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Option: Probabilité et statistiques

par

Mme Hadjer NITA

THEME

Analyse et évaluation de performances de systèmes de files d'attente avec feedback de clients

Soutenue devant le jury

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Mrs Hadjer NITA

THEME

Performance Analysis and Evaluation of a Queuing system with customer feedback

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In front of a jury composed of

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Mrs. NITA Hadjer

⋇ *Dedication* ⋇

To my dearest parents.

Mrs. NITA Hadjer

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General Introduction

The origin of studies on waiting phenomena dates back to 1909–1920 with A.K. Erlang's work [8] on the Copenhagen telephone network. The mathematical theory then developed thanks to the contributions of Palm, Kolmogorov, Khintchine, Pollaczek,... and currently has extended to many fields of application such as inventory management, telecommunications in general, the reliability of complex systems,... And this is due to the quality of the results provided by this theory and to the fact that the problems related to waiting in a service center are omnipresent in our days whose examples are not lacking:

– waiting at a counter (cash desk in a supermarket, administration),
– urban or air traffic,
– telephone networks,
– circulation of coins in a workshop,
– programs in a computer system,

In queueing theory, a traditional queue can be described as a system in which customers arrive according to an arrival process, to be served by a service facility according to a service process. However, in practice different behaviors of the server(s) and customers can be identified. A customer can permanently leaves the system without being served for various reasons. In a balking scenario, customers refuse to enter the queue as it has reached a certain length. Another case is the reneging of the impatient customer. Other than customer balking, a reneging customer joins the service queue. If the perceived wait time exceeds the customer's expectations, then that customer leaves the system. Regarding the server(s), other than the server activity period, the server may be unavailable for a random period resulting from many factors. In some cases, unavailability may be the result of a server failure, which means that the system needs to be repaired and put back into service. It can also be a deliberating action to use the server's idle time for different purposes and in this case, the server is said to be on vacation.

Another customer behavior that can be distinguished in practice is that described by the notion of "feedback", introduced in general to exploit waiting situations where all customers request the main service and someone needs to request another service. Certain situations of the waiting system a customer served may request an additional service by re-joining the tail of the queue with a probability, or leave the system permanently with complementary probability. This latest phenomenon is well known in queuing theory under the terminology *Bernoulli feedback* of the customers. Indeed, This feedback is due to the dissatisfaction of this customer due to the inappropriate quality of service. After obtaining partial or incomplete service, the customer retry their request for another service. In communication with the computer, the data transmission protocol is repeated sometimes because of the occurrence of an error. This is the case of dissatisfaction with the quality of service. The return (feedback) phenomena of a customer in a queue has been the subject of several works. This kind of system was initially considered by Tackas [26] and subsequently by several authors [6, 30, 24, 21].

The aim of this thesis is the analysis of a particular case of the $M/M/1$ system where the probability of feedback depends on the number of customers in the system. Also, we consider the parametric estimation of the characteristics of the waiting model $M/M/1/N$ queue with Bernoulli feedback. From Monte Carlo simulation to study the effect of the estimation of the starting parameters of the waiting system in question on the statistical properties of its performance measurement estimators obtained via the plug-in method. Finally, analysis of the $GI/GI/1$ system with Bernoulli feedback when the probability of the latter phenomenon depends on the number of customers in the system.

Even though these systems are effective at faithfully describing many real situations, there are still many models that haven't been studied in the literature. However, Due to the complexity of the analysis of such systems theoretically, the analytical results are generally difficult to obtain. To overcome the problem, we use discrete event simulation, we have sought to provide bounds for its characteristics through those of simple queuing systems (i.e. whose characteristics exist in the literature).

In addition of the present introduction, this thesis consists of three chapters, a general conclusion and a bibliography.

In the first chapter, a brief review of the queue theory was introduced to understand the notion of stochastic model, the theory of queuing system, the stability of waiting systems.

In the second chapter, an M/M/1 queue with Bernoulli feedback under FCFS discipline is considered. After getting unacceptable service, with probability β'_n, the customer may combine the system as a Bernoulli feedback to order another regular service, or he leaves the system definitively with probability β_n. Numerical and graphical illustrations of different results obtained are exhibits.

In the third chapter, our aim is the analysis of $GI/GI/1$ queuing system with Bernoulli feedback witch its probability depends on the number of customers in the system by the discrete event simulation technique. The obtained results (numerical and graphical) mainly highlight the effect of the distribution of inter-arrivals times, the distribution of service times, the probability of Feedback and the traffic intensity on the stationary characteristics of the system in question and allowed us to draw important conclusions on the behavior of these characteristics.

Chapter 1

Fundamental concepts of queuing systems theory

Introduction

Waiting lines are the most frequently encountered problems in everyday life. For example, queue at a cafeteria, library, bank, etc. Common to all of these cases are the arrival of objects requiring service and the attendant delays when the service mechanism is busy. Waiting lines cannot be eliminated completely, but suitable techniques can be used to reduce the waiting time of an object in the system. A long waiting line may result in a loss of customers to an organization. Waiting time can be reduced by providing additional service facilities, but it may increase the idle time of the service mechanism.

Queuing theory is a form of probability that pertains to the study of waiting lines (queues). This is for a system with a steady inflow of units (customers) and a specified number of servers (service facilities). The analyst wants to know if the number of service facilities in the system is adequate to handle the inflow of demands. The goal is to calculate various performance measures of the system. These include the probability that a server is immediately available to a new arrival, the average number of units in the queue, in the system, and the corresponding times in the queue and system.

The waiting line models help the management in balancing between the cost associated with waiting and the cost of providing service. Thus, queuing or waiting line models can be applied in such situations where decisions have to be taken to minimize the waiting time with minimum investment cost. A flow of customers from an infinite/finite population towards the service facility forms. A queue (waiting line) on account of the lack of capability to serve them all at a time. The queues may be of persons waiting at a doctor's clinic or railway booking office; these may be of machines waiting to be repaired ships in the harbor waiting to be unloaded or letters arriving at a typist's desk. In the absence of a perfect balance between the service facilities and the customers, waiting is required either for the service facilities or for the customer's arrival.

By the term 'customer' we mean the arriving unit that requires some service to be performed. The customer may be persons, machines, vehicles, parts, etc. Queues (waiting lines) stand for several customers waiting to be serviced. The queue does not include the customer being serviced.

The process or system that performs the services to the customer is termed by service channel or service facility.

The word queue comes from the French interpretation of the Latin cauda, meaning a tail. According the Funk and Wagnall's New International Dictionary, a queue is "a line of persons or vehicles waiting in order of their arrival". The world queue is the command way to refer to a line in England.

Queuing Theory is a collection of mathematical models of various queuing systems that take as inputs parameters of the above elements and that provide quantitative parameters describing the system performance.

Because of the random nature of the processes involved the queuing theory is rather demanding and all models are based on very strong assumptions (not always satisfied in practice). Many systems (especially queuing networks) are not soluble at all, so the only technique that may be applied is simulation.

Nevertheless, queuing systems are practically very important because of the typical trade-off between the various costs of providing service and the costs associated with waiting for the service (or leaving the system without being served). High-quality fast service is expensive, but costs caused by customers waiting in the queue are minimal. On the other hand, long queues may cost a lot because customers (machines e.g.) do not work while waiting in the queue or customers leave because of long queues. So a typical problem is to find an optimum system configuration (e.g. the optimum number of servers). The solution may be found by applying queuing theory or by simulation. Queuing systems are widely studied and there is extensive literature on this topic (see [20, 11, 27, 18, 10]).

1.1 Queuing System

The mechanism of a queuing process is very simple. Customers arrive at the service counter and are attended to one or more of the servers. As soon as a customer is served, he departs from the system. Thus a queuing system can be described as composed of customers arriving for service, waiting for service if it is not immediate, and if having waited for service, leaving the system after being served.

The detailed character station of a queuing system is defined by its characteristics discussed in the following section.

1.1.1 Elements of Queuing Systems

In the context of queue theory, improving queue analysis relies primarily on selecting the appropriate model taking into account the following characteristics:

- **Population of Customers** can be considered either limited (closed systems) or unlimited (open systems). The unlimited population represents a theoretical model of systems with a large number of possible customers (a bank on a busy street, a motorway petrol station). An example of a limited population may be several processes to be run (served) by a computer or a certain number of machines to be repaired by a serviceman. It is necessary to take the term "customer" very generally. Customers may be people, machines of various natures,

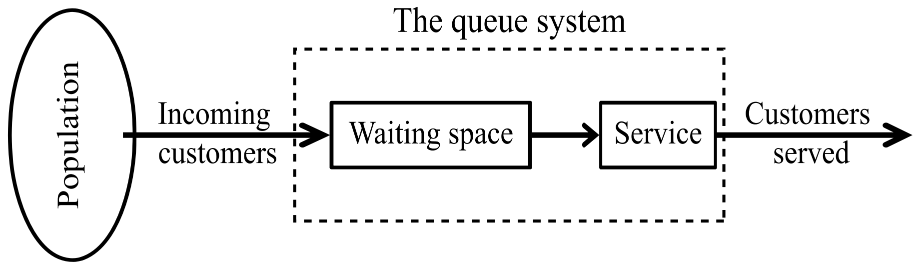

Figure 1.1: The elements of a single queue queuing system

computer processes, telephone calls, etc.

- **Arrival** defines the way customers enter the system. Mostly the arrivals are random with random intervals between two adjacent arrivals. Typically the arrival is described by a random distribution of intervals also called *Arrival Pattern.*

- **Queue** represents a certain number of customers waiting for service (of course the queue may be empty). Typically the customer being served is considered not to be in the queue. Sometimes the customers form a queue literally (people waiting in a line for a bank teller). Sometimes the queue is an abstraction (planes waiting for a runway to land). There are two important properties of a queue: Maximum Size and *Queuing Discipline.*
  Maximum Queue Size (also called *System capacity*) is the maximum number of customers that may wait in the queue (plus the one(s) being served). The queue is always limited, but some theoretical models assume an unlimited queue length. If the queue length is limited, some customers are forced to renounce without being served.

- **Service** represents some activity that takes time and that the customers are waiting for. Again take it very generally. It may be a real service carried on persons or machines, but it may be a CPU time slice, a connection created for a telephone call, being shot down by an enemy plane, etc. Typically a service takes random time. Theoretical models are based on random distribution of service duration also called Service Pattern. Another important parameter is the number of servers. Systems with one server only are called Single Channel Systems, and systems with more servers are called Multi Channel Systems.

- **Output** represents the way customers leave the system. Output is mostly ignored by theoretical models, but sometimes the customers leaving the server enter the queue again ("round robin" time-sharing systems).

Customer's Behaviour

- **Balking**. A customer may not like to join the queue due to the long waiting line.
- **Reneging**. A customer may leave the queue after waiting for some time due to impatience.
- **Collusion**. Several customers may cooperate and only one of them may stand in the queue.
- **Jockeying**. When there are many queues, a customer may move from one queue to another in the hope of receiving the service quickly.

Server's Behaviour

- **Failure**. The service may be interrupted due to the failure of a server.
- **Changing service rates**. A server may speed up or slow down, depending on the number of customers in the queue. For example, when the queue is long, a server may speed up in response to the pressure. On the contrary, it may slow down if the queue is very small.
- **Batch processing**. A server may service several customers simultaneously, a phenomenon known as batch processing.

1.1.2 Kendall Classification of Queuing Systems

The Kendall classification of queuing systems (1953) exists in several modifications. The most comprehensive classification uses 6 symbols:

A/B/s/q/c/p
where:

A is the arrival pattern (distribution of intervals between arrivals).
B is the service pattern (distribution of service duration).
s is the number of servers.
q is the queuing discipline (FIFO, LIFO, ...). Omitted for FIFO or if not specified.
c is the system capacity. Omitted for unlimited queues.
p is the population size (number of possible customers). Omitted for open systems.
These symbols are used for arrival and service patterns:
M is the Poisson (Markovian) process with exponential distribution of intervals or service duration respectively.

Em is the Erlang distribution of intervals or service duration.
D is the symbol for deterministic (known) arrivals and constant service duration.
G is a general (any) distribution.
GI is a general (any) distribution with independent random values.
Examples:
D/M/1 = Deterministic (known) input, one exponential server, one unlimited FIFO or unspecified queue, unlimited customer population.
M/G/3/20 = Poisson input, three servers with any distribution, maximum number of customers 20, unlimited customer population.
D/M/1/LIFO/10/50 = Deterministic arrivals, one exponential server, the queue is a stack of the maximum size 9, the total number of customers 50.

1.1.3 Characteristics of Queuing systems

A queuing system is specified completely by the following five basic characteristics:

The Input Process: It expresses the mode of arrival of customers at the service facility governed by some probability law. The number of customers emanates from finite or infinite sources. Also, the customers may arrive at the service facility in batches of fixed size or variable size or one by one. In the case when more than one arrival is allowed to enter the system simultaneously, (entering the system does not necessarily mean entering into service), the input is said to occur in bulk or batches.

It is also necessary to know the reaction of a customer upon entering the system. A customer may decide to wait no matter how long the queue becomes, or if the queue is too long to suit him, may decide not to enter it. If a customer decides not to enter the queue because of its huge length, he is said to have balked. On the other hand, a customer may enter the queue, but after some time loses patience and decides to leave. In this case, he is said to have reneged. In the case when there are two or more parallel queues, the customer may move from one queue to another for his economic gains, that is jockeying for position.

The final factor to be considered regarding the input process is how the arrival pattern changes with time. The input process which does not change with time is called a stationary input process. If it is time dependent then the process is termed as transient.

The Queue Discipline: It is a rule according to which customers are selected for service when a queue has been formed. The most common discipline is the "first come, first served" (FCFS), or "first in, first out" (FIFO) rule under which the customers are serviced in the strict order of their arrival. Other queue disciplines include the "last in, first out" (LIFO) rule according to which the last arrival in the system is serviced first, the "selection for service in random order" (SIRO) rule according to which the arrivals as serviced randomly irrespective of their arrivals in the system; and a variety of priority schemes-according to which a customer's service is done in preference over some other customer's service.

Under priority discipline, the service is of two types. In the first, which is called preemptive, the customers of high priority are given service over the low priority customer. In the second type, called the non-preemptive, a customer of low priority is serviced before a customer of high priority is entertained for service.

In the case of parallel channels "fastest server rule" (FSR) is adopted. For its discussion, we suppose that the customers arrive before parallel service channels. If only one service channel is free, then the incoming customer is assigned to the free service channel. However, it will be more efficient to assume that an incoming customer is to be assigned a server with the largest service rate among the free ones.

The Service Mechanism: This means the arrangement of server-s facility to serve the customers. If there are infinite numbers of servers then all the customers are served instantaneously on arrival and there will be no queue.

If the number of servers is finite, then the customers are served according to a specific order. Further, the customers may be served in batches of fixed size or variable size rather than individually by the same server, such as a computer with parallel processing or people boarding a bus. The service system in this case is termed a bulk service system.

Sometimes, the service rate may also depend on the number of customers, waiting for service. For example, when the queue becomes longer, a server may work faster or, conversely, may become less efficient. The situation in which service depends upon the number of waiting customers is referred to as a state-dependent system.

The Capacity of the System: Some of the queueing processes admit the physical limitation to the amount of waiting room so that when the waiting line reaches a certain length, no further customers are allowed to enter until space becomes available by a service completion. Such types of situations are referred to as finite source queues, that is, there is a finite limit to the maximum queue size. The queue can also be viewed as one with forced balking

Where a customer is forced to balk if he arrives at a time when queue size is at its limit.

Service Channels: When there are several service channels available to provide service, much depends upon their arrangements. They may be arranged in parallel or in series or a more complex combination of both, depending on the design of the system's service mechanism.

By parallel channels, we mean a number of channels providing identical service facilities so that several customers may be serviced simultaneously. Further, customers may wait in a single queue until one of the service channels is ready to serve, as in a barber shop where many chairs are considered as different service channels; or customers may form separate queues in front of each service channel as in the case of supermarkets.

For series channels, a customer must pass successively through all the ordered channels before service is completed. The situations may be seen in public offices where parts of the service are done at different service counters.

A queuing system is called a one-server model when the system has one server only, and a multiple-server model when the system has several parallel channels each with one server.

1.1.4 Performance measure of a queueing system

The rate of arrival of customers is λ. This means that the expected duration of two successive arrivals is $E(X) = 1/\lambda$.

The customer service rate is noted μ. This means that the expected length of service is $E(Y) = 1/\mu = \beta_1$. Traffic intensity is expressed as follows:

$$\rho = \frac{\lambda}{\mu} = \frac{E(Y)}{E(X)} \tag{1.1}$$

where X is the law of inter-arrivals and Y is the law of service.

The stationary distribution of the introduced stochastic process allows to obtain the operating characteristics of the system, such as a customer's wait time (the time that a customer spends in the queue), the length of a customer's sojourn in the system (consisting of waiting time and service time), the response time of the system, the occupancy rate of the service devices, the duration of the period of operation (the time interval during which there is always at least one customer in the system); and the following performance measures:

- L_s: The mean number of customers in the system,
- L_q: The mean number of customers in the queue,
- W_s: The mean waiting time in the system,
- W_q: The mean time in the queue.

These values are linked by the following relationships:

- $L_s = \lambda W_s$,
- $L_q = \lambda W_q$,
- $L_s = L_q + \frac{\lambda}{\mu}$,
- $W_s = W_q + \frac{1}{\mu}$.

where λ is the customer arrival rate in the system and μ the service rate.

The first two are called "Little formulas". It should be noted that these formulas are only valid under the ergodicity condition check of the system $\rho = \frac{\lambda}{\mu} < 1$. These formulas simply express the fact that, on a stationary basis, the average number of customers in the queue is equal to the customer arrival rate multiplied by the average customer wait time. They recall a Poissonian behavior of the length of the queue in a steady state. These relationships are valid for all $G/G/C/K$.

1.2 Mathematical analysis of a queue system

The mathematical study of a queue system is usually done by the introduction of a stochastic process, appropriately defined. The main focus is on the number of $X(t)$ customers in the system at time $t(t \geq 0)$. Based on the quantities that define the system, the aim is to determine:

State probabilities $P_n(t) = P(X(t) = n)$, which define the transient regime of the stochastic process $X(t), t \geq 0$. The functions $P_n(t)$ depend on the initial state or the initial distribution of the process.

The steady state of the stochastic process is defined by:

$$\pi_n = \lim_{t \to \infty} P_n(t) = P(X(+\infty) = n) = P(X = n), (n = 0, 1, 2, ...). \tag{1.2}$$

$(\pi_n)_{n \geq 0}$ is called stationary distribution of the process $X(t), t \geq 0$.

The explicit calculation of the transient regime is generally difficult, if not impossible, for most of the models given. We therefore simply determine the steady state.

1.2.1 Markovian models

They characterize systems in which the two main stochastic quantities inter-arrival time and service time are exponentially distributed independent random variables ($M/M/1$ model). The absence of memory property of the exponential law facilitates the study of these models. The mathematical study of such systems is done by the introduction of an appropriate stochastic process. This process is often $\{X(t), t \geq 0\}$ defined as the number of customers in the system at the time t. The temporal evolution of the Markovian process $\{X(t), t \geq 0\}$ is completely defined because of the absence of memory property.

1.2.2 Non-Markovian models

In the absence of exponentiality or rather when one deviates from the hypothesis of exponentiality of one of the two stochastic quantities: the time of inter-arrivals and the duration of service, or by taking into account certain specificities of the problems by introducing additional parameters, a non-Markov model is obtained. The combination of all these factors makes the mathematical study of the model very difficult, if not impossible. One then tries to return to a judiciously chosen Markov process using one of the following analytical methods:

Erlang step method: Its principle is to approximate any probability law having a rational Laplace transform by a Cox law (mixture of exponential laws), the latter has the property of absence of memory by stages.

Induced Markov Chain Method: This method, elaborate by Kendall [7], is often used. It consists of choosing a sequence of moments $t_0, t_1, t_2, ..., t_n$ (deterministic or random) such as the induced chain $\{X_n, n \geq 0\}$, when $X_n = X(t_n)$, be markovian and homogeneous.

Auxiliary variable method: It consists of completing the information on the process $\{X(t), t \geq 0\}$ in such a way as to give it the character Markovian. Thus, we come back to the study of the process $\{X(t), A(t_1), A(t_2), ...A(t_n)\}$. The variables $A(t_k), k \in \{1, 2, ..., n\}$ are called auxiliary.

Fictitious event method: The principle of this method is to introduce fictitious events that make it possible to give a probabilistic interpretation to Laplace transforms and random variables describing the studied system.

Simulation: It is a process of artificial imitation of a real process given on the computer. It allows us to study the most complex systems, predict their behaviors, and calculate their characteristics. The results obtained are only approximate but can be used with good precision. This technique is based on the generation of random variables according to the laws governing the system.

1.3 Some classical queue systems

1.3.1 M/M/1 queue

The $M/M/1$ queue system is the most basic in queue theory. The flow of arrivals is the poissonian parameter λ and the service time is the exponential parameter μ.

Transitional regime

Let $X(t)$ be the number of customers present in the system at time $t(t \geq 0)$. Thanks to the fundamental properties of the Poisson process and the exponential law, $X(t)$ is a homogeneous Markov process.

The state probabilities $P_n(t) = P[X(t) = n]$ can be calculated by the differential equations of Kolmogorov below, knowing the initial conditions of the process.

$$\begin{cases} P_n'(t) = -(\lambda + \mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t), \\ P_0'(t) = -\lambda P_0(t) + \mu P_1(t). \end{cases} \tag{1.3}$$

Steady state

Under the ergodicity condition of the $\rho < 1$ or $\lambda < \mu$ system, for which the stationary regime exists, it is necessary to obtain the stationary probabilities $\pi_n = \lim_{t\to\infty} P_n(t)$. We have

$$\pi_n = (1 - \rho)\rho^n, \forall n \in N, \tag{1.4}$$

$\rho = \lambda/\mu$ is called system load or duty cycle. It also represents the probability of occupancy of the service station, since it is equal to $1 - \pi_0$. $\pi = \pi_{nn\geq 0}$ is called stationary distribution, it follows a geometric law.

Characteristics of the system

– The mean number of customers in the system is:

$$L_s = E(X) = \sum_{n\geq 0} n\pi_n = (1-\rho)\sum_{n\geq 0} n\rho^n = \frac{\rho}{1-\rho}. \tag{1.5}$$

– The mean number of customers in the queue is:

$$L_q = \sum_{n\geq 1}(n-1)\pi_n = \frac{\rho^2}{1-\rho}. \tag{1.6}$$

– The mean waiting time in the system is:

$$W_s = \frac{\rho}{\lambda(1-\rho)}. \tag{1.7}$$

– The mean time in the queue is:

$$W_q = \frac{\rho^2}{\lambda(1-\rho)}. \tag{1.8}$$

Remark 1.3.1. The mean stay time in the system and the mean wait time in the queue are obtained from Little's formulas or system distributions.

1.3.2 $M/GI/1$ queue

The inflow of arrivals in the system $M/GI/1$ is poissonian of parameter λ and service-time is distributed according to a general law G of average $1/\mu$. The particularity of this system is that, unlike the case $M/M/1$, the process $X(t)$ is not Markovian.

There are several methods of analyzing these systems (see paragraph 1.2.2). In this thesis, we will limit ourselves to "the method of the induced Markov chain.

Induced Markov Chain and Transition Probabilities

Either X_n: The number of customers in the $M/GI/1$ system at the end of the n^{th} service customer. Note by $G(t)$ the distribution of the length of service and by λ the parameter of the exponential distribution governing the duration between two consecutive arrivals.

The process $\{X_n, n \geq 0\}$ is a Markov chain, transition operator $P = [P_{ij}]_{i,j\geq 0}$, where:

$$P_{i,j} = \begin{cases} P_j & \text{if } i = 0; \\ P_{j-i+1} & \text{if } i \geq 1; \end{cases} \tag{1.9}$$

with

$$P_k = \int \frac{e^{-\lambda t}(\lambda t)^k}{k!} dG(t), k = 0, 1, 2, ... \tag{1.10}$$

Indeed, if A_n is the number of customers who enter the system during the n^{th} service, we have:

$$X_{n+1} = X_n - \delta_n + A_{n+1}, \tag{1.11}$$

Where

$$\delta_n = \begin{cases} 1, & \text{if } X_n > 0; \\ 0, & \text{if } X_n = 0. \end{cases} \tag{1.12}$$

This shows that X_{n+1} depends only on X_n and A_{n+1} and not on X_{n-1}, X_{n-2}, ... This means that the sequence $\{X(t), t \geq 0\}$ is markovian, where $X(t)$ is the number of customers in the system at time t.
Moreover, $Pr(A_n = k/t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$, because the number of A_n customers that enter the system, is distributed according to a parameter Poisson law (λt). According to the total probability theorem,

$$Pr(A_n = k) = P_k = \int \frac{e^{-\lambda t}(\lambda t)^k}{k!} dG(t), \quad \text{where } P_k > 0, \quad \text{(k=1,2,...)}. \tag{1.13}$$

Steady state

The stationary regime of the system exists and it is identical to the stationary state of the induced Markov chain X_n if $\rho = \lambda/\mu < 1$. Finding the stationary distribution $\pi = (\pi_0, \pi_1, ...)$ will generally not be possible. However, we can calculate the corresponding generator function $\Pi(z)$ (see[17])

$$\Pi(z) = G^*(\lambda - \lambda z) \frac{(1-\rho)(1-z)}{G^*(\lambda - \lambda z) - z}, \tag{1.14}$$

where G^* represents the Laplace transform of the duty time probability density, and z is a complex number checking $|z| \leq 1$. This formula is called the 1st Pollaczek-Khintchine formula. Its inversion, to find Π, is often difficult and requires numerical methods.

Characteristics of the system

- The average number of customers in the system: This amount can be determined, the steady state, using the relationship:

$$E(X) = \lim_{Z \to 1} \Pi'(Z) \tag{1.15}$$

  Nevertheless, this calculation is complicated. However, it can be easily obtained by using the relationship:

$$X_{n+1} = X_n - \delta_n + A_{n+1}, \tag{1.16}$$

  This is achieved by:

$$L_s = E(X_n) = \rho + \frac{\rho^2 + \lambda^2 Var(Y)}{2(1-\rho)}, \tag{1.17}$$

  Where $V(Y)$ is the variance of the random variable Y.
- The average number of customers in the queue is:

$$L_q = \frac{\rho^2 + \lambda^2 Var(Y)}{2(1-\rho)}. \tag{1.18}$$

  Using the Little formula, we obtain:

– The mean waiting time in the system is:

$$W_s = \frac{1}{\mu} + \lambda(\frac{Var(Y) + \frac{1}{\mu^2}}{2(1-\rho)}). \tag{1.19}$$

– The mean waiting time in the queue is:

$$W_q = W_s - \frac{1}{\mu} = \lambda(\frac{Var(Y) + \frac{1}{\mu^2}}{2(1-\rho)}). \tag{1.20}$$

1.3.3 $GI/M/1$ queue

The $GI/M/1$ queue system is the "dual" of the $M/GI/1$ system. In this case, the inter-arrival times of the customers are a series of random variables distributed according to a common general law G, average $1/\lambda$, and the service times are independent and identiquement distributed according to an exponential law of parameter μ. The same methods cited in paragraph 1.2.2, may be used for this non-markovian system. We will limit ourselves to the method of the induced Markov chain.

Induced Markov Chain

It can be shown that the two-dimensional process $\{X(t), \delta(t)\}$, where $\delta(t)$ represents the time since the last arrival before t, is a Markovian process. As in the $M/GI/1$ case, this process can be simplified to a one-dimensional process by considering it at particular times. Indeed, by choosing the moments t_n of the arrival of the n^{th} customer, it is clear that the random variables $\delta(t_n)$ are null. We will therefore have to study the Markov chain in discrete time $X_n = X(t_n) =$ "number of customers in the system just before the arrival of the n^{th} customer".

Let $\pi_k = \lim_{t\to\infty} P(X(t) = k)$, and $p_k = \lim_{n\to\infty} P(X_n = k)$ be the stationary probability of the chain X_n.

Unlike the $M/GI/1$ system, in the $GI/M/1$ queue system, the equality between π_k and $p_k, k \in \mathbb{N}$, is generally not achieved. Furthermore, the ergodicity condition of the induced X_n Markov chain is the same as that of the stability of the $GI/M/1$ system ($\rho = \frac{\lambda}{\mu} < 1$).

Transition probabilities

Let be $P_{i,j} = P[X_{n+1} = j/X_n = i], i, j \in \mathbb{N}$, the transition probabilities of the X_n induced Markov chain. It is easy to verify that the transition matrix $P = [P_{ij}]_{i,j\in\mathbb{N}}$, has the following form:

$$P = \begin{pmatrix} \beta_1 & \alpha_0 & 0 & 0 & 0 & \cdots \\ \beta_2 & \alpha_1 & \alpha_0 & 0 & 0 & \cdots \\ \beta_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \cdots \\ \beta_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \cdots \\ & & \ddots & & & \end{pmatrix},$$

where,

$$\beta_{i+1} = \int_0^\infty \sum_{k=i+1}^{\infty} e^{-\mu t} \frac{(\mu t)^k}{k!} dG(t), \qquad i \geq 0, \tag{1.21}$$

and,

$$\alpha_k = \int_0^\infty e^{-\mu t} \frac{(\mu t)^k}{k!} dG(t), \qquad k \geq 0, \tag{1.22}$$

Steady state

We are now able to find the ergodic probabilities, p_k ($k \in \mathbb{N}$), of the induced X_n Markov chain. The ergodicity condition is $\rho < 1$. It can be easily verified that [17]:

$$p_k = (1-\sigma)\sigma^k, \qquad k = 0, 1, 2..., \tag{1.23}$$

where σ is the only solution to the equation:

$$\sigma_k = G^*(\mu - \mu\sigma) \int_0^\infty e^{-(\mu-\mu\sigma)t} dG(t), \tag{1.24}$$

G^* being the Laplace transform of the probability density of times between customer arrivals. It can be shown that $0 < \sigma < 1$. Thus, the number of customers in the system $GI/M/1$ at the time of occurrence of arrival is distributed according to a geometric law.

If $\pi_k = \lim_{t\to\infty} P(X(t) = k)$ then the following relationships can be easily verified:

$$\pi_k = pp_{k-1}, \quad k = 1, 2, ... \;\; and \;\; \pi_0 = 1 - \rho. \tag{1.25}$$

These relationships confirm the effectiveness of the induced Markov chain method. It can then be seen that all the stationary characteristics of the queue system $GI/M/1$, can be deduced from the stationary characteristics of the Markov chain X_n (although the stationary probabilities of the two processes $X(t)$ and X_n are different).

Characteristics of the system

It was noted that the study of the $GI/M/1$ queue system is simpler than that of the $M/GI/1$ system. In this case, it is enough to find the value of σ to deduce all the characteristics of this system. Indeed:

– The average number of customers in the system is easily obtained by the formula:

$$\begin{aligned} L_s &= E(X) = \sum_{k\geq 0} k\pi_k = \sum_{k\geq 0} k\rho p_{k-1} = \sum_{k\geq 0} \rho(k+1)p_k \\ &= \rho \sum_{k\geq 0} kp_k + \rho \sum_{k\geq 0} p_k \\ &= \frac{\rho}{1-\sigma}. \end{aligned} \tag{1.26}$$

– The average number of customers in the queue is given by:

$$L_q = \frac{\rho\sigma}{1-\sigma}. \tag{1.27}$$

Using the Little formula, we obtain:

– The Mean waiting time in the system is:

$$W_s = \frac{L}{\lambda} = \frac{1}{(1-\sigma)\mu}. \tag{1.28}$$

– The average number of customers in the system when a customer arrives:
You can also get the average number of customers in the system that a customer finds upon arrival (L_a). This quantity, unlike $M/GI/1$, is different from L_s.

$$L_a = \sum_{k\geq 1} kp_k = \frac{\sigma}{(1-\sigma)}. \tag{1.29}$$

We notice that $L_a/L = \frac{\sigma}{\rho}$. Therefore, we cannot give a priori any comparison between these two values (since $\rho < 1$ and $\sigma < 1$).

Chapter 2

Queuing systems with Bernoulli Feedback of customers

Introduction

Feedback queues are useful for modeling many phenomena, for example, in telecommunications, a telephone call can generate several tasks to be processed. Such tasks can sometimes be considered as feedback. Also, in communication with the computer, the data transmission protocol is repeated sometimes due to the occurrence of an error. Protocol data transmissions are sometimes repeated, this happens frequently because of poor service. For example, in industry, we have the reproduction of the wrongly composed product. In many queuing situations, customers may be served multiple times for certain reasons. This feedback is due to the dissatisfaction of this customer due to the inappropriate quality of service. After obtaining partial or incomplete service, the customer retries their request for the service(banks, post offices...).

Several real situations can be modeled as queuing systems with feedback where the customer can be referred to the system for another service. In telecommunications, protocol data transmissions are sometimes repeated. This happens frequently because of poor service. In industry (the reproduction of a poorly composed product) is an example of queues with feedback. In Queue models with feedback have been extensively studied by a large number of researchers.

Takacs [26] studied queue with feedback, to determine the stationary process for the queue size, the first two moments of the distribution function of the total time spent in the system by a customer, and the distribution of the customers in the system.

Also, a single server queue with state dependent feedback was studied by [6] D'Avignon and Disney, they studied the $M/G/1$ queue, they supposed that units after being served either immediately join the queue again with some probability or depart permanently with the complementary probability. Such a probability is conditioned upon whether or not the previous unit fed back, upon the increments in the queue length between two consecutive service completions, and upon the length of service received. For such a system, the stationary queue length and departure processes are characterized. The busy period distribution is also worked out. Studies on queue length, the total sojourn time and the waiting time for an $M/G/1$ queue with Bernoulli feedback were provided by Vanden Berg and Boxma [30].

In [1], the authors studied the $M/G/1$ queue with feedback, according to some feedback policy which customers visit the queue a fixed number of times before departure when the Bernoulli policy implies a geometric distribution for the number of visits to the queue by a customer.

Santhakumaran and Thangaraj [24] considered a single server feedback queue with impatient and feedback customers, they studied $M/M/1$ queueing model for queue length at arrival epochs and obtained results for stationary distribution, mean and variance of queue length.

Choudhury and Paul [5] analyzed the file $M/G/1$ with two phases of heterogeneous servers and bernoulli feedback, where the server provides first phase of regular service to all the customers. As soon as the first phase of service of a customer is completed, it may leave the system or may immediately go for second phase of optional service in one additional service channel. However, after receiving the first phase or second phase of unsuccessful service by an unit, then it may immediately join the tail of the original queue as feedback customer to have another regular service. They derive the queue size distribution at random epoch and at a service completion epoch and derive the distribution of response time and busy period.

Krishna Kumar et al. [12] considered a generalized $M/G/1$ feedback queue in which customers are either "positive" or "negative".

Thangaraj, and Vanitha [28] obtain transient solution of $M/M/1$ feedback queue with catastrophes using continued fractions. The steady-state solution, moments under steady state and busy period analysis are calculated.

Salehirad and Badamchizadeh [23] studied the $M/G/1$ queue with k phases of heterogeneous services and feedback.

Ayyapan et al. [4] used matrix geometric method to study $M/M/1$ retrial queueing system with loss and feedback under non preemptive priority service in which two types of customers arrive in a Poisson process with arrival rate λ_1 for low priority customers and λ_2 for high priority customers. These customers are identified as primary calls. The service times follow an exponential distribution with parameters μ_1 and μ_2 for both types of customers respectively. The concept feedback is introduced for low priority customers. If the server is free at the time of the arrival of low priority customer, then the arriving call begins to be served immediately by the server. After completion of service, if the low priority customer dissatisfied then he may re-join the orbit with probability q and with probability (1-q) he leaves the system. This is called feedback [4, 10, 12] in queueing theory.

Also, Sharma and Kumar [15] gave the stationary solution of the $M/M/K$ queue with feedback, abandonment and retention.

Arivudainambi and Godhandaraman [3] considered a batch arrival queueing system with two phases of service, feedback and K optional vacations under a classical retrial policy.

Melikov et al. [19] presented the numerical analysis of a queue system with feedback. The feedback occurs as returning a part of serviced calls to get a new service. The probability of returning to orbit depends on the number of busy channels. Both models with finite and infinite orbits are examined. Both exact and approximate methods to calculate the characteristics of the system are developed.

Kumar and Taneja [16] made an analysis of a queuing system with provision of service by one or more out of three servers depending upon the feedback, one of which is centrally connected to the other two servers, customers can return for service at most once. Equations have been derived to find the average queue length using the probability generator technique.

In [9], the authors studied the strategic customer behavior in an $M/M/1$ feedback queue, they analyzed the decision of rejoining the system as a noncooperative game among the customers. They showed that there existed a unique symmetric Nash equilibrium threshold strategy and they proved that this strategy is evolutionarily stable.

In Nita et al.[21] involved estimating the characteristics of the $M/M/1/N$ waiting model with Bernoulli feedback using parametric estimation. By using the Monte-Carlo simulation study, we have illustrated how the statistical properties of the performance measures estimates obtained using the plug-in method can be affected by estimating the starting parameters of the considered waiting system. In addition, several types of convergence (bias, variance, MSE, in law) of these performance measure estimators have also been showing by simulation, Nita et al.[21] have used several statistical techniques, namely: Parametric estimation, compliance tests and the box plot. A simulation study carried out in this direction illustrates that the impact of estimating starting parameters on the estimated performance measures of the system is strongly dependent on both the estimated starting parameter and the sample size.

2.1 Markovians queues with Feedback constant

We consider an $M/M/1$ queue system with feedback constant. The latter can model a single window where each customer receives a service whose duration is an exponential parameter variable μ, and the process of arriving customers in the queue is a rate Poisson process λ (the number of customers $N(t)$ arriving during a time interval $[0,t]$ follows a Poisson distribution). After obtaining a service, with a probability $\beta' = 1 - \beta$; the customer can join the system as a Bernoulli customer feedback to receive another additional service. Otherwise, it permanently leaves the system, with a probability β; (where $\beta' + \beta = 1$).

This system can be schematized as shown in Figure 2.3. .

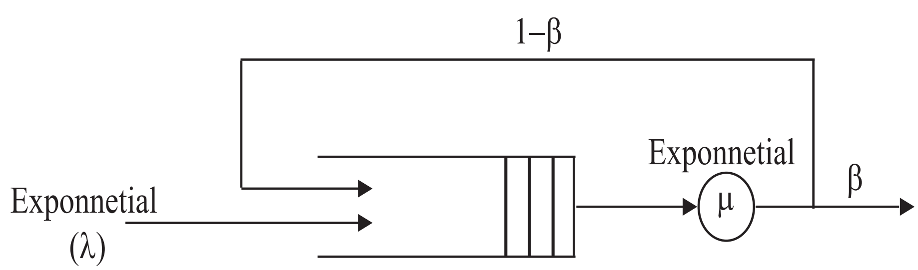

Figure 2.1: Representation $M/M/1$ queue with Bernoulli feedback.

Note that the $N(t)$ arrival process is fully described by:

$$P[N(t) = k] = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, ...(\text{parameter poisson law } \lambda t) \tag{2.1}$$

The duration of inter-arrivals T are exponentially distributed by density function:

$$a(t) = \lambda e^{-\lambda t}, \quad \forall t > 0.$$

The duration of service S lives are exponentially distributed by density function:

$$b(t) = \mu e^{-\mu t}, \quad \forall t > 0.$$

Let $L(t)$ be the number of customers in the system at the time t. The probabilities of state of the system in transient mode are given as follows:

$$P_t = P[L(t) = n] \quad et \quad P_{ij}(dt) = P(L(t+dt) = j/L(t) = i),$$

with

$$P_{ij}(dt) = \begin{cases} 1 - \lambda dt, & if \quad \text{i=j=0}, \\ \lambda dt, & if \quad \text{j=i+1 and j} \geq 1, \\ \beta\mu dt, & if \quad \text{j=i-1 and j} \geq 0, \\ 1 - (\lambda + \beta\mu)dt, & if \quad \text{i=j and j} \geq 1, \\ 0, & if not. \end{cases}$$

The system can be described by the following differential system (Chapman-Kolmogorov equation):

$$\begin{cases} P_0(t+dt) = P_0(t)(1 - \lambda dt) + P_1(t)\beta\mu dt, \\ P_n(t+dt) = P_{n-1}(t)\lambda dt + P_n(t)(1 - (\lambda + \beta\mu)dt) + P_{n+1}(t)\beta\mu dt, \quad n \geq 1. \end{cases}$$

The probabilities of steady-state system status are as follows:

$$\begin{cases} \lambda P_0 = \beta\mu P_1, & if \quad \text{n=0}, \\ (\lambda + \beta\mu)P_n = \lambda P_{n-1} + \beta\mu P_{n+1}, & if \quad \text{n} \geq 1. \end{cases}$$

- **The server utilization rate:** by definition, the utilization rate is the probability for the queue server to be busy:

$$U = \rho = \frac{\lambda}{\beta\mu}. \tag{2.2}$$

- **The probability of n customers in the system at the time of entry is:**

$$\pi_n = \pi_0 \rho^n, \quad n = 0,\ 1,\ 2, ... \ \text{avec}: \ \pi_0 = 1 - \rho. \tag{2.3}$$

- **Average number of customers in system and queue:**

$$L_s = \frac{\rho}{1-\rho} \quad , \quad L_q = \frac{\rho^2}{1-\rho}. \tag{2.4}$$

- **Average stay and waiting time :**

$$W_s = \frac{1}{\beta\mu - \lambda} \quad , \quad W_q = \frac{\lambda}{\beta\mu(\beta\mu - \lambda)}. \tag{2.5}$$

Remark 2.1.1. The "memory-less" property of the exponential law (describing a service time) means that the probability for the service to end before the overtime t_0 knowing that it started t_1 times earlier, does not depend on the time t_1 the customer has already placed in service.

2.2 Analysis of an $M/M/1$ queue with dependent Bernoulli feedback

In this section, we will consider an expectation model that can be very appropriate for modeling several real situations in various areas, such as systems communication and telecommunications, manufacturing systems, computing, etc. The model in question is the model $M/M/1$

queueing system with Bernoulli feedback, with probability $\beta'_n = 1 - \beta_n$, the customer may join the system as a Bernoulli feedback to command another service, or he quits the system with probability β_n, with n is the number of customers in the system. Feedback represents the case where after getting partial or incomplete service, the customer retries for service. This usually happens because of non-satisfactory quality of service. Rework in industrial operations is an example of a queue with feedback.

Our goal is to analyze the impact of the feedback variant on the queue characteristics, in order to reach this goal, we derive the steady-state equations by the Markov process method, we extract performance measures of certain particular cases of the probability of feedback of the preceding system.

This chapter is organized as follows. Firstly, we describe the model $M/M/1$ with Bernoulli feedback. Then, we derive the steady-state equations by the Markov process method and calculate some performance measures of the particular cases of the model. Finally, some numerical examples are presented to demonstrate how the feedback influences the system $M/M/1$.

2.2.1 Model description

Let's consider an $M/M/1$ queueing model with Bernoulli feedback and the following assumptions :

- Arrivals occur in a Poisson stream with an average arrival rate of λ.
- The customers are served on a first-come, first-served ($FCFS$) discipline. Once a customer's service commences, the service always proceeds to completion. In addition, the service times are assumed to be distributed according to an exponential distribution with rate μ.
- After getting unsatisfactory service, with probability $\beta'_n = 1 - \beta_n$, with n is the number of customers in the system, $0 < \beta_n \leq 1$ the customer may rejoin the system at the end of the queue as a Bernoulli feedback customer to receive another regular service. Otherwise, it leaves the system definitively, i.e. with probability β_n (where $\beta'_n + \beta_n = 1$).
  Alternatively, customer feedback may be modeled by a monotonic increasing function β_n.
- For the rate service, we do not distinguish between regular arrival and feedback.
- The inter-arrival times and the service times are independently, identically and exponentially distributed.

The model can be illustrated by the schema presented in Figure3.1 .

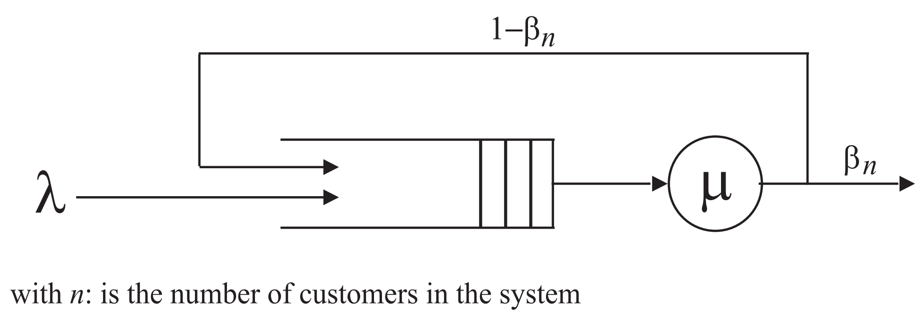

Figure 2.2: $M/M/1$ queue with dependent Bernoulli feedback

2.2.2 Steady-State Solution

In this section, we drive the steady-state probabilities by using the Markov process method based on the principal recursive approach.

To this end, let us define the following notations which are used throughout the section. Let $N(t)$ denote the number of customers in the system at time t due to the fundamental properties of the Poisson process and the exponential law, N(t) is a homogeneous Markov process, $P_n(t)$ denote the time dependent probabilities that there are "n" customers in the system at time "t":

$$P_n(t) = P(N(t) = n), \; n \in N. \tag{2.6}$$

So, at steady-state P_n; $n \geq 0$, will be the probability that there are n customers in the system.

By applying the Markov process theory, it is easy to formulate the equations of the state-transition-probabilities that governs our process $\{(N(t), t \geq 0\}$ and, therefore, we can deduce the state balance equations that are given in the expressions

$$P_0(t+dt) = (1-\lambda dt)P_0(t) + \mu_1 dt P_1(t), \quad n = 0, \tag{2.7}$$

$$P_n(t+dt) = \lambda dt P_{n-1}(t) + (1-(\lambda+\mu_n)dt)P_n(t) + \mu_{n+1} dt P_{n+1}(t), n \geq 1, \tag{2.8}$$

with $\mu_n = \mu\beta_n$.

So, the differential-difference equations of the queueing model are given as:

$$\frac{P_0(t)}{dt} = -\lambda P_0(t) + \mu_1 P_1(t), \qquad n = 0, \tag{2.9}$$

$$\frac{P_n(t)}{dt} = \lambda P_{n-1}(t) - (\lambda+\mu_n)P_n(t) + \mu_{n+1}P_{n+1}(t), \qquad n \geq 1. \tag{2.10}$$

2.2.3 Performance measures of the system

In this section, we clear the expressions for some useful performance measures of the proposed system when β_n has a particular form. Indeed, we extract the explicit form for some important measures of the effectiveness of the queuing model. By using its stationary distribution. Also, we identify the stability condition and the form of stationary probability of the system described in the previous section 2.2.1.

2.2.4 Particular cases

In this passage, we provide the characteristics of the preceding system for certain particular forms of the probability of feedback, β_n, with ρ, $0 < \alpha \leq 1$ and $\delta = \alpha\rho$,$I_{\{.\}}$ is the indicator function.

– Case $\beta_n = \alpha\frac{n}{(n+1)}$

The same steps used in the last case $\beta_n = \beta$, we get the equations of equilibrium states:

$$\lambda P_0 = \mu\alpha P_1, \qquad n = 0, \tag{2.11}$$

$$(\lambda + \mu\alpha\frac{n}{(n+1)})P_n = \lambda P_{n-1} + \mu\alpha\frac{n}{(n+1)}P_{n+1}, \qquad n \geq 1. \tag{2.12}$$

Using the recursive method, we find the following equations:

$$P_1 = \delta P_0, \qquad n = 0, \tag{2.13}$$

$$P_n = (n+1)\delta^n P_0, \qquad n \geq 1. \tag{2.14}$$

– The expression of ρ_n:
Using the definition of ρ_n, for $n \geq 1$ we find the following :

$$\rho_n = \prod_{i=1}^{n} \frac{\lambda}{\mu\beta_i} = \left(\alpha\frac{\lambda}{\mu}\right)^n \prod_{i=1}^{n}\left(\frac{i+1}{i}\right) \quad (2.15)$$

$$= \delta^n \frac{(n+1)!}{n!} \quad (2.16)$$

$$= (n+1)\delta^n. \quad (2.17)$$

– The expression of P_0:
Using the expression

$$P_0 = \left[\sum_{n=0}^{\infty} \rho_n\right]^{-1}. \quad (2.18)$$

under the stability condition $\delta < 1$, we get the following expression for the probability P_0:

$$P_0 = \left[\sum_{n=0}^{\infty} \delta_n\right]^{-1} = \left[\sum_{n=0}^{\infty}(n+1)\delta^n\right]^{-1} \quad (2.19)$$

$$= \left[\frac{\delta}{(1-\delta)^2} + \frac{1}{1-\delta}\right]^{-1} \quad (2.20)$$

$$(2.21)$$

It clear that if $\delta < 1$, then P_0 exist and is given by:

$$P_0 = \left[\frac{1}{(1-\delta)^2}\right]^{-1} = (1-\delta)^2. \quad (2.22)$$

– The expression of the mean number of customers in the system:
By definition we have

$$L_s = \sum_{n=0}^{\infty} nP_n, \quad (2.23)$$

so,

$$L_s = \sum_{n=0}^{\infty} n\rho_n P_0$$

$$= \sum_{n=0}^{\infty} n(n+1)\delta^n(1-\delta)^2$$

$$= (1-\delta)\left[\sum_{n=0}^{\infty} n^2\delta^n(1-\delta) + \sum_{n=0}^{\infty} n\delta^n(1-\delta)\right]$$

it is easy to see the first and second moment of geometric law, we find

$$L_s = \frac{2\delta}{1-\delta}. \quad (2.24)$$

– The expression of the mean number of customers in the queue:

$$\begin{aligned} L_q &= L-(1-P_0) && (2.25)\\ &= \frac{(3-\delta)\delta^2}{1-\delta}. && (2.26)\end{aligned}$$

– Case $\beta_n = \alpha\frac{n(n+2)}{(n+1)^2}$
  – The expression of ρ_n:
    By definition the quantity ρ_n is given as follow:

$$\rho_n = \begin{cases} 1, & \text{if } n=0;\\ \prod_{i=1}^{n} \frac{\lambda}{\mu\beta_i}, & \text{if } n \geq 1.\end{cases} \qquad (2.27)$$

If we substituting β_i by its expression in this latest then we have:

$$\begin{aligned} \rho_n &= \delta^n \prod_{i=1}^{n}\left(\frac{(i+1)^2}{i(i+2)}\right) && (2.28)\\ &= \delta^n \prod_{i=1}^{n}\left(\frac{i+1}{i}\right)\prod_{i=1}^{n}\left(\frac{i+1}{i+2}\right) && (2.29)\\ &= \frac{2(n+1)}{n+2}\delta^n. && (2.30)\end{aligned}$$

  – The expression of P_0:

$$\begin{aligned} P_0 &= \left[\sum_{n=0}^{\infty}\delta_n\right]^{-1}\\ &= \left[2\sum_{n=0}^{\infty}(1-\frac{1}{n+2})\delta^n\right]^{-1}\\ &= \left[2(\sum_{n=0}^{\infty}\delta^n - \sum_{n=0}^{\infty}\frac{1}{n+2}\delta^n)\right]^{-1}.\end{aligned}$$

The first sum of the latest expression is a geometric series, so if $\delta < 1$ then

$$\sum_{n=0}^{\infty}\delta^n = \frac{1}{1-\delta}, \qquad (2.31)$$

and under the same condition, $\delta < 1$, P_0 and exist the second sum can be simplified as follow: consequently,

$$P_0 = \left[\frac{2}{1-\delta}+\frac{2}{\delta}+\frac{2}{\delta^2}\log(1-\delta)\right]^{-1}. \qquad (2.32)$$

– The expression of the mean number of customers in the system:

$$L_s = \sum_{n=0}^{\infty} nP_n \tag{2.33}$$
$$= 2P_0 \sum_{n=0}^{\infty} \frac{n(n+1)}{n+2}\delta^n \tag{2.34}$$
$$= 2P_0 \sum_{n=0}^{\infty} \left(n-1+\frac{2}{n+2}\right)\delta^n \tag{2.35}$$
$$\tag{2.36}$$

so

$$L_s = 2P_0\left[\frac{\delta}{(1-\delta)^2} - \frac{1}{1-\delta} - \frac{2}{\delta} - \frac{2}{\delta^2}\log(1-\delta)\right]. \tag{2.37}$$

– The expression of the mean number of customers in the queue:

$$L_q = L-(1-P_0) \tag{2.38}$$
$$= 2P_0\left[\frac{\delta}{(1-\delta)^2} - \frac{1}{1-\delta} - \frac{2}{\delta} - \frac{2}{\delta^2}\log(1-\delta) + \frac{1}{2}\right] - 1. \tag{2.39}$$

– Case $\beta_n = \frac{n}{N}$
  – The expression of ρ_n:
  In this case the quantity ρ_n is defined as follow:

$$\rho_n = \begin{cases} 1 & \text{if } n=0; \\ \frac{(N\rho)^n}{n!}, & \text{if } 1 \le n < N; \\ \frac{\rho^n N^N}{N!}, & \text{if } n \ge N; \end{cases} \tag{2.40}$$

and the stability condition of the system is $\rho < 1$
  – The expression of P_0:

$$P_0 = \left[\sum_{n=0}^{\infty}\rho_n\right]^{-1} = \left(\sum_{n=0}^{N}\rho_n + \sum_{n=N+1}^{\infty}\rho_n\right)^{-1} \tag{2.41}$$
$$= \left(\sum_{n=0}^{N}\frac{(N\rho)^n}{n!} + \frac{N^N}{N!}\sum_{n=N+1}^{\infty}\rho^n\right)^{-1} \tag{2.42}$$
$$= \left(\sum_{n=0}^{N}\frac{(N\rho)^n}{n!} + \frac{N^N}{N!}\frac{\rho^{N+1}}{1-\rho}\right)^{-1}. \tag{2.43}$$

  – The expression of the mean number of customers in the system:

$$L_s = \sum_{n=0}^{\infty} nP_n = P_0\left[\sum_{n=0}^{N} n\rho_n + \sum_{n=N+1}^{\infty} n\rho_n\right]. \tag{2.44}$$

We have, on the one hand, the quantity $\sum_{n=0}^{N} n\rho_n$ is the sum of a finite number of terms that can easily be calculated numerically and on the other hand,

$$\sum_{n=N+1}^{\infty} n\rho_n = \frac{N^N}{N!}\sum_{n=N}^{\infty} n\rho^n \tag{2.45}$$

then,

$$L_s = \left[\sum_{n=0}^{N} n\frac{(N\rho)^n}{n!} + \frac{N^N}{N!}\frac{\rho^{N+2}}{(1-\rho)^2} + \frac{N^N}{N!}\frac{(N+1)\rho^{N+1}}{1-\rho}\right]P_0. \tag{2.46}$$

– The expression of the mean number of customers in the queue:

$$L_q = \left[\sum_{n=0}^{N} n\frac{(N\rho)^n}{n!} + \frac{N^N}{N!}\frac{\rho^{N+2}}{(1-\rho)^2} + \frac{N^N}{N!}\frac{(N+1)\rho^{N+1}}{1-\rho} + 1\right]P_0 - 1. \tag{2.47}$$

2.2.5 Numerical application

The objective of this section is to present illustrative numerical examples of the behavior of the stationary characteristics of the $M/M/1$ queue with dependent feedback described in Section 2.2.3 according to the starting parameters defining it, namely: the traffic intensity $\rho(=\lambda/\mu)$, and the shape and value of the probability β_n.

To meet our objective, for the numerical calculations, we have set the starting parameters in question as follows:

– The traffic intensity $\rho \in \{0.10, 0.15, 0.20, ..., 0.85\}$.
– The probability $\beta_n \in \{1,\ 0.9,\ \frac{\alpha n}{n+1},\ \frac{\alpha n(n+2)}{(n+1)^2},\ \frac{n}{N}\}$ with
  ◦ $\alpha \in \{0.900,\ 0.950, 1.000\}$,
  ◦ $N \in \{2,\ 3,\ 4,\ 5,\ 10\}$.

The figures 2.3–2.6 represent a sample of the graphical results provided, for the various parameters above, by the computer program that we implemented under the Matlab environment.

The results obtained perfectly illustrate the effect of the traffic intensity and the probability β_n and its shape on the characteristics of the system, where we note (particularly) that:

- The effect of the traffic intensity, ρ, on the queue length, L_s, and the probability that the system is empty, P_0, remains the same as the case of a queue without feedback and the case where the feedback is done with a constant probability where, as the traffic intensity ρ increases L_s increases hence the probability P_0 decreases.
- The effect of the probability β_n on the quantities L_s and P_0 appears clearly for high values for traffic intensity. Indeed, the impact of β_n on L_s and P_0 is all the more important when ρ increases, so that it is negligible for very low values of ρ.

The fact that an increase in ρ leads to an increase in L_s, so due to the considerable number of customers in the system, served customers tend to leave the system rather than request additional service at the end of their services (β_n is increasing according to n). Therefore,

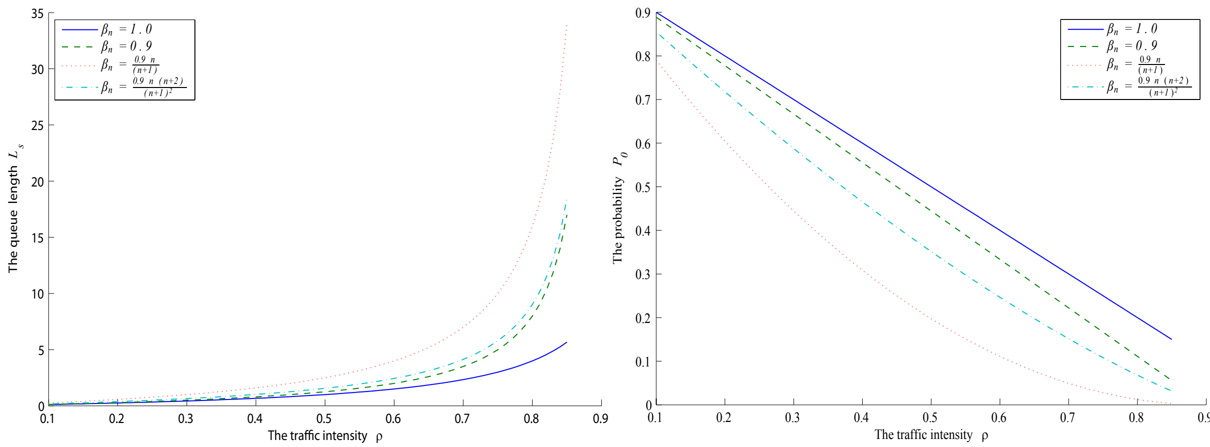

Figure 2.3: Variation of L_s and P_0 according the couple (β_n, ρ).

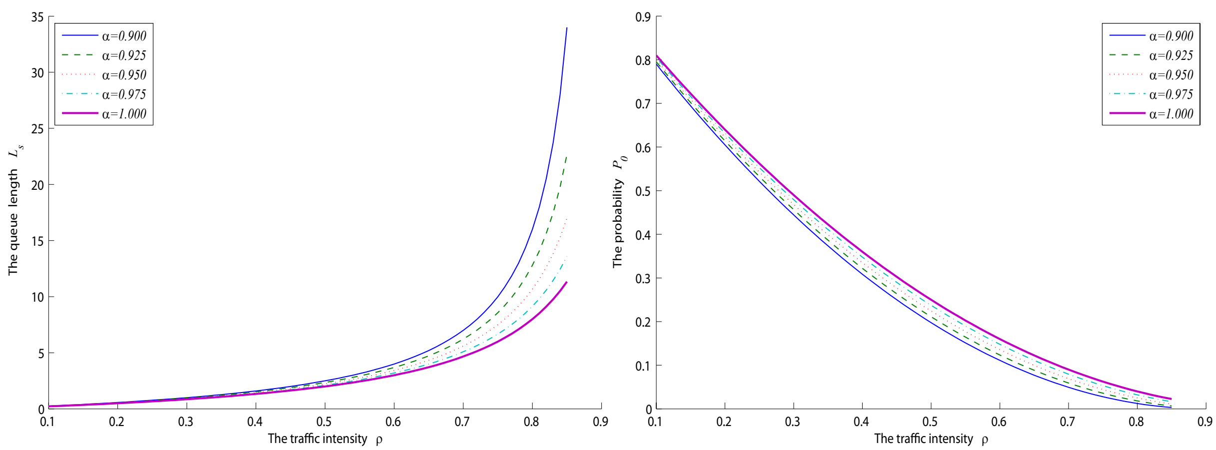

Figure 2.4: Variation of L_s and P_0 according the couple (β_n, ρ): case $\beta_n = \frac{\alpha n}{n+1}$

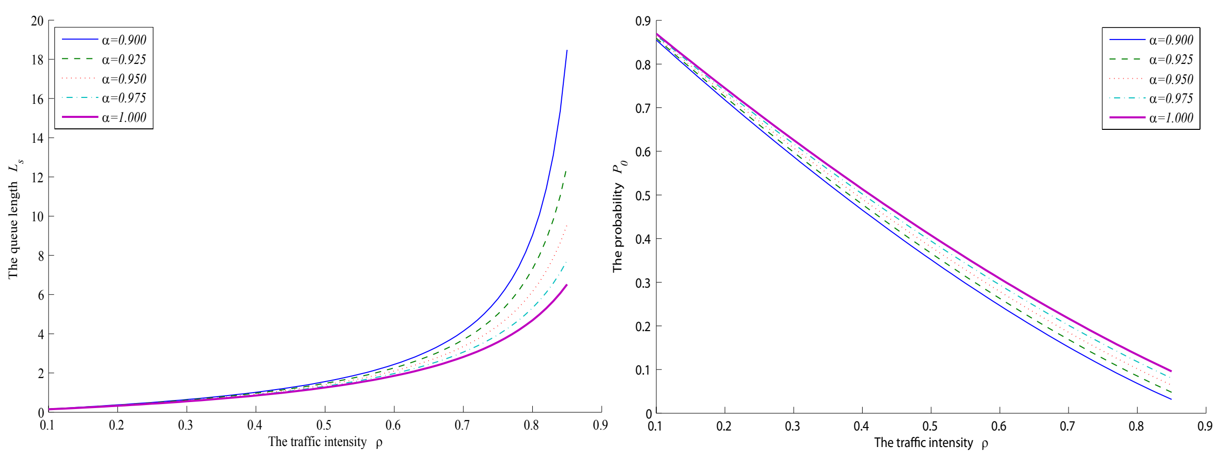

Figure 2.5: Variation of L_s and P_0 according the couple (β_n, ρ): case $\beta_n = \frac{\alpha n(n+1)}{(n+1)^2}$.

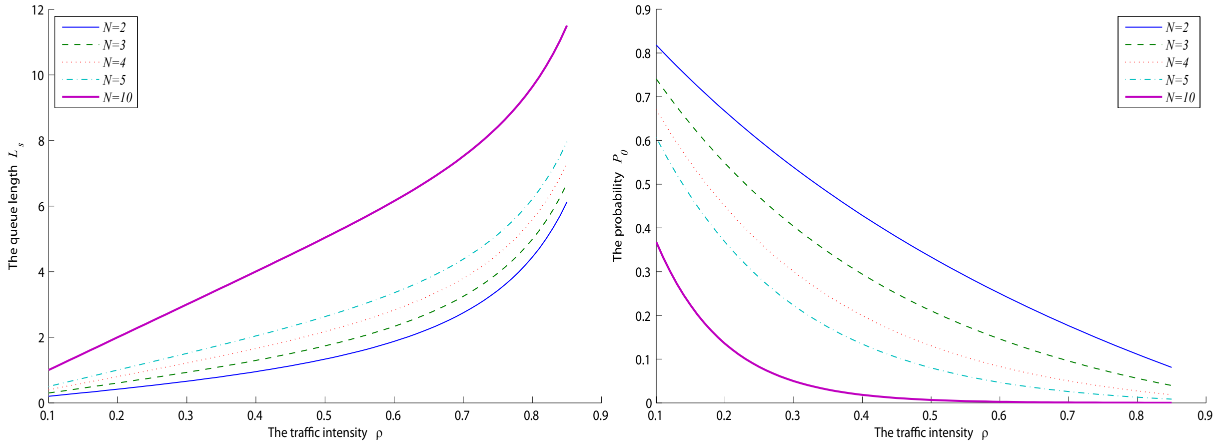

Figure 2.6: Variation of L_s and P_0 according the couple (β_n, ρ): case $\beta_n = \frac{n}{N}$.

as ρ increases and tends to 1, our system tends to behave like a fixed Bernoulli feedback system. This can be justified by :

$$\lim_{\rho \to 1} L_s(\rho) = \infty \Rightarrow \lim_{L_s \to \infty} \beta_{L_s}.$$

- If we have two probabilities $\beta_n^{(1)}$ and $\beta_n^{(2)}$ such that $\beta_n^{(1)} \leq \beta_n^{(2)}$ then for a fixed traffic intensity ρ, we have:

✓ $L_s(\beta_n^{(1)}) \geq L_s(\beta_n^{(2)}) \geq L_s(\beta_n = 1) = \frac{\rho}{1-\rho}$,

✓ $P_0(\beta_n^{(1)}) \leq P_0(\beta_n^{(2)}) \leq P_0(\beta_n = 1) = 1 - \rho$.

Concluding remarks

In this chapter, we considered the analysis and performance evaluation of the $M/M/1$ queue with Bernoulli feedback of the customers whose feedback probability, $\overline{\beta}_n$ $(= 1 - \beta_n)$, is a decreasing function of the number of customers in the system.

The theoretical analysis carried out allowed us to identify the stability condition of such kind of a queue. However, the rest of the (theoretical) analysis brings us to think that, a priori, one cannot obtain a general and explicit form of the characteristics of the system in question. We get the expressions for some useful performance measures of the proposed system when β_n has a particular form.

On the other hand, the numerical study carried out allowed us to note that it is easy to surround the characteristics of such a queue using those of an $M/M/1$ queue whose feedback of the customers obeys a particular and/or simpler probability. Indeed, the first bounds (lower and upper) were obtained for the mean queue length, the probability that the system is empty, the mean sojourn time of a customer in the system, and the mean cumulative time waiting of a customer.

Chapter 3

Analysis of performance measures of a $GI/GI/1$ queue with Bernoulli feedback

Introduction

Our aim, in this chapter, is to evaluate the efficiency of the queuing system with Bernoulli feedback when we move away from the Markov process hypothesis and/or that the feedback probability dependents on the number of customers in the system. Indeed, we propose the analysis of the queuing system $GI/GI/1$ with customer feedback depending on the state of the system.

The literature lacks an exact analysis of a $GI/GI/1$ queuing system with dependent feedback due to its complexity. To achieve our objective and analyze this type of system, we used the discrete event simulation technique.

This chapter is organized as follows: Firstly, we give a description of the model $GI/GI/1$ with Bernoulli feedback. Then, we make a Numerical application that contains: the simulation of the $GI/GI/1$ waiting system, validation of the simulation model, effect of the intensity ρ and the probability β_n, effect of arrivals and service process distributions. Finally, we finish our chapter with a conclusion.

3.1 Model Description

Let's consider an $GI/GI/1$ queuing system with Bernoulli feedback that works under the following assumption:

- The arrival process: The arrival time follows a general distribution and it is denoted by the random variable A with the distribution function $A(x)$, with the mean $E(A) = \frac{1}{\lambda}$.
- The regular service process: The server instantly starts the regular service for the new customers when they arrive at the server in its idle state. The service time follows a general distribution and it is denoted by the random variable S with the distribution function $S(x)$ with mean $E(S) = \frac{1}{\mu}$.
- Feedback rule: After getting unsatisfactory service, the customer may rejoin the system as a Bernoulli feedback customer to receive another regular service with probability $\beta'_n = 1 - \beta_n$, $(n \geq 1)$. Otherwise, it leaves the system definitively, with probability β_n, (where $\beta'_n + \beta_n = 1$, and n is the number of customers in the system). Alternatively, the customer feedback may be modeled by a monotonic decreasing function β'_n.

- All processes (arrivals and service times) are independent. The service discipline is assumed to be first come first served (FCFS) for all customers (ordinary and feedback).

Finally, the considered system can be presented in its reduced form as follows:

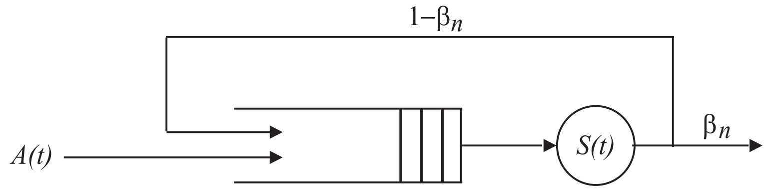

Figure 3.1: $GI/GI/1$ queue with dependent feedback

3.2 Numerical application

We are going to simulate the queuing system of type $GI/GI/1$ with customer feedback considering particular cases of this system, namely: $M/GI/1$, $GI/M/1$, and the system $M/M/1$. More specifically, we will consider the following seven systems (see Table 3.1).

System	$M/M/1$	$GI/M/1$	$M/GI/1$	$GI/GI/1$	$GI/M/1$	$M/GI/1$	$GI/GI/1$
Notation	S_1	S_2	S_3	S_4	S_5	S_6	S_7
Arrival	$Exp(\lambda)$	$Wbl(\lambda_1, \lambda_2)$	$Exp(\lambda)$	$Wbl(\lambda_1, \lambda_2)$	$Er_2(\lambda)$	$Exp(\lambda)$	$Er_2(\lambda)$
Service	$Exp(\mu)$	$Exp(\mu)$	$Wbl(\mu_1, \mu_2)$	$Wbl(\mu_1, \mu_2)$	$Exp(\mu)$	$Er_2(\mu)$	$Er_2(\mu)$

Table 3.1: Particular cases of $GI/GI/1$.

Where Exp denotes exponential law, Wbl denotes Weibull law and Er_2 is the Erlang law of order 2.

Moreover, we assumed in the case of the seven above systems that the customers served can permanently leave the system with a probability β_n, where n is the number of customers in the system at the instant of its service-end, or request possibly another service (feedback) with a probability $1 - \beta_n$. For the form of the latter, we considered the following situations (see Figure 3.2):

- $\beta_n = 1,\ \forall n \in \mathbb{N}^*$, this system is without customer.
- $\beta_n = 0.7,\ \forall n \in \mathbb{N}^*$, this system is with Bernoulli feedback that does not depend on the number of customers in the system.
- $\beta_n = \frac{n}{n+1}$, for $n \in \mathbb{N}^*$, this system is with customer feedback depending on the number of customers in the system.
- $\beta_n = 1 - e^{-n}$, for $n \in \mathbb{N}^*$, this system is with customer feedback exponentially depends on the number of customers in the system.

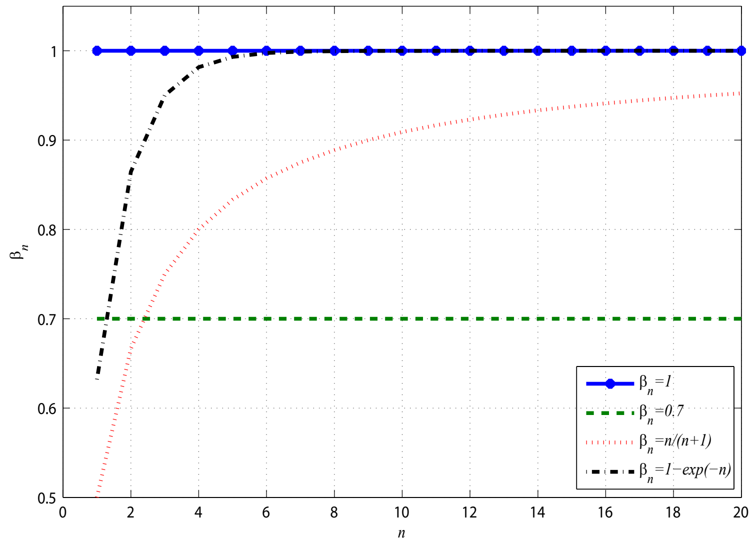

Figure 3.2: Variation of β_n according to the number of customers in the system.

3.2.1 Simulation of $GI/GI/1$ waiting system with feedback

As pointed out in the introduction of this chapter, the exact analytical analysis of a $GI/GI/1$ waiting system with dependent feedback is non-existent in the literature. To this end, in order to analyze this type of system, we will use the discrete event simulation approach.

It is to underline that our aim in this analysis is to provide estimates for the stationary performance measures of the system in question and to study the influence of its starting parameters on this performance. More specifically, the objective is to answer the following questions:

1. What is the impact of the traffic intensity on system characteristics?
2. How does the probability of feedback and its form affect the characteristics of the system?
3. How do the law of inter-arrivals times and that of service times affect the characteristics of the system?

After a careful analysis of the functioning of the system, we have implementer a simulation program, under the MATLAB environment, which can imitate the behavior of the considered system.

3.2.2 Validation of the simulation model

Recall that in the simulation study, the proposed simulation model cannot be used if it has not been behind for validated. In this sense, the objective of the present passage is to check whether the mean value of a characteristic of the system quantified based on the fixed starting parameter, coincides (in the statistical sense) with the exact value. This verification is carried out using the compliance tests (for more details see for example [25, 22]).

In this sense, the theoretical (exact) stationary characteristics of the system S_1, for $\beta_n \in \{1, 0.7\}$ and $\rho \in \{1/15, 3/15, 5/15, ..., 10/15\}$ are presented in Table 3.3. The estimations of the characteristics of this same system returned by the consued simulator on 100 simulations (samples) when the simulation time is fixed to $T = 1000$ of the system are stored in Table 3.3.

	ρ	0.0667	0.2000	0.2667	0.4000	0.4667	0.6000	0.6667
$\beta_n = 1$	L	0.0714	0.2500	0.3636	0.6667	0.8750	1.5000	2.0000
	P_0	0.9333	0.8000	0.7333	0.6000	0.5333	0.4000	0.3333
$\beta_n = 0.7$	L	0.1053	0.4000	0.6154	1.3333	2.0000	6.0000	20.000
	P_0	0.9048	0.7143	0.6190	0.4286	0.3333	0.1429	0.0476

Table 3.2: Performance of the $M/M/1$ system with fixed feedback .

Where, L denotes the mean number of customers in the queue and P_0 is the probability that there are no customers in the system. To validate our model, we will use the Student test that allows us to validate the model. The validation of the simulation model will be done by exploiting the waiting system $M/M/1$ with Bernoulli feedback as a witness and this fact that we have the exact expressions of its performance measures see chapter 2. So, for fixed starting parameters of the system, we can compute with exactitude its performance measures. For this, we will use the Student test to check the equality between the exact analytical results (theoretical) m_{th} and those obtained by the simulation m. That is to say to carry out the following conformity test:

$$H_0 \text{ "}m = m_{th}\text{" } \; Vs \;\; H_1 \text{ "}m \neq m_{th}\text{"}. \tag{3.1}$$

In this case, it suffices to calculate the realization of the statistic T given by:

$$T_{n-1} = \frac{(\overline{X} - m_{th})\sqrt{n-1}}{S}, \tag{3.2}$$

with:

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Thus to take the decision "not to reject H_0", it is sufficient to affirm the following inequality:

$$T_{n-1} \geq t_{(n-1,\frac{\alpha}{2})}, \tag{3.3}$$

where $t_{(n-1,\frac{\alpha}{2})}$ is the Student's law quantile at the threshold and check for a risk threshold α the inequality (3.3), to decide whether or not to reject the H_0 hypothesis. Note that this is equivalent to checking :

$$m_{th} \in [B_{inf}, B_{sup}]\,,$$

where B_{inf} (resp. B_{sup}) is the lower (resp. upper) bound of the confidence interval, at a threshold $1-\alpha$, of the considered characteristic X.

According to the results ranked in the Tables 3.2 and 3.3, we note that all the theoretical values belong to the confidence intervals, designed at a risk threshold of 5%, obtained by the simulation. To this effect, the hypothesis H_0 of the test (3.1) will not be rejected for a decision risk $\alpha = 5\%$. This means that the simulation model that we designed reproduces perfectly the behavior of the $GI/GI/1$ waiting system with feedback, hence the validation of the proposed model. So, this allows us to exploit this simulator to evaluate the performance of the rest of the systems described in Table 3.1.

3.2.3 Effect the traffic intensity ρ and the probability β_n

Our goal in this section is to answer the first and second questions asked earlier in Section 3.2.1. To do this we were interested to the estimators of L, P_0, $\overline{\beta}$ and N.

3.2.3.1 Simulation results

The estimation by $IC_{95\%}$ of the characteristics of this same system returned by the consued simulator on 1000 simulations are presented in Table 3.3 where their mean and variance are presented in Figures 3.3–3.6.

		$\beta_n = 1$		$\beta_n = 0.7$		$\beta_n = 1 - 1/(n+1)$		$\beta_n = 1 - exp(-n)$	
	$\rho = \lambda/\mu$	B_{inf}	B_{sup}	B_{inf}	B_{sup}	B_{inf}	B_{sup}	B_{inf}	B_{sup}
	0.0667	0.0711	0.0716	0.1047	0.1053	0.1425	0.1433	0.1106	0.1113
	0.1333	0.1535	0.1542	0.2351	0.2364	0.3072	0.3087	0.2342	0.2353
	0.2000	0.2494	0.2505	0.4000	0.4023	0.4985	0.5005	0.3725	0.3740
	0.2667	0.3634	0.3650	0.6137	0.6172	0.7247	0.7279	0.5304	0.5325
$L(\beta_n, \rho)$	0.3333	0.4986	0.5007	0.9064	0.9121	0.9966	1.0007	0.7127	0.7155
	0.4000	0.6645	0.6673	1.3315	1.3405	1.3311	1.3364	0.9275	0.9308
	0.4667	0.8746	0.8785	1.9895	2.0063	1.7497	1.7575	1.1856	1.1905
	0.5333	1.1397	1.1456	3.1762	3.2163	2.2778	2.2883	1.5023	1.5096
	0.6000	1.4985	1.5081	5.9326	6.0408	2.9931	3.0101	1.9181	1.9262
	0.6667	1.9967	2.0108	19.6267	20.6489	3.9789	4.0060	2.4747	2.4885
	0.0667	0.9332	0.9336	0.9048	0.9052	0.8708	0.8715	0.8973	0.8978
	0.1333	0.8664	0.8669	0.8089	0.8097	0.7504	0.7513	0.8001	0.8009
	0.2000	0.7997	0.8003	0.7132	0.7142	0.6397	0.6406	0.7093	0.7103
	0.2667	0.7328	0.7335	0.6184	0.6195	0.5373	0.5386	0.6236	0.6245
$P_0(\beta_n, \rho)$	0.3333	0.6664	0.6671	0.5234	0.5248	0.4448	0.4459	0.5430	0.5441
	0.4000	0.5997	0.6006	0.4275	0.4288	0.3594	0.3605	0.4682	0.4693
	0.4667	0.5327	0.5336	0.3333	0.3347	0.2836	0.2846	0.3982	0.3991
	0.5333	0.4661	0.4672	0.2374	0.2391	0.2173	0.2183	0.3332	0.3342
	0.6000	0.3990	0.4003	0.1423	0.1440	0.1594	0.1605	0.2724	0.2733
	0.6667	0.3320	0.3332	0.0471	0.0488	0.1109	0.1118	0.2165	0.2175
	0.0667			0.6997	0.7012	0.5165	0.5180	0.6498	0.6515
	0.1333			0.6993	0.7003	0.5346	0.5357	0.6685	0.6696
	0.2000			0.6992	0.7000	0.5552	0.5559	0.6888	0.6897
	0.2667			0.6996	0.7003	0.5767	0.5775	0.7091	0.7099
$\overline{\beta}(\beta_n, \rho)$	0.3333	1	1	0.6997	0.7004	0.5999	0.6005	0.7305	0.7312
	0.4000			0.6995	0.7002	0.6252	0.6258	0.7529	0.7535
	0.4667			0.6999	0.7005	0.6516	0.6522	0.7761	0.7767
	0.5333			0.6997	0.7003	0.681	0.6821	0.7997	0.8003
	0.6000			0.6999	0.7004	0.7142	0.7149	0.8250	0.8255
	0.6667			0.6998	0.7002	0.749	0.7504	0.8511	0.8517
	0.0667			1.4262	1.4292	1.9304	1.9360	1.5350	1.5389
	0.1333			1.4279	1.4299	1.866	1.8704	1.4934	1.4958
	0.2000			1.4287	1.4303	1.7989	1.8013	1.4499	1.4518
	0.2667			1.4279	1.4294	1.7317	1.7340	1.4086	1.4101
$N(\beta_n, \rho)$	0.3333	1	1	1.4278	1.4292	1.6653	1.6670	1.3676	1.3689
	0.4000			1.4281	1.4295	1.5979	1.5994	1.3271	1.3282
	0.4667			1.4276	1.4287	1.5332	1.5346	1.2874	1.2885
	0.5333			1.4280	1.4291	1.4660	1.4673	1.2496	1.2505
	0.6000			1.4277	1.4288	1.3989	1.4001	1.2113	1.2121
	0.6667			1.4278	1.4288	1.3326	1.3339	1.1741	1.1749

Table 3.3: Simulation results of $M/M/1$ system without and with dependent feedback.

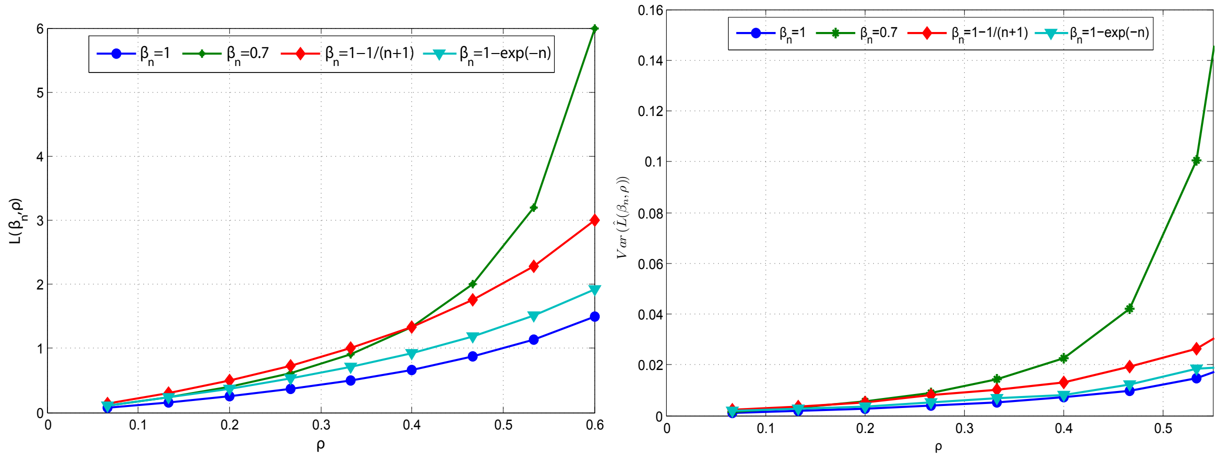

Figure 3.3: Variation of L and $Var(\hat{L})$ according to the couple (β_n, ρ).

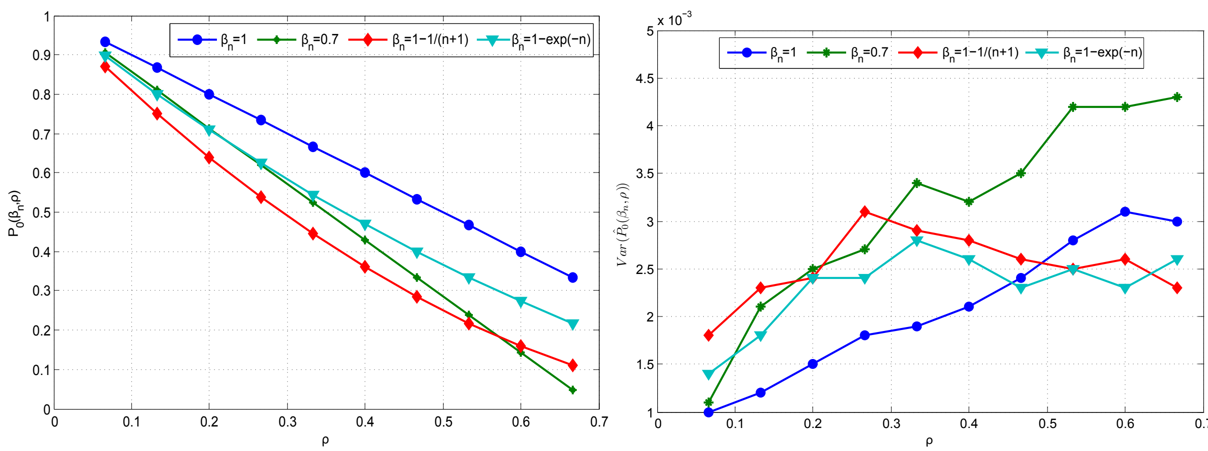

Figure 3.4: Variation of P_0 and $Var(\hat{P}_0)$ according to the couple (β_n, ρ).

3.2.3.2 Discussion and interpretation of the results

The answer to the first and second question asked previously in the Section 3.2.1 will be obtained after the analysis and interpretation of the results stored in Table 3.3 and those presented in figures 3.3–3.6, and we were interested in estimating the characteristics of this system, namely: the average number of customers in the system L, the probability that the system is empty P_0 ($U = 1 - P_0$ is system load), the average probability of no feedback of a customer $E(\beta_n) = \overline{\beta}$, and the average number of services received by the same customer $\overline{N}$.

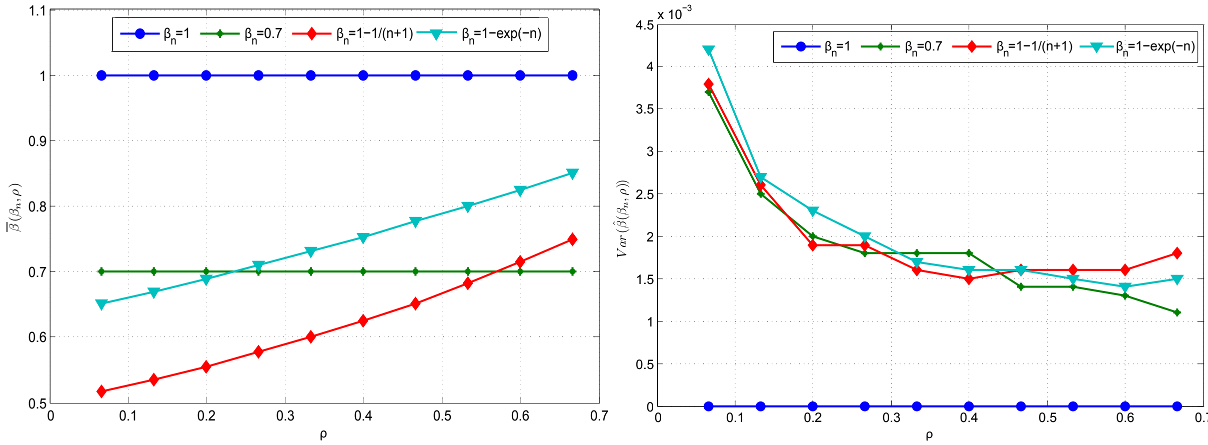

Figure 3.5: Variation of $\overline{\beta}$ and $Var(\hat{\beta})$ according to the couple (β_n, ρ).

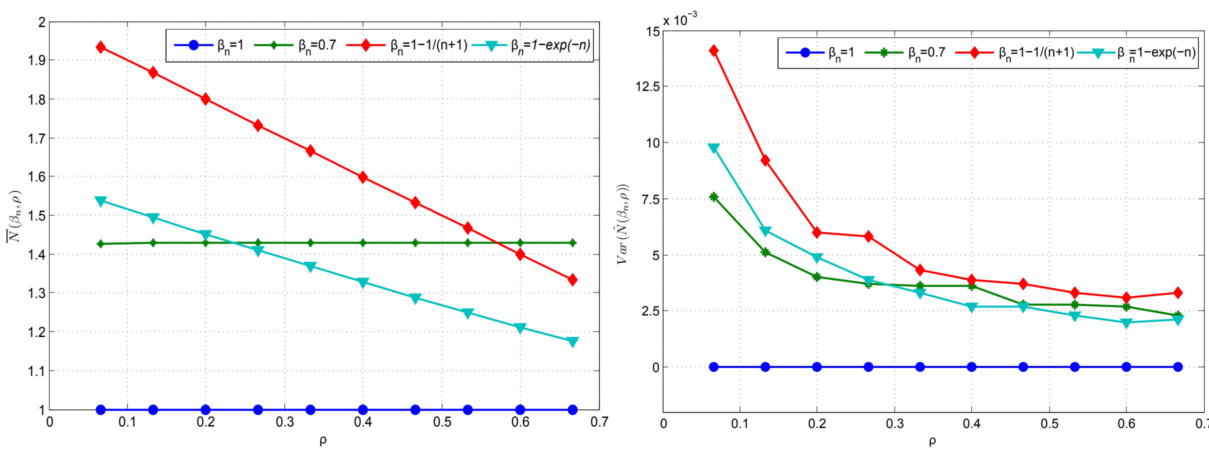

Figure 3.6: Variation of N and $Var(\hat{N})$ according to the couple (β_n, ρ).

The effect of β_n and ρ on L: According to the results presented in the lines associated with $L(\beta_n, \rho)$ in the Table 3.3 and the graphical results presented in Figure 3.3, we notice that the increase in the intensity of the traffic ρ generates a clear increase in the average number of customers in the system L. Moreover, the growth of L as a function of ρ depends also on the form of the feedback probability β_n. Indeed, we note that L is more considerable in the case of $\beta_n = 0.7$, followed by the case where $\beta_n = n/(n+1)$ and $\beta_n = 1 - e^{-n}$, so that L is lower for the case where $\beta_n = 1$ (without feedback). The observation made on the behavior of L as a function of ρ and β_n remains true for $Var(\hat{L})$. Thus, increasing the length of $IC_{95\%}(L) = \left[\overline{L} \pm 0.196\sqrt{Var(\hat{L})}\right]$.

It should be noted that the behavior of L and $Var(\hat{L})$, as a function of ρ coincides with the notions of queuing theory, where it is stated that the process of the number of customers in the infinite capacity system tends to become a non-stationary process when the system load approaches 1 or even non-stationary when the load of the system U is greater than 1 ($L = \infty$ and $Var(L) = \infty$).

The effect of β_n and ρ on P_0: According to the results presented in lines associated with $P_0(\beta_n, \rho)$ of Table 3.3 and those presented in Figure 3.4, we see that the increase of ρ generates the decrease of P_0. This can be explained by the fact that for a large load of the system, the latter tends to be permanently active. That is to say U tends towards 1, consequently P_0 tends towards 0 ($U = 1 - P_0$). While the behavior of P_0 as a function of β_n is exactly the opposite of that of L. The $Var(\hat{P}_0)$ globally tends to be an increasing function according to ρ but not in a regular way. This makes, in some sense the length of the $IC_{95\%}(\hat{P}_0)$ a random variable.

The effect of β_n and ρ on $\overline{\beta}$: According to the results presented in lines associated with $\overline{\beta}(\beta_n, \rho)$ of Table 3.3 and the graphical one that presented in Figure 3.5, we note that as ρ increases $\overline{\beta}$ tends towards 1 in the case $\beta_n = n/(n+1)$ and $\beta_n = 1 - e^{-n}$, this can be explained by the fact that the increase of ρ generates increasing L (see discussion of results $L(\rho, \beta_n)$) therefore served customers tend to leave the system rather than to request an additional service because there is a considerable number of customers in the waiting room at the end of its service (β_n is increasing function according to n). We notice in this case that the $Var(\overline{\beta})$ is practically independent of the form of β_n consequently, the length of the $IC_{95\%}(\overline{\beta})$ is practically a constant when we change β_n.

The effect of β_n and ρ on N: According to the results presented in lines associated with $N(\beta_n, \rho)$ of Table 3.3 and those presented in Figure 3.6, we note on the one hand that for a constant feedback probability ($\beta_n = 1$ and $\beta_n = 0.7$) the number of services received by the same customer is always the same, which is quite natural. Whereas for the case of $\beta_n = n/(n+1)$ and $\beta_n = 1 - e^{-n}$, as ρ increases the system tends to become a system without feedback ($N(\rho)$ tends to 1 when ρ tends to 1). The latter results from the fact

$$\lim_{\rho \to 1} L(\rho) = \infty \Rightarrow \lim_{L \to \infty} \beta_L = 1.$$

On the other hand, the $Var(N)$ gets closer to 0 as ρ increases which can be interrupted by the convergence of N towards a constant when ρ tends towards 1. Thus, the length of the $IC_{95\%}(N)$ decreases as a function of ρ and tends to be zero when the latter tends to 1.

In conclusion, the results obtained in this application indicate that the different characteristics considered $(L, P_0, \overline{\beta}$ and $N)$ depend closely on the value of ρ and on the form of β_n. In addition, they provide us with a finding of great importance which can be summarized as follows:

If for two forms of probabilities feedback $1-\beta_n^{(1)}$ and $1-\beta_n^{(2)}$ of leaving the system, we have $\beta_n^{(1)} \leq \beta_n^{(2)},\ \forall n \geq 1$, then

$$L(\beta_n = 1, \rho) \leq L(\beta_n^{(2)}, \rho) \leq L(\beta_n^{(1)}, \rho).$$

Therefore

$$P_0(\beta_n^{(1)}, \rho) \leq P_0(\beta_n^{(2)}, \rho) \leq P_0(\beta_n = 1, \rho).$$

$$\overline{\beta}(\beta_n^{(1)}, \rho) \leq \overline{\beta}(\beta_n^{(2)}, \rho) \leq \overline{\beta}(\beta_n = 1, \rho)).$$

$$N(\beta_n = 1, \rho) \leq N(\beta_n^{(2)}, \rho) \leq N(\beta_n^{(1)}, \rho).$$

3.2.4 Effect of arrivals and service process distributions

In this section, our objective is to answer the third question fixed previously in section 3.2.1, namely the question concerning the analysis of the effect of the distribution of the arrivals process and that of the service process on the stationary characteristics of the $GI/GI/1$ system with dependent-state feedback.

3.2.4.1 Simulation results

The execution of our simulator for the different systems considered previously $(S_1 - S_7)$ for a simulation duration of 10000 time units provided us with the empirical averages of the four characteristics: L, P_0, $\overline{\beta}$ and N. A sample of the obtained results on the seven systems for some parameters as that the first application (see section 3.2.1) are stored in Table 3.4.

β_n	CaractÃ©ristique	ρ	S_1	S_2	S_3	S_4	S_5	S_6	S_7
$\beta_n = 0.7$	$L(\beta_n, \rho, S_i)$	0.2	0.4014	0.3233	0.3626	0.3015	0.3420	0.3802	0.3275
		0.4	1.3389	0.9522	1.0786	0.7420	1.0631	1.1930	0.9406
		0.6	5.9491	3.9203	4.2682	2.3116	4.5389	5.0988	3.6591
	$P_0(\beta_n, \rho, S_i)$	0.2	0.7136	0.7143	0.7141	0.7144	0.7145	0.7141	0.7139
		0.4	0.4281	0.4286	0.4295	0.4290	0.4286	0.4298	0.4286
		0.6	0.1441	0.1438	0.1438	0.1430	0.1434	0.1430	0.1439
	$\overline{\beta}(\beta_n, \rho, S_i)$	0.2	0.7002	0.6997	0.6995	0.7000	0.6999	0.6997	0.7003
		0.4	0.6999	0.7002	0.7003	0.7003	0.6997	0.7000	0.7002
		0.6	0.7002	0.7003	0.7003	0.7003	0.7001	0.7002	0.7000
	$N(\beta_n, \rho, S_i)$	0.2	1.4281	1.4292	1.4296	1.4287	1.4287	1.4291	1.4281
		0.4	1.4288	1.4281	1.4280	1.4280	1.4292	1.4285	1.4281
		0.6	1.4280	1.4279	1.4279	1.4279	1.4282	1.4280	1.4284
$\beta_n = 1 - 1/(n+1)$	$L(\beta_n, \rho, S_i)$	0.2	0.5004	0.4440	0.4570	0.4171	0.4600	0.4788	0.4419
		0.4	1.3322	1.1328	1.1313	0.9500	1.1917	1.2258	1.0896
		0.6	2.9953	2.4916	2.3949	1.8822	2.6442	2.6796	2.3319
	$P_0(\beta_n, \rho, S_i)$	0.2	0.6402	0.6199	0.6393	0.6168	0.6255	0.6388	0.6244
		0.4	0.3601	0.3212	0.3544	0.3068	0.3350	0.3571	0.3297
		0.6	0.1596	0.1232	0.1485	0.0988	0.1366	0.1549	0.1275
	$\overline{\beta}(\beta_n, \rho, S_i)$	0.2	0.5555	0.5265	0.5546	0.5219	0.5347	0.5544	0.5327
		0.4	0.6246	0.5896	0.6197	0.5769	0.6017	0.6220	0.5969
		0.6	0.7142	0.6842	0.7045	0.6657	0.6943	0.7094	0.6879
	$N(\beta_n, \rho, S_i)$	0.2	1.8001	1.8993	1.8031	1.9163	1.8701	1.8037	1.8772
		0.4	1.6009	1.6961	1.6138	1.7334	1.6620	1.6076	1.6753
		0.6	1.4000	1.4614	1.4193	1.5022	1.4402	1.4096	1.4535

$\beta_n = 1 - exp(-n)$	$L(\beta_n, \rho, S_i)$	0.2	0.3738	0.3370	0.3398	0.3203	0.3459	0.3550	0.3332
		0.4	0.9282	0.7898	0.7699	0.6697	0.8295	0.8469	0.7540
		0.6	1.9248	1.5391	1.4385	1.1167	1.6506	1.6623	1.4112
	$P_0(\beta_n, \rho, S_i)$	0.2	0.7095	0.6950	0.7092	0.6922	0.6994	0.7098	0.6982
		0.4	0.4689	0.4366	0.4659	0.4248	0.4476	0.4670	0.4442
		0.6	0.2726	0.2368	0.2667	0.2147	0.2501	0.2704	0.2434
	$\overline{\beta}(\beta_n, \rho, S_i)$	0.2	0.6888	0.6552	0.6880	0.6496	0.6653	0.6884	0.6625
		0.4	0.7526	0.7097	0.7492	0.6956	0.7242	0.7514	0.7192
		0.6	0.8252	0.7867	0.8182	0.7637	0.8002	0.8221	0.7932
	$N(\beta_n, \rho, S_i)$	0.2	1.4518	1.5264	1.4536	1.5395	1.5032	1.4526	1.5095
		0.4	1.3287	1.4090	1.3347	1.4377	1.3808	1.3309	1.3904
		0.6	1.2117	1.2711	1.2222	1.3094	1.2497	1.2164	1.2607

Table 3.4: Simulation results of the $GI/GI/1$ system with dependent feedback.

3.2.4.2 Analysis and interpretation of results

From the results stored in Table 3.4, we can see that the characteristics of the seven systems $S_1 - S_7$ are closely dependent on the starting parameters defining this system, namely: the intensity of the traffic ρ, the distribution of inter-arrival times, the distribution of service times as well as the form of the probability β_n. Indeed, we note that

– The form of β_n influences the values of the characteristics considered, and this in the same way as in the first application where for medium and high values of the traffic intensity we have:

$$L(\beta_n = 1, \rho) \leq L(\beta_n = 1 - e^{-n}, \rho) \leq L(\beta_n = n/(n+1), \rho) \leq L(\beta_n = 0.7, \rho).$$

– The distribution of the inter-arrival times and those of the service times influence the four characteristics considered in the study. Moreover, their influences are more apparent as the system load increases. On the other hand, for low system load, our results lead us to believe that the characteristics of the 7 systems are practically the same. Therefore, when the load of the system is weak, the evaluation (approximation) of the characteristics of the $GI/GI/1$ system with feedback (whose exact analysis is non-exist), via the $M/M/1$ system is now justified.

– Overall, when the system load increases, the $GI/GI/1$ system behaves like a $GI/GI/1$ system without feedback, this is the fact that:

$$\lim_{\rho \to 1} L(\beta_n, \rho) = \infty \Rightarrow \begin{cases} N \text{ and } \overline{B} \text{ tends to } 1, \\ Var\left(\hat{N}\right) \text{ and } Var\left(\hat{\beta}\right) \text{ tends to } 0. \end{cases}$$

Conclusion

In this chapter, the analysis of the model retained by the discrete event simulation approach gave us an overview of the influence of the intensity of the traffic, the distributions of the processes of the inter-arrivals and the service, as well as the probability of feedback and its form on some performance measures of the system considered in the study.

In this work, we analyze the waiting systems: $M/M/1$, $M/GI/1$, $GI/M/1$ with Bernoulli feedback. We were interested (L, P_0, $\overline{\beta}$, N) when the variation in the estimates of the latter characteristics was also considered.

After the development of a discrete event simulation model for the system $GI/GI/1$ with dependent feedback, its validation was made concerning the Markovian systems $M/M/1$ without feedback and $M/M/1$ with fixed Bernoulli feedback, for which theoretical results exist. The simulator was used to measure the different performances of the $GI/GI/1$ system for different load rates varying between 1/15 and 10/15 and for inter-arrivals and services which follow a Weibull law of parameters (μ_1, μ_2) or a second order Erlang law with parameter μ.

In this study, the analysis of the model retained by the simulation approach gave us the possibility of carrying out a comparison between the results obtained by the latter and those of the theoretical analysis when the latter exists. In addition, the results obtained allowed us to identify how the form of the feedback probability, the traffic intensity, as well as the laws of the inter-arrival and service processes influence the characteristics of the system.

General conclusion

Queuing systems with bernoulli feedback are encountered in several areas. The study of such systems is certainly very important for practical applications, because the feedback and mechanism of impatient customers in the system has a significant influence on the main performance measures of the system.

In this thesis, for a good understanding of the different notions related to the problem addressed, a synthesis on the queueing theory was presented. Also, some systems of classic queues and basic concepts of queuing system and their characteristics, methods to analyze markovian and non markovian models are exposed.

In addition, we studied an $M/M/1$ queue model with bernoulli feedback depending on the number of customers in the system. Using the stationary state markov chain method, we were able to obtain analytical results in special cases and some performance measures of the model in question of this latest cases, such as average number of customers in the system, ...etc.

The exact analysis by markov chain for this type of $M/M/1$ system is difficult to study, and the digital application shows we can identify the characteristics of this $M/M/1$ system. In the second numerical application we tried to find the same results as found in $M/M/1$. Despite the effectiveness of such systems in faithfully describing many real situations, the fact remains that a large class of models has not been studied in the literature. To resolve the problem, we employ discrete event simulation and attempt to constrain its characteristics using those of simple queueing systems, which have characteristics that exist in the literature.

Our focus in the second chapter is on analyzing the model that was retained using the discrete event simulation approach. An overview of how traffic intensity, inter-arrival processes, and service impact each other, as well as the probability of feedback and its form on some performance measures of the system considered in the study. We analyze the waiting systems: $M/M/1$, $M/GI/1$, $GI/M/1$ with Bernoulli feedback. Our aim was to investigate the variation in the estimations of the latter characteristics when taking into account (L, P_0, $\overline{\beta}$, N).

The simulator was used to measure the different performances of the $GI/GI/1$ system for different load rates varying between 1/15 and 10/15 and for inter-arrivals and services which follow a Weibull law of parameters (μ_1, μ_2) or a second order Erlang law with parameter μ.

By analyzing the model retained by the simulation approach, we gained valuable information in this study. The possibility of comparing the results obtained by the latter and those of the theoretical analysis exists. Moreover, we have been able to determine how the feedback probability distribution, traffic intensity, inter-arrival and service process laws impact the system's characteristics through our obtained results.

In terms of continuity of this work, several research perspectives can be envisaged, including:

- Establish an analytical bounds for performance measures of a queuing system with dependent feedback.
- Complete analysis of the $M/M_2/1$ system with feedback (modelling, resolution and numerical illustration).
- Complete analysis of the $M/M_2/1$ system with feedback when customers are impatient.
- Through an economic analysis, the parameters, the characteristics and the optimal functioning policy of a system modeled by an $M/M_2/1$ waiting model with feedback.

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Abstract

The main objective of this thesis is to analyze differences queuing systems with Bernoulli feedback when the probability of this latest phenomena depends on the number of customers in the system. At first, we have considered an $M/M/1$ queue with Bernoulli feedback, where we analyzed a particular cases of them. Secondly, we consider the parametric estimation of the characteristics of the waiting model $M/M/1/N$ queue with Bernoulli feedback. A simulation study was carried out with the objective of analyzing the effect of estimating the starting parameters of the waiting system in question on the statistical properties of its performance measurement estimators obtained via the plug-in method. Where a through discrete event simulation technique, we have analyzed the $GI/GI/1$ queue system with Bernoulli feedback. Numerical and graphic analyzes are carried out to show the effect of the distribution of inter-arrivals times, the distribution of service times, the probability of Feedback, and the traffic intensity on the stationary characteristics of the system in question and allowed us to draw important conclusions on the behavior of these characteristics.
Key words : queuing system; Bernoulli feedback; performance measures; simulation.

Résumé

L'objectif principal de cette thèse est d'analyser différents systèmes de files d'attente avec Bernoulli feedback lorsque la probabilité de ce dernier phénomène dépend du nombre de clients dans le système. Dans un premier temps, nous considérons une file d'attente $M/M/1$ avec Bernoulli feedback, nous analysons un cas particulier d'entre eux. Dans un deuxième temps, nous considérons l'estimation paramétrique des caractéristiques de la file d'attente $M/M/1/N$ avec Bernoulli feedback. Une étude de simulation a été réalisée dans l'objectif d'analyser l'effet de l'estimation des paramètres de départ du système d'attente en question sur les propriétés statistiques de ses estimateurs de mesure de performance obtenus via la méthode plug-in. Ensuite, nous analysons le système de file d'attente $GI/GI/1$ avec les commentaires de Bernoulli. Nous avons eu recours à la technique de simulation d'événements discrets. Des analyses numériques et graphiques sont effectuées pour montrer l'effet de la distribution des temps d'inter-arrivées, la distribution des temps de service, la probabilité de Feedback, et l'intensité du trafic sur les caractéristiques stationnaires du système en question et nous ont permis de tirer des conclusions importantes sur le comportement de ces caractéristiques.
Mots clés: système de file d'attente; Bernoulli feedback; mesures du performances; simulation.