{N}} \sigma_Z^2\right]\hat{\pi}_I\ .
\end{cases} \tag{3.93}$$

So, the equilibrium reinsurance-investment strategies are

$$\begin{cases}
\hat{\pi}_R(t, x, x_1) = -\dfrac{\theta_0 \lambda_N \mu_Y}{\left[\sigma_0{}^2 + \lambda_N \sigma_Z^2\right]}\left[\dfrac{(\psi_1(t) - \eta S(t) + \eta\psi_1^2(t))}{-\eta S(t)}(x + \Theta x_1) + \dfrac{\left(P_1(t) Q_1(t) - \ \frac{1}{2}\psi_2(t)\right)}{-S(t)}\right], \\
\hat{\pi}_I(t, x, x_1) = -\dfrac{\rho(t)}{\left[\sigma^2(t) + \lambda_{\tilde{N}} \sigma_Z^2\right]}\left[\dfrac{(\psi_1(t) - \eta S(t) + \eta\psi_1^2(t))}{-\eta S(t)}(x + \Theta x_1) + \dfrac{\left(P_1(t) Q_1(t) - \ \frac{1}{2}\psi_2(t)\right)}{-S(t)}\right].
\end{cases} \tag{3.94}$$

Hence , we compare with our assumptions to find finally the expressions of $c_0(t)$, $c_1(t)$, $k_0(t)$ and $k_1(t)$ as following

$$\begin{cases} c_0(t) = & -\dfrac{\theta_0\lambda_N\mu_Y}{\sigma_0^2+\lambda_N\sigma_Z^2}\dfrac{\psi_1(t)-\eta S(t)+\eta\psi_1^2(t)}{-S(t)\eta}, \\ k_0(t) = & -\dfrac{\theta_0\lambda_N\mu_Y}{\sigma_0^2+\lambda_N^2\sigma_Z^2}\dfrac{\psi_1(t)Q_1(t)-\ \frac{1}{2}\psi_2(t)}{-S(t)}, \\ c_1(t) = & -\dfrac{\rho(t)}{\sigma^2(t)+\lambda_{\tilde{N}}\sigma_Z^2}\dfrac{\psi_1(t)-\eta S(t)+\eta\psi_1^2(t)}{-S(t)\eta}, \\ k_1(t) = & -\dfrac{\rho(t)}{\sigma^2(t)+\lambda_{\tilde{N}}\sigma_Z^2}\dfrac{\psi_1(t)Q_1(t)-\ \frac{1}{2}\psi_2(t)}{-S(t)}. \end{cases} \tag{3.95}$$

Where

$$\begin{aligned} \frac{(\psi_1(t)-\eta S(t)+\eta\psi_1^2(t))}{-S(t)\eta} &= -\frac{\psi_1(t)}{\eta S(t)}-\frac{\psi_1^2(t)}{S(t)}+1 \\ &= \frac{-1}{\eta}e^{\int_t^T -\varsigma(u)du}-e^{\int_t^T (A(u)-\varsigma(u))du}+1 \\ &= \frac{-1}{\eta}\left[e^{\int_t^T -\varsigma(u)du}+\eta e^{\int_t^T (A(u)-\varsigma(u))du}-\eta\right], \end{aligned}$$

and

$$\begin{aligned} \frac{\psi_1(t)Q_1(t)-\ \frac{1}{2}\psi_2(t)}{-S(t)} = -&\left[e^{-\int_t^T \varsigma(u)du}\int_t^T e^{\int_s^T A(u)du}\left(B\left(s\right)+C\left(s\right)\chi(s)\right)ds\right. \\ &\left.-\int_t^T e^{-\int_t^s \varsigma(u)du}ds.\right] \end{aligned}$$

∎

Remark 3.9

The corresponding equilibrium value function is given by

$$\begin{aligned} &V(t,x,x_1) = h(t,x,x_1,x,x_1)+\frac{\eta(x,x_1)}{2}g^2(t,x,x_1) \\ &= \psi_1(t)\ (x+\Theta x_1)+Q_1(t)-\frac{\eta}{2\left(x+\Theta x_1\right)}\left[S(t)\left(x+\Theta x_1\right)^2+\psi_2(t)(x+\Theta x_1)+Q_2(t)\right] \\ &+\frac{\eta}{2\left(x+\Theta x_1\right)}\left(\psi_1(t)(x+\Theta x_1)+Q_1(t)\right)^2. \end{aligned} \tag{3.96}$$

Conclusion

In this thesis, we have investigated about two stochastic optimal control problems that, in various ways, are time inconsistent in the sense that they do not admit a Bellman optimality principle. In the second chapter, we have developped a theory addressing a broad class of time-inconsistent stochastic control problems characterized by stochastic differential delayed equations (SDDEs), indicating the absence of a Bellman optimality principle. The approach involves framing these problems within a game theoretic framework and seeking subgame perfect Nash equilibrium strategies. For a general controlled process with delay and a reasonably broad objective functional, we have extended the standard Bellman equation into a system of nonlinear equations. This extension has facilitated the determination of both the equilibrium strategy and the equilibrium value function. Importantly, to exemplify the theory's applicability, we have delved into specific example such mean-variance portfolio with state dependent risk aversion problem with delay. In the third chapter, we have studied an optimal investment and reinsurance problem, in stochastic delayed model incorporating jumps in the financial market and the instantaneous capital inflow into or outflow from the insurers current wealth , under mean- variance with state dependent risk aversion model. It is well-known that the optimal control problems with delay are complicated to solve in general since the objective functional may depend on the initial path in a complicated way. So, inspired from the previous literatures, we have made some restrictive conditions on the past dependence of the performance functional. As well as we, on the delay parameters which reduce the original problem from an infinite-dimensional to finite dimensional optimization problem.

Future work: Many interesting problems remain open. For example, Solving consumption-investment time inconsistency problems in stochastic differential equations with delay.

Appendix

Theorem 3.10 (*Feynman-Kac formula*)

We consider the partial differential equation:

$$\begin{aligned}
&\frac{\partial}{\partial t}u\left(t,x\right)+\mu\left(t,x\right)\frac{\partial}{\partial x}u\left(t,x\right)+\frac{1}{2}\sigma^{2}\left(t,x\right)\frac{\partial^{2}}{\partial x^{2}}u\left(t,x\right)-V\left(t,x\right)u\left(t,x\right)\\
&+f\left(t,x\right)\\
&=0,
\end{aligned} \tag{A.1}$$

defined for all $x \in \mathbb{R}$ and $t \in [0, T]$, subject to the terminal condition

$$u\left(T,x\right)=\psi(x), \tag{A.2}$$

where μ, σ, f, ψ, V are known functions, T is a parameter and $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ is the unknown. Then the Feynman-Kac formula expresses $u(t, x)$ as a conditional expectation under the probability measure Q

$$u\left(t,x\right)=\mathbb{E}^{Q}\left[e^{-\int_{t}^{T}V(\tau,X_{\tau})d\tau}\psi(X_{T})+\int_{t}^{T}e^{-\int_{t}^{\tau}V(s,X_{s})ds}f\left(X_{\tau},\tau\right)d\tau\mid X_{t}=x\right], \tag{A.3}$$

where X is an Itô process satisfying

$$dX_{t}=\mu\left(t,X_{t}\right)dt+\sigma\left(t,X_{t}\right)dW_{t}^{Q}, \tag{A.4}$$

where W_t^Q a Wiener process (also called Brownian motion) under Q.

Proof: A proof that the above formula is a solution of the differential equation is long, difficult and not presented here. It is however reasonably straightforward to show that, if a solution exists, it must have the above form. The proof of that lesser result is as follows: Let $u(t, x)$ be the solution to the above partial differential equation. Applying the product rule for itô processes to the process

$$Y(s) = \exp\left(-\int_t^s V(\tau, X_\tau)\, d\tau\right) u(s, X_s) + \int_t^s \exp\left(-\int_t^r V(\tau, X_\tau)\, d\tau\right) f(r, X_r)\, dr, \tag{A.5}$$

we get

$$\begin{aligned} dY_s &= d\left(\exp\left(-\int_t^s V(\tau, X_\tau)\, d\tau\right)\right) u(s, X_s) + \exp\left(-\int_t^s V(\tau, X_\tau)\, d\tau\right) du(s, X_s) \\ &\quad + d\left(\exp\left(-\int_t^s V(\tau, X_\tau)\, d\tau\right)\right) du(s, X_s) + d\left(\int_t^s \exp\left(\int_t^r V(\tau, X_\tau)\, d\tau\right) f(r, X_r)\, dr\right). \end{aligned} \tag{A.6}$$

Since

$$d\left(\exp\left(-\int_t^s V(\tau, X_\tau)\, d\tau\right)\right) = -V(s, X_s)\left(\exp\left(-\int_t^s V(\tau, X_\tau)\, d\tau\right)\right) ds, \tag{A.7}$$

the third term is $O(dtdu)$ and can be dropped, we also have that

$$d\left(\int_t^s \exp\left(\int_t^r V(\tau, X_\tau)\, d\tau\right) f(r, X_r)\, dr\right) = \exp\left(-\int_t^s V(\tau, X_\tau)\, d\tau\right) f(s, X_s)\, ds. \tag{A.8}$$

Applying itô's lemma to $du(s, X_s)$, it follows that

$$\begin{aligned} dY_s &= d\left(\exp\left(-\int_t^s V(\tau, X_\tau)\, d\tau\right)\right)\left(-V(s, X_s) u(s, X_s) + f(s, X_s) + \mu(s, X_s)\frac{\partial u}{\partial X}\right. \\ &\quad \left. + \frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2}{\partial X^2} u(s, X_s)\, ds\right) + \exp\left(-\int_t^s V(\tau, X_\tau)\, d\tau\right)\sigma(s, X_s)\frac{\partial u}{\partial X} dW. \end{aligned} \tag{A.9}$$

The first term contains, in parentheses, the above partial differential equation and is therefore zero. What remains is:

$$dY_s = d\left(\exp\left(-\int_t^s V(\tau, X_\tau)\, d\tau\right)\right)\sigma(s, X_s)\frac{\partial u}{\partial X} dW \tag{A.10}$$

Integrating this equation from t to T, we conclude that

$$Y(T) - Y(t) = \int_t^T \exp\left(-\int_t^s V(\tau, X_\tau)\, d\tau\right)\sigma(s, X_s)\frac{\partial u}{\partial X} dW. \tag{A.11}$$

Upon taking expectations conditioned on $X_t = x$, and observing that the right side is an it itô integral which has expectation zero, it follows that

$$\mathbb{E}[Y(T) \mid X_t = x] = \mathbb{E}[Y(t) \mid X_t = x] = u(t, x). \tag{A.12}$$

The desired result is obtained by observing that:

$$\mathbb{E}\left[Y(t) \mid X_t = x\right] = \mathbb{E}\left[\exp\left(-\int_t^T V(\tau, X_\tau)\, d\tau\right) u(T, X_T) \right.$$
$$\left. + \int_t^T \exp\left(\int_t^r V(\tau, X_\tau)\, d\tau\right) f(r, X_r) \mid X_t = x\right], \tag{3.2}$$

finally,

$$u(t, x) = \mathbb{E}\left[\exp\left(-\int_t^T V(\tau, X_\tau)\, d\tau\right) \psi(X_T) + \int_t^T \left(\exp - \int_t^T V(\tau, X_\tau)\, d\tau\right) f(s, X_s)\, ds \mid X_t = x\right]. \tag{A.13}$$

■

lemma (Gronwall's lemma)

Let $X(t)$ and $f(t)$ be nonnegative continuous functions on $0 \leq t \leq T$, for which the inequality

$$X(t) \leq c + \int_0^t f(s) X(s)\, ds, \qquad t \in [0, T]$$

holds, where $c \geq 0$ is a constant. Then

$$X(t) \leq c \exp\left(\int_0^t f(s)\, ds\right), \qquad t \in [0, T]$$

Bibliography

[1] Alia, I., Chighoub, F., & Sohail, A. (2016). A characterization of equilibrium strategies in continuous-time mean–variance problems for insurers. Insurance: Mathematics and Economics, 68, 212-223.

[2] Alia, I., Chighoub, F., & Sohail, A. (2016). The maximum principle in time-inconsistent LQ equilibrium control problem for jump diffusions. Serdica Math. J, 42, 103-138.

[3] Alia, I., Chighoub, F., Khelfallah, N., & Vives, J. (2021). Time-consistent investment and consumption strategies under a general discount function. Journal of Risk and Financial Management, 14(2), 86.

[4] Andersson,D. Djehiche.B.(2010). A maximum principle for SDE's of mean-field type, Applied Math. and Optimization, **63**(3) , 341-356.

[5] Bahlali, D., & Chighoub, F. (2024). A general time-inconsistent stochastic optimal control problem with delay. Studies in Engineering and Exact Sciences, 5(2), e6922-e6922.

[6] Bajeux-Besnainou, I. Portait, R. (1998). Dynamic asset allocation in a mean-variance framework, Management Science, **44**(11) , 79–95.

[7] Bai, L., & Zhang, H. (2008). Dynamic mean-variance problem with constrained risk control for the insurers. Mathematical Methods of Operations Research, 68(1), 181-205.

[8] Bai, L., Cai, J., & Zhou, M. (2013). Optimal reinsurance policies for an insurer with a bivariate reserve risk process in a dynamic setting. Insurance: Mathematics and Economics, 53(3), 664-670.

[9] Basak, S., & Chabakauri, G. (2010). Dynamic mean-variance asset allocation. The Review of Financial Studies, 23(8), 2970-3016.

[10] Bäuerle, N. (2005). Benchmark and mean-variance problems for insurers. Mathematical Methods of Operations Research, 62(1), 159-165.

[11] Bellman.R. (1957). Dynamic Programming, Princeton University Press, Princeton, New Jersey .

[12] Bi, J., & Cai, J. (2019). Optimal investment–reinsurance strategies with state dependent risk aversion and VaR constraints in correlated markets. Insurance: Mathematics and Economics, 85, 1-14.

[13] Bi, J., Meng, Q., & Zhang, Y. (2014). Dynamic mean-variance and optimal reinsurance problems under the no-bankruptcy constraint for an insurer. Annals of Operations Research, 212(1), 43-59.

[14] Bielecki,T.R. Jin,H. Pliska,S. Zhou.X. (2005). Continuous-time mean-variance portfolio selection with bankruptcy prohibition, Mathematical Finance, **15**(2), 213–244.

[15] Björk, T., Murgoci, A.(2008). *A general theory of Markovian time-inconsistent stochastic control problems*, SSRN: 1694759.

[16] Björk, T., Murgoci, A., & Zhou, X. Y. (2014). Mean–variance portfolio optimization with state-dependent risk aversion. Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics, 24(1), 1-24

[17] Björk, T., Khapko, M., & Murgoci, A. (2017). On time-inconsistent stochastic control in continuous time. Finance and Stochastics, 21, 331-360.

[18] Bouaicha, N.E.H. Chighoub, F. Alia,I. Sohail, A.(2022). Conditional LQ time-inconsistent Markov-switching stochastic optimal control problem for diffusion with jumps, Modern Stochastics: Theory and Applications, **9**(2), 157-205.

[19] Bouaicha, N.E.H. (2022). Nash equilibrium strategies of an inconsistent stochastic control problem, Doctorate thesis

[20] Browne, S. (1995). Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. Mathematics of operations research, 20(4), 937-958

[21] Cao, Y., & Wan, N. (2009). Optimal proportional reinsurance and investment based on Hamilton–Jacobi–Bellman equation. Insurance: Mathematics and Economics, 45(2), 157-162

[22] C. Karnam, J. Ma, J. Zhang.(2017). Dynamic approaches for some time inconsistent problems, The Annals of Applied Probability, **27**, 3435–3477

[23] Chang, M. H., Pang, T., & Yang, Y. (2011). A stochastic portfolio optimization model with bounded memory. Mathematics of Operations Research, 36(4), 604-619.

[24] Chen, L., & Wu, Z. (2010). Maximum principle for the stochastic optimal control problem with delay and application. Automatica, 46(6), 1074-1080.

[25] Czichowsky, C. (2013). Time-consistent mean-variance portfolio selection in discrete and continuous time. Finance and Stochastics, 17(2), 227-271.

[26] Dai, M. Xu, Z. Zhou.X. (2010). Continuous-time Markowitz's model with transaction costs, SIAM Journal on Financial Mathematics, **1** , 96–125.

[27] David Promislow, S., & Young, V. R. (2005). Minimizing the probability of ruin when claims follow Brownian motion with drift. North American Actuarial Journal, 9(3), 110-128.

[28] David, D. (2008). Optimal control of stochastic delayed systems with jumps.

[29] Delong, Ł., & Gerrard, R. (2007). Mean-variance portfolio selection for a non-life insurance company. Mathematical Methods of Operations Research, 66(2), 339-367.

[30] Du, H., Huang, J., & Qin, Y. (2013). A stochastic maximum principle for delayed mean-field stochastic differential equations and its applications. IEEE Transactions on Automatic Control, 58(12), 3212-3217.

[31] Ekeland, I., Lazrak, A., (2008). *Equilibrium policies when preferences are time inconsistent.* ArXiv:0808.3790v1

[32] Ekeland, I., & Pirvu, T. A. (2008). Investment and consumption without commitment. Mathematics and Financial Economics, 2(1), 57-86.

[33] Elsanosi, I., & Larssen, B. (2001). Optimal consumption under partial observations for a stochastic system with delay. Preprint series. Pure mathematics http://urn. nb. no/URN: NBN: no-8076.

[34] Kanwal, R. P. (1997). Linear integral equations: Theory and technique. New York: Springer.

[35] F.E. Kydland, E. Prescott. (1997). Rules rather than discretion: The inconsistency of optimal plans, Journal of Political Economy, **85**, 473–492.

[36] Federico, S., Goldys, B., & Gozzi, F. (2010). HJB equations for the optimal control of differential equations with delays and state constraints, I: regularity of viscosity solutions. SIAM Journal on Control and Optimization, 48(8), 4910-4937.

[37] Federico, S. (2011). A stochastic control problem with delay arising in a pension fund model. Finance and Stochastics, 15(3), 421-459.

[38] Guan, G., & Liang, Z. (2014). Optimal reinsurance and investment strategies for insurer under interest rate and inflation risks. Insurance: Mathematics and Economics, 55, 105-115.

[39] Loewenstein, G. Prelec, D.(1992). Anomalies in intertemporal choice: Evidence and an interpretation, The Quarterly Journal of Economics, **107**(2), 573–597.

[40] Gu, M., Yang, Y., Li, S., & Zhang, J. (2010). Constant elasticity of variance model for proportional reinsurance and investment strategies. Insurance: Mathematics and Economics, 46(3), 580-587.

[41] Hjgaard, B., & Taksar, M. (1998). Optimal proportional reinsurance policies for diffusion models. Scandinavian Actuarial Journal, 1998(2), 166-180.

[42] Hipp, C., & Plum, M. (2000). Optimal investment for insurers. Insurance: Mathematics and Economics, 27(2), 215-228.

[43] Hu, Y., Jin, H., & Zhou, X. Y. (2012). Time-inconsistent stochastic linear–quadratic control. SIAM journal on Control and Optimization, 50(3), 1548-1572.

[44] Hu, Y., Jin, H., & Zhou, X. Y. (2017). Time-inconsistent stochastic linear-quadratic control: characterization and uniqueness of equilibrium. SIAM Journal on Control and Optimization, 55(2), 1261-1279.

[45] Larssen, B., & Risebro, N. H. (2003). When are HJB-equations in stochastic control of delay systems finite dimensional? (2003): 643-671

[46] Lasry, J.M. Lions,P.L. (2007). Mean-field games, Japanese Journal of Mathematics, **2**, 229–260.

[47] Li, J. (2012). Stochastic maximum principle in the mean-field controls, Automatica, **48**, 366–373.

[48] Loewenstein, G. Prelec, D.(1992). Anomalies in intertemporal choice: Evidence and an interpretation, The Quarterly Journal of Economics, **107**(2), 573-597.

[49] L.Pedersen, Peskir, J.G. (2017). Optimal mean-variance portfolio selection, Mathematics and Financial Economics, **11**, 137-160.

[50] Li, Z., Zeng, Y., & Lai, Y. (2012). Optimal time-consistent investment and reinsurance strategies for insurers under Heston's SV model. Insurance: Mathematics and Economics, 51(1), 191-203

[51] Lim, A. E. Zhou, B. X. (2002). Quadratic hedging and mean-variance portfolio selection with random parameters in a complete market, Mathematics of Operations Research, **27**(1), 101-120.

[52] Li, D.Ng, W.(2000). Optimal dynamic portfolio selection: Multi-period mean-variance formulation, Mathematical Finance, **10**, 387–406.

[53] Li, D., Rong, X., & Zhao, H. (2015). Time-consistent reinsurance-investment strategy for an insurer and a reinsurer with mean-variance criterion under the CEV model. Journal of Computational and Applied Mathematics, 283, 142-162.

[54] Luo, S., & Taksar, M. (2011). On absolute ruin minimization under a diffusion approximation model. Insurance: Mathematics and Economics, 48(1), 123-133.

[55] Markowitz, H.M. (1952). Portfolio selection, Journal of Finance, **7**, 77-91.

[56] Mohammed, S. E. A.(1996). *Stochastic differential equations with memory: theory, examples and applications.* In Progress in probability, Stochastic analysis and related topics 6. The geido workshop. Birkhauser.B.

[57] Phelps, E. S., Pollak, R. A.(1968). *On second-best national saving and game-equilibrium growth.* The Review of Economic Studies, 35, , 185-199

[58] Pollak, R.(1968). *Consistent planning.* The Review of Financial Studies, 35, 2 , 201-208.

[59] Powell, W. (2011) Approximate Dynamic Programming, John Wiley and Sons.

[60] Ramsey F. P. (1928). *A Mathematical Theory of Saving.* Economic Journal 38, 543-559.

[61] Rubinstein, M. (2002). Markowitz's" portfolio selection": A fifty-year retrospective. The Journal of finance, 57(3), 1041-1045.

[62] Richardson, H.R. (1989). A minimum variance result in continuous trading portfolio optimization, Management Science, **35** (9), 1045-1055.

[63] ∅ksendal, B, & Sulem, A. (2001). A maximum principle for optimal control of stochastic systems with delay, with applications to finance. In Optimal control and partial differential equations (Paris, 4 December 2000) (pp. 64-79).

[64] ∅ksendal, B., & Sulem, A. (2005). Stochastic Control of jump diffusions (pp. 39-58). Springer Berlin Heidelberg.

[65] Sheng, Li (2021). Optimal time-consistent investment-reinsurance strategy for state-dependent risk aversion with delay and common shocks. Communications in Statistics-Theory and Methods, 1-38.

[66] Shi,J. (2013). Two different approaches to stochastic recursive optimal control problem with delay and applications. arXiv preprint arXiv:1304.6182.

[67] Shen, Y., & Siu, T. K. (2013). The maximum principle for a jump-diffusion mean-field model and its application to the mean–variance problem. Nonlinear Analysis: Theory, Methods & Applications, 86, 58-73.

[68] Shen, Y., Meng, Q., & Shi, P. (2014). Maximum principle for mean-field jump-diffusion stochastic delay differential equations and its application to finance. Automatica, 50(6), 1565-1579.

[69] Shen, Y., & Zeng, Y. (2014). Optimal investment-reinsurance with delay for mean–variance insurers: A maximum principle approach. Insurance: Mathematics and Economics, 57, 1-12.

[70] Strotz, R. H. (1955). Myopia and inconsistency in dynamic utility maximization. The review of economic studies, 23(3), 165-180.

[71] Solano, J. and Navas, J.(2010). *Consumption and portfolio rules for time-inconsistent investors.* European Journal of Operational Research, 201, 860-872.

[72] Xu, L., Wang, R., & Yao, D. (2008). On maximizing the expected terminal utility by investment and reinsurance. Journal of Industrial & Management Optimization, 4(4), 801.

[73] Yang, H., & Zhang, L. (2005). Optimal investment for insurer with jump-diffusion risk process. Insurance: Mathematics and Economics, 37(3), 615-634.

[74] Yong, J.(2011). *A deterministic linear quadratic time-inconsistent optimal control problem.* Mathematical Control and Related Fields, 1, 83-118.

[75] Yong, J.(2013). Linear quadratic optimal control problems for mean-field stochastic differential equations: Time-consistent solutions, SIAM J. Control Optim., **51**(4), 2809–2838

[76] Yong, J., X. Y. Zhou. (1999). Stochastic Controls: Hamiltonian Systems and HJB Equations, Springer-Verlag, New York.

[77] Yong, J.(2012). *Time-inconsistent optimal control problems and the equilibrium HJB equation.* Mathematical Control and Related Fields, 2 (3), 271-329.

[78] Y. Hu, H. Jin, X. Y. Zhou. (2012).Time-inconsistent stochastic linear quadratic control, SIAM J. Control Optim., **50**(3), 1548-1572.

[79] Zeng, Y., Li, Z., & Liu, J. (2010). Optimal strategies of benchmark and mean-variance portfolio selection problems for insurers. Journal of Industrial & Management Optimization, 6(3), 483.

[80] Zeng, Y., & Li, Z. (2011). Optimal time-consistent investment and reinsurance policies for mean-variance insurers. Insurance: Mathematics and Economics, 49(1), 145-154.

[81] Zeng, Y., Li, Z., & Lai, Y. (2013). Time-consistent investment and reinsurance strategies for mean–variance insurers with jumps. Insurance: Mathematics and Economics, 52(3), 498-507.

[82] Zhou.X. Y, D. Li. (2000). Continuous-time mean-variance portfolio selection: A stochastic LQ framework, Appl. Math. Optim., **42**, 19-33